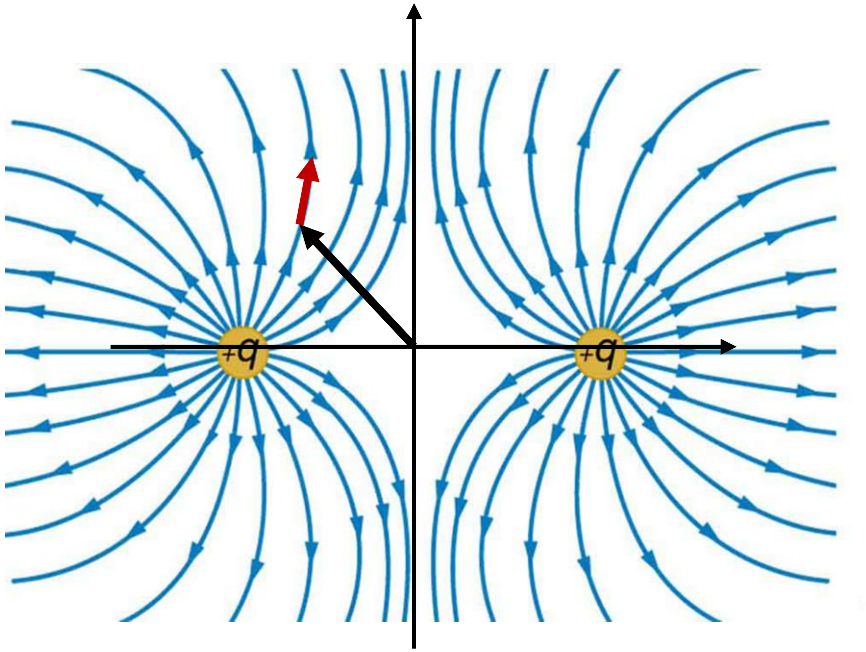
第一章 复习课

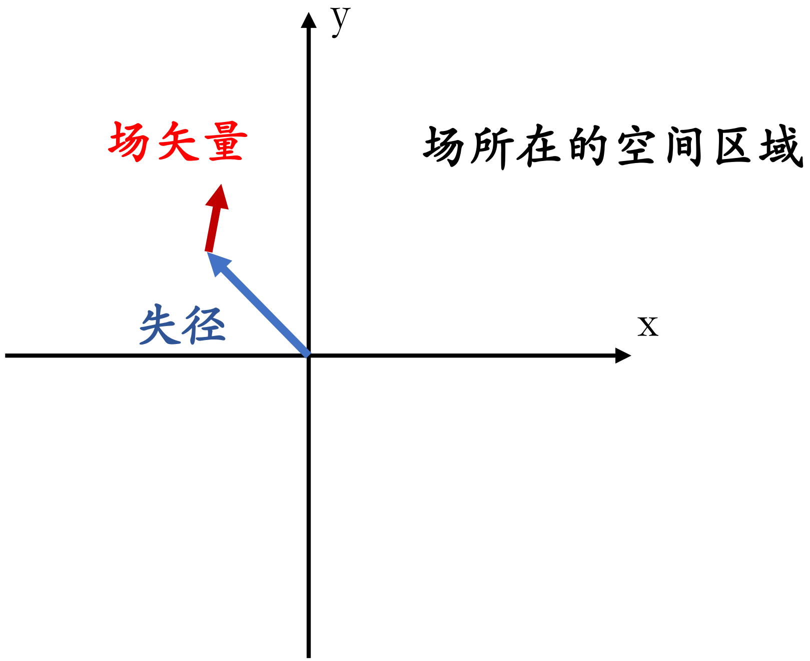
射频技术与软件研究所

内 容

1. 矢量、失径（位置矢量）、场矢量
2. 矢量的点积（/点乘）、叉积（叉乘）
3. 矢量的坐标变换
4. 矢量微分
5. 标量场的方向导数和梯度
6. 通量和散度
7. 环量和旋度
8. 拉普拉斯算子
9. 散度定理
10. 斯托克斯定理

1. 矢量、失径（位置矢量）、场矢量





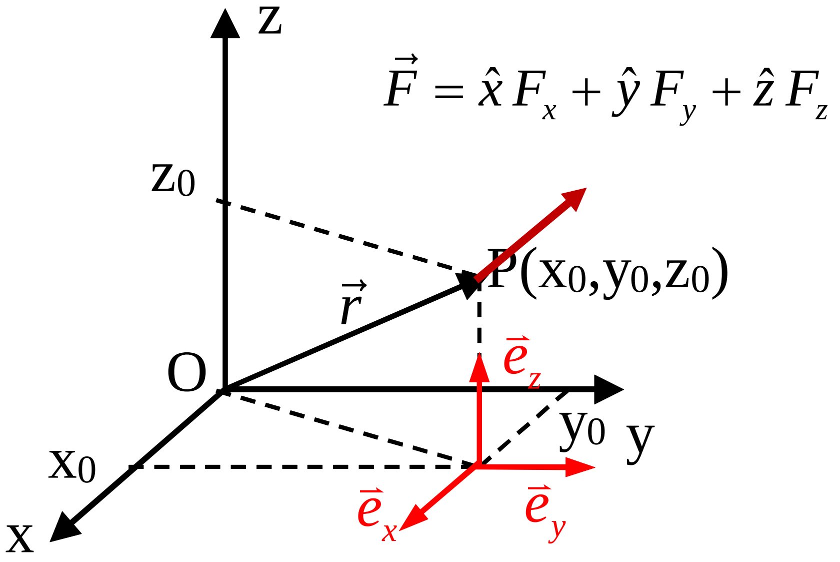
如何表述场矢量?

直角坐标系:

单位矢量: $\hat{e}_1 = \hat{x}, \quad \hat{e}_2 = \hat{y}, \quad \hat{e}_3 = \hat{z}$

矢径 $\vec{r} = \hat{x}\, x_0 + \hat{y}\, y_0 + \hat{z}\, z_0$

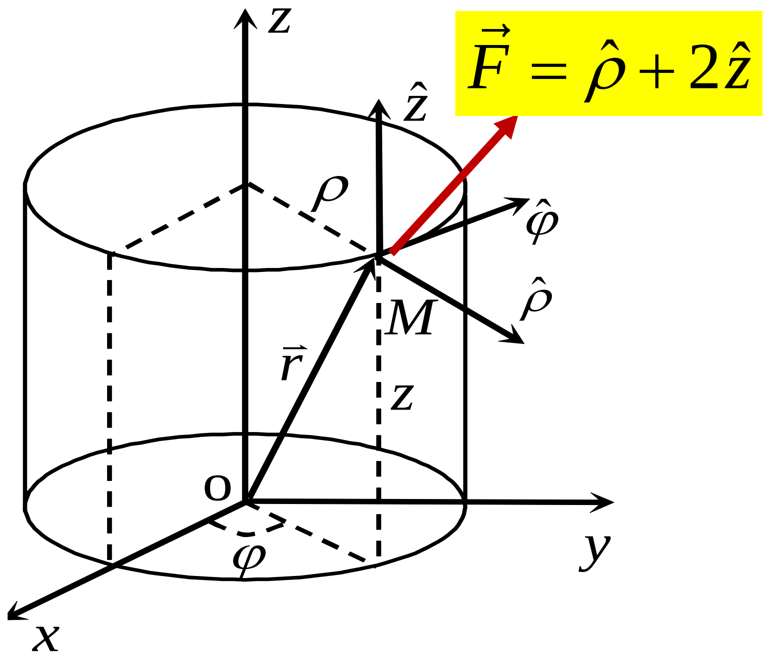
场矢量: $\vec{F} = \hat{x}\, F_x + \hat{y}\, F_y + \hat{z}\, F_z$



例如: 在(1, 1, 1)处的存在一矢量 $\vec{F} = \hat{x} + 2\hat{y} + 3\hat{z}$

矢量的书写: $\vec{F}$ or $\boldsymbol{F}$ 单位矢量 $\hat{F}$

如何表述场矢量?



圆柱坐标系:

单位矢量: $\hat{e}_1=\hat{\rho}\quad \hat{e}_2=\hat{\phi}\quad \hat{e}_3=\hat{z}$

矢径: $\vec{r}=\rho\hat{\rho}+\hat{z}\,z$

场矢量: $\vec{F}=\hat{\rho}F_\rho+\hat{\phi}F_\phi+\hat{z}\,F_z$

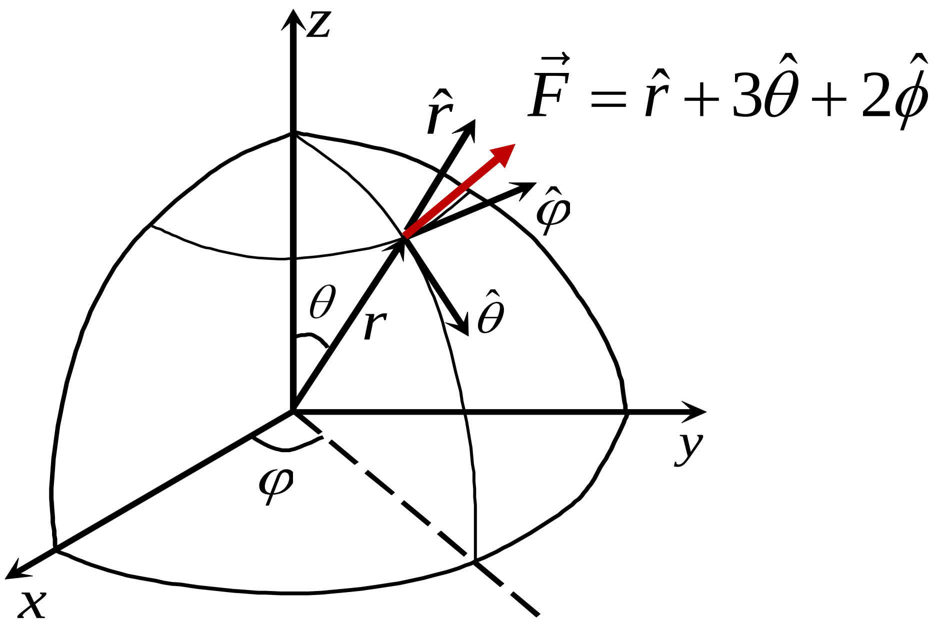
位于柱坐标系（$1,\frac{\pi}{4},2$）处的 $\vec{F}=\hat{\rho}+\hat{\phi}+2\hat{z}$

球坐标系:

矢径: $\vec{r} = r\hat{r}$

单位矢量: $\hat{e}_1 = \hat{r}; \hat{e}_2 = \hat{\theta}; \hat{e}_3 = \hat{\phi}$

场矢量: $\vec{F} = \hat{r}F_r + \hat{\theta}F_\theta + \hat{\phi}F_\phi$



球坐标系

场矢量: $\vec{F} = \hat{r}F_r + \hat{\theta}F_\theta + \hat{\phi}F_\phi$

位于球坐标系（$2, \frac{\pi}{4}, \frac{\pi}{6}$）处的 $\vec{F} = \hat{r} + 3\hat{\theta} + 2\hat{\phi}$

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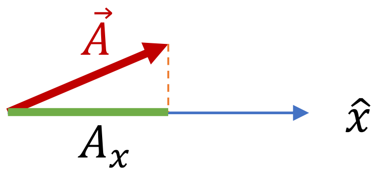
2. 矢量的点积（/点乘）、叉积（叉乘）

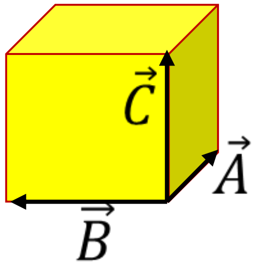
$$\vec{A}\cdot\vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\cdot\hat{x} = A_x$$



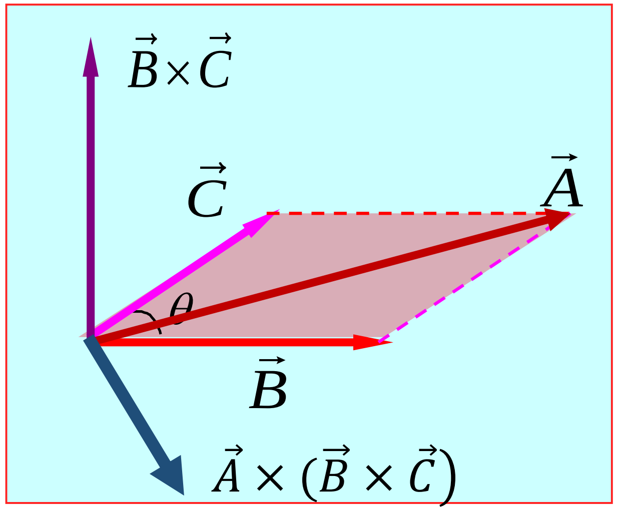
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$



$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

"back-cab" rule

bac | cab



作业1.4：证明对于非零矢量A、B、C，如果$\mathbf{A}\cdot\boldsymbol{B}=\boldsymbol{A}\cdot\boldsymbol{C}$和$\mathbf{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$，则：$\boldsymbol{B}=\boldsymbol{C}$。

$$\mathbf{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$$

非零矢量A**叉乘：**

$$\mathbf{A}\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}\times(\mathbf{A}\times\mathbf{C})$$

$$\mathbf{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$$

“back-cab” rule

$$\mathbf{A}(\mathbf{A}\cdot\mathbf{B})-\mathbf{B}(\mathbf{A}\cdot\mathbf{A})=\mathbf{A}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{A})$$

$$\mathbf{A}\cdot\boldsymbol{B}=\boldsymbol{A}\cdot\boldsymbol{C}$$

$$-\mathbf{B}(\mathbf{A}\cdot\mathbf{A})=-\mathbf{C}(\mathbf{A}\cdot\mathbf{A})$$

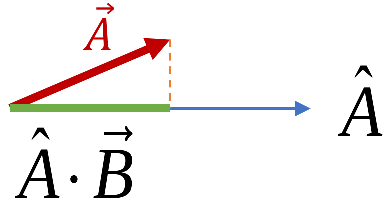
A模值非零

$$\mathbf{B}=\mathbf{C}$$

作业1.4：证明对于非零矢量A、B、C，如果$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{A}\cdot\boldsymbol{C}$和$\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}$，则：$\boldsymbol{B}=\boldsymbol{C}$。

