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Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space

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SUMMARY

We study the local asymptotic stability of undirected formations of single-integrator and double-integrator modeled agents based on interagent distance control. First, we show that n-dimensional undirected formations of single-integrator modeled agents are locally asymptotically stable under a gradient control law. The stability analysis in this paper reveals that the local asymptotic stability does not require the infinitesimal rigidity of the formations. Second, on the basis of the topological equivalence of a dissipative Hamiltonian system and a gradient system, we show that the local asymptotic stability of undirected formations of double-integrator modeled agents in n-dimensional space is achieved under a gradient-like control law. Simulation results support the validity of the stability analysis. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: formation control; distance-based control; graph rigidity; Hamiltonian systems; gradient systems

1. INTRODUCTION

We study the stability of formations of mobile agents under interagent distance control. Because the interagent distances are controlled to achieve the desired formation, such an approach might be called distance-based formation control. The distance-based formation control has recently attracted research interest, because it is assumed that agents do no share a common coordinate system, and thus, it is of interest in theoretical and practical aspects.

In the literature, gradient control laws have been dominantly employed for formation control under the distance-based problem setup. The local asymptotic stability of undirected formations of single integrators has been addressed in [1–3]. The stability of cycle-free persistent formations has been studied in [1, 4]. Triangular formations have been studied in [2, 5, 6], and tree formations have been studied in [7].

Because the existing results have primarily focused on the formations of single-integrator modeled agents in the plane, we study stability of formations of single-integrator and double-integrator modeled agents in general n-dimensional space in this paper. Accordingly, the contributions of this paper can be summarized as follows. First, we show that undirected formations of single-integrator modeled agents in n-dimensional space are locally asymptotically stable under a gradient control law. Although the dynamics of the agents is described as a gradient system, the well-known stability analysis results on gradient systems [8] are not straightforwardly applicable to the agent case because the equilibrium points are not isolated. Focusing on the dynamics of interagent distances, we provide an elegant approach for the stability analysis of distance-based

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formations. Second, the stability analysis in this paper reveals that any rigid formations are locally asymptotically stable under the gradient control law. Whereas the stability analyses found in [1–3] require desired formations to be infinitesimally rigid, we relax the condition in this paper. Finally, we prove the local asymptotic stability of undirected formations of double-integrator modeled agents in n-dimensional space under a gradient-like control law. Formation dynamics of the agents under the control law can be described as a dissipative Hamiltonian system. On the basis of the topological equivalence of the dissipative Hamiltonian system and a gradient system, which has been studied in [9], we show that the local asymptotic stability of the undirected formations of double-integrator modeled agents is achieved. Although the author of [10] has addressed the stability of undirected formations of double-integrator modeled agents by means of the LaSalle invariance principle, as pointed out in [1], it is not certain whether the principle can be applied because the equilibrium set of the formation dynamics is not compact.

The outline of this paper is as follows. In Section 2, mathematical background is reviewed and formation control problems are formulated. In Sections 3 and 4, stability of undirected formations of single-integrator and double-integrator modeled agents is analyzed, respectively. Simulation results are provided in Section 5. Conclusion is then given in Section 6.

2. PRELIMINARIES

The set of nonnegative (respectively, positive) real numbers is denoted by $\bar{\mathbb{R}}_+$ (respectively, $\mathbb{R}_+$). Given a set S, $|S|$ denotes the cardinality of S. Given a vector x, $\|x\|$ denotes the Euclidean norm of x. Given a matrix A, $\mathrm{Im}(A)$ denotes the image of A. The rank of A is denoted by $\mathrm{Rank}(A)$. The matrix I_n denotes the n-dimensional identity matrix. Given two matrices A and B, $A \otimes B$ denotes the Kronecker product of the matrices.

2.1. *Basic notions on graphs*

A undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as a pair of a finite set of nodes $\mathcal{V}$ and a finite set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$ for all $i, j \in \mathcal{V}$. Note that an undirected graph always has even number of edges. We often refer to an undirected graph as a graph if there is no ambiguity. If $(i,j) \in \mathcal{E}$, then the node i (respectively, j) is the sink (respectively, source) node of the edge. The neighbor set $\mathcal{N}_i$ of the node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$.

For the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the edge set $\mathcal{E}$ can be partitioned as $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$ such that $\mathcal{E}_+$ and $\mathcal{E}_-$ are disjoint and $(i,j) \in \mathcal{E}_+$ if and only if $(j,i) \in \mathcal{E}_-$. Let $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E}_+ = \{\epsilon_{+,1}, \ldots, \epsilon_{+,M}\}$. Then, we define $H_+ = [h_{+,ij}] \in \mathbb{R}^{N \times M}$ as

$$h_{+,ij} := \begin{cases} 1, & \text{if } i \text{ is the sink node of } \epsilon_{+,j}, \\ -1, & \text{if } i \text{ is the source node of } \epsilon_{+,j}, \\ 0, & \text{otherwise,} \end{cases}$$

which corresponds to the incidence matrix of the directed graph $(\mathcal{V}, \mathcal{E}_+)$.

