# Mathematics for Computer Applications

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September 15, 2018

# SECTION 1

## MODULE 3 : Set, Relation and Function

**Definition 1.0.1**

A Set is any well defined collection of objects called the elements or members of the set.

**Examples**

(a) The rivers in India
(b) The vowels of alphabets

**Notation**

Sets are denoted by capital letters

$$A, B, C, \cdots$$

Elements of sets are denoted by lower case letters

$$a, b, c, \cdots$$

(i) We write $a \in A$ to denote that $a$ is an element of the set $A$.

(ii) The notation $a \notin A$ denotes that $a$ is not an element of the set $A$.

There are several ways to describe a set.

One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces.

For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

Example

The set $O$ of odd positive integers less than 10 can be expressed by

$$O = \{1, 3, 5, 7, 9\}$$

.

Another way to describe a set is to use **set builder** notation.

We characterize all those elements in the set by stating the property or properties they must have to be members.

**Example**

(i) The set $O$ of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

(ii) The set $B = \{1, 4, 9, 16, 25\}$ can be written as

$$B = \{x \mid x = n^2 \text{ where } n \text{ is a natural number less than or equal to } 5\}$$

**Definition 1.0.2**

(i) A set with finite number of elements is called Finite set.

(ii) An infinite set is a set with infinite number of elements.

(iii) A set which contains no elements at all is called Null set or Empty set or Void set.

(iv) A set which has only one element is called a Singleton set.

**Example**

1. The set of months in a year.
2. The set of students in a class.
3. The set of all integers.
4. $D = \{x \mid x^2 + 4 = 0,\ x \text{ is real}\}$
5. $T = \{a\}$

**Definition 1.0.3**

The set $A$ is a subset of $B$ if and only if every element of $A$ is also an element of $B$. We use the notation

$$A \subseteq B$$

to indicate that $A$ is a subset of the set $B$.

For every set $A$,

$(i)$ $\emptyset \subseteq A$ and

$(ii)$ $A \subseteq A$.

**Showing Two Sets are Equal**

To show that two sets $A$ and $B$ are equal, show that $A \subseteq B$ and $B \subseteq A$.

## Operations on Set

Let $U$ be the Universal Set

(1) Union

$$A \cup B = \{x \ : \ x \in A \ or \ x \in B\}$$

(2) Intersection

$$A \cap B = \{x \ : \ x \in A \ and \ x \in B\}$$

(3) Complements

$$A^c \ \ or \ \ A' = \{x \ : \ x \in U \ and \ x \notin A\}$$

(4) Relative complement

$$A - B = \{x \ : \ x \in U \ and \ x \notin B\}$$

(5) Symmetric Difference

$$A \Delta B = (A \cup B) - (A \cap B) = \{x \ : \ x \text{ belongs to exactly one of } \ A \ and \ B\}$$

## Algebra of Sets

(i) Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(ii) Complement Laws

$$A \cup A' = U \qquad A \cap A' = \emptyset$$

$$U' = \emptyset \qquad \emptyset' = U$$

(iii) DeMorgan's Laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

## Definition 1.0.4

Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$, denoted by

$$A \times B,$$

is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \ and \ b \in B\}.$$

## Properties of Cartesian product

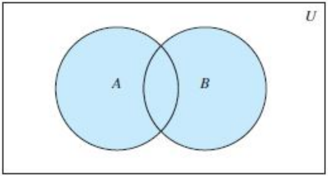
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$

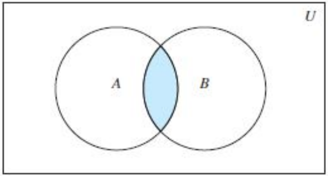
$$(A - B) \times C = (A \times C) - (B \times C)$$

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$

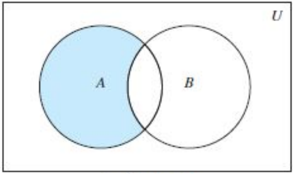
## Venn Diagrams



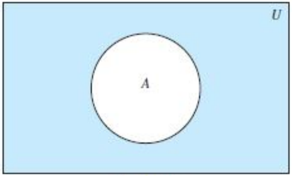

$A \cup B$ is shaded.



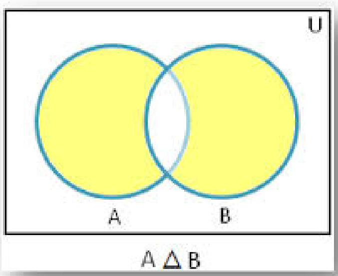

$A \cap B$ is shaded.




$A - B$ is shaded.




$\overline{A}$ is shaded.


U
A
B
A △ B

## Definition 1.0.5

Let $A$ be a set. If there are exactly $n$ distinct elements in $A$ where $n$ is a nonnegative integer, we say that $A$ is a finite set and that n is the cardinality of $A$.

The cardinality of $A$ is denoted by $|A|$ or $n(A)$.

## Properties

$$|(A \cup B)| = |A| + |B| - |A \cap B|$$

$$|(A - B)| = |A| - |A \cap B|$$

$$|(B - A)| = |B| - |A \cap B|$$

$$|(A \cup B \cup C)| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

**Definition 1.0.6**

If $A$ is any set, then the collection of all subsets of $A$ is called Power set of $A$.