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{A}\cdot\boldsymbol{C}\ \Rightarrow\ \hat{A}\cdot\vec{B}=\hat{A}\cdot\vec{C}$$

$$A\neq 0$$



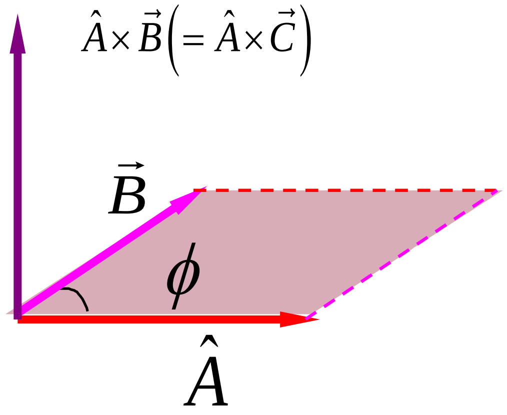
$$\hat{A}:=\hat{x}$$

定义：$\hat{A}\times\vec{B}:=\hat{z}$

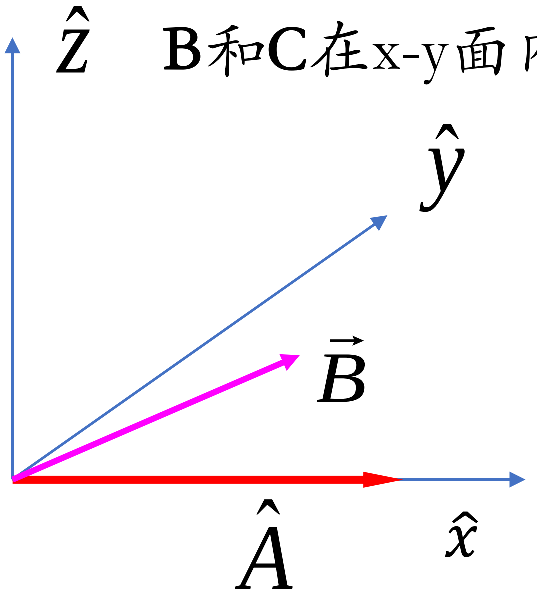
$$\hat{y}=\hat{z}\times\hat{x}$$

$$\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{A}\times\boldsymbol{C}\ \Rightarrow\ \hat{A}\times\vec{B}=\hat{A}\times\vec{C}$$

$$A\neq 0$$



B和C在x-y面内



$$\hat{A}\cdot\vec{B}\Rightarrow\hat{x}\cdot\vec{B}=B_x$$

$$\hat{A}\cdot\vec{C}\Rightarrow\hat{x}\cdot\vec{C}=C_x$$

$$\hat{A}\cdot\vec{B}=\hat{A}\cdot\vec{C}\Rightarrow B_x=C_x$$

$$\hat{A}\times\vec{B}=\hat{x}\times\vec{B}=\hat{z}B\sin\phi=\hat{z}B_y$$

$$\hat{A}\times\vec{C}=\hat{x}\times\vec{C}=\hat{z}C\sin\phi'=\hat{z}C_y$$

$$\hat{A}\times\vec{B}=\hat{A}\times\vec{C}\Rightarrow B_y=C_y$$

则：$\boldsymbol{B}=\boldsymbol{C}$

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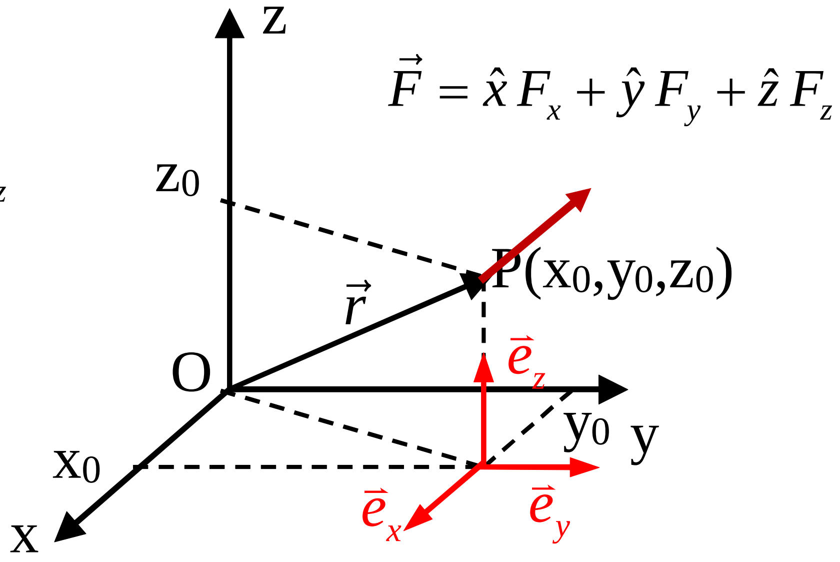
位于直角坐标系（1, 1, 1）处的 $\vec{F} = \hat{x}$ 如何在圆柱系中表示?



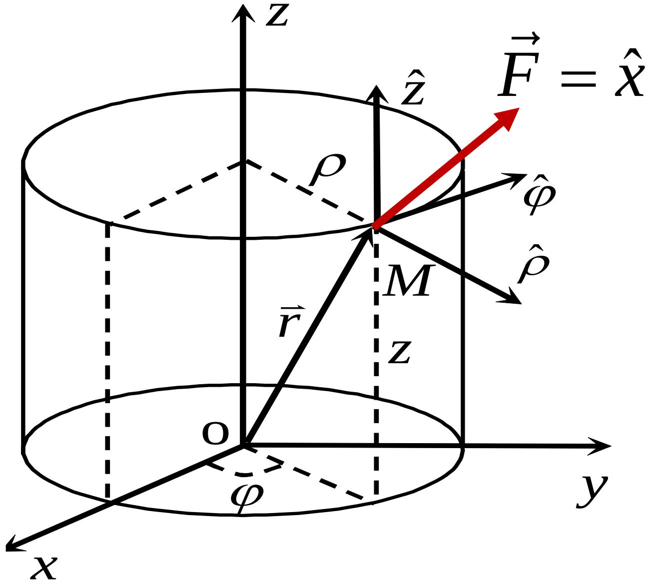
即给出：位于圆柱坐标系（ρ, ϕ, z）处的 $\vec{F} = \hat{\rho}F_\rho + \hat{\phi}F_\phi + \hat{z}F_z$

$z = 1$ $\quad \rho = ?$ $\quad \hat{\rho} = ?$ $\quad F_\rho = ?$

$F_z = 0$ $\quad \phi = ?$ $\quad \hat{\phi} = ?$ $\quad F_\phi = ?$

$$\rho = \sqrt{x_0^2 + y_0^2} = \sqrt{2}$$

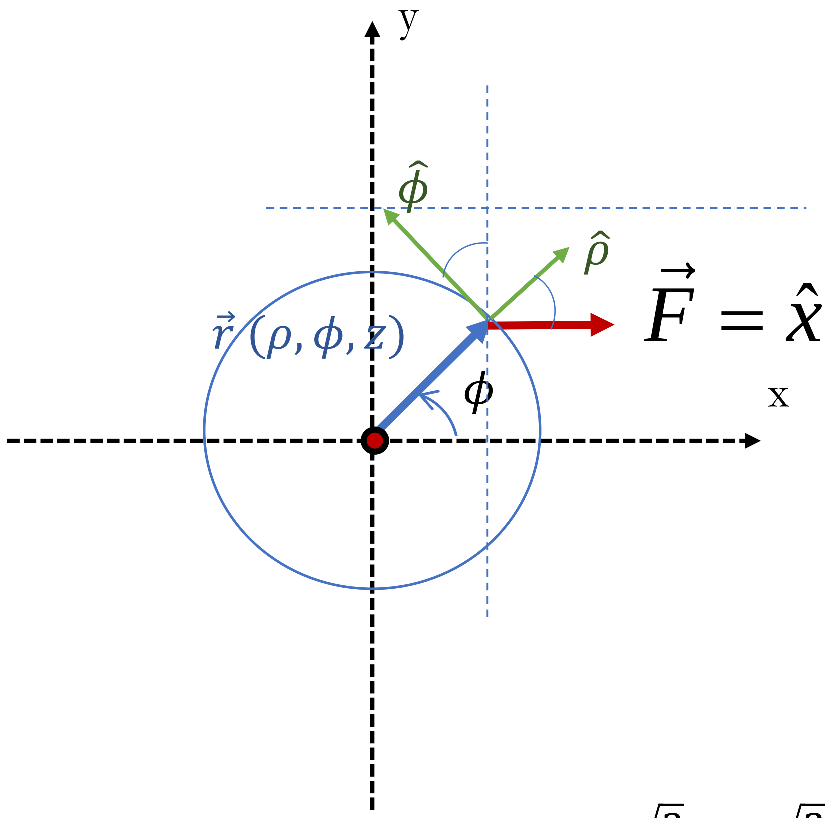
$$\phi = \arctan\left(\frac{y_0}{x_0}\right) = \frac{\pi}{4}$$



方法一：利用点积求投影

$F_\rho = ?$ $\boxed{F_\rho = \vec{F}\cdot\hat{\rho}}$ $\hat{\rho} = ?$

$F_\phi = ?$ $\boxed{F_\phi = \vec{F}\cdot\hat{\phi}}$ $\hat{\phi} = ?$



$$F_\rho = \vec{F}\cdot\hat{\rho} = \hat{x}\cdot\left(\frac{\sqrt{2}}{2}\hat{x} + \frac{\sqrt{2}}{2}\hat{y}\right) = \frac{\sqrt{2}}{2}$$

$$F_\phi = \vec{F}\cdot\hat{\phi} = \hat{x}\cdot\left(-\frac{\sqrt{2}}{2}\hat{x} + \frac{\sqrt{2}}{2}\hat{y}\right) = -\frac{\sqrt{2}}{2}$$

$$\hat{\rho} = (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) = \frac{\sqrt{2}}{2}\hat{\mathbf{x}} + \frac{\sqrt{2}}{2}\hat{\mathbf{y}}$$

$$\hat{\phi} = (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) = -\frac{\sqrt{2}}{2}\hat{\mathbf{x}} + \frac{\sqrt{2}}{2}\hat{\mathbf{y}}$$

$$\vec{F} = \hat{\rho}F_\rho + \hat{\phi}F_\phi + \hat{z}F_z = \frac{\sqrt{2}}{2}\hat{\rho} - \frac{\sqrt{2}}{2}\hat{\phi}$$

位于圆柱坐标系（$\sqrt{2}, \frac{\pi}{4}, 1$）处的 $\vec{F} = \frac{\sqrt{2}}{2}\hat{\rho} - \frac{\sqrt{2}}{2}\hat{\phi}$

方法二：直接分解

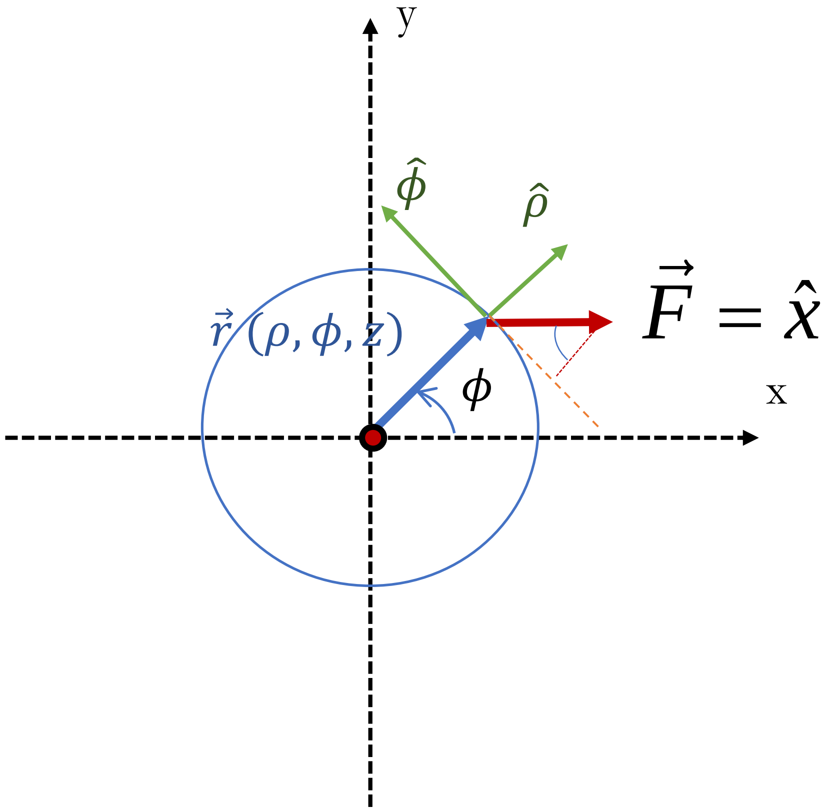
$$\rho = \sqrt{x_0^2 + y_0^2} = \sqrt{2}$$

$$\phi = \arctan\left(\frac{y_0}{x_0}\right) = \frac{\pi}{4}$$

$$\vec{F} = \hat{x} \qquad \hat{x} = -\sin\phi\hat{\phi} + \cos\phi\hat{\rho} = \frac{\sqrt{2}}{2}\hat{\rho} - \frac{\sqrt{2}}{2}\hat{\phi}$$

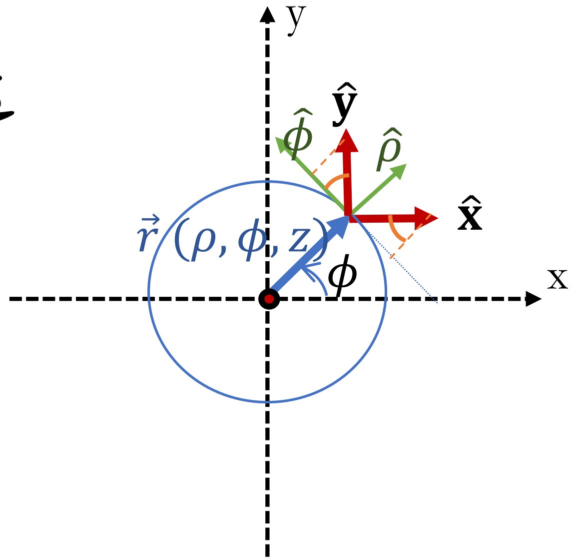
$$\vec{F} = \frac{\sqrt{2}}{2}\hat{\rho} - \frac{\sqrt{2}}{2}\hat{\phi}$$

位于圆柱坐标系（$\sqrt{2}, \frac{\pi}{4}, 1$）处的 $\vec{F} = \frac{\sqrt{2}}{2}\hat{\rho} - \frac{\sqrt{2}}{2}\hat{\phi}$



作业：1.6，把下面矢量函数转化为圆柱坐标系表达

$$\boldsymbol{A}=\hat{\mathbf{x}}(x+y)+\hat{\mathbf{y}}(y-x)+\hat{\mathbf{z}}z$$



(ρ,ϕ,z) $\rho=\sqrt{x^2+y^2}$ $\phi=arccos(x/\rho)$

$$x=\rho\cos\phi$$
$$y=\rho\sin\phi$$

$$\hat{\rho}=(cos\phi\hat{\mathbf{x}}+sin\phi\hat{\mathbf{y}})$$

$$\hat{\phi}=(-sin\phi\hat{\mathbf{x}}+cos\phi\hat{\mathbf{y}})$$

$$A_\rho=\mathbf{A}\cdot\hat{\rho}=(\hat{\mathbf{x}}(x+y)+\hat{\mathbf{y}}(y-x)+\hat{\mathbf{z}}z)\cdot(cos\phi\hat{\mathbf{x}}+sin\phi\hat{\mathbf{y}})$$
$$=(\rho cos\phi+\rho sin\phi)cos\phi+(\rho sin\phi-\rho cos\phi)sin\phi=\rho$$

$$A_\phi=\mathbf{A}\cdot\hat{\phi}=(\hat{\mathbf{x}}(x+y)+\hat{\mathbf{y}}(y-x)+\hat{\mathbf{z}}z)\cdot(-sin\phi\hat{\mathbf{x}}+cos\phi\hat{\mathbf{y}})$$
$$=-(\rho cos\phi+\rho sin\phi)sin\phi+(\rho sin\phi-\rho cos\phi)cos\phi=-\rho$$

$$\mathbf{A}=A_\rho\hat{\rho}+A_\phi\hat{\phi}+\hat{\mathbf{z}}z=\rho\hat{\rho}-\rho\hat{\phi}+\hat{\mathbf{z}}z$$

$$\hat{x}=-\sin\phi\hat{\phi}+\cos\phi\hat{\rho}$$
$$\hat{y}=\cos\phi\hat{\phi}+\sin\phi\hat{\rho}$$

$$\boldsymbol{A}=\hat{\mathbf{x}}(x+y)+\hat{\mathbf{y}}(y-x)+\hat{\mathbf{z}}z$$
$$=\left(-\sin\phi\hat{\phi}+\cos\phi\hat{\rho}\right)(\rho\cos\phi+\rho\sin\phi)+\left(\cos\phi\hat{\phi}+\sin\phi\hat{\rho}\right)(\rho\sin\phi-\rho\cos\phi)+\hat{\mathbf{z}}z$$
$$=\hat{\phi}\left[-\sin\phi(\rho\cos\phi+\rho\sin\phi)+\cos\phi(\rho\sin\phi-\rho\cos\phi)\right]$$
$$+\hat{\rho}\left[\cos\phi(\rho\cos\phi+\rho\sin\phi)+\sin\phi(\rho\sin\phi-\rho\cos\phi)\right]$$
$$+\hat{\mathbf{z}}z$$
$$=-\rho\hat{\phi}+\rho\hat{\rho}+\hat{\mathbf{z}}z$$

课堂练习：已知场矢量见下式，求在坐标P点处，该矢量场在球坐标系中的表示。

$$\vec{F}=x\hat{x}+2z^{2}\hat{y}+xz\hat{z} \qquad \mathrm{P}(1,1,\sqrt{2}/2)$$

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4. 矢量微分

变化的矢量

失性函数的导数的运算法则

$$\vec{F}'(t)=\frac{d\vec{F}(t)}{dt}=\frac{d}{dt}\left(F_x(t)\hat{x}+F_y(t)\hat{y}+F_z(t)\hat{z}\right)=\hat{x}\frac{d}{dt}F_x(t)+\hat{y}\frac{d}{dt}F_y(t)+\hat{z}\frac{d}{dt}F_z(t)$$

$$\frac{d}{dt}\left(f(t)\vec{F}(t)\right)=f(t)\frac{d\vec{F}(t)}{dt}+\vec{F}(t)\frac{df(t)}{dt}$$