2.2. *Graph rigidity*

For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E}_+ = \{\epsilon_{+,1}, \ldots, \epsilon_{+,M}\}$, let $p_i \in \mathbb{R}^n$ be the point that is assigned to $i \in \mathcal{V}$. Then, $p = \left[p_1^T \cdots p_N^T\right]^T \in \mathbb{R}^{nN}$ is said to be a realization of $\mathcal{G}$ in $\mathbb{R}^n$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^n$. By ordering edges in $\mathcal{E}_+$, an edge function $g_{\mathcal{G}} : \mathbb{R}^{nN} \to \mathbb{R}^M$ associated with $(\mathcal{G}, p)$ is defined as

$$g_{\mathcal{G}}(p) := \frac{1}{2}\left[\cdots \|p_i - p_j\|^2 \cdots\right]^T, \ \forall (i,j) \in \mathcal{E}_+. \tag{1}$$

The rigidity of frameworks is then defined as follows:

Definition 2.1 ([11]*)*
A framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^n$ if there exists a neighborhood U_p of $p \in \mathbb{R}^{nN}$ such that $g_{\mathcal{G}}^{-1}(g_{\mathcal{G}}(p)) \cap U_p = g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p)) \cap U_p$, where $\mathcal{K}$ is the complete graph on N-nodes. Further, the framework $(\mathcal{G}, p)$ is globally rigid in $\mathbb{R}^n$ if $g_{\mathcal{G}}^{-1}(g_{\mathcal{G}}(p)) = g_{\mathcal{K}}^{-1}(g_{\mathcal{K}}(p))$.

Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are said to be equivalent if $g_{\mathcal{G}}(p) = g_{\mathcal{G}}(q)$, that is, $\|p_i - p_j\| = \|q_i - q_j\|$ for all $(i, j) \in \mathcal{E}_+$. Further, they are said to be congruent if $\|p_i - p_j\| = \|q_i - q_j\|$ for all $i, j \in \mathcal{V}$. Thus, the framework $(\mathcal{G}, p)$ is rigid if there exists a neighborhood U_p of $p \in \mathbb{R}^{nN}$ such that, for any $q \in U_p$, if $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are equivalent, then they are congruent.

Let m be the dimension of convex hull of $\{p_1, \ldots, p_N\}$. The framework $(\mathcal{G}, p)$ is then said to be infinitesimally rigid in $\mathbb{R}^n$ if $\mathrm{Rank}(\partial g_{\mathcal{G}}(p)/\partial p) = nN - (m+1)(2n-m)/2$.

2.3. Problem statement

Consider the following N single-integrator modeled agents in n-dimensional space.

$$\dot{p}_i = u_i,\ i = 1, \ldots, N, \tag{2}$$

where $p_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the coordinates of the position and the control input, respectively, of agent i with respect to the global Cartesian coordinate system ${}^g\sum$. We assume that the agents do not necessarily share a common sense of orientation. Because of the absence of a common sense of orientation, agent $i \in \{1, \ldots, N\}$ maintains its own local Cartesian coordinate system, whose origin is located at p_i and orientation is not aligned with ${}^g\sum$. Further, the orientations of the local coordinate systems are not aligned with each other. We denote the local coordinate system of agent i by ${}^i\sum$. By adopting a notation in which superscripts are used to denote coordinate systems, the position dynamics of the agents can be written as

$$\dot{p}_i^i = u_i^i,\ i = 1, \ldots, N, \tag{3}$$

where $p_i^i \in \mathbb{R}^n$ and $u_i^i \in \mathbb{R}^n$ are the coordinates of the position and the control input, respectively, of agent i with respect to ${}^i\sum$.

The sensing topology among the agents is modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We refer to the graph as the sensing graph of the agents. We then assume that each agent measures the relative positions of its neighboring agents with respect to its own local coordinate system. Thus, the following measurements are available to agent $i \in \mathcal{V}$,

$$p_{ji}^i := p_j^i - p_i^i \equiv p_j^i,\ \forall j \in \mathcal{N}_i, \tag{4}$$

where p_j^i is the coordinate of the position of agent j with respect to ${}^i\sum$.

For a given realization $p^* = \left[p_1^{*T} \cdots p_N^{*T}\right]^T \in \mathbb{R}^{nN}$, we define the desired formation E_{p^*} of the agents as the set of formations that are congruent to p^*:

$$E_{p^*} := \left\{p \in \mathbb{R}^{nN} : \|p_j - p_i\| = \|p_j^* - p_i^*\|,\ \forall i, j \in \mathcal{V}\right\}. \tag{5}$$

Then, the formation control problem for the single-integrator modeled agents is stated as follows:

Problem 2.1
For N single-integrator modeled agents (2) in n-dimensional space, suppose that the sensing graph of the agents is given by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Given a realization $p^* \in \mathbb{R}^{nN}$, design a decentralized control law on the basis of measurements (4) such that E_{p^*} is asymptotically stable under the decentralized control law.