The power set of $A$ is denoted by $P(A)$.

Notation $P(A) = \{T \ : \ T \in A\}$.

Note $\emptyset$ and $A$ are both elements of $P(A)$.

If $A$ is a finite set with $n$ elements, then the power set of $A$ contains $2^n$ elements.

$$|P(A)| = 2^n$$

## Relation

The word relation is used to indicate a realtionship between two objects

- A business and its telephone number
- An employee and his or her salary
- A positive integer and one that it divides
- A program and a variable it uses
- A computer language and a valid statement in this language

## Definition 1.0.7

Let $A$ and $B$ be sets. A relation from $A$ to $B$ is a subset of $A \times B$.

## Notation

We use the notation $aRb$ to denote that $(a, b) \in R$ and

$a \not R b$ to denote that $(a, b) \notin R$.

Moreover, when $(a, b)$ belongs to $R$, $a$ is said to be related to $b$ by $R$.

## Example

Let

$$A = \{0, 1, 2\} \text{ and } B = \{a, b\}.$$

Then

$$\{(0, a), (0, b), (1, a), (2, b)\}$$

is a relation from $A$ to $B$.

Here $0Ra$, but that $1 \not R b$.

## Example

- Let $A$ be the set $\{1,2,3,4\}$. Which ordered pairs are in the relation

$$R = \{(a,b) \mid a \text{ divides } b\} \ ?$$

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

- Consider these relations on the set of integers:
  - $R_1 = \{(a,b) \mid a \leq b\}$,
  - $R_2 = \{(a,b) \mid a > b\}$,
  - $R_3 = \{(a,b) \mid a = b \ or \ a = -b\}$,
  - $R_4 = \{(a,b) \mid a = b\}$,
  - $R_5 = \{(a,b) \mid a = b+1\}$,
  - $R_6 = \{(a,b) \mid a+b \leq 3\}$.

Which of these relations contain each of the pairs

$$(1,1),(1,2),(2,1),(1,-1), \text{ and } (2,-2)?$$

**Definition 1.0.8**

1. The set $\{a \in A \,:\, (a,b) \in R \text{ for some } b \in B\}$ is called domain of $R$.

   It is denoted by $Dom(R)$
2. The set $\{b \in B \,:\, (a,b) \in R \text{ for some } a \in A\}$ is called range of $R$.

   It is denoted by $Ran(R)$

Let $|A| = m,\ |B| = n$, then

$$\text{Total Numer of Distinct relation from a set } A \text{ to a set } B = 2^{mn}$$

**Definition 1.0.9**

Let $R$ be any relation from a set to a set $B$. The inverse of $R$ is denoted by $R^{-1}$ is relation from $B$ to $A$ which consists of those ordered pairs which when reversed belongs to $R$.

## Example

Let $A = \{2, 3, 5\}$ and $B = \{6, 8, 10\}$ and define binary relation $R$ from $A$ to $B$ as follows :

$$(x, y) \in R \iff x \mid y.$$

Write $R, R^{-1}, Dom(R)$ and $Ran(R)$

Properties of Relation

**Definition 1.0.10**

A relation $R$ on a set $A$ is called

- reflexive if $(a, a) \in R$ for every element $a \in A$.
- symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
- antisymmetric if $a = b$ whenever for all $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$,
- transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for all $a, b, c \in A$.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of the above relations are
(i) Reflexive
(ii) Symmetric
(iii) antisymmetric
(iv) transitive ?

## Example

Let $A$ be the set of natural numbers and $xRy$ if and only if $|x-y| \leq 7$. Is $R$ reflexive, symmetric and transitive ?

**Solution** : For any $x \in N$, $|x-x| = 0 \leq 7$.
So, $xRx$ for all $x \in N$.

Hence $R$ is reflexive.

Now,

$$\begin{aligned} xRy &\implies |x-y| \leq 7 \\ &\implies |-(y-x)| \leq 7 \\ &\implies |(y-x)| \leq 7 \\ &\implies yRx. \end{aligned}$$

Hence $R$ is symmetric.

The relation $R$ is not transitive because

$$2R5 \text{ as } |2-5| \leq 7$$

and

$$5R11 \text{ as } |5-11| \leq 7$$

but

$$2 \not R\, 11 \text{ since } |2-11| \nleq 7$$

### Definition 1.0.11

A relation $R$ on a set $A$ is called an Equivalence relation if it is reflexive, symmetric and transitive.

### Example

Let $R$ be the relation on the set of integers such that $aRb$ if and only if $a - b$ is a multiple 6. Show that $R$ is an equivalence relation.

**Solution** : Let $x \in Z$.
Then $x - x = 0$ and 0 is divisible by 6. Therefore $xRx$ for all $x \in Z$.

Hence $R$ is reflexive.

Now,

$$\begin{aligned} xRy &\implies (x-y) \text{ is divisible by } 6 \\ &\implies -(x-y) \text{ is divisible by } 6 \\ &\implies y-x \text{ is divisible by } 6 \\ &\implies yRx. \end{aligned}$$

Hence $R$ is symmetric.

