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# Rigid Graph Control Architectures for Autonomous Formations

## APPLYING CLASSICAL GRAPH THEORY TO THE CONTROL OF MULTIAGENT SYSTEMS

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For millions of years, nature has presented examples of collective behavior in groups of insects, birds, and fish. This behavior has arisen to permit sophisticated functions of the group that cannot be achieved by individual members [1], [2]. Collective behavior serves needs such as foraging for food, defense against predators, aggression against prey, and mating. Fish and birds particularly, as part of their group behavior, often display *formation-type* behavior. In this type of behavior, the relative positions of the fish or birds are preserved, and the formation moves as a cohesive whole. From time to time, a formation may split, rearrange itself in a minor way, perhaps to remove a burden on one or more members of the formation, or rearrange itself in a major way, perhaps for obstacle avoidance, predator avoidance, or merging with another formation.

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With this inspiration, formations of robots (see Figure 1), underwater vehicles, and autonomous or piloted airborne vehicles (see Figure 2) are being deployed to tackle problems in both the civilian and military spheres, such as bush-fire control, surveillance, and underwater exploration. For various reasons, a formation of vehicles may constitute a more effective sensor than a single vehicle. First, multiple vehicles can be used to synthesize a large antenna for receiving electromagnetic or acoustic signals, allowing better resolution or higher sensitivity than might be possible with an antenna carried by a single vehicle. Accurate knowledge and control of the agents' relative positions of the formation is essential for this application. A second reason is that some tasks inherently require multiple sensors with known relative positions. For example, in three dimensions, when distances to an object of interest are measured, to determine the position of that object, at least four distance measurements from noncoplanar sensors with known positions are needed to uniquely localize

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the object, that is, to determine its position. A third reason is that multiple sensors may have individually differing functionalities, which in aggregate give a new functionality to the sensor formation. For example, a group of unmanned airborne vehicles may include radio-frequency direction-sensing sensors on some vehicles and optical sensors on others, to allow target localization and target identification by the entire formation. A fourth reason is that small, mobile sensors are often less expensive to deploy than a single large sensor. Lastly, multiple mobile sensors can cover a region of interest more quickly than a single sensor when the sensing range is too small to allow the whole region to be scanned from one position.

Sometimes multiple reasons apply at the same time. For example, the agents of the formation may localize using angle information, but given sensor noise and visibility limitations, more agents need to be used to make up the formation than if there were no noise, and all agents had 360° visibility. Formations in naturally or artificially hazardous environments may also require larger numbers of agents to cope with outages.

From a control point of view, tasks are specified at both the level of the whole formation, determining, for example, way points for a path that the center of gravity of the formation is commanded to follow, as well as for the individual agents of the formation, such as maintaining their relative positions, or shifting from one formation shape to another formation shape. Certainly in formations occurring in nature, and commonly in synthetic formations, there is no single master agent exercising control over all other agents. Control tasks in some way are handled on a decentralized basis.

In meeting these objectives, various systems problems arise. The most basic problem is defining practical architectures for control, communications, and sensing. Suitable architectures, however, cannot be defined independently of one another. An overarching requirement is that architectures be scalable. The scalability requirement imposes

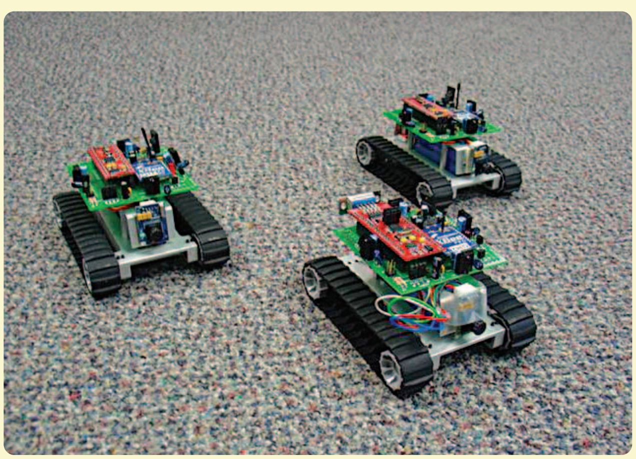

**FIGURE 1** Robot formation. The distance between each robot pair remains constant during a move of the whole formation due to active control of inter-robot distances. (Photo taken at the Australian National University.)

**FIGURE 2** Formation of jet fighters. The distance between each aircraft pair remains constant during flight. For this purpose, the pilots actively maintain a subset of interaircraft distances and relative bearings. In determining its path, each aircraft cannot take into account the relative positions of the five remaining aircraft since the load on the pilots would almost certainly be excessive. (Picture reproduced with the kind permission of Marcel Grand, www.swissjet.com.)

the need for significant decentralization of information and controller structures. For example, in a formation of birds, no one bird can be expected to watch all of the other birds and compute its own trajectory using even partial knowledge of the trajectories of all of the other birds. Hence the amount of sensing, communication, and control computation by any one agent must be limited.

This article does not traverse the whole problem domain sketched above, but only one portion. Most of this article describes the types of sensing and control architectures that are needed to maintain the shape of a formation, while the formation moves as a cohesive whole. As already intimated, it is often desirable that the formation maintain its shape while executing an overall change of location, or change from one shape to another, perhaps to avoid an obstacle. Again, a formation may need to split into multiple subformations or merge with a distinct formation. The architectures need to be such that these tasks can be performed.

Within this limited domain, we impose a further limitation. To maintain a formation shape, agents must sense some aspect of the formation geometry, in other words, an agent needs to measure at least one geometrically relevant variable involving at least some of the other agents in the formation, to apply control to correct errors in formation shape. Mostly, distances or angles are sensed. For example, if agents $i$, $j$, and $k$ are commanded to maintain a triangle, this task could be accomplished by sensing and correcting distances between agents, or by sensing and maintaining one distance and two angles. Additional sensed measurements in alternative situations include, for example, bearing relative to north, inclination/declination relative to the horizon, and time-difference-of-arrival at two agents of a pulse transmitted from a third agent.

In many cases, biological organisms can sense more complicated variables than those mentioned above. One of our biologist colleagues conjectures that birds in a formation effectively project the bearing angles of other birds located within a visual cone onto a hemisphere, and at the same time sense perhaps one distance. With this ability, the formation would have a scale, and distances would be fixed, which would not be the case if angles alone were to be sensed. In this article, we confine our attention to the use of distance measurements.

The key tools we use are graph theoretic. Rather than presenting control laws for individual agents in a formation, which typically serve multiple objectives such as shape control, collision avoidance, and motion to a desired end point, we focus on the higher level question of defining the architecture behind these control laws. Considerable work on formation control laws includes [3]–[11]. Some of these references make substantial use of graph-theoretic ideas. For example, [11] establishes an interface between graph-theoretic rigidity concepts along with the dynamics of the agents and the cohesive motion of the whole formation, using a particular set of distributed control schemes.

Most formation-shape laws require sensing of the relative positions of neighboring agents, and thus more variables are sensed than are controlled. Relative positions can be determined if some angles as well as distances can be sensed, or if one-hop and two-hop neighbor distances can be sensed. Also, it turns out that the graph-theoretic tools we describe can often be applied to formations with controlled (as opposed to sensed) variables that include both angles and distances [12]–[14].

Imposing direction constraints on edges relative to a global coordinate basis (whether or not these edges are subject to length constraints) leads to minor modifications of much that follows, except that such constraints not only contribute to maintaining formation shape but also to maintaining the formation's orientation relative to the global coordinate basis. On the other hand, imposing angle constraints between two edges with a common vertex leads to difficulties since it is not straightforward to introduce into the theory the fact that the angles around the vertices of a polygon are not independent [12], [13]. Nevertheless, if polygonal angle constraints are not present because of the way the graph associated with the formation is structured, then angle and distance constraints can be mixed [14].