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A formation control problem for double-integrator modeled agents is similarly formulated. We consider the following N double-integrator modeled agents in n-dimensional space:

$$\begin{cases} \dot{p}_i = v_i, \\ \dot{v}_i = u_i, \end{cases} \quad i = 1, \ldots, N, \tag{6}$$

where $p_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^n$, and $u_i \in \mathbb{R}^n$ denote the position, the velocity, and the control input, respectively, of agent i with respect to ${}^g\sum$. Then, we assume that each agent measures its own velocity and the relative positions of its neighbors with respect to its own local coordinate system. Further, given a realization $p^* = \left[p_1^{*T} \cdots p_N^{*T}\right]^T \in \mathbb{R}^{nN}$, we define the desired formation E_{p^*,v^*} of the agents as

$$E_{p^*,v^*} := \left\{\left[p^T \ v^T\right]^T \in \mathbb{R}^{2nN} : \|p_j - p_i\| = \|p_j^* - p_i^*\|, \ v = 0, \ \forall i, j \in \mathcal{V}\right\}.$$

Then, the formation control problem for the double-integrator modeled agents is stated as follows:

Problem 2.2
For N double-integrator modeled agents (6) in n-dimensional space, suppose that the sensing graph of the agents is given by a undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Given a realization $p^* \in \mathbb{R}^{nN}$, design a decentralized control law on the basis of the velocities of the agents and measurements (4) such that E_{p^*,v^*} is asymptotically stable under the decentralized control law.

3. UNDIRECTED FORMATIONS OF SINGLE INTEGRATORS

3.1. Gradient control law

Consider the single-integrator modeled agents under the assumptions of Problem 2.1. For each agent i, let us define a local potential $\phi_i : \mathbb{R}^{n(|\mathcal{N}_i|+1) \to \bar{\mathbb{R}}_+}$ as follows:

$$\phi_i\left(p_i^i, \ldots, p_j^i, \ldots\right) := \frac{k_p}{2} \sum_{j \in \mathcal{N}_i} \gamma\left(\|p_j^i - p_i^i\|^2 - \|p_j^* - p_i^*\|^2\right), \tag{7}$$

where $k_p > 0$ and $\gamma : \mathbb{R} \to \bar{\mathbb{R}}_+$ is a function that satisfies the following assumption:

Assumption 3.1
The function $\gamma : \mathbb{R} \to \bar{\mathbb{R}}_+$ satisfies the following conditions:

- *Positive definiteness*: $\gamma(x) \geqslant 0$ for any $x \in \mathbb{R}$ and $\gamma(x) = 0$ if and only if $x = 0$;
- *Analyticity*: γ is analytic in a neighborhood of 0.

On the basis of the local potential function ϕ_i, a control law for agent i can be designed as

$$\begin{aligned} u_i^i &= -\nabla_{p_i^i} \phi_i\left(p_i^i, \ldots, p_j^i, \ldots\right) \\ &\equiv -\left[\frac{\partial \phi_i\left(p_i^i, \ldots, p_j^i, \ldots\right)}{\partial p_i^i}\right]^T \\ &= -\left[\frac{k_p}{2} \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma\left(\tilde{d}_{ji}\right)}{\partial \tilde{d}_{ji}} \frac{\partial \tilde{d}_{ji}}{\partial p_i^i}\right]^T \\ &= k_p \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma\left(\tilde{d}_{ji}\right)}{\partial \tilde{d}_{ji}} p_j^i, \end{aligned} \tag{8}$$

where $\tilde{d}_{ij} := \|p_j^i - p_i^i\|^2 - \|p_j^* - p_i^*\|^2$. It is worth noting that the gradient control law (8) is decentralized in the sense that it can be implemented in the local coordinate systems of the agents by using only the measurements (4).

Although the gradient control law (8) is implemented in ${}^i\sum$ in practice, it is convenient to represent the dynamics agents with respect to ${}^g\sum$ for stability analysis. Thus, we represent (8) with respect to ${}^g\sum$ by a suitable coordinate transformation. Let $\mathcal{B}^g$ be the standard basis for ${}^g\sum$ and $\mathcal{B}^i$ for ${}^i\sum$. Further, let $[\mathcal{I}_n]_{\mathcal{B}^g}^{\mathcal{B}^i}$ be the $\mathcal{B}^i - \mathcal{B}^g$ basis representation of the n-dimensional identity linear transformation. Then, we have the following relationship [12]:

$$p_j - p_i = [I_n]_{\mathcal{B}^g}^{\mathcal{B}^i} p_j^i, \tag{9a}$$

$$u_i = [I_n]_{\mathcal{B}^g}^{\mathcal{B}^i} u_i^i. \tag{9b}$$

On the basis of (9), the gradient control law (8) can be represented with respect to ${}^g\sum$ as follows:

$$\begin{aligned} u_i &= [\mathcal{I}_n]_{\mathcal{B}^g}^{\mathcal{B}^i} u_i^i \\ &= [\mathcal{I}_n]_{\mathcal{B}^g}^{\mathcal{B}^i} k_p \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma(\tilde{d}_{ji})}{\partial \tilde{d}_{ji}} p_j^i \\ &= k_p \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma(\tilde{d}_{ji})}{\partial \tilde{d}_{ji}} (p_j - p_i), \end{aligned}$$

which shows that

$$u_i = -\nabla_{p_i} \phi_i(p_i, \ldots, p_j, \ldots). \tag{10}$$

Note that $\tilde{d}_{ij} = \|p_j^i - p_i^i\|^2 - \|p_j^* - p_i^*\|^2 \equiv \|p_j - p_i\|^2 - \|p_j^* - p_i^*\|^2$.