Let $xRy$ and $yRz$. Then $(x-y)$ is divisible by 6 and $(y-z)$ is divisible by 6.

$$\begin{aligned} &\implies (x-y)+(y-z) \text{ is divisible by } 6 \\ &\implies (x-z) \text{ is divisible by } 6 \\ &\implies xRz \end{aligned}$$

Hence $R$ is transitive.

Thus $R$ is an equivalence relation.

**Example**

- Let $R$ be the relation on the set of real numbers such that $aRb$ if and only if $a - b$ is an integer. Is R an equivalence relation?
- Does the divides relation on the set of positive integers an equivalence relation ?
- Let $R$ be the relation on the set of integers such that $aRb$ if and only if $a - b$ is a multiple of 3. Is $R$ an equivalence relation ?

## Definition 1.0.12

A relation $R$ on a set $A$ is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive.

A set $A$ together with a partial ordering $R$ is called a partially ordered set, or poset, and is denoted by $(A, R)$.

Members of $A$ are called elements of the poset.

## Example

(1) Show that the set of all positive integers $Z^+$ under divisibility relation form poset.

**Solution** : Since $a \mid a$ for all $a \in Z^+$, $\mid$ is reflexive.

$a \mid b$ and $b \mid a$ impies $a = b$, $\mid$ is antisymmetric.

$a \mid b$ and $b \mid c$ impies $a \mid c$, $\mid$ is transitive.

Hence $\mid$ is a partial order on $Z^+$ and $(Z^+, \mid)$ is a poset.

(2) Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set $A$.

**Solution** : Because $A \subseteq A$ whenever $A$ is a subset of $S$, $\subseteq$ is reflexive.

It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$.

Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$.

Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset.

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

**a)** $\{(0, 0), (2, 2), (3, 3)\}$

**b)** $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

**c)** $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

**d)** $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$

**e)** $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

## Representing Relations Using Matrices

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let $R$ be the relation from $A$ to $B$ containing $(a, b)$ if $a \in A, b \in B$, and $a > b$. What is the matrix representing $R$ if $a_1 = 1, a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

**Solution** : Here

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

the matrix for $R$ is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The $1s$ in $M_R$ show that the pairs $(2, 1), (3, 1)$, and $(3, 2)$ belong to $R$. The $0s$ show that no other pairs belong to $R$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation $R$ represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

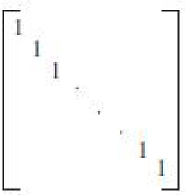
**Solution** :

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

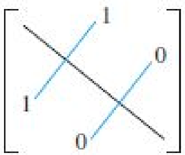
# Relation and their Properties

## Zero-One Matrix for Reflexive, Symmetric and Antisymmetric Relation

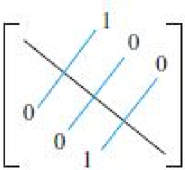
Reflexive




Symmetric




AntiSymmetric



**Example**

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}?$$

Is $R$ reflexive, symmetric, and/or antisymmetric?

## Union, Intersection, Inverse, Complement and Composition

The matrix representing the union of the relations $R_1$ and $R_2$ has a 1 in the positions where either $M_{R_1}$ or $M_{R_2}$ has a 1. The matrix representing the intersection of the relations has a 1 in the positions where both $M_{R_1}$ and $M_{R_2}$ have a 1. Thus, the matrices representing the union and intersection of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \text{ and } M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

$$M_{R_1}^{-1} = M_{R_1}^{T}$$

$$M_{R_1}^{'} = \text{ Change 0 into 1 and 1 into 0 in matrix of } M_{R_1}$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ \ R_1 \cap R_2,\ \ R_2^{-1},\ \ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

What are the matrices representing $R_1 \cup R_2, \;\; R_1 \cap R_2, \;\; R_2^{-1}, \;\; R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ?

What are the matrices representing $R_1 \cup R_2, \; R_1 \cap R_2, \; R_2^{-1}, \; R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\qquad$ $M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ?

What are the matrices representing $R_1 \cup R_2$, $R_1 \cap R_2$, $R_2^{-1}$, $R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices
$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ?
What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?
**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_2^{-1}} =$$

### Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_2^{-1}} = M_{R_2}^T =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_2^{-1}} = M_{R_2}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_2^{-1}} = M_{R_2}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$M_{R_1'} =$$

## Example

Suppose that the relations $R_1$ and $R_2$ on a set $A$ are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

What are the matrices representing $R_1 \cup R_2,\ R_1 \cap R_2,\ R_2^{-1},\ R_1^{'}$?

**Solution** : The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_2^{-1}} = M_{R_2}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$M_{R_1^{'}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

## Composition of Relations

The zero-one matrices for $SoR$,

$$M_{SoR} = M_R \odot M_S$$

$$M_{R^2} = M_{RoR} = M_R \odot M_R$$

## Example

Find the matrix representing the relations $SoR$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}?$$

**Solution** : The matrix for $SoR$ is

$M_{SoR}$

$= M_R \odot M_S$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (0 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 0) & (0 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

## Digraph and Hasse Diagram

Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow.

We use such pictorial representations when we think of relations on a finite set as directed graphs, or digraphs.

### Definition 1.0.13

A directed graph, or digraph, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs).