单一变量

$$\frac{d}{dt}\left(\vec{F}(t)\cdot\vec{E}(t)\right)=\vec{E}(t)\cdot\frac{d\vec{F}(t)}{dt}+\vec{F}(t)\cdot\frac{d\vec{E}(t)}{dt}\qquad\frac{d}{dt}\left(\vec{F}(t)\times\vec{E}(t)\right)=\vec{F}(t)\times\frac{d\vec{E}(t)}{dt}+\frac{d\vec{F}(t)}{dt}\times\vec{E}(t)$$

多元变量

$$\frac{\partial\vec{F}(x,y,z)}{\partial x}=\frac{\partial F_x(x,y,z)}{\partial x}\hat{x}+\frac{\partial F_y(x,y,z)}{\partial x}\hat{y}+\frac{\partial F_z(x,y,z)}{\partial x}\hat{z}$$

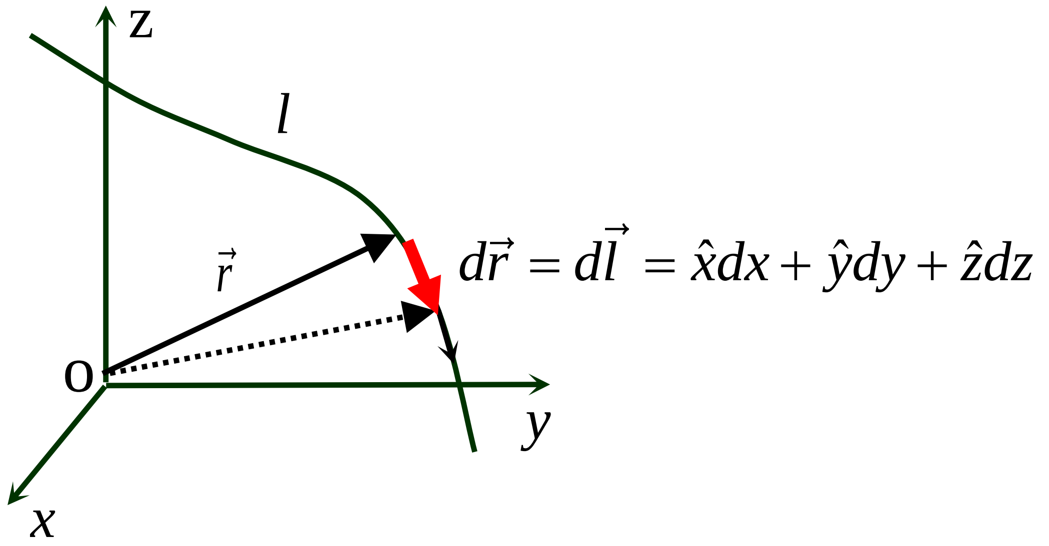
全微分 $$d\vec{F}=\frac{\partial\vec{F}}{\partial x}dx+\frac{\partial\vec{F}}{\partial y}dy+\frac{\partial\vec{F}}{\partial z}dz$$

矢径的全微分 $$d\vec{r}=\frac{\partial\vec{r}}{\partial x}dx+\frac{\partial\vec{r}}{\partial y}dy+\frac{\partial\vec{r}}{\partial z}dz$$

矢径函数 $\vec{r}$ 的微分

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

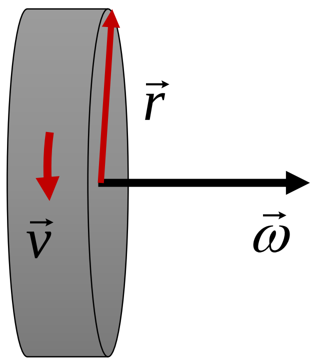


模值 $|d\vec{r}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

弧微分 $dl = \pm\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

所以 $\dfrac{d\vec{r}}{dl} = \dfrac{|d\vec{r}|}{|dl|}\hat{e}_l = \hat{e}_l$ $\hat{e}_l$ 是曲线 l 上指向正向的切线方向的单位矢量

课堂练习：已知车轮旋转的角速度为常矢，求轮带外缘的旋转运动的加速度。



$$\vec{v} = \vec{\omega} \times \vec{r}(t) \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

拉梅系数：弧微分和坐标变量微分的比例系数。

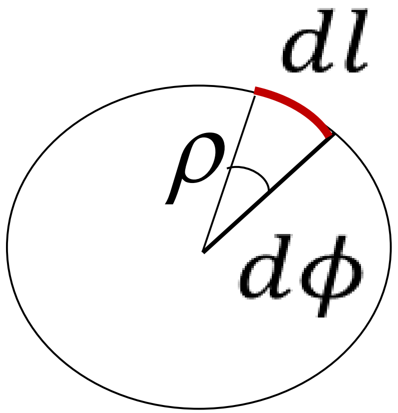
对于直角坐标系：$d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz = \left|\frac{\partial \vec{r}}{\partial x}\right|\hat{x}dx + \left|\frac{\partial \vec{r}}{\partial y}\right|\hat{y}dy + \left|\frac{\partial \vec{r}}{\partial z}\right|\hat{z}dz$ $\quad \left|\frac{\partial \vec{r}}{\partial x}\right| = \left|\frac{\partial \vec{r}}{\partial y}\right| = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$

对于曲面坐标系：$\vec{r} = \vec{r}(u_1, u_2, u_3)$

拉梅系数（$Lame'$）或度量因子

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|\hat{u}_1 du_1 + \left|\frac{\partial \vec{r}}{\partial u_2}\right|\hat{u}_2 du_2 + \left|\frac{\partial \vec{r}}{\partial u_3}\right|\hat{u}_3 du_3$$

拉梅系数：$h_i = \left|\frac{\partial \vec{r}}{\partial u_i}\right|$



$$d\vec{l}_1 = \hat{u}_1 dl_1 = \hat{u}_1 h_1 du_1$$

$h_1 du_1 = dl_1$

非长度　长度

例如：$\rho d\phi = dl$

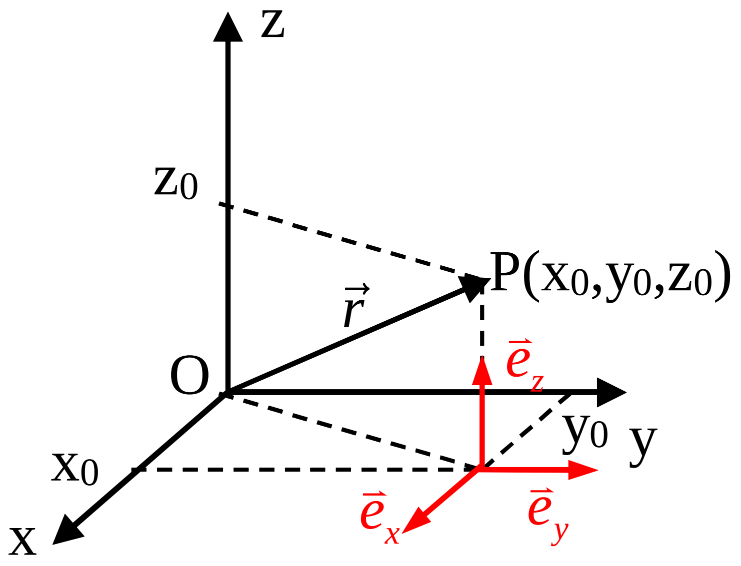
矢量积分常用到的矢量微分：线元和面元

矢径 $\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$

单位矢量 $\hat{x}, \quad \hat{y}, \quad \hat{z}$

拉梅系数 $h_1 = \left|\frac{\partial \vec{r}}{\partial x}\right| = |\hat{x}| = 1$

$h_1 = 1 \quad , \quad h_2 = 1 \quad , \quad h_3 = 1$



线元 $d\vec{l} = dl_1\hat{x} + dl_2\hat{y} + dl_3\hat{z} \qquad dl_1 = dx, dl_2 = dy, dl_3 = dz \qquad d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

面元 $d\vec{s} = ds_1\hat{x} + ds_2\hat{y} + ds_3\hat{z} \qquad ds_1 = dy\,dz, ds_2 = dx\,dz, ds_3 = dx\,dy \qquad d\vec{x} = dydz\hat{x} + dzdx\hat{y} + dydx\hat{z}$

体元 $d\tau = dxdydz$

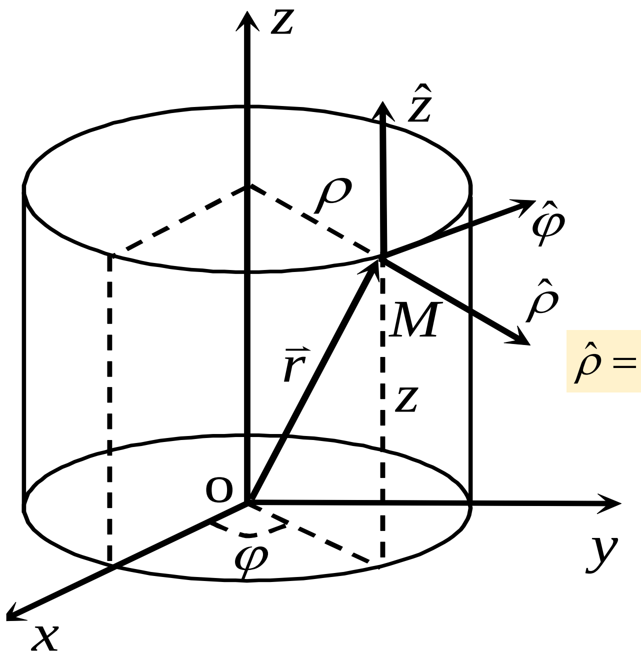
圆柱坐标系的线元和面元表示:

矢径　$\vec{r} = \rho\hat{\rho} + \hat{z}\,z$

单位矢量　$\hat{e}_1 = \hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\hat{e}_2 = \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \hat{e}_3 = \hat{z}$

拉梅系数　$h_1 = \left|\dfrac{\partial\vec{r}}{\partial\rho}\right| = |\hat{\rho}| = 1 \quad h_2 = \left|\dfrac{\partial\vec{r}}{\partial\phi}\right| = \left|\rho\dfrac{\partial\hat{\rho}}{\partial\phi}\right| = ?$



$\hat{\rho} = \hat{x}\ \cos\phi + \hat{y}\ \sin\phi$

圆柱坐标系

$h_1 = 1 \quad , \quad h_2 = \rho \quad , \quad h_3 = 1 \qquad d\vec{l}_1 = \hat{e}_1 dl_1 = \hat{e}_1 h_1 du_1$

矢量线元　$d\vec{l} = \hat{\rho}d\rho + \hat{\varphi}\rho d\varphi + \hat{z}dz$

矢量面元　$d\vec{s} = \hat{\rho}ds_1 + \hat{\varphi}ds_2 + \hat{z}ds_3 = \hat{\rho}\rho d\varphi dz + \hat{\varphi}d\rho dz + \hat{z}\rho d\rho d\varphi$

体元　$d\tau = \rho d\rho d\varphi dz$

球坐标系坐系的线元和面元表示:

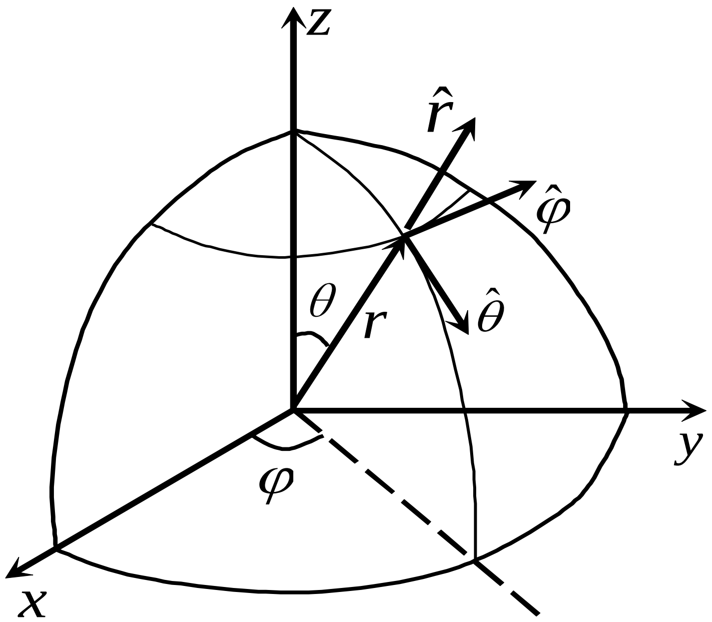
$$\hat{r}=\cos\theta\hat{z}+\sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}$$

矢径 $\vec{r}=r\hat{r}$

单位矢量: $\hat{e}_1=\hat{r};\hat{e}_2=\hat{\theta};\hat{e}_3=\hat{\phi}$

拉梅系数 $h_3=\left|\dfrac{\partial\vec{r}}{\partial\phi}\right|=?$

$h_1=1,\ h_2=r,\ h_3=r\sin\theta$



球坐标系

矢量线元 $d\vec{l}=\hat{r}dr+\hat{\theta}rd\theta+\hat{\varphi}r\sin\theta\, d\varphi$

矢量面元 $d\vec{s}=\hat{r}ds_1+\hat{\theta}ds_2+\hat{\varphi}ds_3=\hat{r}r^2\sin\theta d\theta d\varphi+\hat{\theta}r\sin\theta drd\varphi+\hat{\varphi}rdrd\theta$

体元 $d\tau=r^2\sin\theta drd\theta d\varphi$

课堂练习：已知矢量 $\vec{F}=3y\hat{x}+(3x-y^2)\hat{y}$ ，计算沿直线路径A-C-B-A的积分。

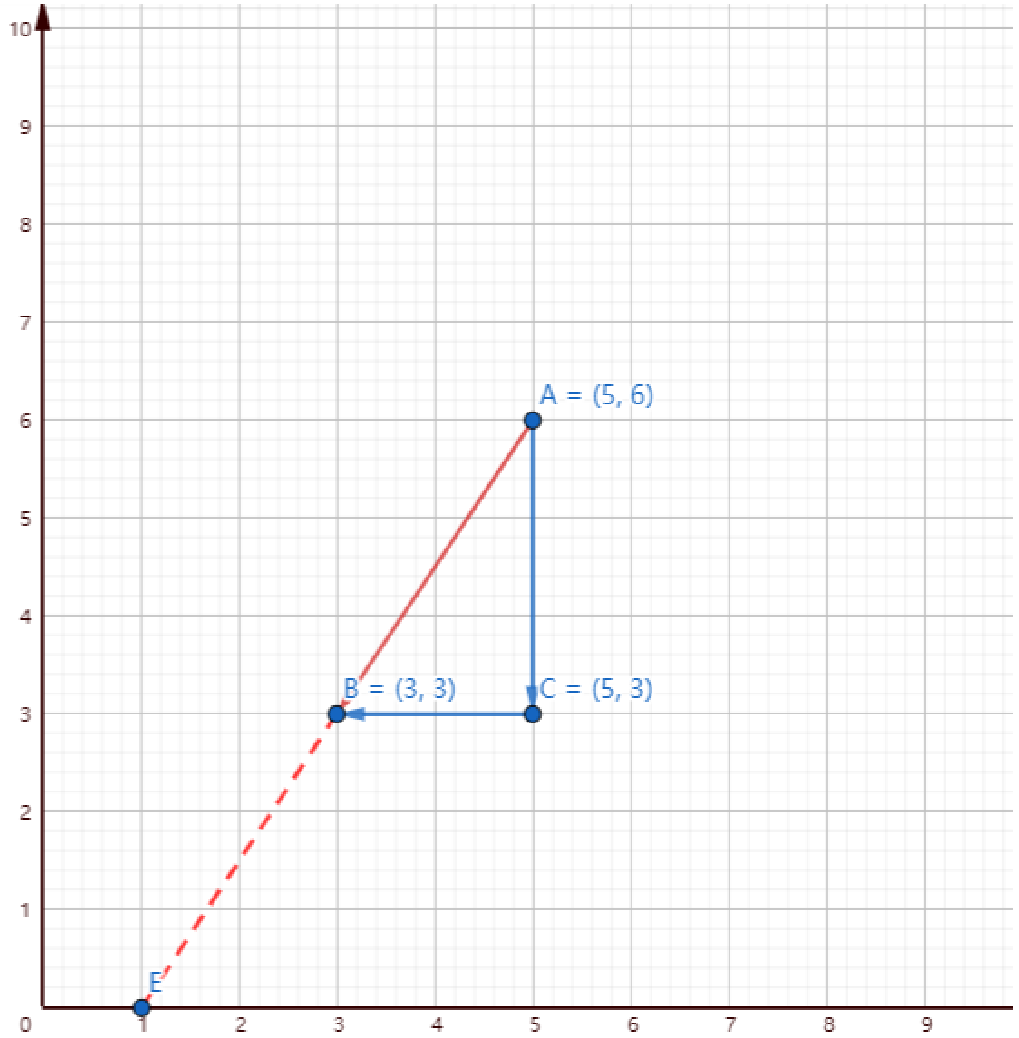
$$d\vec{l}=d\vec{r}=dx\hat{x}+dy\hat{y}$$