3.2. *Stability analysis*

In the following paragraphs, we first show that the overall dynamics of the agents can be described as a gradient system. Although the properties of gradient systems are well known [8], the existing results are not directly applicable to the gradient system because its critical points are not isolated in general. Further, the noncompactness of the desired formation E_{p^*} makes stability analysis complicated. Note that E_{p^*} contains all realization congruent to p^*. To overcome this challenging point, we then describe the overall dynamics of the agents by the interagent displacements, thereby defining the desired formation as a compact set. This allows us to investigate the stability of the desired formation based on a property of gradient systems.

We define a set E'_{p^*}, which is the set of realizations that are equivalent to $(\mathcal{G}, p^*)$, as follows:

$$E'_{p^*} := \left\{ p \in \mathbb{R}^{nN} : \|p_j - p_i\| = \|p_j^* - p_i^*\|,\ \forall (i,j) \in \mathcal{E}_+ \right\}. \tag{11}$$

Obviously, $E_{p^*} \subseteq E'_{p^*}$. In the case that $(\mathcal{G}, p^*)$ is rigid, for any $\bar{p} \in E_{p^*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that $E_{p^*} \cap U_{\bar{p}} = E'_{p^*} \cap U_{\bar{p}}$. In the following, we show the local asymptotic stability of E'_{p^*}. The local asymptotic stability of E_{p^*} is then followed by the rigidity of $(\mathcal{G}, p^*)$.

To show that the overall dynamics of the agents can be described as a gradient system, we define a global potential function ϕ for the agents as

$$\phi(p) := \frac{k_p}{2} \sum_{(i,j) \in \mathcal{E}_+} \gamma\left(\|p_j - p_i\|^2 - \|p_j^* - p_i^*\|^2\right). \tag{12}$$

On the basis of the fact that

$$u_i = -\nabla_{p_i} \phi_i(p_i, \ldots, p_j, \ldots) \equiv -\nabla_{p_i} \phi(p),$$

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the overall dynamics of the agents under the gradient control law (10) can be described as the following gradient system:

$$\dot{p} = -\nabla\phi(p). \tag{13}$$

Although the stability property of an isolated equilibrium point of gradient systems is well-known [8], the desired formation E'_{p^*} consists of nonisolated points.

The noncompactness of E'_{p^*} makes stability analysis complicated. To avoid the complicatedness, we describe the dynamics by interagent displacements. We define the link $e \in \mathbb{R}^{nM}$ of $(\mathcal{G}, p)$ as

$$e = \left[e_1^T \cdots e_M^T\right]^T := \left(H_+^T \otimes I_n\right) p, \tag{14}$$

which has been introduced in [3]. From the definition (14), e belongs to the column space of $H_+^T \otimes I_n$, that is, $e \in \operatorname{Im}\left(H_+^T \otimes I_n\right)$. The space $\operatorname{Im}\left(H_+^T \otimes I_n\right)$ is referred to as the link space associated with the framework $(\mathcal{G}, p)$. We then define a function $v_{\mathcal{G}} : \operatorname{Im}\left(H_+^T \otimes I_n\right) \to \mathbb{R}^M$ as

$$v_{\mathcal{G}}(e) := \frac{1}{2}\left[\|e_1\|^2 \cdots \|e_M\|^2\right]^T,$$

which can be regarded as the edge function $g_{\mathcal{G}}$ parameterized in the link space. That is, without loss of generality, we can assume that $g_{\mathcal{G}}(p) \equiv v_{\mathcal{G}}\left(\left(H_+^T \otimes I_n\right) p\right)$. Define

$$D(e) := \operatorname{diag}(e_1, \ldots, e_M)$$

to obtain

$$\begin{aligned}\frac{\partial g_{\mathcal{G}}(p)}{\partial p} &= \frac{\partial v_{\mathcal{G}}(e)}{\partial e}\frac{\partial e}{\partial p}\\ &= [D(e)]^T\left(H_+^T \otimes I_n\right),\end{aligned}$$

which leads to

$$\begin{aligned}\dot{p} = -\nabla\phi(p) &\equiv -\left[\frac{\partial\phi(p)}{\partial p}\right]^T\\ &= -\left[\frac{\partial\phi(p)}{\partial\tilde{d}}\frac{\partial\tilde{d}}{\partial p}\right]^T\\ &= -\left[\frac{\partial\phi(p)}{\partial\tilde{d}}\frac{\partial g_{\mathcal{G}}(p)}{\partial p}\right]^T\\ &= -k_p\left(H_+ \otimes I_n\right) D(e)\Gamma\left(\tilde{d}\right),\end{aligned}$$

where $\tilde{d} := \left[\|e_1\|^2 - \|e_1^*\|^2 \cdots \|e_M\|^2 - \|e_M^*\|^2\right]^T$, $e^* = \left[e_1^{*T} \cdots e_M^{*T}\right]^T := \left(H_+^T \otimes I_n\right) p^*$, and

$$\Gamma\left(\tilde{d}\right) := \left[\frac{\partial\gamma\left(\tilde{d}_1\right)}{\partial\tilde{d}_1} \cdots \frac{\partial\gamma\left(\tilde{d}_M\right)}{\partial\tilde{d}_M}\right]^T.$$