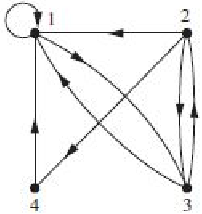
The vertex $a$ is called the initial vertex of the edge $(a, b)$, and the vertex $b$ is called the terminal vertex of this edge.

## Example

(1) The directed graph of the relation

$$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$$

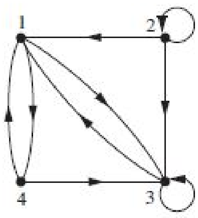
on the set $\{1,2,3,4\}$



(2) The directed graph of the relation

$$R = \{(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,3),(4,1),(4,3)\}$$

on the set $\{1,2,3,4\}$

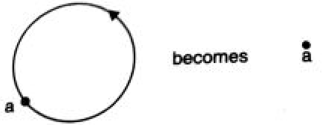


Now we represent partial order on a set $A$ by means of diagram which is known as Hasse Diagram.

## Constructing a Hasse Diagram

We can represent a finite poset using this procedure: Start with the directed graph for this relation.

- As partial ordering is reflexive, a loop $(a, a)$ is present at every vertex $a$. Remove these loops.



- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity.

  That is, remove all edges $(x, y)$ for which there is an element $z \in A$ such that $xRz$ and $zRx$.

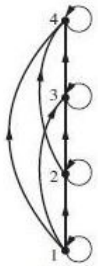

- Finally, arrange each edge so that its initial vertex is below its terminal vertex. Remove all the arrows on the directed edges, because all edges point upward toward their terminal vertex.

## Example

Draw the Hasse diagram for the set $A = \{1, 2, 3, 4\}$ with partial order $\leq$, starting from digraph.

**Solution** : Here $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

First we draw digraph of the given relation



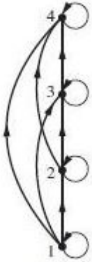

(a)

### Example

Draw the Hasse diagram for the set $A = \{1, 2, 3, 4\}$ with partial order $\leq$, starting from digraph.

**Solution** : Here $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

First we draw digraph of the given relation




(a)

Now we remove loops from the digraph

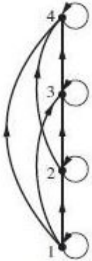

(b)

## Example

Draw the Hasse diagram for the set $A = \{1, 2, 3, 4\}$ with partial order $\leq$, starting from digraph.

**Solution** : Here $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

First we draw digraph of the given relation




(a)

Now we remove loops from the digraph


(b)

Finally we remove transitive edges


(c)

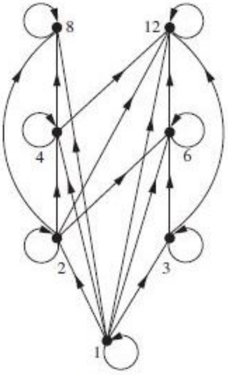
## Example

Draw the Hasse diagram for the set $A = \{1, 2, 3, 4, 6, 8, 12\}$ with partial order $|$, starting from digraph.

**Solution** : Here

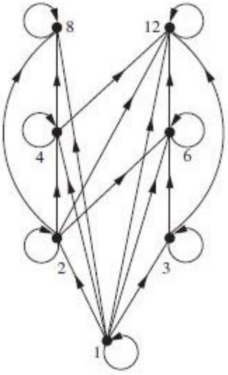
$$R = \left\{ \begin{array}{c} (1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8), \\ (2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12) \end{array} \right\}$$

First we draw digraph of the given relation



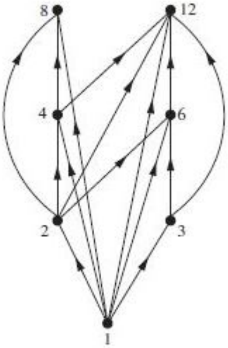

(a)

First we draw digraph of the given relation



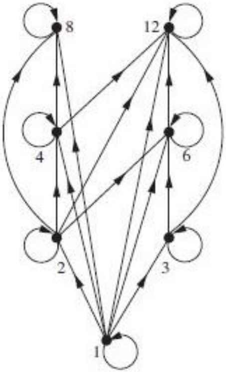

(a)

Now we remove loops from the digraph



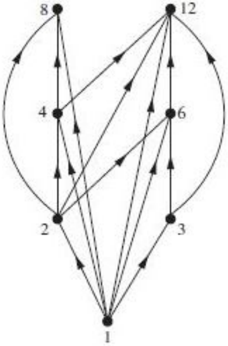

(b)

First we draw digraph of the given relation



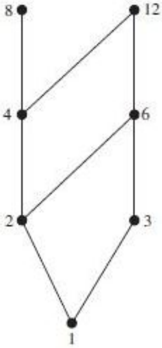

(a)

Now we remove loops from the digraph




(b)

Finally we remove transitive edges




(c)

**Definition 1.0.14**

- An element of a poset is called maximal if it is not less than any element of the poset.

  That is, $a$ is maximal in the poset $(A, \preceq)$ if there is no $b \in A$ such that $a \preceq b$.

- An element of a poset is called minimal if it is not greater than any element of the poset.

  That is, $a$ is minimal if there is no element $b \in A$ such that $b \preceq a$.