## OUTLINE OF THE ARTICLE

In the next two sections, we describe how some aspects of a formation architecture can be described using graphs. A clear dividing line between the two sections is associated with the type of graphs we are using. Consider two agents $j$ and $k$ in a formation. Suppose that the distance between these agents is actively maintained, and that one or both of the agents in question can sense that distance. Then the task of maintaining the distance might be given jointly to $j$ and $k$, implying that both agents would have to sense the distance. Alternatively, the task might be given to $j$ alone, or $k$ alone, in which case only one agent would need to sense the distance. If the distance-sensing task is given to $j$ alone, then $k$ would be unconscious of $j$. In the case of joint responsibility, it turns out that undirected graphs are an appropriate tool, in which case the graph has an undirected edge between vertices corresponding to agents $j$ and $k$. When responsibility is given to $j$ alone or $k$ alone, the graph is directed and has a directed edge from $j$ to $k$ when $j$ is responsible for maintaining distance, and from $k$ to $j$ when $k$ is responsible for maintaining distance. These two sections describe conditions, in two and three dimensions, that must be satisfied by architectures that preserve formation shape during formation movement.

In a following section, we address operations with formations, including merging, splitting, and closing ranks. Closing ranks is the task of repairing a formation when one or more agents are lost. Generally speaking, a minor rearrangement of the architecture is needed.

The penultimate section considers these tasks in more detail. Two broad conceptual approaches can be adopted.

For the merging problem we contemplate two formations, both capable of maintaining cohesive motion, and we ask the following question: If the formations are brought into proximity with one another and thought of as a single formation whose agent set is the union of the agent sets of the individual formations, which additional interagent distances must be sensed and controlled (beyond those already sensed and controlled in the individual formations) in order that the new single formation be capable of cohesive motion? This question provides a natural way to look at the problem. In the second conceptual approach we view each of the two formations as a super agent, or meta-agent, whose internal structure is not fundamentally relevant. Then we identify rules for assigning edges between two meta-agents in order that the combination be capable of maintaining cohesive motion. We offer some concluding remarks on future directions in the final section.

## FORMATIONS AND UNDIRECTED RIGID GRAPHS

Rigid graph theory [15]–[17] is a key tool for analyzing the property of formation rigidity [8], [12], [18], [19]. In this framework, agents are modeled as points, and agent pairs whose interagent distance is actively constrained to be constant can be thought of as being joined by bars whose lengths enforce the interagent distance constraints. The formation can therefore be modeled by a graph where vertices represent point-like agents, and interagent distance constraints are abstracted as weighted edges. Rigid graph theory is concerned with properties of graphs that ensure that the formation modeled by the graph is rigid. Roughly speaking, a formation is rigid if its only smooth motions are those corresponding to translation or rotation of the whole formation. For a precise definition, see [15]–[17].

Figure 3 shows several examples of two-dimensional formations, two of which are rigid and one of which is not rigid. In a nonrigid formation, part of the formation can flex or move, while the rest of the formation remains stationary, and all distance constraints remain satisfied. The notion of rigidity conforms to our intuition and corresponds to the rigidity of frameworks in civil and mechanical engineering. For further information about the history of rigidity, see "Origins of Rigid Graph Theory."

### *Rigidity and Minimal Rigidity*

Two key tools are often used to analyze rigidity. The first is linear algebra. Given knowledge of the positions of the agents at one time, it is possible to construct a matrix, known as the *rigidity matrix* [15], [16], whose size and rank can be used to determine whether or not the formation is rigid. Since the size and rank of the rigidity matrix turn out to be the same for almost all positions of the agents, rigidity matrices have the same rank when formed from two formations differing only in terms of the relative positions (such as differences of the position vectors) of those agent pairs with distance constraints, except for very special or nongeneric sets of the agent positions. (An extreme example of a very special set of agent positions is when all agents are collinear). For further information, see "What Are Rigidity Matrices?" The concept is valid in both two and three dimensions.

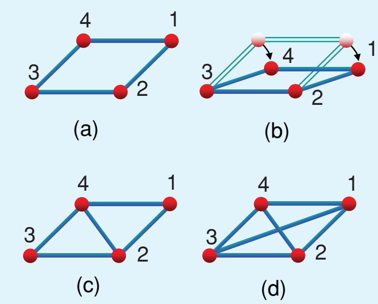


**FIGURE 3** Rigid and nonrigid formations. The formation represented in (a) is not rigid, since it can be deformed by a smooth motion without affecting the distance between the agents connected by edges, as shown in (b). The formations represented in (c) and (d) are rigid, since they cannot be deformed. The formation represented in (c) is minimally rigid, because the removal of any edge renders it nonrigid. Finally, (d) is not minimally rigid, since any single edge can be removed without losing rigidity.

## Origins of Rigid Graph Theory

Perhaps the earliest recognized contribution to what is now graphical rigidity theory came from Euler in the form of a conjecture concerning polyhedra. Then 150 years or so later, perhaps motivated by Euler, Cauchy made a major contribution in proving that two strictly convex polyhedra with congruent faces are congruent as three-dimensional entities. This result essentially concerns plate and hinge structures, rather than rod and pin-joint structures. Minor errors in this work took more than 100 years to be fixed. Cauchy's result was modified for three-dimensional rod-and-pin-joint structures by Asimow and Roth [S1], who proved that if a three-dimensional rod and pin-joint structure is the skeleton of a strictly convex polygon that is rigid, then all of its faces must be triangular. Between Cauchy's work and that of Asimow and Roth, contributions include the exploration of properties by Maxwell of what we today term a rigidity matrix and its link with rigidity; the textbook of Henneberg [S2], which includes a description of growing and decomposing rigid graphs, although the claims and procedures are incomplete or vague for graphs associated with three-dimensional structures; and a combinatoric characterization of graphs of two-dimensional rigid structures [20]. Results for graphs of two-dimensional structures remain more developed than those for three-dimensional structures.


[S1] L. Asimow and B. Roth, "The rigidity of graphs," *Trans. Amer. Math. Soc.*, vol. 245, pp. 279–289, 1978.

[S2] L. Henneberg, *Die Graphische Statik der starren Systeme*. Leipzig: B.G. Teubner, 1911.

As discussed in "What Are Rigidity Matrices?" and as a refinement of the above observation, if the rigidity matrix rank test implies rigidity for one set of agent positions, then rigidity is guaranteed for almost all sets of agent positions with the same graph. Thus the property of rigidity is "almost" independent of agent positions. Therefore, it is natural to seek a test for rigidity that does not depend on vertex positions. Such a test is available for two dimensions, but not for three dimensions.

In two dimensions rigidity can also be characterized in purely combinatorial terms, without appeal to the rigidity matrix. In particular, counting-type conditions related to the graph can be used to ascertain the rigidity or nonrigidity of a generic formation corresponding to the graph. A key result is Laman's theorem [20], for which no three-dimensional analog exists. In three dimensions, necessary conditions and sufficient conditions (but not conditions that are both necessary and sufficient) are known for a graph to correspond to a formation that is rigid for generic values of the constrained interagent distances [15]. For further information, see "What Does Laman's Theorem Say?"

The *Henneberg construction* [15], [21] deals with the iterative construction of rigid formations. As for the characterization of rigidity, the Henneberg construction procedure for two-dimensional formations is better developed than that for three-dimensional formations. Before describing the result, we note the concept of *minimal rigidity* [15]. A formation is *minimally rigid* if it is rigid and if no single interagent distance constraint can be removed without

## What Are Rigidity Matrices?

Consider a graph $G = (V, E)$ modeling a formation in two dimensions of $|V|$ vertices and $|E|$ edges. The notation $|V|$ means the cardinality of the set $V$. Let the coordinates of vertex $j$ be $(x_j, y_j)$. The rigidity matrix, which is defined with an arbitrary ordering of the vertices and edges, has $2|V|$ columns and $|E|$ rows. Each edge gives rise to a row, and, if an edge links vertices $j$ and $k$, then the nonzero entries of the corresponding row of the matrix are in columns $2j$, $2j+1$, $2k$, and $2k+1$, and are, respectively, $x_j - x_k$, $y_j - y_k$, $x_k - x_j$, and $y_k - y_j$. For example, for the graphs of Figure 3(a) and (c), the rigidity matrices are

$$R = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 & 0 & 0 \\ 0 & 0 & x_2 - x_4 & y_2 - y_4 & 0 & 0 & x_4 - x_2 & y_4 - y_2 \end{bmatrix}.$$

At some special positions for the vertices of a graph, rigidity is lost, typically when certain collections of vertices are collinear in two dimensions or coplanar in three dimensions. Call such situations nongeneric. The key result is the following theorem [15], [16].