Then, the gradient system (13) can be described in the link space as follows:

$$\begin{aligned}\dot{e} &= \left(H_+^T \otimes I_n\right)\dot{p}\\ &= -k_p\left(H_+^T \otimes I_n\right)(H_+ \otimes I_n)D(e)\Gamma\left(\tilde{d}\right).\end{aligned} \tag{15}$$

Further, the set E'_{p^*} can be parameterized in the link space as follows:

$$E'_{e^*} := \left\{e \in \operatorname{Im}\left(H_+^T \otimes I_n\right) : \|e_i\| = \|e_i^*\|,\ \forall i = 1, \ldots, M\right\}.$$

As remarked in [3], E'_{e^*} is compact, whereas E'_{p^*} is not. We exploit the compactness of E'_{e^*} in the proof of Theorem 3.2.

To analyze the stability of E'_{e^*}, we define $V : \text{Im}\left(H_+^T \otimes I_n\right) \to \bar{\mathbb{R}}_+$ as

$$V(e) := \sum_{i=1}^{M} \frac{1}{2}\gamma\left(\|e_i\|^2 - \|e_i^*\|^2\right).$$

The time derivative of V can be arranged as

$$\begin{aligned}\dot{V}(e) &= \frac{\partial V(e)}{\partial e}\dot{e} \\ &= -k_p \frac{\partial V(e)}{\partial e}\left(H_+^T \otimes I_n\right)(H_+ \otimes I_n)\, D(e)\Gamma\left(\tilde{d}\right) \\ &= -k_p \underbrace{\left[D(e)\Gamma\left(\tilde{d}\right)\right]^T (H_+ \otimes I_n)^T}_{=-[\nabla\phi(p)]^T} \underbrace{(H_+ \otimes I_n) D(e)\Gamma\left(\tilde{d}\right)}_{=-\nabla\phi(p)} \\ &= -k_p\|\nabla\phi(p)\|^2 \\ &\leqslant 0,\end{aligned}$$

which shows the local stability of E'_{e^*}. Then, the local asymptotic stability of E'_{e^*} can be ensured by showing the existence of a neighborhood $U_{E'_{e^*}}$ of E'_{e^*} such that, for any $e \in U_{E'_{e^*}}$, if $e \notin E_{e^*}$, then $\dot{V}(e) < 0$.

The following inequality, which is known as the Lojasiewicz theorem, is useful for the stability analysis of gradient systems.

Theorem 3.1 ([13]*)*
Suppose that $f : D \subseteq \mathbb{R}^{n_f} \to \mathbb{R}$ is a real analytic function in a neighborhood of $z \in D$. There exist constants $k_f > 0$ and $\rho_f \in [0, 1)$ such that

$$\|\nabla f(x)\| \geqslant k_f \|f(x) - f(z)\|^{\rho_f}$$

in some neighborhood of z.

On the basis of Theorem 3.1, we obtain the following lemma:

Lemma 3.1
For any $\bar{p} \in E'_{p^*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that, for any $p \in U_{\bar{p}}$ and $p \notin E'_{p^*}$, $\|\nabla\phi(p)\| > 0$.

Proof
Because γ is analytic in some neighborhood of 0, for any $\bar{p} \in E'_{p^*}$, there exists a neighborhood of $\bar{p}$ such that ϕ is analytic in the neighborhood. Thus, it follows from Theorem 3.1 that there exist $k_\phi > 0$, $\rho_\phi \in [0, 1)$, and a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that

$$\|\nabla\phi(p)\| \geqslant k_\phi\|\phi(p) - \phi(\bar{p})\|^{\rho_\phi} = k_\phi\|\phi(p)\|^{\rho_\phi},$$

for all $p \in U_{\bar{p}}$. Further, $\phi(p) = 0$ if and only if $p \in E'_{p^*}$ by the positive definiteness of γ. Thus, for any $p \in U_{\bar{p}}$ and $p \notin E'_{p^*}$, $\|\nabla\phi(p)\| > 0$. □

The local asymptotic stability of E'_{p^*} is then ensured on the basis of Lemma 3.1 as follows:

Theorem 3.2
The set E'_{p^*} is locally asymptotically stable with respect to (13).

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Proof
We prove this theorem by showing that E'_{e*} is locally asymptotically stable with respect to (15). To show the local asymptotic stability of E'_{e*}, we construct a neighborhood of E'_{e*} such that $\dot{V}(e) \geqslant 0$ for any e in the neighborhood and $\dot{V}(e) = 0$ if and only if $e \in E'_{e*}$.