$$\oint \vec{F}\cdot d\vec{l}=\int_A^C \vec{F}\cdot d\vec{l}+\int_C^B \vec{F}\cdot d\vec{l}+\int_B^A \vec{F}\cdot d\vec{l}$$

$$AC: x=5; y=6:3$$

$$CB: x=5:3; y=3$$

$$BA: y=\frac{3}{2}(x-1)$$



内　容

5.标量场的方向导数和梯度

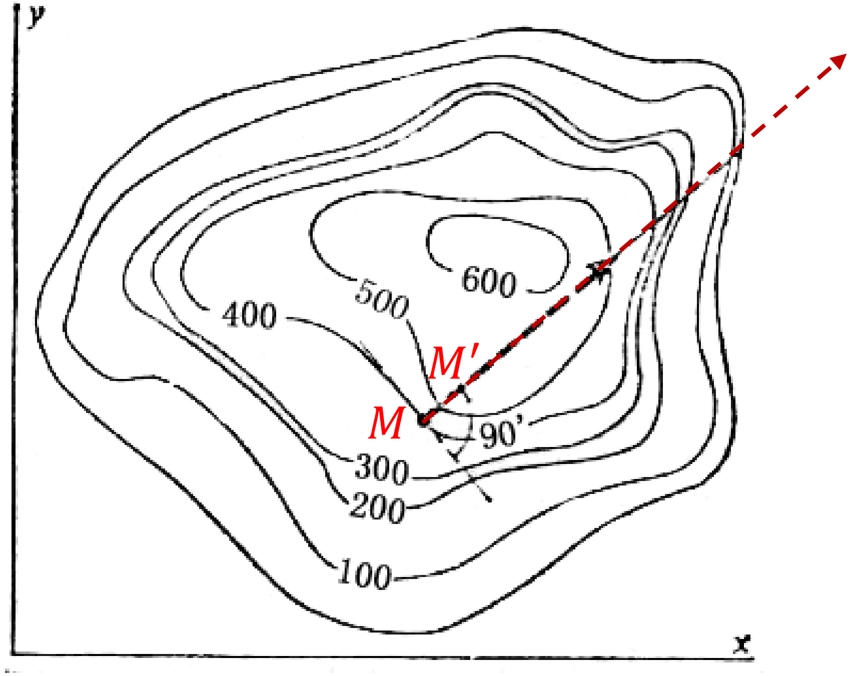
定义式: $\left.\dfrac{\partial f}{\partial l}\right|_M = \lim\limits_{\Delta l \to 0} \dfrac{f(M') - f(M)}{\Delta l}$

意义: 函数f在给定点处沿某个方向对距离的变化率

$\partial f / \partial l > 0$ f在给定点处沿 $\vec{l}$ 方向增加

$\partial f / \partial l < 0$ f在给定点处沿 $\vec{l}$ 方向减小

$\partial f / \partial l = 0$ f在给定点处沿 $\vec{l}$ 方向不变



计算公式:由全增量公式

$$f(M') - f(M) = \Delta f = \frac{\partial f}{\partial u_1}\Delta u_1 + \frac{\partial f}{\partial u_2}\Delta u_2 + \frac{\partial f}{\partial u_3}\Delta u_3 + \omega \Delta l$$

代入定义式中 $$\frac{\Delta f}{\Delta l}=\frac{\partial f}{\partial u_1}\frac{\Delta u_1}{\Delta l}+\frac{\partial f}{\partial u_2}\frac{\Delta u_2}{\Delta l}+\frac{\partial f}{\partial u_3}\frac{\Delta u_3}{\Delta l}+\omega$$

$$\left.\frac{\partial f}{\partial l}\right|_M=\lim_{\Delta l\to 0}\left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\boxed{\frac{h_1\Delta u_1}{\Delta l}}+\frac{1}{h_2}\frac{\partial f}{\partial u_2}\boxed{\frac{h_2\Delta u_2}{\Delta l}}+\frac{1}{h_3}\frac{\partial f}{\partial u_3}\boxed{\frac{h_3\Delta u_3}{\Delta l}}+\omega\right)$$

$h_i\Delta u_i \longrightarrow$ Δl 在坐标曲线上u_i的投影

$h_i\Delta u_i/\Delta l \longrightarrow$ $\vec{l}$ 对坐标单位矢量 $\hat{e}_i$的方向余弦

若 α，β，γ 用分别表示M点处 $\vec{l}$ 与$\hat{e}_1$，$\hat{e}_2$，$\hat{e}_3$的夹角，则

$$\frac{h_1\Delta u_1}{\Delta l}=\cos\alpha\ ;\quad \frac{h_2\Delta u_2}{\Delta l}=\cos\beta\ ;\quad \frac{h_3\Delta u_3}{\Delta l}=\cos\gamma$$

方向余弦 ➡ 表征射线l的方向

代入上式，考虑当$\Delta l\to 0$ 时，$\omega\to 0$，略去下标M，得

$$\frac{\partial f}{\partial l}=\frac{1}{h_1}\frac{\partial f}{\partial u_1}\cos\alpha+\frac{1}{h_2}\frac{\partial f}{\partial u_2}\cos\beta+\frac{1}{h_3}\frac{\partial f}{\partial u_3}\cos\gamma$$

★引入梯度

将方向导数表达式写成

$$\frac{\partial f}{\partial l}=\left(\hat{e}_1\frac{1}{h_1}\frac{\partial f}{\partial u_1}+\hat{e}_2\frac{1}{h_2}\frac{\partial f}{\partial u_2}+\hat{e}_3\frac{1}{h_3}\frac{\partial f}{\partial u_3}\right)\cdot(\hat{e}_1\cos\ \alpha+\hat{e}_2\cos\ \beta+\hat{e}_3\cos\ \gamma)\ \boxed{=\nabla f\cdot\hat{e}_l}$$

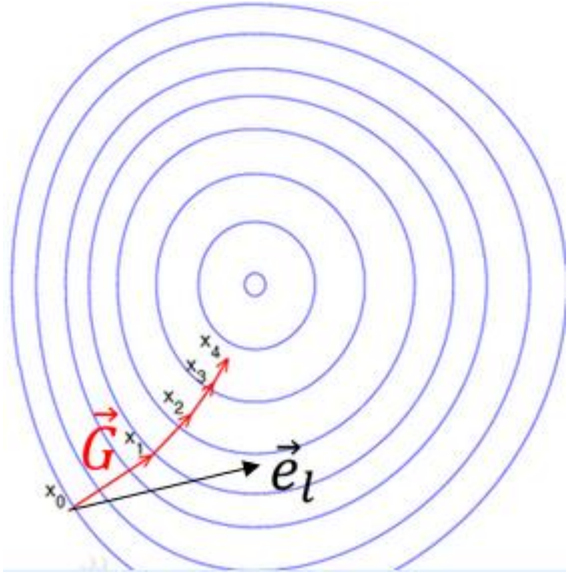
其中 $\hat{e}_l=\hat{e}_1\cos\alpha+\hat{e}_2\cos\beta+\hat{e}_3\cos\gamma \longrightarrow \vec{l}$ 方向上的单位矢量

$$\nabla f=\hat{e}_1\frac{1}{h_1}\frac{\partial f}{\partial u_1}+\hat{e}_2\frac{1}{h_2}\frac{\partial f}{\partial u_2}+\hat{e}_3\frac{1}{h_3}\frac{\partial f}{\partial u_3} \longrightarrow$$ f 在 M 点处的梯度

★定义式 $\mathrm{grad}\ f=\nabla f=\left.\frac{df}{dl}\hat{e}_l\right|_{\max}$

方向　　函数在该点变化率最大的方向

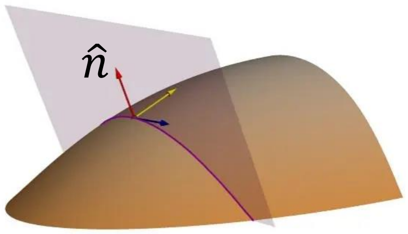
模值　　最大变化率



★性质

①方向导数等于梯度矢量在该方向上的投影

$$\frac{\partial f}{\partial l} = (\nabla f)\cdot \hat{e}_l$$



常见的表示: $\frac{\partial \Phi}{\partial n} = (\nabla \Phi)\cdot \hat{n}$

②场中每一点 M 处的梯度，垂直于过该点等值面，且总指向函数 $f(\vec{r})$ 增大的方向

★哈密顿（Hamilton）算子

①矢量微分算符 $\nabla = \hat{e}_1 \frac{1}{h_1}\frac{\partial}{\partial u_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial}{\partial u_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial}{\partial u_3}$

梯度表示为 $\mathrm{grad}\ f = \nabla f$

②性质：同时具有矢量和微分的性质

③运算法则

$$\nabla c = 0$$

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

$$\nabla fg = g\nabla f + f\nabla g$$

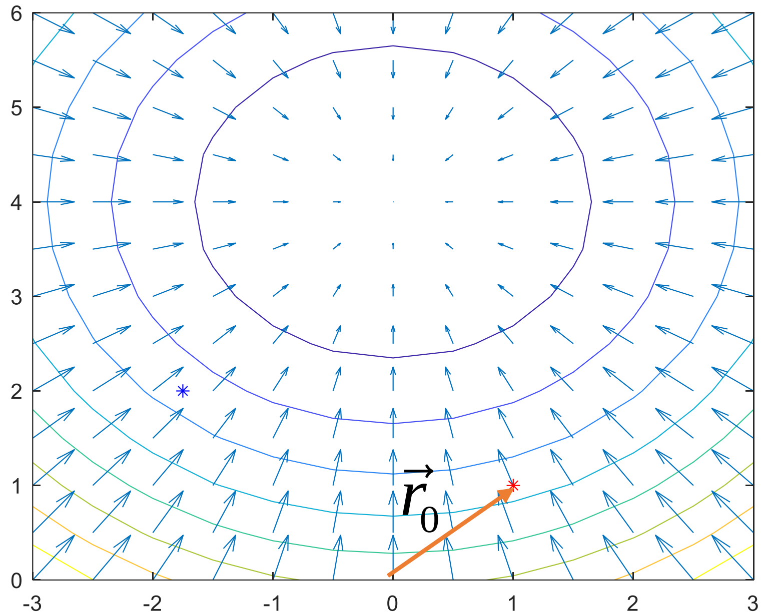
$$\nabla f(g) = \frac{df}{dg}\nabla g$$

复合函数求导法则

梯度概念的使用：最速下降法

已知优化的目标函数(Objective Function)如下，以最速下降法做出优化。设初始值为（1，1）。

$$OF: f(x,y) = x^2 + (y-4)^2$$



梯度：$-\nabla f = -\nabla\left[x^2 + (y-4)^2\right] = -2x\hat{x} - 2(y-4)\hat{y}$

起点：$\vec{r}_0 = \hat{x} + \hat{y}$

下一点：$\vec{r}_1 = \vec{r}_0 + \delta(-\nabla f)\big|_{r_0} = \hat{x} + \hat{y} - \delta 2\hat{x} + \delta 6\hat{y} = (1-2\delta)\hat{x} + (1+6\delta)\hat{y}$

$$f(x_1,y_1) = (1-2\delta)^2 + (1+6\delta-4)^2 = (1-2\delta)^2 + 9(2\delta-1)^2$$

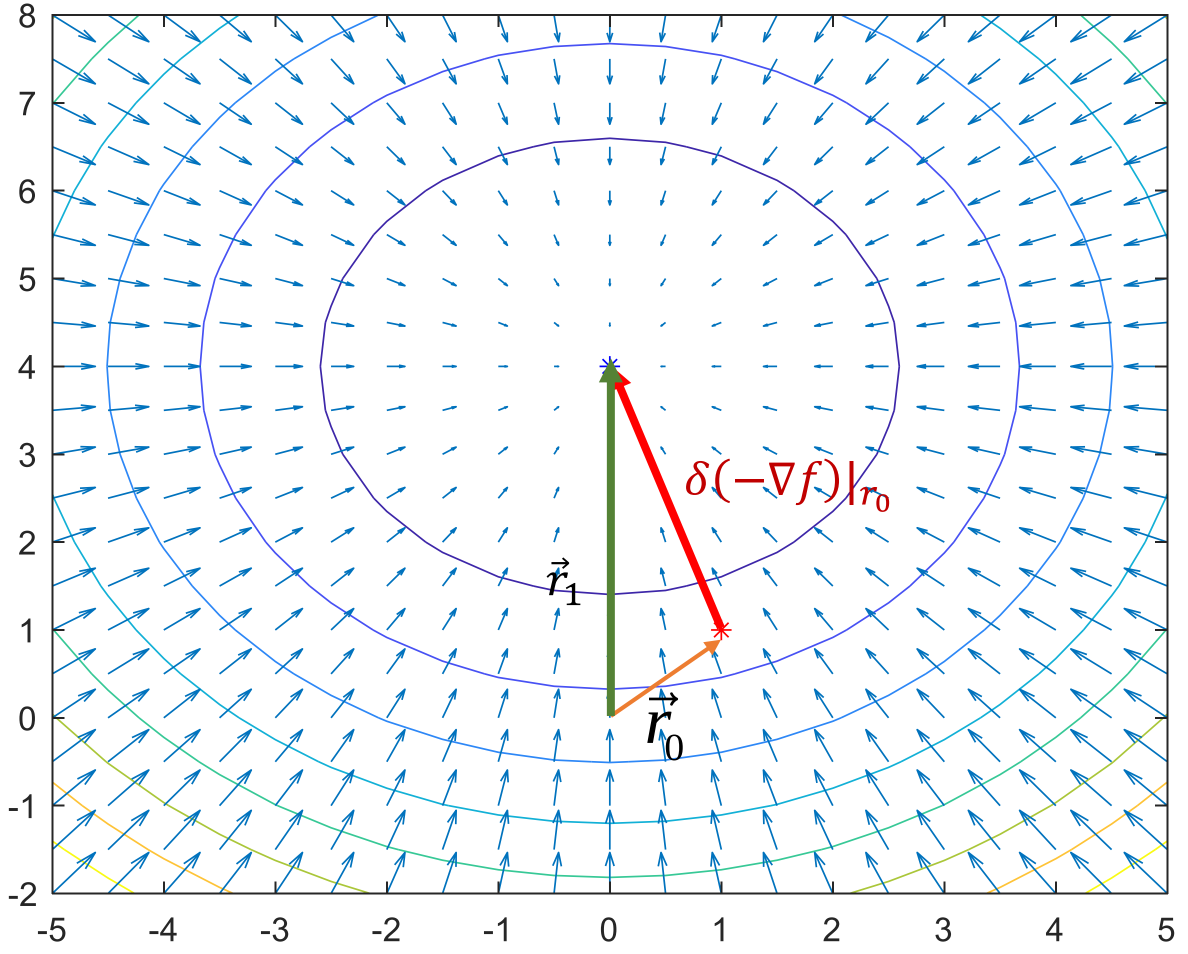
$$1 + 4\delta^2 - 4\delta + 9(4\delta^2 - 4\delta + 1) = 40\delta^2 - 40\delta + 10$$

$$f\left(x_1, y_1\right) = 40\delta^2 - 40\delta + 10 \longrightarrow \text{最小值}$$

$$\frac{df}{d\delta} = 0 \Rightarrow 80\delta - 40 = 0 \Rightarrow \delta = 0.5$$