***Rigidity Matrix Theorem***

A graph $G = (V, E)$ modeling a formation in two dimensions of $|V|$ vertices and $|E|$ edges is rigid if and only if, for generic vertex positions, the rank of the rigidity matrix $R$ is $2|V| - 3$.

It is easy to verify this result for the examples in Figure 3, where the rank of the rigidity matrix of Figure 3(a) is four and, for Figure 3(c), is five.

When the rank of the rigidity matrix $R$ is $2|V| - 3$, the dimension of its kernel or null space is three. A vector in the kernel corresponds to agent velocities when the formation is translating or rotating. In two dimensions, two independent translations and one rotation are possible. When the kernel dimension is greater than three, independent motions in addition to translation and rotation are possible, corresponding to some kind of flexing.

The rigidity matrix theorem extends easily to graphs that model formations in three dimensions. In this case, three columns of the matrix are associated with each vertex, and rigidity corresponds to $R$ having a rank of $3|V| - 6$ for generic vertex positions.

The rigidity matrix of a physical structure conveys information about how loads on a structure translate into forces on its members. Rigidity matrices with three zero singular values and with a small nonzero singular value appear to correspond to two-dimensional physical structures where small loads can cause large deflections.

causing the formation to lose rigidity. A graph is minimally rigid if almost all formations to which the graph corresponds are minimally rigid. Minimal rigidity can be described in two and three dimensions with the rigidity matrix, is characterizable in two dimensions with Laman's theorem, and, in three dimensions, is the subject of necessary conditions and distinct sufficiency conditions on the graph determined by a formation. Given a graph with edge set $E$ and vertex set $V$, which is known to be rigid, it is additionally minimally rigid in two or three dimensions if and only if $|E| = 2|V| - 3$ or $|E| = 3|V| - 6$, respectively, where $|E|$ and $|V|$ are the numbers of edges and vertices of the graph. See Figure 3 for an illustration.

The two-dimensional Henneberg construction comprises the application at each step of one of two possible operations, known as vertex addition and edge splitting. The procedure begins with a minimally rigid graph, and the application of either operation creates a minimally rigid graph with one additional vertex. Minimally rigid graphs can be progressively built up this way. For further information and an illustration, see "Henneberg Construction" and Figure 4.

A by-product of the associated theory is the fact that every two-dimensional minimally rigid graph is constructible from a primitive comprising a two-vertex single-edge graph by using a sequence of these operations. Second, each two-dimensional minimally rigid graph can be deconstructed by inverse operations to yield a sequence of minimally rigid graphs, each of which has one less vertex than its predecessor in the sequence, terminating with a two-vertex, single-edge graph.

In three dimensions, operations that are analogous to vertex addition and edge splitting can be defined, but whether these analogously defined operations are necessary and sufficient to build and deconstruct all minimally rigid graphs is still a matter of conjecture. Examples of such conjectures are provided in [15].

## What Does Laman's Theorem Say?

Laman's theorem [20] requires the idea of an induced subgraph of a graph $G = (V, E)$. Let $V'$ be a subset of $V$. Then the subgraph of $G$ induced by $V'$ is the graph $G' = (V', E')$ where $E'$ includes all those edges of $E$ that are incident on a vertex pair in $V'$.

***Laman's Theorem***

A graph $G = (V, E)$ modeling a formation in two dimensions with $|V|$ vertices and $|E|$ edges is rigid if and only if there exists a subgraph $G' = (V, E')$ with $2|V| - 3$ edges such that for every subset $V''$ of $V$, the induced subgraph $G'' = (V'', E'')$ of $G'$ satisfies $|E''| \le 2|V''| - 3$.

Figure 3 shows some simple graphs for which the claim of the theorem can be checked.

***Partial Extension of Laman's Theorem to Three Dimensions***

If a graph $G = (V, E)$ modeling a formation in three dimensions of $|V|$ vertices and $|E|$ edges is rigid then a) there exists a subgraph $G' = (V, E')$ with $3|V| - 6$ edges such that, for every subset $V''$ of $V$, the induced subgraph $G'' = (V'', E'')$ of $G'$ satisfies $|E''| \le 3|V''| - 6$, and b) if $|E''| = 3|V''| - 6$, then $G''$ is 3-connected (equivalently, every pair of vertices of $G''$ is connected by three paths that pairwise have no vertices in common except the end vertices).

## Henneberg Construction

The Henneberg construction is a technique for growing minimally rigid graphs, starting from a graph with two vertices and one edge joining the vertices [15]. At each step in the procedure, illustrated in greater detail below, either a vertex and two edges are added, or a vertex and three new edges are added, while one existing edge is removed. The new edges are incident on the new vertex. These operations are termed *vertex addition* and *edge splitting*, respectively.

Let $G = (V, E)$ be a graph modeling a formation in two dimensions. The vertex addition operation is the following. A new graph $G' = (V', E')$ is formed, where a vertex $v$ is adjoined as well as two edges from $G$ to $v$, so that $V' = V \cup \{v\}$ and $E' = E \cup \{(v, j), (v, k)\}$ for some $j, k \in V$. Vertex addition is illustrated in Figure 4.

The edge splitting operation is the following. A new graph $G' = (V', E')$ is formed where a vertex $v$ is adjoined as well as three edges from $G$ to $v$, while an edge of $G$ is removed. More precisely, $V' = V \cup \{v\}$ and $E' = E \cup \{(v, j), (v, k), (v, m)\} \setminus \{e\}$ for some $j, k, m \in V$ with at least two of the vertices $j, k, m$ adjacent in $G$, and the edge $e$ either $(j, k)$, $(j, m)$, or $(k, m)$. This operation is illustrated in Figure 4.

These vertex-addition and edge-splitting operations are enough to grow every minimally rigid graph. It is also possible to deconstruct a minimally rigid graph by removing one vertex and a net count of two edges at each step. In the reverse edge-splitting step, one vertex and three edges are removed, and one new edge is inserted. The two vertices on which the new edge is incident are among the set of three vertices on which the three removed edges are initially incident, the removed vertex being excluded from this set. Arbitrarily choosing these two vertices from the three candidates is not generally possible. In growing and deconstructing a minimally rigid graph, all the intermediate graphs are minimally rigid.

This article briefly discusses extensions of the Henneberg construction concept to three-dimensional graphs, directed graphs, and graphs whose agents are replaced by whole formations.

### *Extension to Global Rigidity*

Consider a formation with distinctly labeled agents and with some of the interagent distances known. We wish to understand what alternative formation shapes agents can have when they are positioned consistently with the data.

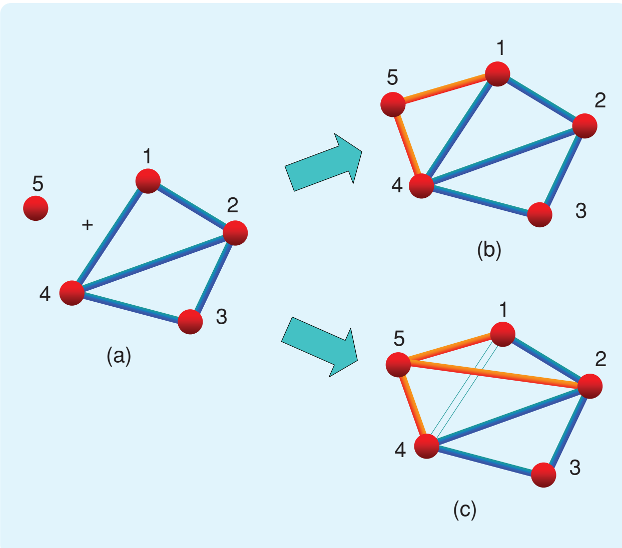


**FIGURE 4** Growing formations using a Henneberg sequence. (a) The four-agent formation can be expanded to include agent 5 by either (b) a vertex addition operation or (c) an edge-splitting operation. In (b), agent 5 is connected to two distinct agents of the formations. In (c) the edge (1,4) is replaced by two edges (1,5) and (5,4), which are incident on the newly added agent. A third connection (5,2) is also added to the latter agent. Both operations lead to a minimally rigid formation provided that the initial formation is minimally rigid. Moreover, every minimally rigid formation can be obtained by performing a sequence of these operations on an initial formation consisting of two connected agents.