It follows from Lemma 3.1 that, for any $\bar{p} \in E'_{p*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that $\|\nabla\phi(p)\| > 0$ for all $p \in U_{\bar{p}}$ and $\xi \notin E'_{p*}$. We take $r^*_p > 0$ such that

$$\left\{p \in \mathbb{R}^{nN} : \|p - \bar{p}\| < r^*_p\right\} \subseteq U_{\bar{p}}. \tag{16}$$

Define

$$U_{E'_{e*}}(r_e) := \left\{ e \in \operatorname{Im}\left(H^T_+ \otimes I_n\right) : \inf_{\eta \in E'_{e*}} \|e - \eta\| < r_e \right\}.$$

Let $r^*_e = \sigma_{\min}\left(H^T_+ \otimes I_n\right) r^*_p$, where $\sigma_{\min}\left(H^T_+ \otimes I_n\right)$ denotes the nonzero smallest singular value of $H^T_+ \otimes I_n$. Then, for any $e \in U_{E'_{e*}}\left(r^*_e\right)$, there exists $\bar{e} \in E'_{e*}$ such that

$$\inf_{\eta \in E'_{e*}} \|e - \eta\| = \|e - \bar{e}\| < r^*_e$$

because E'_{e*} is compact and $\|e - \eta\|$ is a continuous function of η [14]. From the fact that $(e - \bar{e}) \in \operatorname{Im}\left(H^T_+ \otimes I_n\right)$, there always exist $p \in \mathbb{R}^{nN}$ and $\bar{p} \in E'_{p*}$ such that $\left(H^T_+ \otimes I_n\right)(p - \bar{p}) = e - \bar{e}$ and $(p - \bar{p}) \in \operatorname{Im}\left(H^T_+ \otimes I_n\right)$. Because $p - \bar{p}$ belongs to the row space of $H^T_+ \otimes I_n$, we obtain

$$\sigma_{\min}\left(H^T_+ \otimes I_n\right)\|p - \bar{p}\| \leqslant \|e - \bar{e}\| = \|\left(H^T_+ \otimes I_n\right)(p - \bar{p})\|. \tag{17}$$

Thus, we have

$$\|p - \bar{p}\| \leqslant \frac{\|e - \bar{e}\|}{\sigma_{\min}\left(H^T_+ \otimes I_n\right)} < r^*_p,$$

which implies that $p \in U_{\bar{p}}$ from (16). It follows from Lemma 3.1 that if $e \notin E'_{e*}$,

$$\dot{V}(e) = -k_p\|\nabla\phi(p)\|^2 < 0,$$

which implies that E'_{e*} is locally asymptotically stable with respect to (15). Thus, E'_{p*} is locally asymptotically stable with respect to (13). □

Then, the local asymptotic stability of E_{p*} is ensured if $(\mathcal{G}, p^*)$ is rigid.

Theorem 3.3
If $(\mathcal{G}, p^*)$ is rigid, the set E_{p*} is locally asymptotically stable with respect to (13).

Proof
From Theorem 3.2, E'_{p*} is locally asymptotically stable. Because $(\mathcal{G}, p^*)$ is rigid, it follows from the definition of the graph rigidity that, for any $\bar{p} \in E_{p*}$, there exists a neighborhood $U_{\bar{p}}$ of $\bar{p}$ such that $E_{p*} \cap U_{\bar{p}} = E'_{p*} \cap U_{\bar{p}}$. This implies that E_{p*} is locally asymptotically stable with respect to (13). □

Although Theorem 3.3 confirms the local asymptotic stability of E_{p*} under the condition that $(\mathcal{G}, p^*)$ is rigid, it does not ensure the convergence of p to a finite realization in E_{p*}. The convergence property is ensured by the fact that the centroid of an undirected formation is stationary under the gradient control law (8), that is, $(1/N)\sum_{i \in \mathcal{V}} \dot{p}_i = 0$. Because E_{p*} is locally asymptotically stable and the centroid is stationary, p converges to a finite realization in E_{p*}.

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Remark 3.1
Two remarks are in order here. First, we present an elegant approach for the stability analysis of undirected formations in n-dimensional space. Because it is challenging to analyze the stability of the noncompact equilibrium set E_{p^*}, we focus on the link dynamics (15) to obtain the stability analysis result in this section. A property of gradient systems and the link dynamics interplay in the stability analysis. Second, the stability analysis in this section reveals that the local asymptotic stability of E_{p^*} does not necessarily require the infinitesimal rigidity of the framework $(\mathcal{G}, p^*)$. Note that the results in [1–3] require $(\mathcal{G}, p^*)$ to be infinitesimally rigid because the rank condition of the matrix $\partial g_{\mathcal{G}/(p)} \partial p|_{p=p^*}$ is crucial in the proofs of the existing results.

