

402112-2017001

湖南理工学院 2017-2018 学年第 二 学期

2006 级线性代数试题 B 卷

班级 _____ 学号 _____ 姓名 _____ 成绩 _____

一、(10 分) 设 $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 3 & 4 & 10 \end{pmatrix}$, 求 $\det(A - 2A^T)$.

解: $\begin{vmatrix} 2A^T & 0 \\ 0 & A^T \end{vmatrix} = \begin{vmatrix} 2A^T \\ 0 & A^T \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 3 & 4 & 10 \end{vmatrix} = 2 \cdot 1 \cdot 1 \cdot \det(A) = 2 \det(A)$

二、(15 分) 已知 $A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}$, 求 X .

解: 求逆阵 $X^{-1} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 3 \\ 2 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & 0 & 3 \\ -1 & 2 & 3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \times (-1)} \begin{pmatrix} -2 & 0 & -3 \\ -1 & 2 & 3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 2 & 3 \\ -2 & 0 & -3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \times (-1)} \begin{pmatrix} 1 & -2 & -3 \\ -2 & 0 & -3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & -2 & -3 \\ 0 & -4 & -9 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \times (-1/4)} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 9/4 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 9/4 \\ 0 & 0 & -5/2 \end{pmatrix} \xrightarrow{R_3 \times (-2/5)} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 9/4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \times (4/9)} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 2R_2 + 3R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

三、(10 分) 求下列方程组的通解

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - 2x_3 - 2x_4 = 2 \\ -x_1 + x_2 + x_3 - x_4 = 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

求该方程组的基础解系及特解

$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -2 & -2 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 + R_1, R_4 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \times (1/3)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix} \xrightarrow{R_4 - 2R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \times (1/2)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

五、(10分) 已知 $\alpha_1 = (2, 1, 5), \alpha_2 = (3, -1, 2), \alpha_3 = (5, 0, 5), \alpha_4 = (-1, 2, 1), \alpha_5 = (1, 1, 1)$

(1) 求向量 $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ 张成的一个极大无关组

(2) 求该极大无关组在表中其它向量.

$$\text{解: } \begin{pmatrix} 2 & 3 & 5 & -1 & 1 \\ 1 & -1 & 0 & 2 & 1 \\ 5 & 2 & 5 & 1 & 1 \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 5 & 5 & -5 & -1 \\ 0 & 5 & 5 & -5 & -2 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(1) 选 $\alpha_1, \alpha_2, \alpha_5$ 为极大无关组.

(2) 显然 $\alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = \alpha_1 - \alpha_2$

五、(10分) 设 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in P, P_1, P_2, P_3, P_4$ 是 P 中互不相同的向量, 且 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$\text{且 } P_1, P_2, P_3, P_4 \text{ 的坐标依次是 } \begin{pmatrix} 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \text{ 求向量 } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \text{ 的坐标.}$$

为 $(1, -1, 2, -2)$, 求 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 的坐标.

$$\begin{pmatrix} 1 & -1 & 2 & -2 \end{pmatrix}$$

六、(15分) 已知 $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$, 求可逆阵 P , 使 $P^{-1}AP$ 是对角阵.

$$\text{解: } |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = (\lambda-1) \left[(\lambda-2)^2 - 9 \right] = (\lambda-1)(\lambda-5)(\lambda+1)$$

特征值为 $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 5$

$$A - \lambda_1 I = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ 特征向量 } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{\text{行变换}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix} \text{ 特征向量 } \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad P^{-1}P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由 (1) 得 $\alpha_1 = (2, 1, 3)$ 及 $\alpha_2 = (1, 1, -1)$. 求与 α_1, α_2 都正交的向量 α_3 . 有

证 $\alpha_1, \alpha_2, \alpha_3$ 为基. 证 α_1, α_2 为 R^3 的一个基. 证 $\alpha_3 \neq 0$.

证: $\alpha_1 \perp \alpha_2, \alpha_1 \perp \alpha_3, \alpha_2 \perp \alpha_3$ $\Leftrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix} \alpha_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (1)

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \end{pmatrix} \rightarrow \begin{cases} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \end{pmatrix} \begin{matrix} x_3 = 0 \\ x_3 = 0 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \end{pmatrix} \begin{matrix} x_3 = 0 \\ x_3 = 0 \end{matrix} \end{cases}$$

$\alpha_3 = (0, 0, -1)$

证 $\alpha_1, \alpha_2, \alpha_3$ 为基. 证 α_1, α_2 为 R^3 的一个基. 证 $\alpha_3 \neq 0$.

4. (10分) 求可逆线性变换 T , 将二次型

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 - 3x_3^2 + 4x_1x_2$$

化为标准形.

$$f = -(x_1 - 2x_2)^2 + 3x_2^2 - 3x_3^2 = -\frac{1}{3}(x_1 - 2x_2)^2 + \frac{2}{3}x_2^2 - 3x_3^2$$

证: $\begin{cases} x_1 = x_1 - 2x_2 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{1}{3}x_1 + \frac{2}{3}x_2 \\ x_2 = \frac{2}{3}x_2 \\ x_3 = \frac{1}{3}x_3 \end{cases} \rightarrow \begin{cases} x_1 = \frac{1}{3}x_1 \\ x_2 = \frac{2}{3}x_2 \\ x_3 = \frac{1}{3}x_3 \end{cases} \rightarrow \begin{cases} x_1 = \frac{1}{3}x_1 \\ x_2 = \frac{2}{3}x_2 \\ x_3 = \frac{1}{3}x_3 \end{cases}$

故 T 为 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 即可. 证 T 为可逆线性变换. 证 T 为可逆线性变换.

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证: $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R})$ $\Leftrightarrow T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R})$ (1)

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

证 T 为可逆线性变换.

1. (10分) 设 A 为 n 阶实对称阵, 且 $A^2 = 0$. 证 $A = 0$. 证 $A = 0$.

证 $A = 0$.

证: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R})$ $\Leftrightarrow A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 0$ (1)

证 $A = 0$. 证 $A = 0$. 证 $A = 0$. 证 $A = 0$.