  Maximal and minimal elements are easy to spot using a Hasse diagram. They are the ”top” and ”bottom” elements in the diagram.

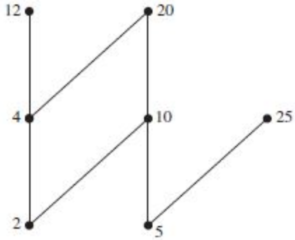
## Example

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?

**Solution :**

$$R = \left\{ \begin{array}{c} (2,2),(2,4),(2,10),(2,12),(2,20),(4,4),(4,12),(4,20),(5,5),(5,10), \\ (5,20),(5,25),(10,10),(10,20),(12,12),(20,20),(20,25) \end{array} \right\}$$

First we draw Hasse diagram of the given relation



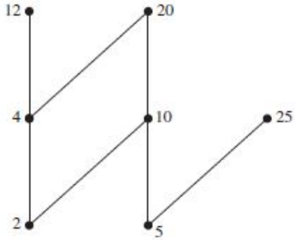
## Example

Which elements of the poset $(\{2,4,5,10,12,20,25\}, \mid)$ are maximal, and which are minimal?

**Solution :**

$$R = \left\{ \begin{array}{c} (2,2),(2,4),(2,10),(2,12),(2,20),(4,4),(4,12),(4,20),(5,5),(5,10), \\ (5,20),(5,25),(10,10),(10,20),(12,12),(20,20),(20,25) \end{array} \right\}$$

First we draw Hasse diagram of the given relation




The maximal elements are

12, 20, and 25

and the minimal elements are

2 and 5.

As this example shows, a poset can have more than one maximal element and more than one minimal element.

**Definition 1.0.15**

- An element $a$ is the greatest element of the poset $(A, \preceq)$ if $b \preceq a$ for all $b \in A$.

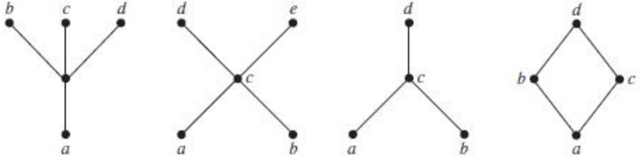
  The greatest element is unique when it exists.

- An element $a$ is the least element of $(A, \preceq)$ if $a \preceq b$ for all $b \in A$.

  The least element is unique when it exists

## Example

Determine whether the posets represented by each of the Hasse diagrams below have a maximal element, minimal element, greatest element, least element

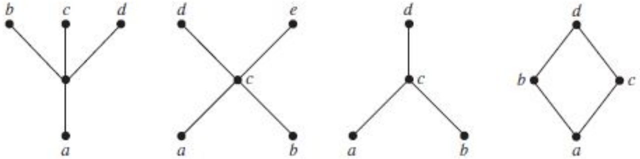



maximal elements

$$b, c, d$$
$$d, e$$
$$d$$
$$d$$

## Example

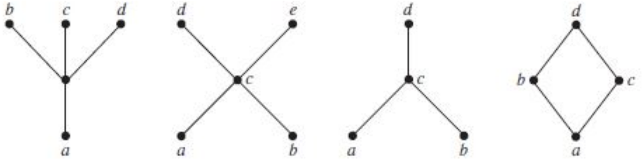
Determine whether the posets represented by each of the Hasse diagrams below have a maximal element, minimal element, greatest element, least element




| maximal elements | minimal elements |
|---|---|
| $b, c, d$ | $a$ |
| $d, e$ | $a, b$ |
| $d$ | $a, b$ |
| $d$ | $a$ |

## Example

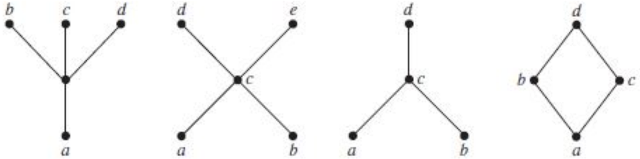
Determine whether the posets represented by each of the Hasse diagrams below have a maximal element, minimal element, greatest element, least element




| maximal elements | minimal elements | greatest elements |
|---|---|---|
| $b, c, d$ | $a$ | none |
| $d, e$ | $a, b$ | none |
| $d$ | $a, b$ | $d$ |
| $d$ | $a$ | $d$ |

## Example

Determine whether the posets represented by each of the Hasse diagrams below have a maximal element, minimal element, greatest element, least element




| maximal elements | minimal elements | greatest elements | least elements |
|---|---|---|---|
| $b, c, d$ | $a$ | none | $a$ |
| $d, e$ | $a, b$ | none | none |
| $d$ | $a, b$ | $d$ | none |
| $d$ | $a$ | $d$ | $a$ |

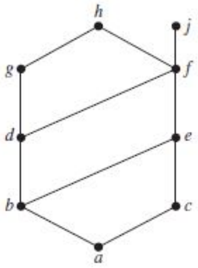
## Definition 1.0.16

Let $A$ be subset of a poset $(S, \preceq)$.