4. UNDIRECTED FORMATIONS OF DOUBLE INTEGRATORS

Consider the double-integrator modeled agents under the assumptions of Problem 2.2. Because each agent measures its own velocity and the relative distances of its neighboring agents, we can design a formation control law for the agents as follows:

$$u_i^i = -k_v v_i^i - k_p \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma \left(\tilde{d}_{ji}\right)}{\partial \tilde{d}_{ji}} p_j^i,$$

which can be represented with respect to ${}^g\sum$ as follows:

$$u_i = -k_v v_i - k_p \sum_{j \in \mathcal{N}_i} \frac{\partial \gamma \left(\tilde{d}_{ji}\right)}{\partial \tilde{d}_{ji}} (p_i - p_j).$$

Defining

$$\psi(p, v) := \frac{1}{2} \sum_{i \in \mathcal{V}} \|v_i\|^2 + \frac{k_p}{2} \sum_{i=1}^{M} \gamma \left(\|p_j - p_i\|^2 - d_{ij}^*\right),$$

the overall dynamics of the agents can be described as a dissipative Hamiltonian system:

$$\begin{aligned} \dot{p} &= v \\ &= \nabla_v \psi, \end{aligned} \tag{18a}$$

$$\begin{aligned} \dot{v} &= -k_v v - k_p (H_+ \otimes I_n) D(e) \tilde{d} \\ &= -k_v \nabla_v \psi - \nabla_p \psi, \end{aligned} \tag{18b}$$

where $k_p > 0$, $k_v > 0$, and $v = \left[v_1^T \cdots v_N^T\right]^T$.

We now consider the following one-parameter family $\mathcal{H}_\lambda$ of dynamical systems, which combines the dissipative Hamiltonian system (18) and a gradient system as follows:

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} &= \left((1-\lambda) \begin{bmatrix} 0 & I_{nN} \\ -I_{nN} & 0 \end{bmatrix} - \begin{bmatrix} -\lambda I_{nN} & 0 \\ 0 & k_v I_{nN} \end{bmatrix} \right) \begin{bmatrix} \nabla_p \psi \\ \nabla_v \psi \end{bmatrix} \\ &= -\underbrace{\begin{bmatrix} \lambda I_{nN} & -(1-\lambda) I_{nN} \\ (1-\lambda) I_{nN} & k_v I_{nN} \end{bmatrix}}_{W_\lambda :=} \begin{bmatrix} \nabla_p \psi \\ \nabla_v \psi \end{bmatrix}, \end{aligned} \tag{19}$$

where $\lambda \in [0, 1]$. The parameterized system (19) continuously interpolates between the Hamiltonian system (18) and a gradient system by means of convex combination. When $\lambda = 0$, the parameterized system (19) reduces into the Hamiltonian system (18). In the case that $\lambda = 1$, (19) reduces into the following gradient system:

$$\dot{p} = -\nabla_p \psi, \tag{20a}$$

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$$\dot{v} = -k_v \nabla_v \psi. \tag{20b}$$

It has been revealed that parameterized systems of the form (19) have the identical equilibrium set and the identical local stability properties for all $\lambda \in [0, 1]$ [9] as stated in the following theorem:

Theorem 4.1 ([9]*)*
For the one-parameter family $\mathcal{H}_\lambda$ of dynamical systems in (19), the following statements hold independently of the parameter $\lambda \in [0, 1]$.

- Equilibrium set: For all $\lambda \in [0, 1]$, the equilibrium set of $\mathcal{H}_\lambda$ is given by the set of critical points of the potential function ψ, that is, $E_{p,v} = \{[p^T\ v^T]^T : \nabla\psi = 0\}$.
- Local stability: For any equilibrium $[p^T\ v^T]^T \in E_{p,v}$ and for all $\lambda \in [0, 1]$, the numbers of the stable, neutral, and unstable eigenvalues of the Jacobian of $\mathcal{H}_\lambda$ are not dependent on λ.

Theorem 4.1 allows us to study the local stability of the formation dynamics (18) by investigating the local stability of the gradient system (20). Because subsystems (20a) and (20b) are decoupled, we investigate the local stability of each subsystem. First, the subsystem (20a) coincides to the gradient system (13), and thus, both systems have the same stability property. Thus, the local stability of E'_{p^*} and E_{p^*}, which are defined in (11) and (5), respectively, follows from Theorems 3.2 and 3.3 with respect to (20a). Second, the only equilibrium point of (20b) is obviously the origin, and it is globally exponentially stable.

We present the main result of this section, which confirms the local asymptotic stability of the desired formation of the double-integrator modeled agents.

Theorem 4.2
If $(\mathcal{G}, p^*)$ is rigid, the set E_{p^*,v^*} is locally asymptotically stable with respect to (18).

The dynamics of the centroid of the agents is given by $(1/N)\sum_{i\in\mathcal{V}} \ddot{p}_i + k_v(1/N)\sum_{i\in\mathcal{V}} \dot{p}_i = 0$, which implies that the centroid converges to a finite point. This ensures the convergence of p to a finite realization.

Remark 4.1
The stability of undirected formations of double-integrator modeled agents has been studied in [10] on the basis of the LaSalle invariance principle. However, it is not certain whether the principle can be applied to E_{p^*,v^*}, which is a noncompact invariant set with respect to (18). Note that the proof of LaSalle's theorem found in [15] requires the compactness of the invariant set.

5. SIMULATION RESULTS

We present the simulation results of formation control of five single-integrator and double-integrator modeled agents in three-dimensional space. The sensing graph for both kinds of agents is depicted in Figure 1. The function γ is defined as $\gamma(x) = (1/2)x^2$, which is popularly adopted in the literature.