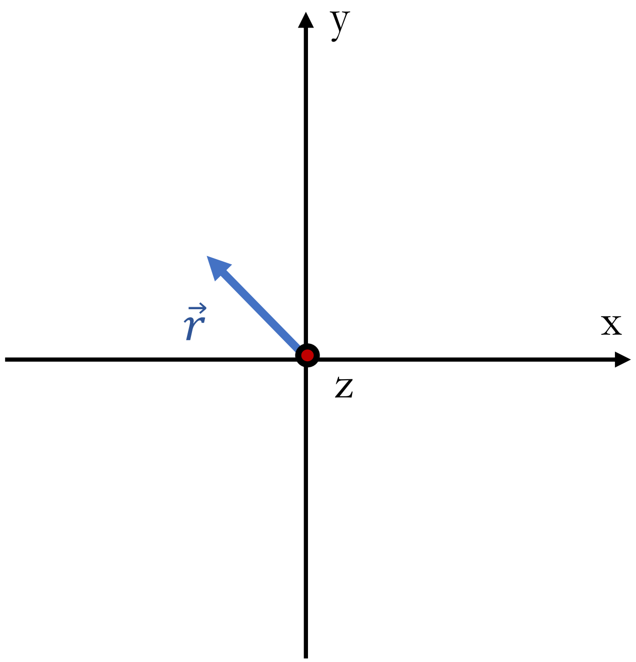
$$\vec{r}_1 = \vec{r}_0 + \delta\left(-\nabla f\right)\Big|_{r_0} = 4\hat{y}$$

$$\left(-\nabla f\right)\Big|_{r_1} = \left(-2\hat{x} - 2\left(y-4\right)\hat{y}\right)\Big|_{r_1} = 0 \quad \text{停止}$$



常见标量场的梯度:

$$\nabla = \hat{e}_1 \frac{1}{h_1}\frac{\partial}{\partial u_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial}{\partial u_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial}{\partial u_3}$$



$$\nabla x = ?$$

$$\nabla \rho = ?$$

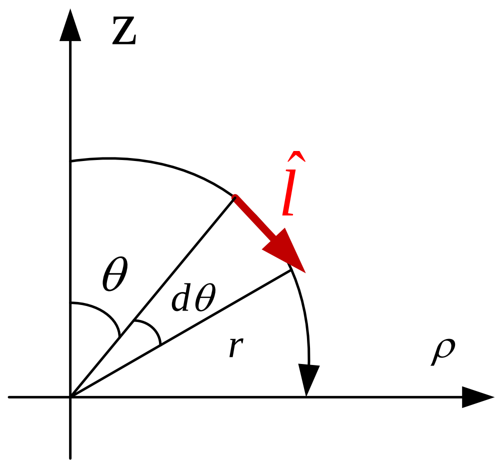
$$\nabla r = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) r = \hat{r}$$

$$\nabla \theta = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \theta = \frac{\hat{\theta}}{r}$$

$$\nabla \phi = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \phi = \frac{\hat{\phi}}{r\sin\theta}$$

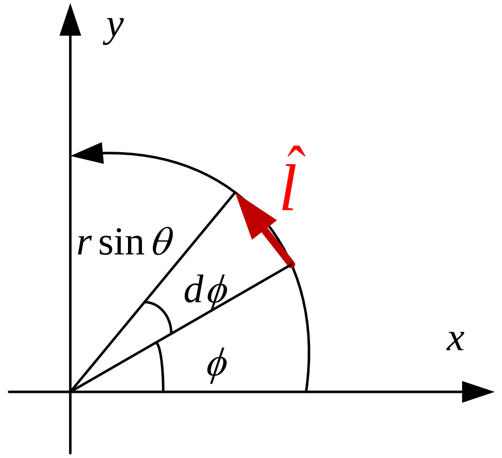
从定义理解~

$$\nabla \frac{1}{r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\nabla r = -\frac{\hat{r}}{r^2} \qquad \nabla f(g) = \frac{df}{dg}\nabla g$$



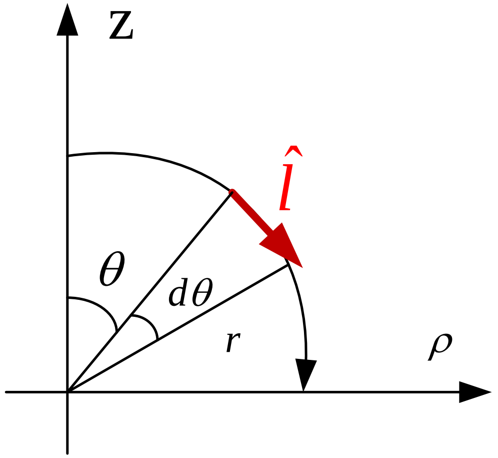
$$\nabla\theta = \frac{\hat{\theta}}{r}$$

$$\left|\nabla f\right| = \left.\frac{\partial f}{\partial l}\right|_{Max} = \lim_{\Delta l \to 0} \frac{\theta - (\theta + d\theta)}{\Delta l} = \lim_{\Delta l \to 0} \frac{d\theta}{r d\theta} = \frac{1}{r}$$

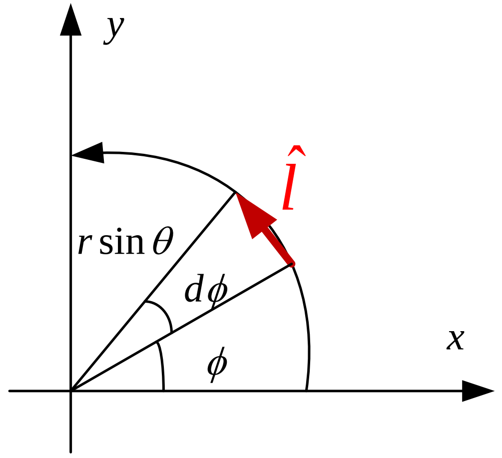


$$\nabla\phi = \frac{\hat{\phi}}{\rho} = \frac{\hat{\phi}}{r\sin\theta}$$

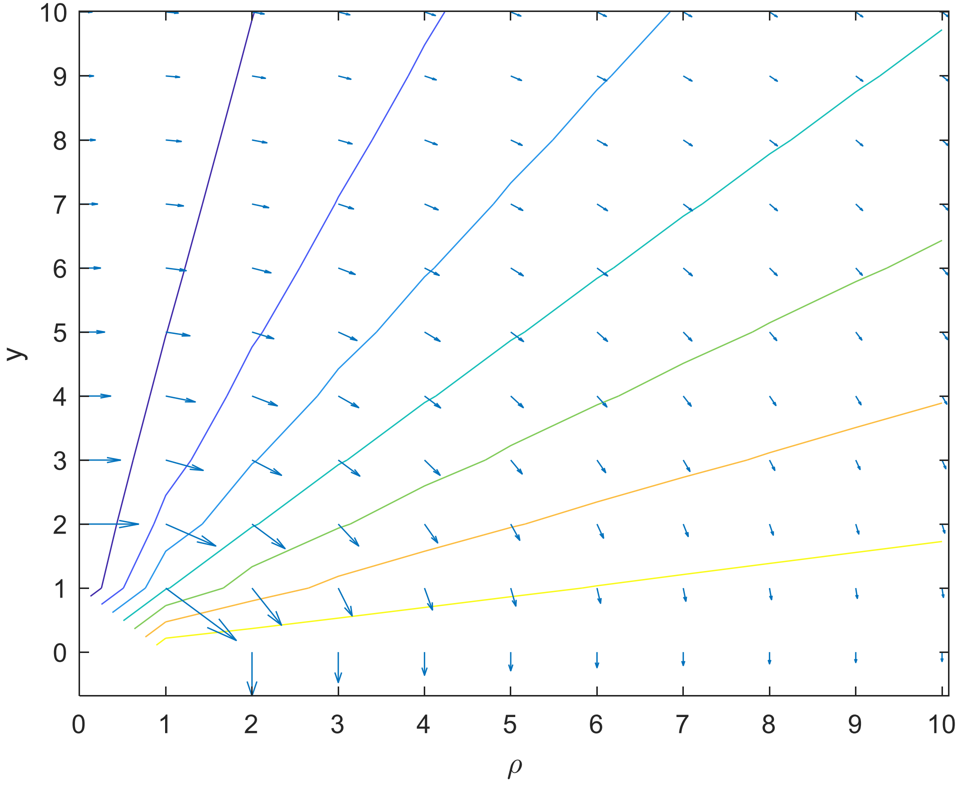
$$\left|\nabla f\right| = \left.\frac{\partial f}{\partial l}\right|_{Max} = \lim_{\Delta l \to 0} \frac{\phi - (\phi + d\phi)}{\Delta l} = \lim_{\Delta l \to 0} \frac{d\phi}{\rho d\phi} = \frac{1}{\rho}$$

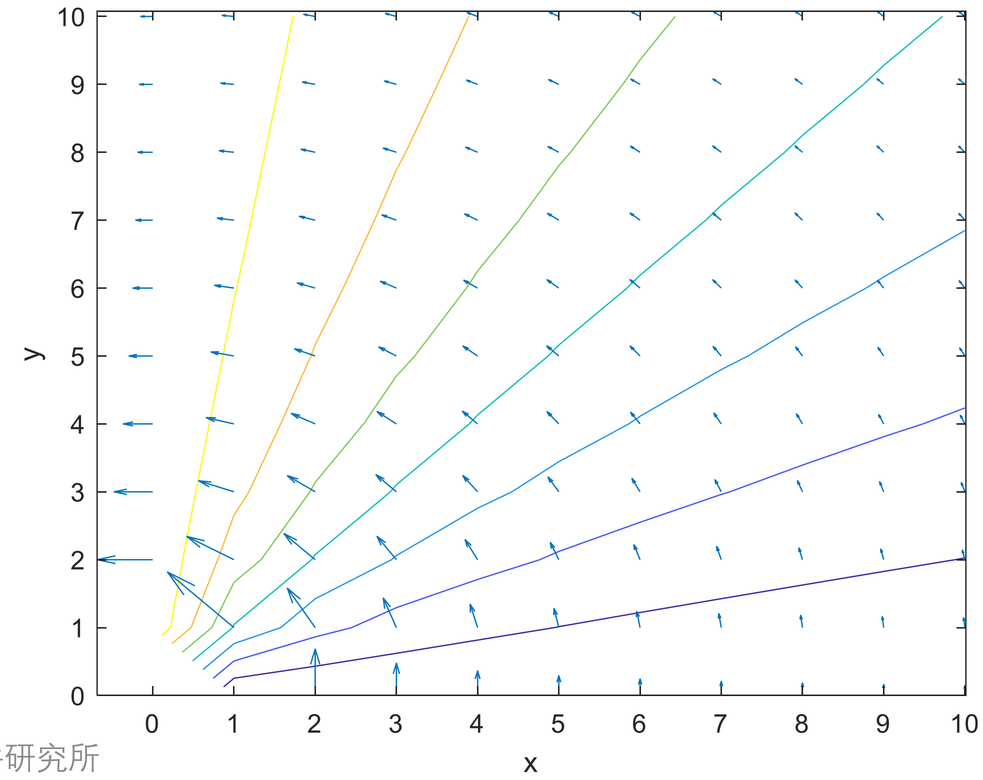


$$\nabla\theta = \frac{\hat{\theta}}{r}$$



$$\nabla\phi = \frac{\hat{\phi}}{\rho}$$





1.8 已知静电场强度 $\boldsymbol{E}$ 可写为标量电势函数的梯度的负值，即 $\boldsymbol{E}=-\nabla V$，试求以下电势函数在 $P(x=1,y=0,z=1)$ 点的电场强度。

a） 直角坐标系下，$V=V_0e^{-x-|z|}\sin\dfrac{\pi y}{4}$；

b） 球坐标系下，~~$V=V_0R\cos\theta$~~。 → $\mathrm{V}=V_0(r\cos\theta)$

利用运算法则计算：

$$\nabla f(g)=\frac{df}{dg}\nabla g$$

$$\nabla V=V_0\left(\sin\frac{\pi y}{4}\nabla e^{-x-|z|}+e^{-x-|z|}\nabla\sin\frac{\pi y}{4}\right)$$

$$=V_0\left(\sin\frac{\pi y}{4}\left(e^{-x-|z|}\nabla\left(-x-|z|\right)\right)+e^{-x-|z|}\cos\frac{\pi y}{4}\nabla\left(\frac{\pi y}{4}\right)\right)$$

$$=V_0\left(\sin\frac{\pi y}{4}\left(e^{-x-|z|}\left(-\hat{x}\mp\hat{z}\right)\right)+e^{-x-|z|}\cos\frac{\pi y}{4}\frac{\pi\hat{y}}{4}\right)$$

$$=V_0e^{-x-|z|}\left(\sin\frac{\pi y}{4}\left(-\hat{x}\mp\hat{z}\right)+\cos\frac{\pi y}{4}\frac{\pi\hat{y}}{4}\right)$$

$$\vec{E}_P=-V_0e^{-2}\left(\frac{\pi\hat{y}}{4}\right)$$

1.8 已知静电场强度 $\boldsymbol{E}$ 可写为标量电势函数的梯度的负值，即 $\boldsymbol{E}=-\nabla V$，试求以下电势函数在 $P(x=1,y=0,z=1)$ 点的电场强度。

a) 直角坐标系下，$V=V_0 e^{-x-|z|}\sin\frac{\pi y}{4}$；

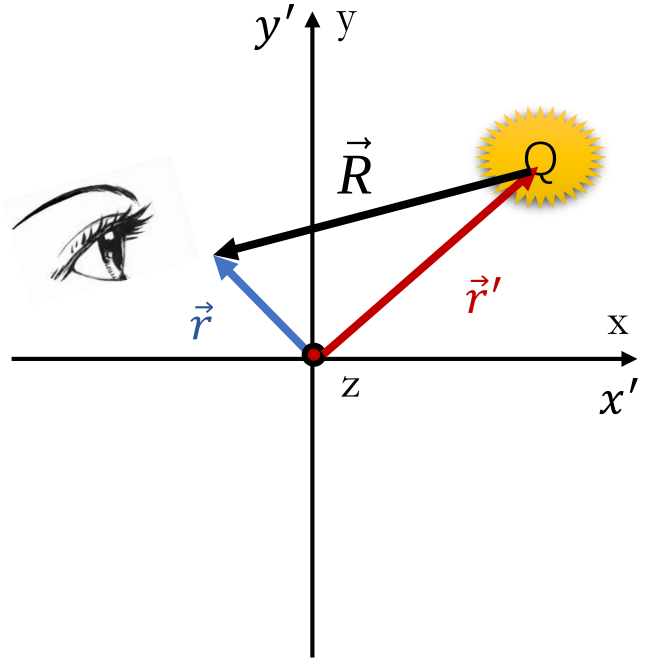
b) 球坐标系下，$\cancel{V=V_0 R\cos\theta}$。 $\boxed{V=V_0(r\cos\theta)}$

$$\nabla V=\nabla\left[V_0\left(r\cos\theta\right)\right]=V_0\left[r\nabla\left(\cos\theta\right)+\cos\theta\nabla r\right]$$

$$=V_0\left[-r\sin\theta\nabla\theta+\cos\theta\hat{r}\right]=V_0\left[-r\sin\theta\frac{\hat{\theta}}{r}+\cos\theta\hat{r}\right]$$

$$=V_0\left[-\sin\theta\hat{\theta}+\cos\theta\hat{r}\right]$$

$$\vec{E}_P=-V_0\left[-\sin\frac{\pi}{4}\hat{\theta}+\cos\frac{\pi}{4}\hat{r}\right]=V_0\left[\sin\frac{\pi}{4}\hat{\theta}-\cos\frac{\pi}{4}\hat{r}\right]$$



场位置： $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

源场位置： $\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$$

$$\vec{R} = \vec{r} - \vec{r}'$$

$$= (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$R = \left|\vec{R}\right| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\nabla R = \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = \hat{R}$$

$$\nabla' R = -\hat{R}$$

$$\nabla\frac{1}{R} = -\frac{\nabla R}{R^2} = -\frac{\hat{R}}{R^2}$$

$$\nabla'\frac{1}{R} = -\frac{\nabla' R}{R^2} = \frac{\hat{R}}{R^2}$$

内　容

1. 矢量、 失径（位置矢量）、 场矢量
2. 矢量的点积（/点乘）、 叉积（叉乘）
3. 矢量的坐标变换
4. 矢量微分
5. 标量场的方向导数和梯度
6. 通量和散度
7. 环量和旋度
8. 拉普拉斯算子
9. 散度定理
10. 斯托克斯定理