- If $u$ is an element of $S$ such that $a \preceq u$ for all elements $a \in A$, then $u$ is called an upper bound of $A$.
- An element $l$ less than or equal to all the elements in $A$. If $l$ is an element of $S$ such that $l \preceq a$ for all elements $a \in A$, then $l$ is called a lower bound of $A$.
- The element $x$ is called the least upper bound of the subset $A$ if $x$ is an upper bound that is less than every other upper bound of $A$.
- The element $y$ is called the greatest lower bound of $A$ if $y$ is a lower bound that is greater than every other lower bound of A.

## Example

Find the lower bound, upper bounds, least upper bounds and greatest lower bound of the subsets $\{a, b, c\}, \{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram below




**Solution** : For $\{a, b, c\}$

Upper bound = $\{e, f, j, h\}$

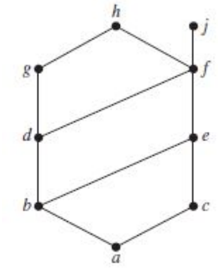
Lower bound = $\{a\}$

Least Upper bound = $\{e\}$

Greatest Lower bound = $\{a\}$

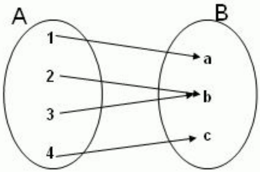
## Example

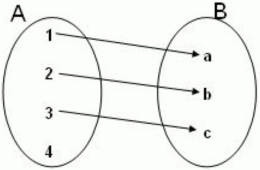
Find the lower bound, upper bounds, least upper bounds and greatest lower bound of the subsets $\{a, b, c\}, \{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram below

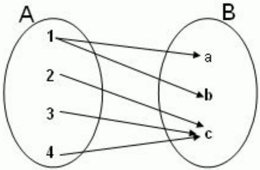



**Solution** : For $\{a, b, c\}$
Upper bound = $\{e, f, j, h\}$
Lower bound = $\{a\}$
Least Upper bound = $\{e\}$
Greatest Lower bound = $\{a\}$

For $\{j, h\}$
No Upper bound
Lower bound = $\{a, b, c, d, e, f\}$
No Least Upper bound
Greatest Lower bound = $\{f\}$




**Definition 1.0.17**

Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of B to each element of $A$.

If $f$ is a function from $A$ to $B$, we write

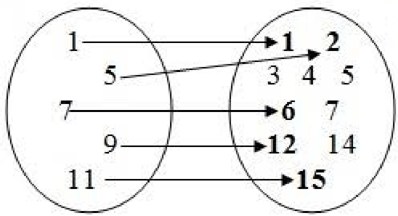
$$f : A \to B.$$

## Definition 1.0.18

If $f$ is a function from $A$ to $B$, we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$.

If $f(a) = b$, we say that $b$ is the image of $a$ and $a$ is a preimage of $b$.

The range, or image, of $f$ is the set of all images of elements of $A$.




Domain

$$= \{1, 5, 7, 9, 11\}$$

Codomain

$$= \{1, 2, 3, 4, 5, 6, 7, 12, 14, 15, \}$$

Range

$$= \{1, 2, 6, 12, 15\}$$

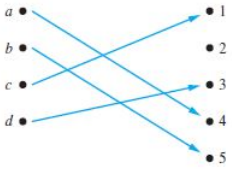
## One-One, Onto, Bijective

### Definition 1.0.19

A function $f : A \to B$ is said to be one-one, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in $A$.

### Example

- Determine whether the function $f$ from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1$, and $f(d) = 3$ is one-to-one



- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

  **Solution** : The function $f(x) = x^2$ is not one-to-one because,

  $$f(1) = f(-1) = 1, \text{ but } 1 \neq -1.$$

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

  **Solution** : The function $f(x) = x^2$ is not one-to-one because,

$$f(1) = f(-1) = 1, \text{ but } 1 \neq -1.$$

- $g : R \to R, \;\; g(x) = 3x - 1.$

  **Solution** :

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

  **Solution** : The function $f(x) = x^2$ is not one-to-one because,

  $$f(1) = f(-1) = 1, \text{ but } 1 \neq -1.$$

- $g : R \to R, \ g(x) = 3x - 1.$

  **Solution** : Let $a, b \in R$, then

  $$g(a) = g(b) \implies 3a - 1 = 3b - 1 \implies a = b.$$

  Hence $g$ is one-one.

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

  **Solution** : The function $f(x) = x^2$ is not one-to-one because,

  $$f(1) = f(-1) = 1, \text{ but } 1 \neq -1.$$

- $g : R \to R, \; g(x) = 3x - 1.$

  **Solution** : Let $a, b \in R$, then

  $$g(a) = g(b) \implies 3a - 1 = 3b - 1 \implies a = b.$$

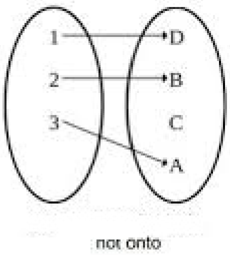
  Hence $g$ is one-one.

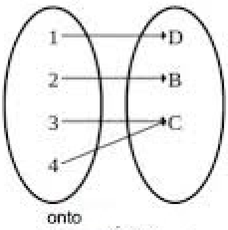
- $h : R \to R, \; h(x) = |x|.$

**Definition 1.0.20**

A function $f : A \to B$ is said to be onto, if every element of $b \in B$ is the image of some element $a \in A$, i.e. $f(a) = b$

$$\text{Range of } f = B$$



## Example

- $f(x) = x^2$ from the set of real numbers to the set of real numbers is not onto.