Note that if the formation $\mathcal{F}$ is consistent with the data, then so is every rotation, translation, and reflection of $\mathcal{F}$. However, the existence of alternative formation shapes that are not obtainable from $\mathcal{F}$ through rotation, translation, and reflection is not clear.

A two-dimensional or three-dimensional formation is *globally rigid* if and only if any two formations corresponding to the distance data differ by a combination of translation, rotation, and reflection [17]. A graph is *globally rigid* if and only if a generic formation corresponding to the graph is globally rigid. Global rigidity is a more demanding concept than rigidity since there exist rigid formations that are not globally rigid, and such formations can be converted to globally rigid formations only by the inclusion of additional distance constraints. Figure 5 gives two examples of two-dimensional formations that, due to two different types of ambiguities described below, are rigid but not globally rigid and yet satisfy the same set of distance constraints.

The notion of flip ambiguity, which is easily grasped, is illustrated in Figure 5(a). Not every minimally rigid graph contains a flip ambiguity. The notion of flex ambiguity needs more explanation. Although the two formations in Figure 5(b) satisfy the same distance constraints, these formations are not congruent. Every formation corresponding to a minimally rigid graph with four or more vertices at generic positions can exhibit flex ambiguity [22]. For example, for the formation of Figure 5(a), the flex ambiguity happens to be the same as the flip ambiguity. Consequently, when only enough distance constraints are present to ensure minimal rigidity, the shape of an associated formation is not uniquely specified by those constraints. Rigidity, whether or not minimal, ensures that if a formation assumes one of the allowed shapes, it cannot deform smoothly from that shape. Minimal rigidity therefore allows retention of shape only in the face of potential smooth deformations, although it does not of itself specify what shape is retained. Global rigidity instead is required to ensure that the shape is unique. For further information, see "Global Rigidity Characterization."

Henneberg-like construction is also possible for two-dimensional globally rigid graphs. Two sets of operations are discussed in [17] and [21].

One application area in which global rigidity is of interest is *sensor network localization* [23]–[25], as we now describe. In the abstraction of a planar sensor network, a set of points corresponding to a set of sensors, which is analogous to a set of agents in a formation, is given together with a set of distances

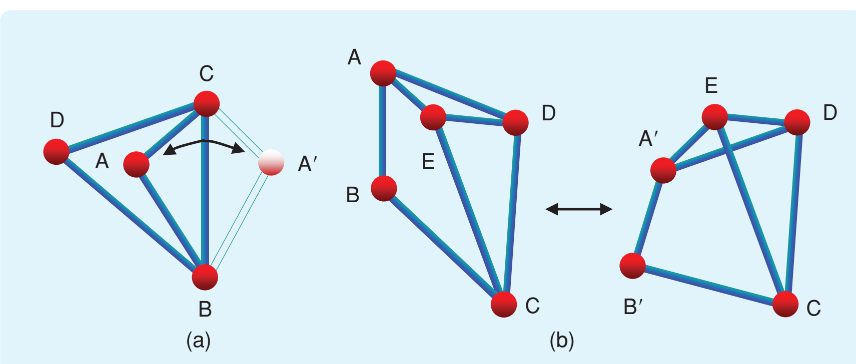


**FIGURE 5** Noncongruent rigid formations with the same set of distance constraints. (a) depicts flip ambiguity since vertex A can be flipped over edge (B,C) to the symmetric position A′ while the distance constraints remain the same. (b) depicts discontinuous flex ambiguity since, by temporarily removing edge (A,D), the edge triple (A,E), (A,B), (B,C) can be flexed to obtain positions A′ and B′ such that the edge length (A′,D) equals the edge length (A,D), and thus all of the distance constraints are maintained. The formations (a) and (b) are thus rigid but not globally rigid. Although no smooth motion can deform these formations without affecting the distances between agents connected by edges, the sets of edges and corresponding distance constraints do not uniquely define the relative positions of the agents. In other words, the distance constraints do not specify the formation shape up to a rotation, translation, or reflection.

between some pairs of points, where the distance information is obtained by exchange of information between sensors in the physical network. Distance data are typically available for pairs of points whose separation is less than some threshold. The associated graph is called a unit disk graph. The sensor network localization problem is to use the distance data to determine the Euclidean coordinates for the sensor positions that are consistent with the distance set. However, using this distance data the Euclidean coordinates are specified only up to translation, rotation, or reflection of the whole set of positions. The Euclidean coordinates, however, become uniquely specified by using further information obtained from designated *anchor* nodes or sensors, whose positions are known absolutely.

Some applications of formations involve localization of objects whose positions are unknown. For example, the following two-dimensional application arises in an unmanned aerial vehicle operational scenario. Suppose agents $A$, $B$, and $C$ are located at known positions, while agents $D$, $E$, and $F$ at are located at unknown positions. Suppose that the interagent distances are known for $A$, $B$, and $C$, and separately for $D$, $E$, and $F$. Suppose further that the distances $AD$, $BE$, and $CF$ are known. Is it possible to localize $D$, $E$, and $F$, and if so, how? The associated graph can be shown to be minimally rigid using, for example, Laman's theorem, and thus the formation is not globally rigid. Generically, therefore, $D$, $E$, and $F$ cannot be localized, although a finite set of possible positions for each of $D$, $E$, and $F$ can be determined, where the positions differ by flex ambiguities.

### Nonminimally Rigid Formations

Globally rigid formations with four or more agents are not minimally rigid. Hence, if formation shape maintenance as opposed to formation shape acquisition is the key problem of interest, there is no particular reason to use enough distance constraints to secure a globally rigid formation, as opposed to a minimally rigid formation.

There are, however, further reasons for using nonminimally rigid formations, where more constraints are imposed than are needed for shape maintenance. For example, a minimally rigid formation lacks protection against loss of a sensor, a communication link, or a control actuator, and, in practice, deliberate use of redundancy is often necessary to achieve robustness. No single accepted quantitative measure of robustness is available to reflect the ability to sustain the loss of an edge or the loss of an agent, or indeed the loss of a multiplicity of edges, agents, or both. Loss of agents is in part covered by the subsequent discussion of closing ranks.

Some problems arise in handling nonminimally rigid formations. For example, imagine a two-dimensional physical structure of bars and joints that is nonminimally rigid. If the structure is to be constructed from a blueprint, then errors in the lengths of the members corresponding to the edges will prevent the structure from fitting together properly. Likewise, if the distances between agents in a formation are measured with some noise and controls operate to try to bring certain distances to specified values, a similar type of inconsistency will be likely to arise. In two dimensions, an agent that is required to maintain specified distances to three of its neighbors may not be able to position itself so that all three measured values are equal to the specified values. Similar problems of coping with inconsistencies introduced by inaccurate measurements also arise in sensor network localization problems, and techniques are becoming available to deal with these problems [26].

## Global Rigidity Characterization

A nice characterization of global rigidity [22] is available for two-dimensional formations and their associated graphs. No extension is known for three-dimensional formations. The two-dimensional characterization uses the terms *redundant rigidity* and *three connectivity*. An undirected graph is termed *redundantly rigid* if it remains rigid after the removal of any single edge. A graph is *connected* if there is a path from any vertex to any other vertex. A graph is *k-connected* if it remains connected when any $k-1$ vertices are removed. Equivalently, between any two vertices of a $k$-connected graph, there are $k$ paths that have no common edge and no common vertex, except for the end vertices [S3].

***Theorem***

A graph $G=(V, E)$ modeling a formation in two dimensions of $|V|$ vertices and $|E|$ edges is globally rigid if and only if it is redundantly rigid and 3-connected.

When a graph is globally rigid, a generic formation corresponding to the graph is also globally rigid.

A variation of Laman's theorem is available that characterizes the redundant rigidity property.

A useful property of a two-dimensional globally rigid formation is that if the positions of three noncollinear agents are known, then the positions of all the other agents can be uniquely determined from the interagent distances corresponding to the edges in the formation graph.