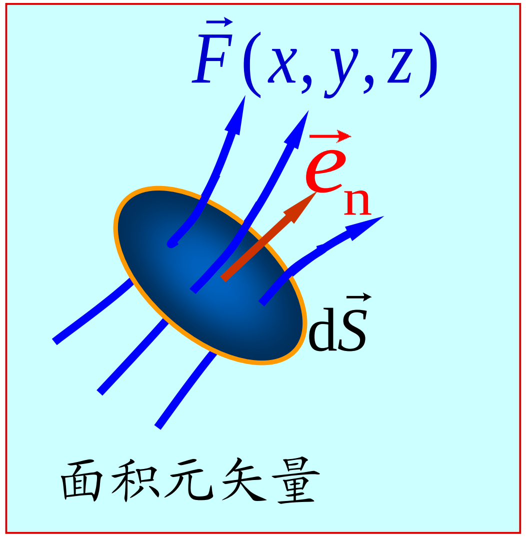
6.通量和散度

通量

★微分定义　$d\Phi = \vec{F}\cdot d\vec{s} = \left|\vec{F}\right|\cos\theta\, ds$

★对于有向曲面S　$\Phi = \int_S \vec{F}\cdot d\vec{S}$

★若曲面S是闭合曲面　$\Phi = \oiint_S \vec{F}(\vec{r})\cdot d\vec{\mathbf{S}}$



★物理意义:

a) 若 $\Phi > 0$ ，闭合面内有正源;

b) 若 $\Phi < 0$ ，闭合面内有负源;

c) 若 $\Phi = 0$ ，闭合面无源。

散度

★定义　　在场空间 $\vec{F}(\vec{r})$ 中任意点M处作一个闭合曲面，所围的体积为$\Delta\tau$，则定义场矢量 $\vec{F}(\vec{r})$ 在 M 点处的散度为：

$$\mathrm{div}\vec{F}(\vec{r}) = \nabla\cdot\vec{F} = \lim_{\Delta\tau\to 0}\frac{\oint_s \vec{F}(\vec{r})\cdot d\vec{\mathbf{S}}}{\Delta\tau}$$

★物理意义　　通量（宏观）➡ 散度（微观）

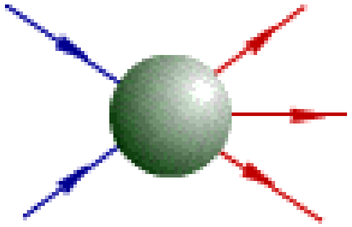
1) 矢量场的散度是一个标量；

2) 矢量场的散度值表示场中一点处的通量对体积的变化率，也就是在该点处单位体积所穿出之通量，也称为源强度。

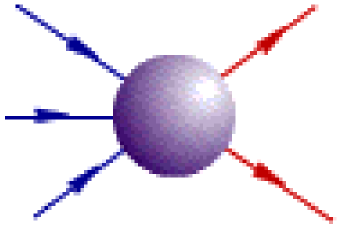
$\nabla\cdot\vec{F}(\vec{r}) = 0$　无源场

$\nabla\cdot\vec{F}(\vec{r}) > 0$　有正源

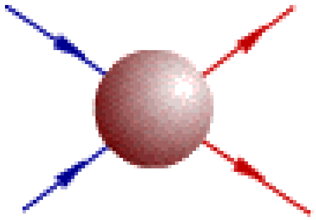
$\nabla\cdot\vec{F}(\vec{r}) < 0$　有负源



（$div\vec{F}(\vec{r}) = \rho > 0$ 正源）

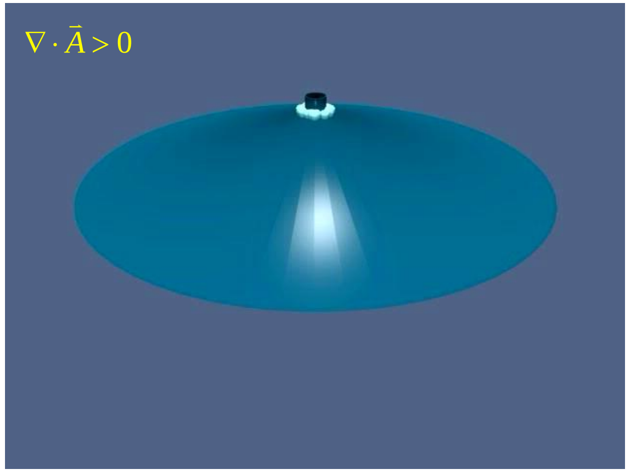


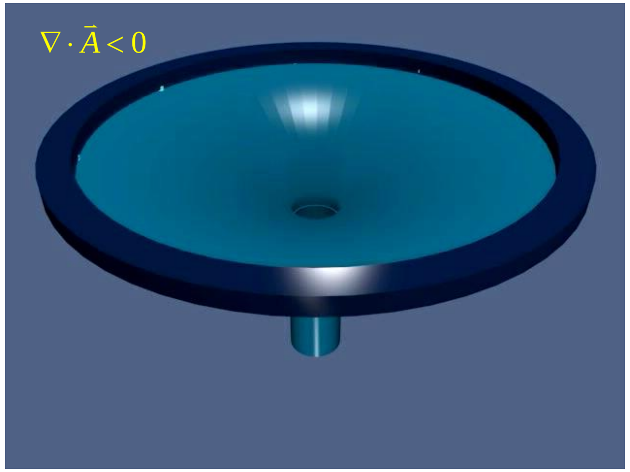
$div\vec{F}(\vec{r}) = \rho < 0$ 负源）



（ $div\vec{F}(\vec{r}) = 0$ 无源 ）

Source





Sink

$$\nabla\cdot\vec{F}=\frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(F_1h_2h_3)+\frac{\partial}{\partial u_2}(F_2h_1h_3)+\frac{\partial}{\partial u_3}(F_3h_1h_2)]$$

★常用坐标系下的散度计算

1) 直角坐标系 $$\nabla\cdot\vec{F}(\vec{r})=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$

2) 圆柱坐标系 $$\nabla\cdot\vec{F}(\vec{r})=\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial F_\varphi}{\partial\varphi}+\frac{\partial F_z}{\partial z}$$

3) 球坐标系 $$\nabla\cdot\vec{F}(\vec{r})=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

散度的运算法则：

$$\nabla\cdot\vec{C}=0$$

$$\nabla\cdot\left(\vec{F}\pm\vec{E}\right)=\nabla\cdot\vec{F}\pm\nabla\cdot\vec{E}$$

$$\nabla\cdot f\vec{F}=f\nabla\cdot\vec{F}+\vec{F}\cdot\nabla f$$

$$\nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_1 h_3) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right]$$

$$\nabla \cdot \hat{x} = ?$$

$$\nabla \cdot \hat{\rho} = ?$$

$$\nabla \cdot \hat{\phi} = ?$$

$$\nabla \cdot \hat{r} = ?$$

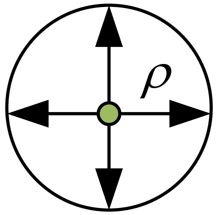
$$\nabla \cdot \hat{\theta} = ?$$

$$\boxed{\begin{aligned} &\nabla \cdot \hat{x} = 0; \\ &\nabla \cdot \hat{\rho} = 1/\rho; \\ &\nabla \cdot \hat{\phi} = 0; \\ &\nabla \cdot \hat{r} = \frac{2}{r}; \\ &\nabla \cdot \hat{\theta} = \frac{\cos\theta}{r\sin\theta}; \end{aligned}}$$

从定义理解~~

$$\nabla \cdot \hat{\rho} = \lim_{\Delta\tau \to 0} \frac{\oint_s \hat{\rho} \cdot d\vec{\mathbf{S}}}{\Delta \tau}$$

$$\nabla \cdot \hat{\rho} = \lim_{\Delta\tau \to 0} \frac{\oint_s \hat{\rho} \cdot d\vec{\mathbf{S}}}{\Delta \tau} = \lim_{\Delta\tau \to 0} \frac{\hat{\rho} \cdot \hat{\rho} 2\pi\rho h}{\pi\rho^2 h} = \lim_{\rho \to 0} \frac{2\rho}{\rho^2} = \lim_{\rho \to 0} \frac{1}{\rho} = \frac{1}{\rho}$$



洛必达法则：“$\frac{0}{0}$”型不定式 $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A$(或$\infty$)

★哈密顿（Hamilton）算子 $\nabla=\vec{e}_1\frac{1}{h_1}\frac{\partial}{\partial u_1}+\vec{e}_2\frac{1}{h_2}\frac{\partial}{\partial u_2}+\vec{e}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}$ 回顾: ∇f

利用哈密顿算子计算散度: $\nabla\cdot\vec{F}=\left[\vec{e}_1\frac{1}{h_1}\frac{\partial}{\partial u_1}+\vec{e}_2\frac{1}{h_2}\frac{\partial}{\partial u_2}+\vec{e}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}\right]\cdot[\vec{e}_1F_1+\vec{e}_2F_2+\vec{e}_3F_3]$

直角坐标系下: $\nabla\cdot\vec{F}=\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\cdot[\hat{x}F_x+\hat{y}F_y+\hat{z}F_z]$

$$
\begin{aligned}
&=\hat{x}\cdot\frac{\partial}{\partial x}(\hat{x}F_x)+\hat{y}\cdot\frac{\partial}{\partial y}(\hat{x}F_x)+\hat{z}\cdot\frac{\partial}{\partial z}(\hat{x}F_x)\\
&+\hat{x}\cdot\frac{\partial}{\partial x}(\hat{y}F_y)+\hat{y}\cdot\frac{\partial}{\partial y}(\hat{y}F_y)+\hat{z}\cdot\frac{\partial}{\partial z}(\hat{y}F_y)\\
&+\hat{x}\cdot\frac{\partial}{\partial x}(\hat{z}F_z)+\hat{y}\cdot\frac{\partial}{\partial y}(\hat{z}F_z)+\hat{z}\cdot\frac{\partial}{\partial z}(\hat{z}F_z)=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}
\end{aligned}
$$

$$\hat{x}\cdot\frac{\partial}{\partial x}(\hat{x}F_x)=\hat{x}\cdot\left[\hat{x}\frac{\partial F_x}{\partial x}+F_x\frac{\partial\hat{x}}{\partial x}\right]=\frac{\partial F_x}{\partial x}$$

注意：对于曲面坐标系按上面的哈密顿算子展开计算是十分繁杂的（<u>因为微分运算不仅要对矢量场各分量进行，而且要对单位方向矢量进行</u>），故一般不用哈密顿算子点乘来计算，而是直接使用微分式。

圆柱坐标系：
$$\nabla = \vec{e}_1 \frac{1}{h_1}\frac{\partial}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial}{\partial u_3} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla \cdot \vec{F} = \left[\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right] \cdot \left[\hat{\rho}F_\rho + \hat{\phi}F_\phi + \hat{z}F_z\right]$$

$$= \hat{\rho}\cdot\frac{\partial}{\partial \rho}\left(\hat{\rho}F_\rho\right) + \frac{1}{\rho}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left(\hat{\rho}F_\rho\right) + \hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\rho}F_\rho\right)$$

$$+ \hat{\rho}\cdot\frac{\partial}{\partial \rho}\left(\hat{\phi}F_\phi\right) + \frac{1}{\rho}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left(\hat{\phi}F_\phi\right) + \hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\phi}F_\phi\right)$$

$$+ \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\boxed{\nabla \cdot \vec{F}(\vec{r}) = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}}$$

$$\hat{\rho}\cdot\frac{\partial}{\partial \rho}\left(\hat{\rho}F_\rho\right) = ? \quad = \hat{\rho}\cdot\left[\hat{\rho}\frac{\partial F_\rho}{\partial \rho} + F_\rho\frac{\partial \hat{\rho}}{\partial \rho}\right] = \frac{\partial F_\rho}{\partial \rho}$$

$$\frac{1}{\rho}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left(\hat{\rho}F_\rho\right) = ? = \hat{\phi}\frac{1}{\rho}\cdot\left[\hat{\rho}\frac{\partial F_\rho}{\partial \phi} + F_\rho\frac{\partial \hat{\rho}}{\partial \phi}\right] = \frac{1}{\rho}\hat{\phi}\cdot\frac{\partial \hat{\rho}}{\partial \phi}F_\rho = \frac{1}{\rho}F_\rho$$

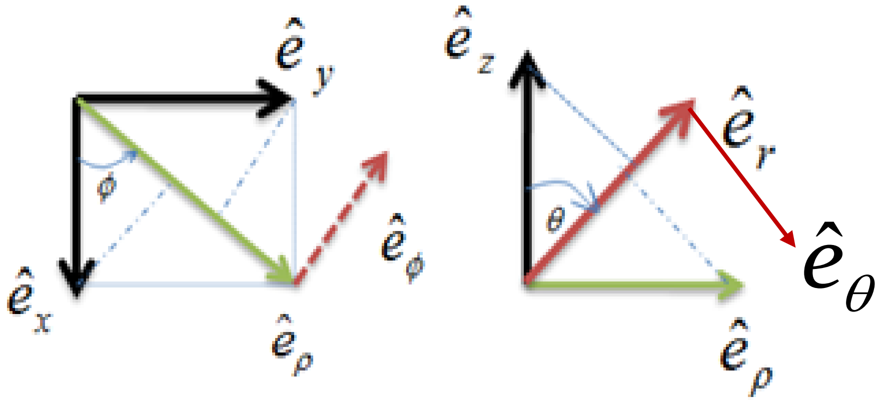
$$\hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\rho}F_\rho\right) = ? \quad = \hat{z}\cdot\left[\hat{\rho}\frac{\partial F_\rho}{\partial z} + F_\rho\frac{\partial \hat{\rho}}{\partial z}\right] = 0$$

$$\hat{\rho}\cdot\frac{\partial}{\partial \rho}\left(\hat{\phi}F_\phi\right) = ? \quad = \hat{\rho}\cdot\left[\hat{\phi}\frac{\partial F_\phi}{\partial \rho} + F_\phi\frac{\partial \hat{\phi}}{\partial \rho}\right] = 0$$

$$\frac{1}{\rho}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left(\hat{\phi}F_\phi\right) = ? = \hat{\phi}\frac{1}{\rho}\cdot\left[\hat{\phi}\frac{\partial F_\phi}{\partial \phi} + F_\phi\frac{\partial \hat{\phi}}{\partial \phi}\right] = \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi}$$

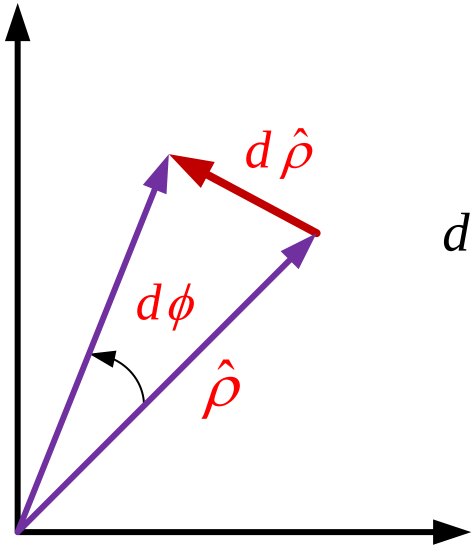
$$\hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\phi}F_\phi\right) = ? \quad = \hat{z}\cdot\left[\hat{\phi}\frac{\partial F_\phi}{\partial z} + F_\phi\frac{\partial \hat{\phi}}{\partial z}\right] = 0$$

$$\frac{\partial\hat{\rho}}{\partial\rho}=0 \qquad \frac{\partial\hat{\phi}}{\partial\rho}=0$$

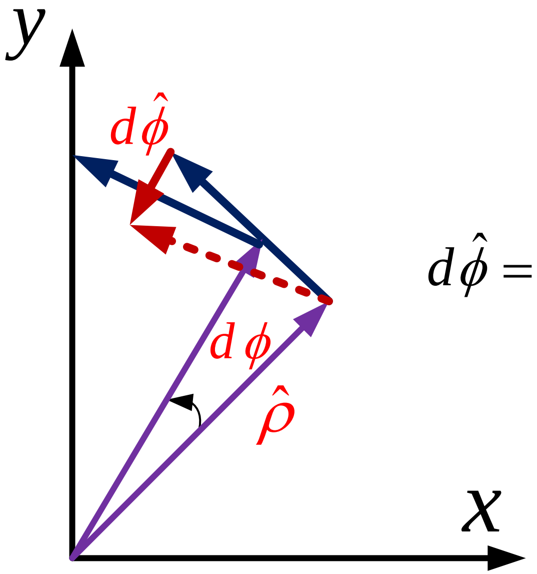


$$\frac{\partial\hat{\rho}}{\partial\phi}=\frac{\partial}{\partial\phi}\left(\cos\phi\hat{x}+\sin\phi\hat{y}\right)=-\sin\phi\hat{x}+\cos\phi\hat{y}=\hat{\phi}$$