  **Solution** : The function $f(x) = x^2$ is not onto because, we cannot find a real number whose square is negative, then the range of $f$ cannot be equal to $R$.

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  **Solution** : The function $f(x) = x^2$ is not onto because, we cannot find a real number whose square is negative, then the range of $f$ cannot be equal to $R$.

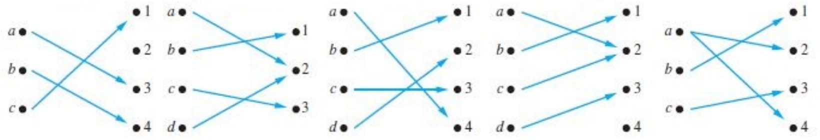
- $g : R \rightarrow R, \;\; g(x) = 3x - 1.$

  **Solution** : Let $b \in R$, then we have to find $a \in$ Domain so that

$$g(a) = b.$$

Let $a = \frac{b+1}{3}$, then we have $g(a) = b$. Hence $g$ is onto.

Determine which of the following are function, one-one function and onto function.




**Definition 1.0.21**

The function $f : A \to B$ is a bijection, if it is both one-one and onto.

**Definition 1.0.22**

Let $f : A \to B$ be a bijection. The inverse function of $f$ is denoted by $f^{-1}$ and $f$ is said to be invertible. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

## Example

- If the function $f : N \to N$ be defined by

$$f(x) = x^2 + 3,$$

then find $f^{-1}(7)$ and $f^{-1}(19)$.

**Solution** : From definition, let $f^{-1}(7) = a$ then

$$f(a) = 7 \implies a^2 + 3 = 7 \implies a^2 = 4 \implies a = 2$$

Therefore, $f^{-1}(7) = 2$
Similarly, we have

$$f^{-1}(19) = 4.$$

### Example

(1) Let $f : Z \to Z$ be such that $f(x) = x + 1$. Is $f$ invertible, and if it is, what is its inverse?

**Solution** : The function $f$ has an inverse because it is one-one and onto.

For inverse function, suppose that $f(x) = y$, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $Z$ that is sent to $y$ by $f$. Consequently,

$$f^{-1}(y) = y - 1.$$

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(2) Let $f$ be the function from $R$ to $R$ with $f(x) = x^2$. Is $f$ invertible?

**Solution** : Because

$$f(-3) = f(3) = 9,$$

$f$ is not one-one. Hence, $f$ is not invertible.

# Types of Function

**Definition 1.0.23**

A function of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_0, a_1, \cdots, a_n$ are real constants and $a_n \neq 0$ is called a polynomial in $x$ of degree $n$.

e.g. $4x^3 + 3x^2 - 6x + 11$

**Definition 1.0.24**

A function of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial in $x$, $g(x) \neq 0$ is called a rational function.

e.g. $F(x) = \frac{x^2+1}{x+4}$

**Definition 1.0.25**

A function $f(x) = a^x$ $(a > 0)$ satisfying the law $a^1 = a$ and $a^x a^y = a^{x+y}$ is called a exponential function.

**Definition 1.0.26**

The inverse of exponential function is called a logarithm function.
If $y = a^x$, $a \neq 1$, then $x = \log_a y$ is called Logarithm function.

## Composition of Functions

### Definition 1.0.27

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The composition of the functions $f$ and $g$, denoted for all $a \in A$ by $f og$, is defined by

$$(f og)(a) = f(g(a)).$$

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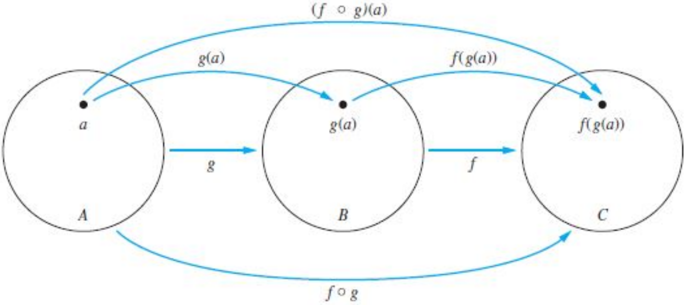
$$(f o g)(a) = f(g(a)).$$



NOTE : The composition $f o g$ cannot be defined unless the range of $g$ is a subset of the domain of $f$.

**Example**

- Let $g$ be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b, g(b) = c$, and $g(c) = a$. Let $f$ be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3, f(b) = 2$, and $f(c) = 1$. What is the composition of $f$ and $g$, and what is the composition of $g$ and $f$ ?

  **Solution** : The composition $f o g$ is defined by

  $$(f o g)(a) = f(g(a)) = f(b) = 2, (f o g)(b) = f(g(b)) = f(c) = 1,$$

  and

  $$(f o g)(c) = f(g(c)) = f(a) = 3.$$

  Note that $g o f$ is not defined, because the range of $f$ is not a subset of the domain of $g$.

- Let $f$ and $g$ be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of $f$ and $g$? What is the composition of $g$ and $f$ ?