[S3] B. Bollobas, *Modern Graph Theory*. New York: Springer-Verlag, 1998.

## FORMATIONS AND DIRECTED GRAPHS

The task of maintaining a prescribed distance between a pair of agents requires control action. The execution of the task could be the responsibility of both agents or one nominated agent of the pair. Modeling using undirected graphs is appropriate in the case of shared responsibility. However, the case of single-agent or unilateral responsibility is captured by assigning a direction to the relevant edge in the graph. A directed edge from vertex $u$ to vertex $v$ appears when agent $u$ has the task of maintaining a specified distance from agent $v$, and agent $v$ is unconstrained in its own motions with respect to the motion of $u$, or differently put, agent $v$ is unconscious of the task that agent $u$ has to execute. In this case, only one of the two agents has to sense the position of the other agent or receive the position information broadcast by the other agent and make decisions on its own. Therefore, and advantageously, both the overall communication complexity in terms of sensed or received information and the overall control complexity for the formation are halved in comparison to the shared responsibility case [27]. The reduction in communication links can mean lower bit rates and reduced difficulties with interference. In the shared responsibility case, a problem can arise when noise is present and two agents fail to share distance estimate information and instead use inconsistent estimates of the distance between each other [3], but this problem cannot arise in the single-agent responsibility case where distance constraints are implemented unilaterally. Many works on formation control conceive of using directed links; in the special case when the associated graph is acyclic, the control problem is more easily grasped intuitively.

### *Constraint Consistence and Persistence of Formations*

We now consider conditions involving unilateral distance constraints that ensure that the motions of a formation are restricted to translations or rotations of the formation as a whole. This problem is examined in [28] for two dimensions and in [19] for three dimensions. We describe the two-dimensional result first. A notion of *persistence* is introduced, which is an amalgam of two conditions, specifically, rigidity as described above, and a further notion of *constraint consistence*. Rigidity says that, if certain interagent distances are maintained, then all interagent distances are maintained when the formation moves smoothly. Constraint consistence is the ability to maintain the specified interagent distances.

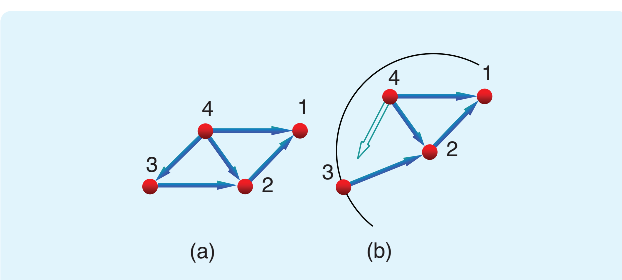


FIGURE 6 Failure to maintain constraint consistence. The rigid formation in (a) is not constraint consistent. In particular, (a) is rigid because if every edge distance constraint were to be enforced, then the shape of the formation would be maintained. However, the direction assignments mean that one agent, namely, agent 4, cannot simultaneously satisfy all the constraints it is responsible for. Agent 3 is indeed responsible for only one constraint and can therefore freely move on a circle centered on agent 2. If agent 3 does so as represented in (b), it becomes impossible for agent 4 to satisfy the three distance constraints that it is responsible for. Such a formation is called nonconstraint-consistent.

To illustrate constraint consistence, consider Figure 6. Suppose agents 1 and 2 are fixed, with agent 2 at its correct distance from agent 1. Suppose also that agent 3 is at its correct distance from agent 2, and agent 4 at its correct distance from agents 1, 2, and 3. Now observe that, since agent 3 has only one distance constraint, it can move, while maintaining its distance from agent 2, on a circle centered at agent 2. Agent 3 is unconscious of the constraint that agent 4 is supposed to maintain on the distance between agents 3 and 4. When agent 3 moves, agent 4 has an impossible task. Agent 4 can be in one of only two possible positions to maintain its correct distances from agents 1 and 2. For generic allowable positions of agent 3, agent 4 cannot maintain the correct distance from agent 3 from either of these two positions. We describe such an arrangement as being *not constraint consistent*. Evidently, too much is being asked of one agent.

In a constraint-consistent formation, no agent is given potentially impossible constraints, in the manner of agent 4 in Figure 6. We note the following key facts about constraint consistence and persistence; see [19] and [28] for precise definitions. First, a two-dimensional graph that has no more than two outgoing edges from every vertex is constraint consistent, although there are constraint consistent graphs that possess vertices having out-degree greater than two. Secondly, a graph can be checked for persistence, that is, rigidity plus constraint consistence, by testing a certain collection of subgraphs for rigidity. For further information, see "Persistence Testing."

### *Securing Persistence*

In light of the above remarks, a new question presents itself. Suppose that a two-dimensional undirected graph is rigid. Can we assign edge directions so that it is constraint consistent and thus persistent? The question in its full generality remains open. However, affirmative answers exist for minimally rigid graphs [28], as well as for graphs with certain structures, including wheel graphs, trilateration graphs, complete graphs, and power graphs of cycle graphs, which are all examined in [27].

The simplest algorithm for assigning directions in a minimally rigid graph is to consider the associated undirected graph and determine a Henneberg sequence that generates it. Then directions can be assigned to the edges that are added at each step, using the rule that any vertex can have no more than two outgoing edges [28], [29]. Such a directed graph is termed minimally persistent. Minimally persistent graphs are precisely those that are minimally rigid and constraint consistent.

Figure 7 shows some direction assignments for wheel graphs and $C^2$-graphs, both structures being attractive for operational autonomous agent formations. Given an arbitrary graph $G$, the graph $G^2$ (the square of $G$) is obtained by adding edges to $G$, linking each vertex to its two-hop neighbors, that is, to those vertices with which a neighbor vertex is shared. A $C^2$-graph is the square of a cycle graph, which is a graph that is simply a cycle. Wheel and $C^2$ graphs have robustness properties, in the form of tolerance of agent or link loss in the formation, corresponding to vertex or edge loss in the graph. A wheel graph with $N$ vertices, which has $2N-2$ edges, can tolerate the loss of any single edge while remaining persistent. A wheel graph can also tolerate the loss of any single vertex other than the central vertex together with the associated edges leaving or entering the lost vertex, in the sense that persistence is retained for the remaining graph. A $C^2$ graph with $N$ vertices, which has $2N$ edges, retains persistence with the loss of any single edge, or with the loss of any single vertex and its incident edges.

### Henneberg Sequence Theory and Persistent Graphs

In view of the use of Henneberg sequence theory in growing and deconstructing undirected graphs in two dimensions, extension of this theory to directed graphs is of interest [29]. The broad conclusion is that the technique can be applied, so long as the primitive operations are modified to allow directed edges in the graphs, and also a further primitive operation is introduced. More than one operation is possible, with the simplest possible operation being edge-reversal, that is, reversing the direction of one edge arriving at a vertex that has one or two degrees of freedom. In two dimensions, a vertex is said to have two, one, or zero degrees of freedom if it has zero, one, or at least two outgoing edges; each outgoing edge uses up one degree of freedom. A direct generalization applies in three dimensions, where an agent can have up to three degrees of freedom.

### Extension of the Persistence Concept to Three-Dimensional Formations

In three dimensions, most of the consistence and persistence ideas described above carry through. However, these extensions involve the behavior of subsets of agents, as opposed to individual agents or vertices. For three and indeed higher dimensions, the concept of structural persistence is required [19]. In three dimensions, checking structural persistence is trivial once persistence has been established. To illustrate structural persistence, Figure 8 depicts a three-dimensional formation with an underlying directed graph, which is persistent since it is rigid and each agent has no more than three outgoing edges. However, agents 1 and 2 are unconstrained, having no outgoing edges, and so in principle can move apart, thus destroying the shape of the formation. Hence, despite the persistence property, this formation is not structurally persistent, and thus does not have a sensing and control architecture that allows retention of its shape.

Every three-dimensional graph that has no more than three outgoing edges from every vertex is constraint consistent, while a graph can be checked for persistence by testing a certain collection of subgraphs for rigidity. If a directed graph is persistent, the graph can be checked for structural persistence. In particular, the graph is structurally persistent if it is persistent and there is at most one vertex of the graph with no outwardly directed edges.