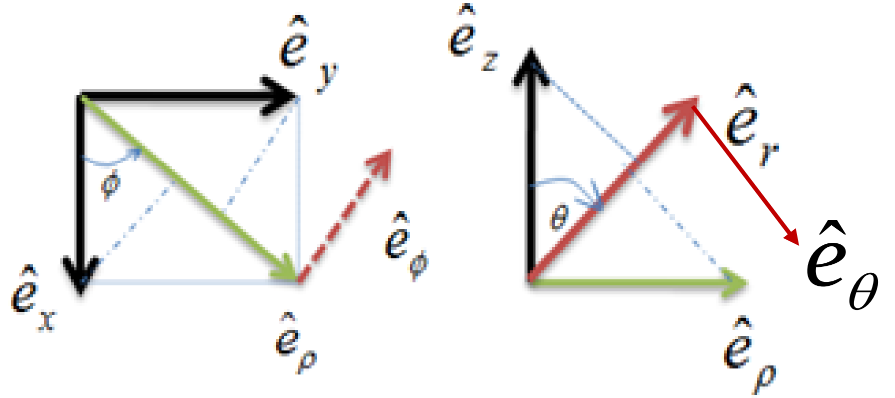
$$\frac{\partial\hat{\phi}}{\partial\phi}=\frac{\partial}{\partial\phi}\left(-\sin\phi\hat{x}+\cos\phi\hat{y}\right)=-\cos\phi\hat{x}-\sin\phi\hat{y}=-\hat{\rho}$$



$$d\hat{\rho}=\hat{\phi}\cdot 1\cdot d\phi \Rightarrow \frac{d\hat{\rho}}{d\phi}=\hat{\phi}$$



$$d\hat{\phi}=-\hat{\rho}\cdot 1\cdot d\phi \Rightarrow \frac{d\hat{\phi}}{d\phi}=-\hat{\rho}$$



$$\frac{\partial \hat{r}}{\partial r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}\right) = -\sin\theta\hat{z} + \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}\right) = -\sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y} = \sin\theta\hat{\phi}$$

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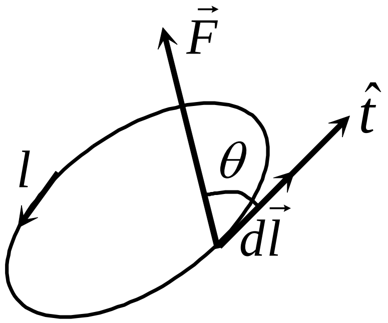
7. 环量和旋度

环量与环量密度

★环量：在矢量场中，矢量沿某一闭合有向曲线 l 的曲线积分，称为该矢量按所取方向沿曲线的环量

$$\Gamma = \oint_l \vec{F} \cdot d\vec{l} = \oint_l F\cos\theta \, dl$$



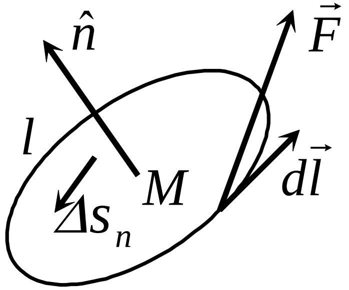
环量

注意：当闭合曲线所围曲面的方向取定后，曲线的方向总是按右旋法则确定。

★环量密度

在矢量场 $\vec{F}(\vec{r})$ 中，围绕空间某点M取一面元ΔS_n，其边界曲线为l，面元法线方向为 $\hat{n}$，当面元以任意方式向M收缩时，则$\vec{F}(\vec{r})$在l上的环量与ΔS_n比值的极限称为在M点处沿$\hat{n}$方向的环量密度。

$$\mu_n = \lim_{\Delta s_n \to 0} \frac{\oint_l \vec{F} \cdot d\vec{l}}{\Delta s_n}$$



环量密度

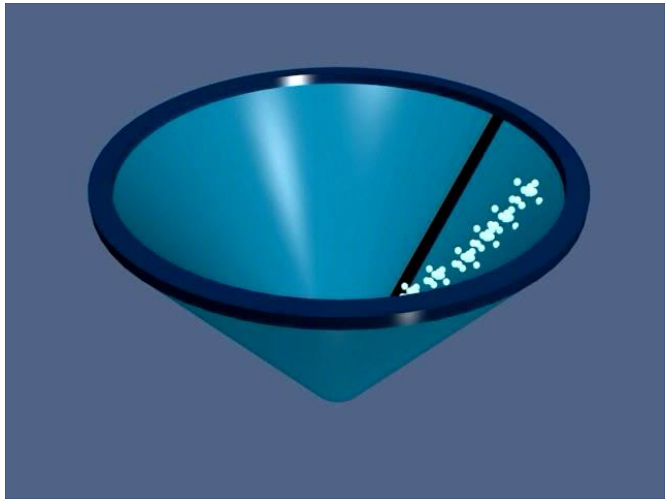
意义：该点给定方向上的单位面积的环量

旋度

★定义式 $$(\mathrm{rot}\,\vec{F})\cdot\hat{n}=\left(\nabla\times\vec{F}\right)\cdot\hat{n}=\lim_{\Delta s_n\to 0}\frac{\oint\vec{F}\cdot d\vec{l}}{\Delta s_n}$$

★特点：旋度矢量的模等于该点的最大环量密度，其方向就是取得该最大环量密度的方向。

$$\left(\nabla\times\vec{F}\right)\cdot\hat{n}=\left|\nabla\times\vec{F}\right|\hat{n}_0\cdot\hat{n}\le\left|\nabla\times\vec{F}\right|$$



★物理意义

1）矢量的旋度为矢量，是空间坐标的函数；

2）在矢量场中，若 $\nabla\times\vec{F}\neq 0$ ， 称为有旋场；

3）矢量在空间某点处的旋度表征矢量场在该点处的旋涡源密度。

旋度计算公式:

$$\nabla\times\vec{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ (F_1h_1) & (F_2h_2) & (F_3h_3) \end{vmatrix}$$

旋度运算法则:

$$\begin{cases} \nabla\times\vec{C}=0 \\ \nabla\times(f\vec{F})=f\nabla\times\vec{F}+\nabla f\times\vec{F} \\ \nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot\nabla\times\vec{A}-\vec{A}\cdot\nabla\times\vec{B} \\ \nabla\times(\vec{A}\times\vec{B})=\vec{A}\nabla\cdot\vec{B}-\vec{B}\nabla\cdot\vec{A}+\left(\vec{B}\cdot\nabla\right)\vec{A}-\left(\vec{A}\cdot\nabla\right)\vec{B} \\ \nabla(\vec{A}\cdot\vec{B})=\vec{A}\times\nabla\times\vec{B}+\vec{B}\times\nabla\times\vec{A}+\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A} \end{cases}$$

$$\nabla\times\vec{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ (F_1h_1) & (F_2h_2) & (F_3h_3) \end{vmatrix}$$

$$\boxed{\begin{aligned} &\nabla\times\hat{x}=?\\ &\nabla\times\hat{\rho}=?\\ &\nabla\times\hat{\phi}=?\\ &\nabla\times\hat{r}=?\\ &\nabla\times\hat{\theta}=? \end{aligned}}$$

$$\boxed{\begin{aligned} &\nabla\times\hat{x}=0;\nabla\times\hat{y}=0;\nabla\times\hat{z}=0\\ &\nabla\times\hat{\rho}=0;\\ &\nabla\times\hat{\phi}=\frac{\hat{z}}{\rho};\\ &\nabla\times\hat{r}=0;\\ &\nabla\times\hat{\theta}=\frac{\hat{\phi}}{r}; \end{aligned}}$$

从定义理解~~ $\left(\nabla\times\hat{\phi}\right)\cdot\hat{n}=\lim\limits_{\Delta s_n\to0}\dfrac{\oint\hat{\phi}\cdot d\vec{l}}{\Delta s_n}$

$$\left|\nabla\times\hat{\phi}\right|=\lim_{\Delta s_n\to0}\frac{\oint\hat{\phi}\cdot d\vec{l}}{\Delta s_n}=\lim_{\rho\to0}\frac{\oint\hat{\phi}\cdot\hat{\phi}d\phi}{\pi\rho^2}=\lim_{\rho\to0}\frac{2\pi\rho}{\pi\rho^2}=\frac{1}{\rho}$$

例题：已知矢径：$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$，和一常失量 $\vec{A}=a\hat{x}+b\hat{y}+c\hat{z}$ 。

求： $\nabla(\vec{r}\cdot\vec{A})$

解1： $\vec{r}\cdot\vec{A}=ax\hat{x}+by\hat{y}+cz\hat{z} \quad \Rightarrow \nabla(\vec{r}\cdot\vec{A})=a\hat{x}+b\hat{y}+c\hat{z}=\vec{A}$

解2： $\nabla(\vec{A}\cdot\vec{B})=\vec{A}\times\nabla\times\vec{B}+\vec{B}\times\nabla\times\vec{A}+\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A}$

$$\nabla(\vec{r}\cdot\vec{A})=\underline{\vec{r}\times\nabla\times\vec{A}+\vec{A}\times\nabla\times\vec{r}+\left(\vec{r}\cdot\nabla\right)\vec{A}}+\left(\vec{A}\cdot\nabla\right)\vec{r}$$

$$=\underline{0+0+0}+\left(a\hat{x}+b\hat{y}+c\hat{z}\right)\cdot\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\vec{r}$$

$$=\left(a\frac{\partial\vec{r}}{\partial x}+b\frac{\partial\vec{r}}{\partial y}+c\frac{\partial\vec{r}}{\partial z}\right)=\left(a\hat{x}+b\hat{y}+c\hat{z}\right)=\vec{A}$$

例题：(r,θ,ϕ) 为球坐标系坐标，计算 $\nabla\times\nabla\left(\dfrac{1}{r}\right)$

$$\nabla\times\nabla\left(\frac{1}{r}\right)=\nabla\times\left(-\frac{\hat{r}}{r^2}\right)=-\frac{1}{r^2}\nabla\times\hat{r}+\nabla\left(-\frac{1}{r^2}\right)\times\hat{r}$$

$$=-\nabla\left(\frac{1}{r^2}\right)\times\hat{r}=\frac{2}{r^3}\nabla r\times\hat{r}=0$$

恒定为零的矢量恒等式

标量场的梯度的旋度恒为零 $\quad \nabla\times(\nabla u)\equiv 0$

矢量场的旋度的散度恒为零 $\quad \nabla\cdot(\nabla\times\vec{F})\equiv 0$

思考：仅仅已知一个矢量的旋度，能否唯一地确定这个矢量场？ $\nabla\times\left(\vec{F}+\nabla\phi+\vec{C}\right)=\nabla\times\vec{F}$

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8. 拉普拉斯算子

标量拉普拉斯算子 $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$

直角坐标系下: $\Delta f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

矢量拉普拉斯算子 $\nabla^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times \nabla \times \boldsymbol{A}$

直角坐标系下: $\nabla^2 \boldsymbol{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$

不同坐标系下的标量拉普拉斯算子计算公式:

直角坐标系 $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

圆柱坐标系 $$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial u}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

球坐标系 $$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

例题： $r \neq 0$, 计算：$\nabla^2 \left(\frac{1}{r}\right)$

$$\nabla^2 \left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = 0$$

$$\nabla \cdot \nabla \left(\frac{1}{r}\right) = \nabla \cdot \left(-\frac{\hat{r}}{r^2}\right) = \nabla \left(-\frac{1}{r^2}\right) \cdot \hat{r} + \nabla \cdot \hat{r} \left(-\frac{1}{r^2}\right)$$

$$= \frac{2}{r^3}\nabla r \cdot \hat{r} + \frac{2}{r}\left(-\frac{1}{r^2}\right) = 0$$

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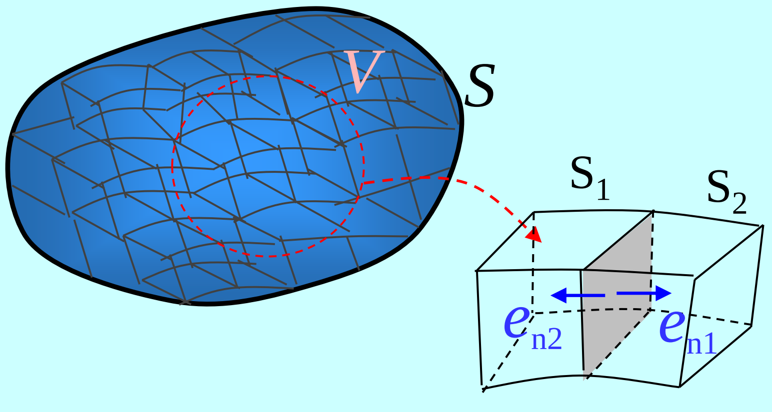
9. 散度定理

★数学表达式 $$\int_\tau \nabla\cdot\vec{F}(\vec{r})d\tau = \oint_S \vec{F}(\vec{r})\cdot d\vec{S}$$

★意义　式中：S为包围$\Delta\tau$的闭合面，面元和体元的坐标表示要一致。

矢量场的散度在给定体积的体积分，等于此矢量场在该体积外表面上的闭合面积分。

★证明 $$\int_V \nabla\cdot\vec{F}\mathrm{d}V = \sum_i \nabla\cdot\vec{F}\mathrm{d}V_i = \sum_i \oint_{S_i} \vec{F}\cdot\mathrm{d}\vec{S}_i$$



体积的剖分

$$= \oint_S \vec{F}\cdot\mathrm{d}\vec{S}$$

补充：δ 函数

1、定义 $\delta\left(x-x'\right)=0 \qquad \left(x\neq x'\right)$

$$\int_a^b \delta(x-x')dx=\begin{cases}1 & (a<x'<b)\\ 0 & (x'<a \;\; or \;\; x'>b)\end{cases}$$

2、性质

① δ 函数是偶函数 $\delta\left(x-x'\right)=\delta\left(x'-x\right)$

② 函数有还原性（筛选性）$\int_{-\infty}^{\infty} f\left(x\right)\delta\left(x-x'\right)dx=f\left(x'\right)$

试证明 $-\frac{1}{4\pi}\nabla^2\frac{1}{R}$ 是 δ 函数

证明：① $\nabla^2\frac{1}{R}=0 \ (\vec{r}\neq\vec{r}')$

$$\nabla^2\frac{1}{R}=\nabla\cdot\left(\nabla\frac{1}{R}\right)=\nabla\cdot\left(-\frac{\nabla R}{R^2}\right)=\nabla\cdot\left(-\frac{\hat{R}}{R^2}\right)=\left(-\frac{1}{R^2}\right)\nabla\cdot\hat{R}+\hat{R}\cdot\nabla\left(-\frac{1}{R^2}\right)$$

$$=-\frac{2}{R^3}+2\hat{R}\cdot\frac{\hat{R}}{R^3}=-\frac{2}{R^3}+\frac{2}{R^3}=0 \ (\vec{r}\neq\vec{r}')$$

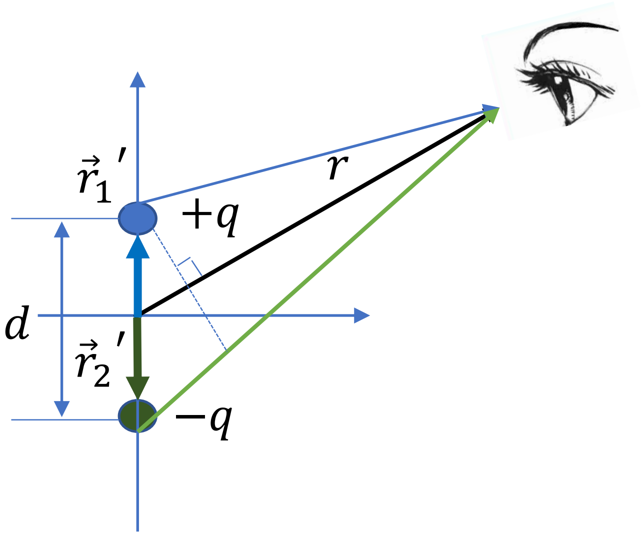
② $\int_\tau -\frac{1}{4\pi}\nabla^2\frac{1}{R}\,d\tau=1$

$$\int_s(\nabla\times\vec{F})\cdot d\vec{s}=\oint_l\vec{F}\cdot d\vec{l}$$

$$\int_\tau -\frac{1}{4\pi}\nabla^2\frac{1}{R}\,d\tau=-\frac{1}{4\pi}\int_\tau\nabla\cdot\nabla\frac{1}{R}\,d\tau=-\frac{1}{4\pi}\int_\tau\nabla\cdot\left(-\frac{\hat{R}}{R^2}\right)d\tau$$

$$=\frac{1}{4\pi}\oint_s\frac{\hat{R}}{R^2}\cdot d\vec{S}=\frac{1}{4\pi}\iint\frac{\hat{R}}{R^2}\cdot\hat{R}R^2\sin\theta d\theta d\phi=\frac{1}{2}\int_0^\pi\sin\theta d\theta=1$$