For the single-integrator modeled agents, we assume that $p_1^* = \left[0\ 0\ 10\sqrt{5}\right]^T$, $p_2^* = [0\ 20\ 0]^T$, $p_3^* = \left[-10\sqrt{3}\ -10\ 0\right]^T$, $p_4^* = \left[10\sqrt{3}\ -10\ 0\right]^T$, and $p_5^* = \left[0\ 0\ -10\sqrt{5}\right]^T$. Thus, the desired interagent distances $\|p_j^* - p_i^*\|$ for all $(i, j) \in \mathcal{E}$ are 30. The components of the initial positions $p_i(0)$ for all $i \in \mathcal{V}$ are randomly perturbed from those of p_i^* by a random variable uniformly distributed on $[-7.5, 7.5]$. Figure 2 shows the formation p and the interagent distance errors of the five single integrators. The formation of the agents converges to the desired formation, and the interagent errors converge to zero as depicted in the figure.

For the double-integrator modeled agents, p^* is given as the same as that for the single-integrator modeled agents. The initial positions are also given as the same as those for the single-integrator modeled agents. The components of the initial velocities of the double-integrator group

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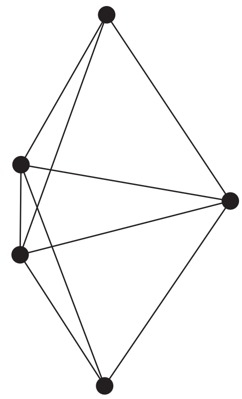
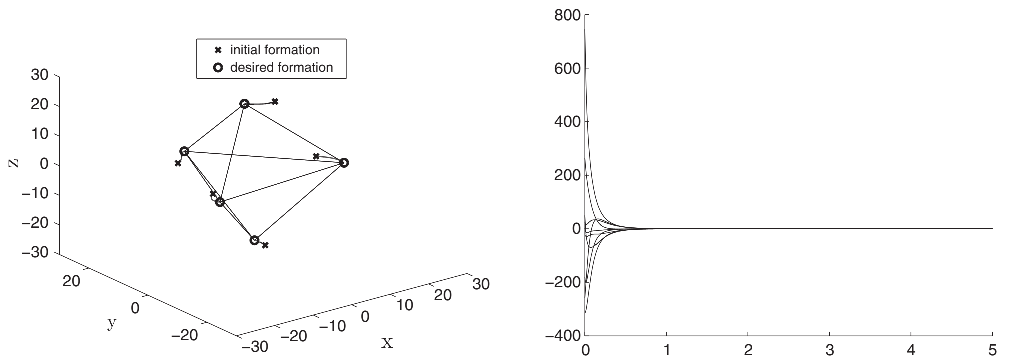
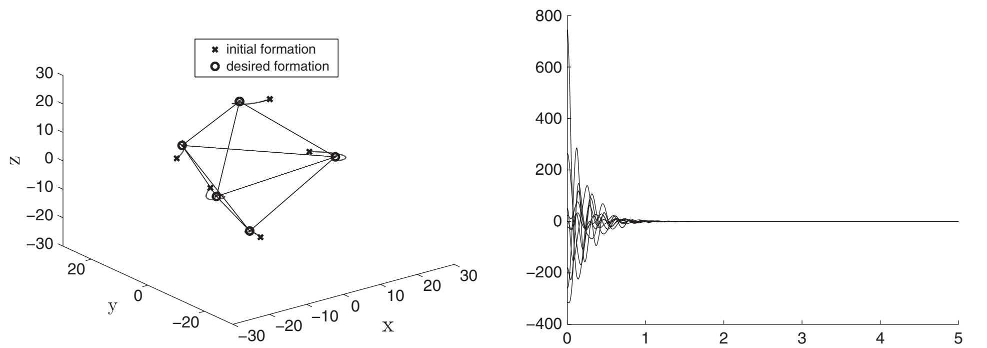


Figure 1. Sensing graph for five agents.



(a) The formation p. (b) The inter-agent distance errors $\tilde{d}_{ij}$ for all $(i,j) \in \mathcal{E}_+$.

Figure 2. Simulation result for five single integrators.



(a) The formation p. (b) The inter-agent distance errors $\tilde{d}_{ij}$ for all $(i,j) \in \mathcal{E}_+$.

Figure 3. Simulation result for five double integrators.

are randomly given by a random variable uniformly distributed on $[-5, 5]$. As depicted in Figure 3, the formation p converges to a finite realization congruent to p^*, and the interagent distance errors converge to zero.

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6. CONCLUSION

In this paper, we have studied the local asymptotic stability of n-dimensional undirected formations of single-integrator and double-integrator modeled agents in the distance-based problem setup. To overcome the challenging stability analysis, we have focused on the link dynamics of the agents, which allows us to utilize the property of gradient systems. This approach was useful for the investigation of the stability of a gradient system having noncompact equilibrium set. We also have revealed that a rigid formation is locally asymptotically stable even if it is not infinitesimally rigid, whereas infinitesimal rigidity has been usually assumed in the literature. On the basis of the topological equivalence of a Hamiltonian system and a gradient system, the local asymptotic stability of the double-integrator modeled agent formation has been proved.

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