If a graph is acyclic and persistent, it has at most one vertex with no outwardly directed edges. Because the persistence property requires that the total number of degrees of freedom summed across all vertices is six, and the only way this number can be achieved in an acyclic graph is by having one vertex with three degrees of freedom, one vertex with two degrees of freedom, one vertex with one degree of freedom, and the remainder with zero degrees of freedom, it follows that there is at most one vertex with no outwardly directed edges, and the graph is then structurally persistent.

## Persistence Testing

In two dimensions, the following test for persistence of a graph is given in [19] and [28].

Let $G$ be a directed graph representing a two-dimensional formation, and let $\{G_1, G_2, \ldots\}$ be the set of undirected graphs that are obtainable from $G$ by deleting edges outgoing from every vertex whose out-degree exceeds two, until exactly two outgoing edges remain for that vertex. Suppose, for example, that $G$ has exactly one vertex $A$ with out-degree three; one vertex $B$ with out-degree four; and the others with out-degree two or less. There are then three possibilities for deleting a single outgoing edge from $A$, and there are six ways to delete two edges from the four edges outgoing from $B$. Thus there are 18 different graphs $G_i$. Then *$G$ is persistent if and only if all undirected versions of the $G_i$ are rigid*.

In three dimensions, the same idea applies, except that outgoing edges in excess of three rather than two are deleted. A strengthened form of persistence, termed *structural persistence*, is useful for working with three-dimensional graphs.

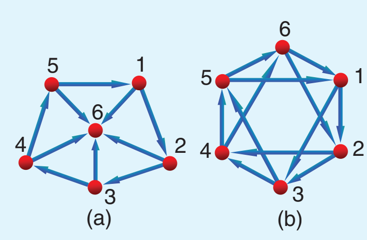


**FIGURE 7** Persistent graphs. (a) depicts a persistent wheel graph, while (b) illustrates a persistent $C^2$ graph, a generalization of a graph comprising a single cycle, obtained by inserting edges between two-hop neighbors of the original cycle. The persistence property follows from the fact that both graphs are rigid as well as the fact that no vertex has an out-degree larger than two. The direction assignment can be generalized to all wheel and $C^2$ graphs. A wheel graph is a convenient abstraction for a formation that has one of its agents as a commander or leader; a $C^2$ graph abstracts the reverse situation, in that all agents have the same task, and none is a leader.

## OPERATIONS WITH FORMATIONS

Various operations on formations can be contemplated. An early examination of formation reshaping is provided in [14], and splitting and merging are addressed in [31]. In [30], several operations involving formations are studied. In particular, the concepts of splitting, merging, and closing ranks are defined for formations that are modeled using undirected graphs.

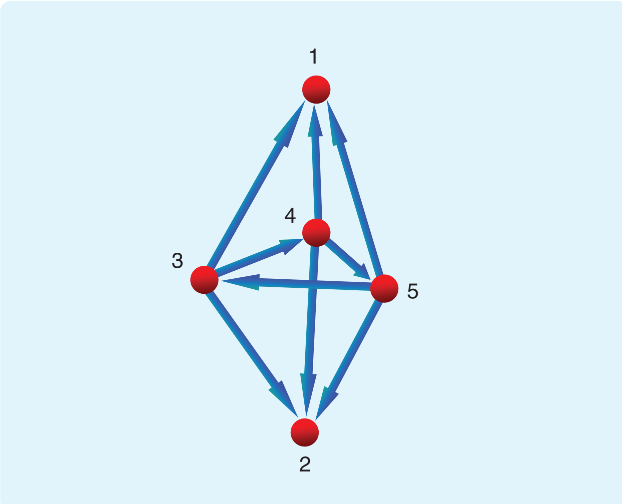


FIGURE 8 Persistence and structural persistence. According to the theoretical characterization of persistence, this three-dimensional formation is persistent because it can be proved to be rigid and none of its agents has an out-degree larger than three. This formation, however, does not qualify as a persistent formation according to the intuitive meaning that we give to persistence. Agents 1 and 2 are indeed not responsible for the maintenance of any distance and can move freely. Suppose that agents 1 and 2 move in such a way that the distance between them changes. Then agents 3, 4, and 5 cannot simultaneously satisfy all of the constraints that they are responsible for, although each of them could individually satisfy its three constraints if the positions of all other agents remained fixed. This paradox is solved by introducing the concept of *structural persistence*, which, in addition to considering the behavior of each agent, considers the behavior of groups of agents. The formation depicted here is persistent but not structurally persistent.

The practical implications of these operations as well as real-life scenarios triggering such operations are discussed in [27] and [30]. An application domain where such scenarios arise is terrain surveillance and target localization using a formation of aerial vehicles [27]. The individual vehicles carry sensors performing tasks such as range determination, bearing determination, or time-difference-of-arrival determination. In this application, acquiring and maintaining certain formation shapes and hence rigidity is essential because of the need for well-established coordination as well as optimality of certain formation geometries for cooperative localization. Formation shape maintenance within a class of allowable shapes may also be required in scenarios such as avoidance of obstacle collision or of entry into no-fly zones; this may be achieved by splitting, and merging back the split formations once the obstacle or no-fly zone is passed. Formation shape adjustment may also be needed for restructuring (closing ranks) to maintain rigidity and form an allowable shape in the event of loss of one or more aerial vehicle agents, or for addition of one or more aerial vehicle agents during a mission to improve surveillance coverage. Apart from rigidity preservation, the successful execution of these maneuvers requires detailed consideration of agent dynamics (since instantaneous changes of control strategy can produce undesired transient behavior), as well as inclusion of collision-avoidance protection. For details see [10].

### *Splitting*

Consider a single rigid formation. Splitting refers to the division of the set of agents into two subsets, along with suppression of the distance constraints between agent pairs when the agents are in different subsets. Reasons for splitting include a change of objective or obstacle avoidance. See Figure 9 for an illustration of the problem. In graph theory terms, the split yields two separate subgraphs, neither of which is guaranteed to be rigid. Introducing additional distance constraints in the separate subformations is thus required to ensure rigidity of each subformation separately. Additionally, we could consider variations of the problem. For example, we could assume that the starting formation is minimally rigid, we could consider two- and three-dimensional formations, and we could consider directed graph versions. We could also consider algorithm complexity, as well as the possibility of imposing computational constraints on individual agents to perform calculations on a decentralized basis for determining where additional distance constraints should be inserted. These problems are largely open.

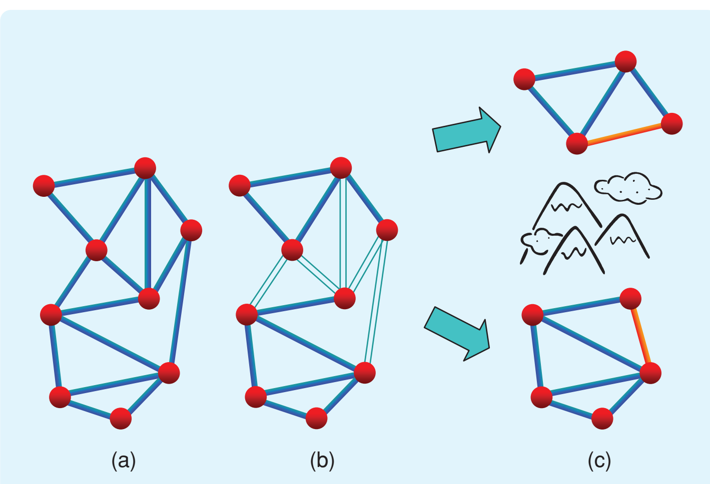


FIGURE 9 Formation splitting to avoid obstacles. Some distance constraints in (a) are removed (b) to separate the agents into two independent groups. The two formations obtained are not necessarily rigid, although rigidity can always be regained as in (c) by adding connections within each formation between agents that, in the original formation, are incident to the removed edges.

### Merging

Consider two rigid formations. Merging requires the determination of additional distance constraints linking agent pairs with one agent drawn from each formation, such that the union of the agents of the two formations, along with the union of the distance constraints in the original formations and the new distance constraints, describes a single rigid formation. Figure 10 illustrates the problem. A technique is given below for dealing with the problems of splitting, merging and closing ranks.