  **Solution** : Both the compositions $fog$ and $gof$ are defined. Moreover,

  $$(fog)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

  and

  $$(gof)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

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- Find $fog$ and $gof$ , where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $R$ to $R$.

  **Solution** :

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  **Solution** :

  $$(fog)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 2$$

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  **Solution** :

  $$(fog)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 2$$

  $$(gof)(x) = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 2 = x^2 + 3$$

**Definition 1.0.28**

The floor function at $x$ is denoted by $\lfloor x \rfloor$ and it is defined as

$$\lfloor x \rfloor = \text{ the largest integer that is less than or equal to } x.$$

The ceiling function at $x$ is denoted by $\lceil x \rceil$ and it is defined as

$$\lceil x \rceil = \text{ the smallest integer that is greater than or equal to } x.$$

**Example**

(1) These are some values of the floor and ceiling functions:

$$\left\lceil \frac{1}{3} \right\rceil = 1, \quad \left\lfloor \frac{1}{3} \right\rfloor = 0$$

$$\left\lceil -\frac{1}{3} \right\rceil = 0, \quad \left\lfloor -\frac{1}{3} \right\rfloor = -1$$

$$\left\lceil 5.7 \right\rceil = 6, \quad \left\lfloor 5.7 \right\rfloor = 5$$

$$\left\lceil -3.1 \right\rceil = -3, \quad \left\lfloor -3.1 \right\rfloor = -4, \qquad \left\lceil 5 \right\rceil = 5, \quad \left\lfloor 5 \right\rfloor = 5$$

(2) Prove or disprove that

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$$

for all real numbers $x$ and $y$.

## SECTION 2

### MODULE 2 : Determinants

## Definition 1.0.29

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the square matrix of order 2. Then the determinant of $A$ is denoted by $det(A)$ or $|A|$ and is evaluated as

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be the square matrix of order 3. Then

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(eh - gf) - b(di - gf) + c(dh - eg)$$

NOTE : If $A = [a]$ then
$$det(A) = det(a) = a.$$

**Example**

Find $det(A)$ if $A$ is given by

(i) $\begin{bmatrix} 2 & -3 \\ 4 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 \\ 6 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -1 & 3 \\ 6 & 4 & 16 \\ 8 & 5 & 8 \end{bmatrix}$ (vi) $\begin{bmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{bmatrix}$

**Solution :**

$$(i)\ \ det(A) = \begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix}$$
$$= 2\times 9 - 4\times(-3) = 18 + 12 = 30$$

$$(iv)\ \ det(A) = \begin{vmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{vmatrix}$$
$$= 1\begin{vmatrix} 6 & 2 \\ 7 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix}$$
$$= 1(6-14) - 5(2-6) + 3(14-18)$$
$$= -8 + 20 - 12 = 0$$

(ii) 18 (iii) 1 (v) $-182$ (vi) 0

Minor and Cofactor

**Definition 1.0.30**

The minor of an element in a determinant is the determinant obtained by suppressing the row and the column in which the particular element occurs.

In the $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

The minor of $a = d$, minor of $b = d$, minor of $c = b$ and minor of $d = a$.

In $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$,

The minor of $a = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$

The minor of $b = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$

The minor of $c = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The minor of $d = \begin{vmatrix} b & c \\ h & i \end{vmatrix}$

The minor of $e = \begin{vmatrix} a & c \\ g & i \end{vmatrix}$

The minor of $f = \begin{vmatrix} a & b \\ g & h \end{vmatrix}$

The minor of $g = \begin{vmatrix} b & c \\ e & f \end{vmatrix}$

The minor of $h = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$

The minor of $i = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$

## Example

(1) Find minor of each element of

(i) $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 6 & 2 \\ 3 & 7 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 6 & 4 & 16 \\ 8 & 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{bmatrix}$

Ans : (i) The minor of $1 = -8,\ 5 = -4,\ 3 = -4,\ 2 = -16,\ 6 = -8,\ 2 = -8,\ 3 = -8,\ 7 = -4, 1 = -4.$

(2) Find value of

$$\begin{vmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -z & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Ans : $1 + x^2 + y^2 + z^2$ , $(x-y)(y-z)(z-x)$

**Definition 1.0.31**

The Cofactor of an element in $i^{th}$ row and $j^{th}$ column is $(-1)^{i+j}$ times its minor.

In $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$,

The cofactor of $a$=$(-1)^{1+1}\times$ The minor of $a = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$

The cofactor of $b$=$(-1)^{1+2}\times$ The minor of $b = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}$

The cofactor of $c$=$(-1)^{1+3}\times$ The minor of $c = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

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The cofactor of $i$=$(-1)^{3+3}\times$ The minor of $i$ = $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$

**Example**

Find cofactor of each element of the determinant $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 0 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

**Solution** :

The cofactor of $0 = (-1)^{1+1}\begin{vmatrix} 0 & 5 \\ 4 & 6 \end{vmatrix} = -20$

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The cofactor of $2 = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = -10$

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The cofactor of $5 = 2$, The cofactor of $2 = 5$, The cofactor of $4 = -2$,
The cofactor of $6 - 2$,

(2) Find cofactor of each element of the determinant $\begin{vmatrix} 3 & a & b \\ -e & 0 & 4 \\ 7 & y & 1 \end{vmatrix}$