### Closing Ranks

Consider a single rigid formation. Suppose that one agent is removed, and, consequently all distance constraints are lost between this agent and the remaining agents of the formation; see Figure 11. The problem is to determine where new distance constraints can be inserted to recover rigidity. In addition, the closing ranks problem can be generalized to contemplate simultaneous removal of two or more agents from a formation, with the removal also of the associated distance constraints.

One way to solve these problems depends on a significant modification of the Henneberg sequence and is based on a *minimal cover* problem, which is introduced and solved in [32]. This minimal cover problem presents a graph that is not rigid and for which a minimum number of new edges must be introduced to render the graph generically rigid. The solution of the minimal cover problem can be applied to solve the problems of formation merging, splitting, and closing ranks. Additionally, the splitting problem is actually a particular case of the closing ranks problem, since one subformation can regard the agents of the other subformation as the lost agents. Further, the closing ranks problem can be solved by introducing new edges between former neighbors of the lost vertices of the graph, that is, by performing a *local* repair as illustrated in Figure 11. Therefore, in the splitting problem, new edges can be restricted to connecting pairs of those vertices in one subformation graph that were previously neighbors of vertices that ended up in the other subformation graph, as illustrated in Figure 9.

The formation operations discussed above can also be contemplated for directed graphs. Some results appear in [33] and [34].

## METAVERTICES, RIGID BODIES, AND METAFORMATIONS

In merging two formations, much of the internal structure is largely irrelevant. For example, if two rigid formations are to be merged in two dimensions, the merging can be done by introducing three distance constraints, each of which involves one agent in each of the two formations, while ensuring that, in each formation, at least two distinct agents are involved in the new distance constraints [35]. This conclusion holds irrespective of the internal structure of the formations. Establishing general rules for connecting formations to form larger formations, while preserving rigidity is thus of interest. Since for this purpose details of the internal connections of the individual formations are unimportant, we term the larger formation a metaformation [33], [36]. In this connection, we note two streams of work.

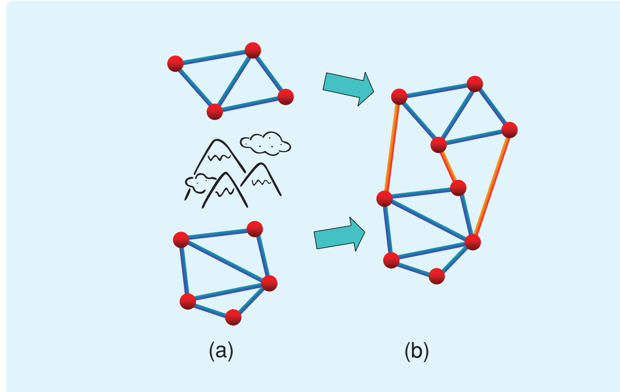


FIGURE 10 Formation merging. (a) Two distinct rigid formations are merged into the larger rigid formation (b) by inserting three interconnecting edges. At least three interconnecting edges are needed, and merging to secure a larger rigid formation can always be achieved using exactly three interconnecting edges. In that case, the formation obtained is rigid if and only if at least two distinct agents in each formation are incident to the new interconnecting edges.

### Rigidity and Two-Dimensional Formations of Formations

*Body-bar systems*, investigated in [37] and [38], conceptually underpin the metaformation view of merging problems. A body is a generalization of a point agent. Any rigid formation of agents can be replaced by a body, a rigid object that in two dimensions has three degrees of freedom, two displacements and one rotation. In contrast, a point agent in two dimensions has two degrees of freedom, both translational. Each body can be deemed to have a set of connection points on its surface, with the property that distances can be constrained between two connection points in distinct bodies by inserting a joining bar. We can imagine a framework comprising a set of bodies, each of which might also be termed a metavertex or meta-agent, with certain distance constraints between them. Usually more than one

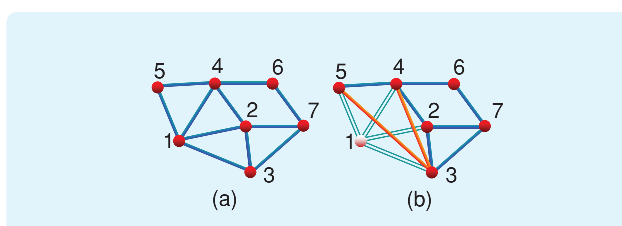


FIGURE 11 Closing ranks. In the rigid formation in (a), agent 1 and all associated links are lost, and new links (5,3) and (4,3) are inserted as in (b) to reestablish rigidity. Agents 6 and 7 are not affected by the process. Rigidity can always be recovered after loss of one or more agents by appropriately connecting agents that are adjacent in the original formation to the lost agent(s).

connection point on the surface of a body is used; for if only one connection point were used, the body or the metavertex would be free to rotate about the connection point and the individual agents in it would then no longer remain at constant distance from the agent at the other end of the distance constraint from the connection point. A key problem is to determine when such a framework, better termed a metaformation given that it is obtained by interconnecting bodies, is rigid. Of course, we want to answer this problem without accounting for the internal structure of the bodies.

Rigidity is characterized for metaformations of bodies in [37] and [38], using both a generalization of the rigidity matrix and a generalization of Laman's theorem for the two-dimensional case. Recall that Laman's theorem; "What Does Laman's Theorem Say?" provides necessary and sufficient conditions for generic rigidity of a graph corresponding to a formation of point agents, and the conditions are of a counting form; a simple adjustment of certain numbers appearing in the statement of Laman's theorem converts the result to a theorem concerning generic rigidity of a graph corresponding to a two-dimensional body-framework. However, as for checking rigidity of a normal graph in three dimensions, there is no known necessary and sufficient counting condition in the style of Laman's theorem for a three-dimensional body-bar framework to be rigid. On the other hand, the rigidity matrix ideas work in three-dimensional space, where the bodies have six degrees of freedom, namely, three translational degrees of freedom and three rotational degrees of freedom [37].

Interconnection of two formations involves the interconnection of two bodies, and the extension of Laman's theorem states that three distance constraints between connection points on each of the two bodies, with at least two connection points involved for each body, implies rigidity of the overall metaformation. This idea is extendable to the problem of merging more than two formations (metavertices) and agents (vertices) [36]. This type of result holds also for directed graphs [33], [34].

### More on Formation Merging

Besides the problem described above of connecting two formations in two dimensions, several other related problems can be considered [35]. These related problems include connecting, by means of insertion of additional edges, two formations in three dimensions to secure minimal rigidity, connecting two formations in two or three dimensions to secure global rigidity, and connecting two formations when they are not disjoint. Nondisjoint formations may have a limited number of common vertices, or a limited number of common edges, or both.

By appealing to various results on rigidity and global rigidity, various conditions are established to solve these problems. These conditions require the insertion of a minimum or exactly specified number of new connections that are incident on a minimum specified number of vertices in each of the two merging formations. We give two examples to illustrate the results.

Consider two globally rigid two-dimensional graphs, with one vertex in common. By adding two new edges with one vertex in each formation, and such that in at least one of the formations the edges are incident on two distinct vertices, we obtain a globally rigid graph. Note that the two added edges cannot be incident on the vertex common to the two initially given graphs.

Consider two minimally rigid graphs of three-dimensional formations, with no vertices in common. Merging requires the addition of at least six new edges, with the new edge set incident in each graph on at least three vertices. Most patterns involving precisely six edges ensure that the merged graph is minimally rigid. Acceptable patterns include every pattern such that each graph has exactly three vertices that the six new edges are incident on. Each graph can have either two new incident edges for each of the three vertices, or one, two, and three edges incident on the three vertices. A further set of patterns can be obtained using any vertex that has two or more new incident edges, by moving one of those incidences to a new vertex in the same graph on which no new edge is yet incident. In this way, up to six vertices in each of the two merging graphs can have a new edge incident on it.

A related result is that if the two three-dimensional graphs are rigid but not necessarily minimally rigid, inserting six new edges using the same incidence rules yields a rigid graph. Such an interconnection can be regarded as minimally rigid from the metaformation point of view, since the issue of whether or not the individual metavertices are minimally rigid internally is irrelevant.

Directed versions of these results are given in [34]. Conditions for securing persistence include the conditions applicable to securing rigidity as discussed above. For example, to merge two minimally persistent graphs in three dimensions into a larger minimally persistent graph, we need to insert six directed interconnection edges that leave vertices with some degrees of freedom in the initial pre-merging graphs, thereby using one degree of freedom for each outgoing edge, although the added edges can arrive at any three or more vertices of the other initial pre-merging graph. In three dimensions, the degrees of freedom of a vertex are three, two, one or zero, according to whether the vertex has, respectively, zero, one, two, three or more outgoing edges. Not every selection of interconnection edges leads to a persistent merged graph, although, if one exists, then a set of interconnection edges that make the merged graph structurally persistent can always be found, even when the initial persistent graphs are not structurally persistent.

If the merged graph must be persistent but not necessarily minimally persistent, we need to add six directed edges that must leave vertices with positive degrees of freedom. Further, the number of new edges leaving a vertex with positive degrees of freedom must be no greater than its degree of freedom count. Other edges, possibly leaving vertices with zero degrees of freedom, can under some conditions also be added, although such additions can always be avoided. As a consequence, at least six degrees of freedom must be available in the two initial graphs. the presence of six degrees of freedom is, however, not guaranteed in nonminimally persistent graphs, in which case the two graphs cannot be merged.

In three dimensions, if only two agents in the two persistent graphs have positive degrees of freedom, then the two graphs cannot be merged into a single persistent graph. However, in every other case, if at least six degrees of freedom are available, six directed interconnection edges can always be chosen to make the merged graph persistent and even structurally persistent.

Figure 12 illustrates merging two persistent two-dimensional formations using directed edges. As with undirected graphs, three new edges incident on at least two agents in each formation can achieve the merging. To ensure persistence of the overall structure, the vertices left by these three new edges must have in the merged structure a maximum out-degree of two. This upper bound restricts the vertices at which outwardly directed edges must be added. In Figure 12(a), all the degrees of freedom of the merged formation end up associated with the upper subformation, which in a sense is a leader. The lower subformation is the follower.

For three-dimensional formations, structural persistence of the merged formation is also desired. Requirements are set out in [34].

### Toward a More Systematic Theory

In two dimensions we have described a variation of Laman's theorem describing the rigidity of a metaformation, obtained by connecting together metavertices or meta-agents. This result leads to the question of whether the concept of a Henneberg sequence can be extended to metaformations. Such a sequence could start with a single metavertex, or rigid formation, and involve the successive addition of meta-agents to the metaformation. Each addition would result in a metaformation that has the minimal number of edges between metavertices so as to guarantee rigidity of the overall metaformation. Indeed, such a construction is available. Analogues to the primitive operations of the standard Henneberg construction, namely, vertex addition and edge splitting, can be constructed. These operations are termed metavertex addition and metaedge splitting, respectively. See, for example, Figure 13. The construction can also be described by building on the results described in the previous section; see [36].

## CONCLUSIONS

This article sets out the rudiments of a theory for analyzing and creating architectures appropriate to the control of formations of autonomous vehicles. The theory rests on ideas of rigid graph theory, some but not all of which are old. The theory, however, has some gaps in it, and their elimination would help in applications. Some of the gaps in the relevant graph theory are as follows. First, there is as yet no analogue for three-dimensional graphs of Laman's theorem, which provides a combinatorial criterion for rigidity in two-dimensional graphs. Second, for three-dimensional graphs there is no analogue of the two-dimensional Henneberg construction for growing or deconstructing

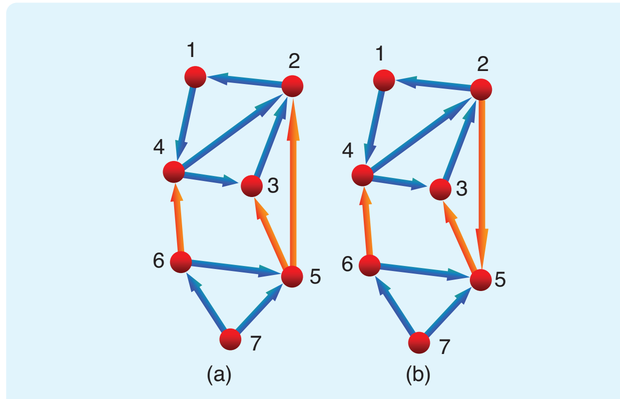


**FIGURE 12** Merging persistent graphs. Two types of merging are illustrated, (a) *leader-follower merging* and (b) *collaborative merging*. In both cases, two persistent formations are merged into a larger persistent formation by inserting three directed interconnecting edges. For the obtained graph to be persistent, it is necessary to insert at least three edges and that the agents that these edges leave do not have an out-degree larger than two in the resulting formation. The formation is then persistent if and only if it is rigid. Under some conditions, alternative interconnecting edges are possible, but using these other possibilities is never necessary.

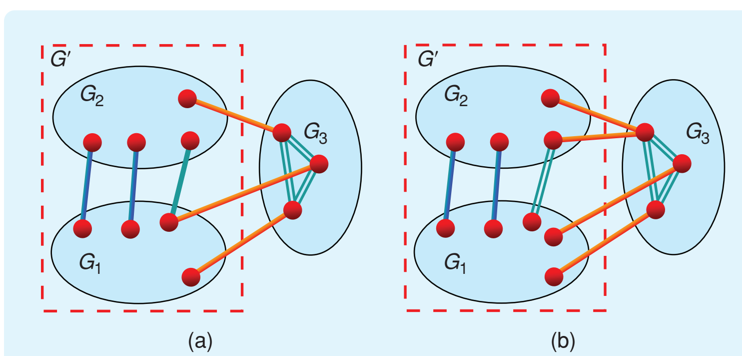


**FIGURE 13** Merging three two-dimensional formations. Two possible ways to merge three rigid two-dimensional formations are illustrated. In the first step, $G'$ is obtained by connecting $G_1$ to $G_2$ with three interconnecting edges. This operation is called *meta-vertex addition*. $G_3$ can then be merged with $G'$ using meta-vertex addition as in (a) or by means of *meta-edge* splitting as in (b). In the latter case, one of the edges connecting $G_1$ to $G_2$ is split into two parts, which are reconnected to an agent in $G_3$. Two additional edges are then inserted to connect $G_3$ to $G'$.

minimally rigid graphs although there are conjectures [15]. Third, global rigidity can easily be characterized for two-dimensional graphs, but not for three-dimensional graphs.

An interesting extension is the need to study graphs associated with formations in dimensions higher than three, because the physical agents being abstracted in the graphs may have orientation as well as position, all in three-dimensional Euclidean space. Generally speaking, results for dimensions higher than three are less well documented than for three dimensions although some can certainly be found [16].

In addition to problems relevant to three-dimensional formations, challenging issues remain in the use of graph theory for two-dimensional formations. Real-world formations are often not minimally rigid but rather are constructed to provide redundancy, in effect an ability to sustain loss of an agent or sensing link. However, the relevant literature lacks a single agreed measure of redundancy of a formation, as well as understanding of which formation shapes with a prescribed number of agents and distance constraints in two- or three-dimensional space are the most robust to agent and distance constraint loss.

At some point, we need to dive below the graph theory and consider the distributed control laws that agents use to maintain the shape of a formation, both in the undirected and the directed graph case. Results are becoming available in this area [39], with much more work having been done on formations modeled by undirected graphs. Work has also been done on acyclic directed formations, but when a cycle is present in a directed graph, the control law design problem is not at all straightforward and much less work has been done.

Such study is expected to reveal which formations are difficult to control, in the sense of requiring accurate sensors and large control signals. The formations may be those whose smallest nonzero singular value of the rigidity matrix is close to zero, and this idea with some generalization may carry over to the case where information is sensed other than or in addition to just distances, such as angles or differences of distances obtained from time-difference-of-arrival information. Further issues relevant at the agent control level include those of individual agent dynamics, the possibility of orientation constraints, and constraints on kinematic quantities, such as speed, acceleration, and angular velocity.

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