利用散射定理求解静电场问题：$\nabla\cdot\vec{E}=\dfrac{\rho}{\varepsilon}$ 例题1.14



$$\nabla\cdot\vec{E}=\frac{q}{\varepsilon}\delta\left(\vec{r}-\vec{r}_1'\right)\Rightarrow\oint\nabla\cdot\vec{E}d\tau=\oint\vec{E}\cdot ds=E_1 4\pi\left|\vec{r}-\vec{r}_1'\right|^2=\frac{q}{\varepsilon}$$

$$\Rightarrow E_1=\frac{q}{4\pi\varepsilon\left|\vec{r}-\vec{r}_1'\right|^2}\Rightarrow\vec{E}_1=\frac{q\left(\vec{r}-\vec{r}_1'\right)}{4\pi\varepsilon\left|\vec{r}-\vec{r}_1'\right|^3}$$

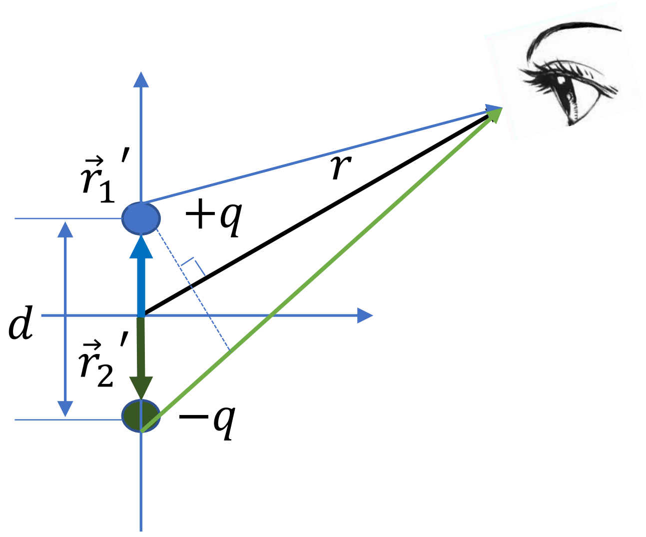
同理：$\vec{E}_2=\dfrac{-q\left(\vec{r}-\vec{r}_2'\right)}{4\pi\varepsilon\left|\vec{r}-\vec{r}_2'\right|^3}$

$r\gg d$

$$\left|\vec{r}-\vec{r}_1'\right|\approx r-\frac{d}{2}\cos\theta;\left|\vec{r}-\vec{r}_2'\right|\approx r+\frac{d}{2}\cos\theta$$

$$\frac{1}{\left|\vec{r}-\vec{r}_1'\right|^3}\approx\frac{1}{\left(r-\frac{d}{2}\cos\theta\right)^3}\approx\frac{1}{r^3\left(1-\frac{d}{2r}\cos\theta\right)^3}\approx\frac{1}{r^3}\left(1+\frac{3d}{2r}\cos\theta\right)$$

$$\frac{1}{\left|\vec{r}-\vec{r}_2'\right|^3}\approx\frac{1}{r^3}\left(1-\frac{3d}{2r}\cos\theta\right)$$



$$\vec{E}_1+\vec{E}_2=\frac{q\left(\vec{r}-\vec{r}_1'\right)}{4\pi\varepsilon\left|\vec{r}-\vec{r}_1'\right|^3}+\frac{-q\left(\vec{r}-\vec{r}_2'\right)}{4\pi\varepsilon\left|\vec{r}-\vec{r}_2'\right|^3}$$

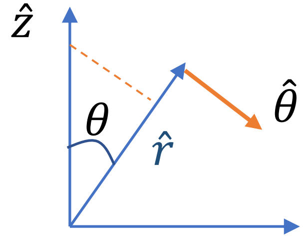
$$=\frac{q}{4\pi\varepsilon r^3}\left[\left(1+\frac{3d}{2r}\cos\theta\right)\left(\vec{r}-\vec{r}_1'\right)-\left(1-\frac{3d}{2r}\cos\theta\right)\left(\vec{r}-\vec{r}_2'\right)\right]$$

$$=\frac{q}{4\pi\varepsilon r^3}\left[\left(1+\frac{3d}{2r}\cos\theta\right)\vec{r}-\left(1+\frac{3d}{2r}\cos\theta\right)\vec{r}_1'-\left(1-\frac{3d}{2r}\cos\theta\right)\vec{r}+\left(1-\frac{3d}{2r}\cos\theta\right)\vec{r}_2'\right]$$

$$=\frac{q}{4\pi\varepsilon r^3}\left[\frac{3d}{r}\cos\theta\vec{r}-\left(1+\frac{3d}{2r}\cos\theta\right)\vec{r}_1'+\left(1-\frac{3d}{2r}\cos\theta\right)\vec{r}_2'\right]$$

$$=\frac{q}{4\pi\varepsilon r^3}\left[3d\cos\theta\hat{r}+\vec{r}_2'-\vec{r}_1'-\left(\frac{3d}{2r}\cos\theta\right)\left(\vec{r}_1'+\vec{r}_2'\right)\right]$$

$$=\frac{q}{4\pi\varepsilon r^3}\left[3d\cos\theta\hat{r}-d\hat{z}\right]$$

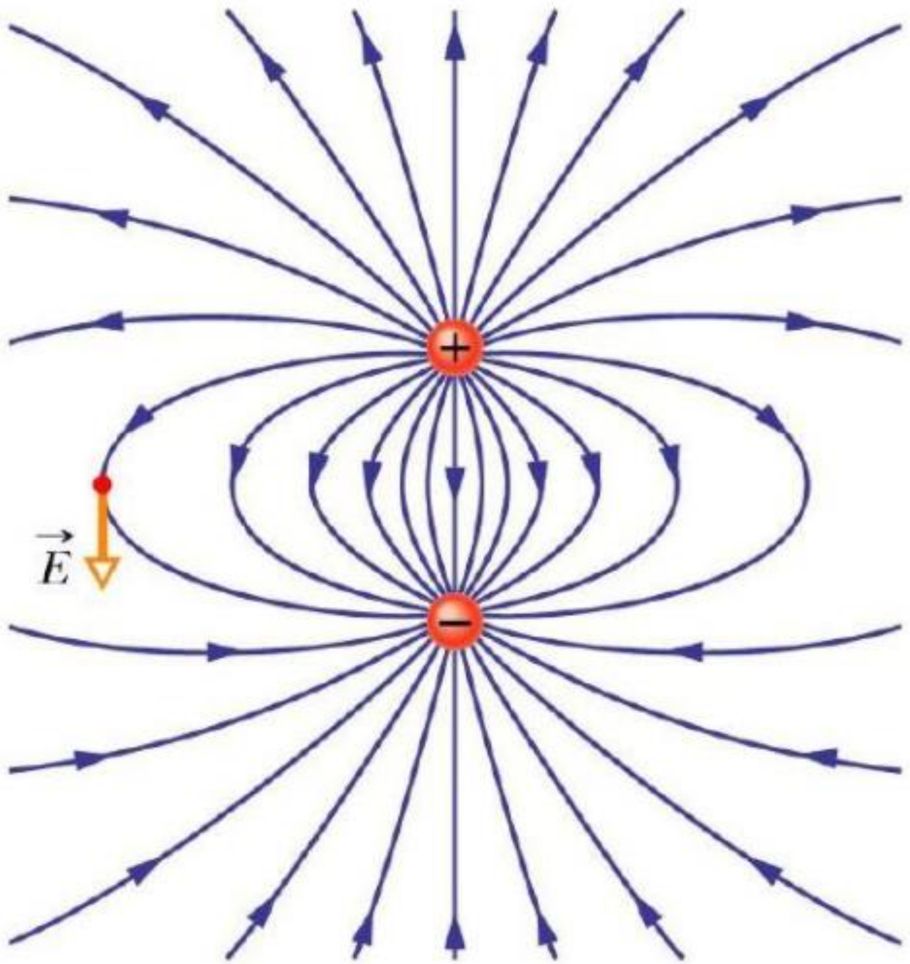


$$\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}$$

$$\because\ \vec{E}_1+\vec{E}_2=\frac{q}{4\pi\varepsilon r^3}\left[3d\cos\theta\hat{r}-d\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\right]$$

$$=\frac{qd}{4\pi\varepsilon r^3}\left[2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right]$$

$p=qd$　电偶极距



$$\vec{E} = \frac{qd}{4\pi\varepsilon r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

内　容

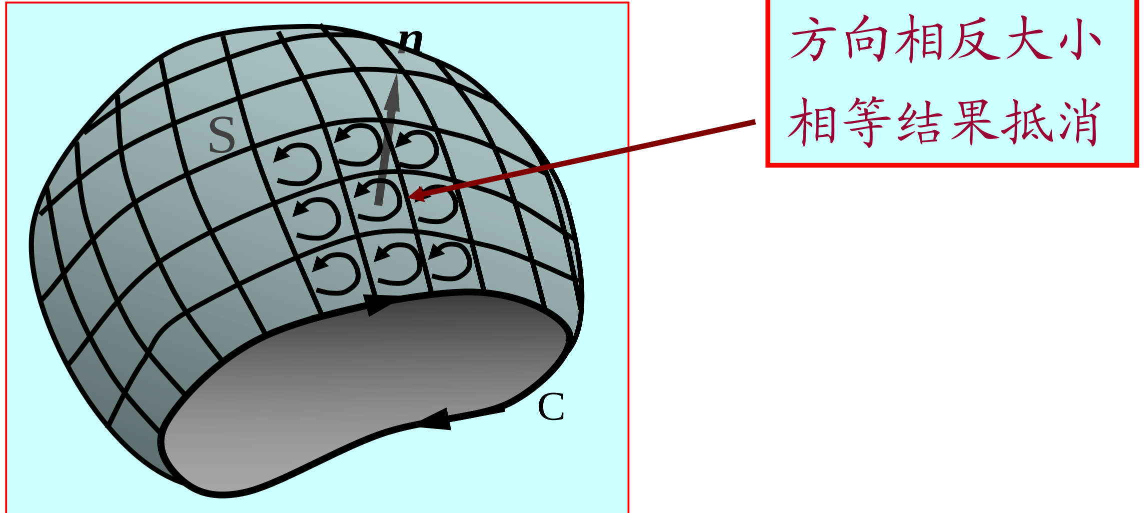
1. 矢量、矢径（位置矢量）、场矢量
2. 矢量的点积（/点乘）、叉积（叉乘）
3. 矢量的坐标变换
4. 矢量微分
5. 标量场的方向导数和梯度
6. 通量和散度
7. 环量和旋度
8. 拉普拉斯算子
9. 散度定理
10. 斯托克斯定理

10、斯托克斯定理

$$\int_s (\nabla\times\vec{F})\cdot d\vec{s} = \oint_l \vec{F}\cdot d\vec{l}$$

意义：矢量场的旋度在曲面上的积分等于该矢量场在限定该曲面的闭合曲线上的线积分。

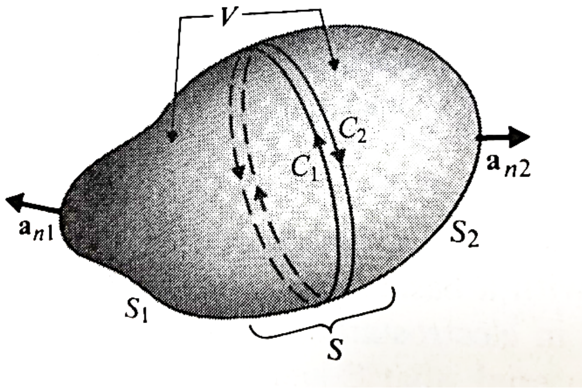
应用：将曲面积分转换为曲线积分，进而简化计算。



曲面的剖分

$$\nabla\cdot(\nabla\times\vec{F})\equiv 0$$

利用算子的积分定理证明上述恒等式成立。



左式取体积分:

$$\int_V \nabla\cdot(\nabla\times\vec{F})\,\mathrm{dv}=\int_S (\nabla\times\vec{F})\cdot\mathrm{d}\vec{\mathrm{s}}$$

$$=\int_{S1} (\nabla\times\vec{F})\cdot\mathrm{d}\vec{\mathrm{s}}_1+\int_{S2} (\nabla\times\vec{F})\cdot\mathrm{d}\vec{\mathrm{s}}_2$$

$$=\int_{C1} \vec{F}\cdot\mathrm{d}\vec{l}_1+\int_{C2} \vec{F}\cdot\mathrm{d}\vec{l}_2$$

$$=\int_{C1} \vec{F}\cdot\mathrm{d}\vec{l}_1-\int_{C1} \vec{F}\cdot\mathrm{d}\vec{l}_1$$

$$=0$$

$$\nabla \times (\nabla u) \equiv 0$$

利用算子的积分定理证明上述恒等式成立。

左式面积分：$\displaystyle \int_S [\nabla \times (\nabla u)] \cdot \mathrm{d}\vec{s} = \int_C \nabla u \cdot \mathrm{d}\vec{l}$

$$\nabla u \cdot \mathrm{d}\vec{l} = \left(\frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z} \right) \cdot \left(dx\hat{x} + dy\hat{y} + dz\hat{z} \right)$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dx = du$$ 全微分

$$\therefore \oint_C \nabla u \cdot \mathrm{d}\vec{l} = \oint_C du = 0$$

回顾之前的练习

作业原题： 对于矢量场 $\vec{F} = xy\hat{x} + (3x - y^2)\hat{y}$

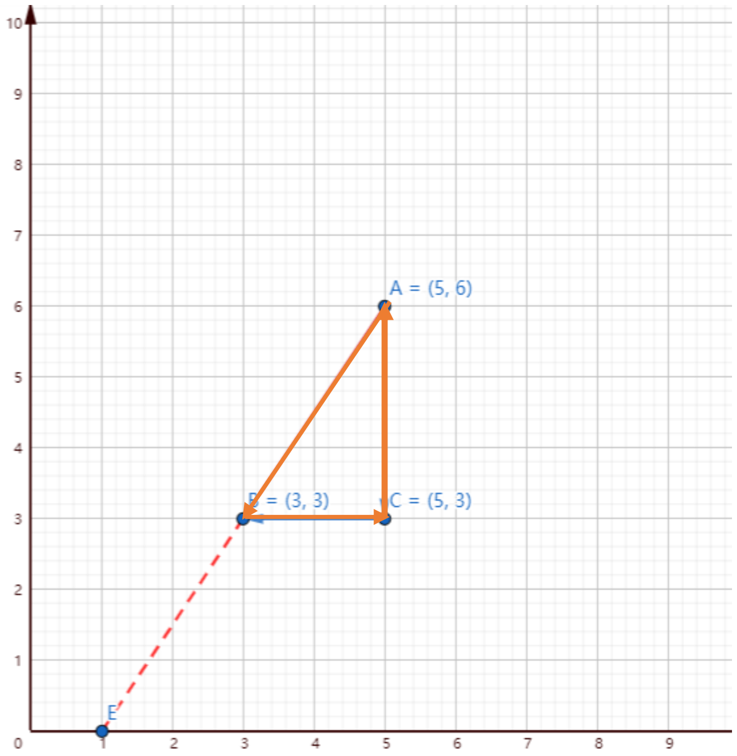
$$\int_A^B \vec{F}\cdot d\vec{l} = -10 \qquad \int_B^C \vec{F}\cdot d\vec{l}_2 = 24 \qquad \int_C^A \vec{F}\cdot d\vec{l}_1 = -18$$

$$\oint \vec{F}\cdot d\vec{l} = \int_A^B \vec{F}\cdot d\vec{l} + \int_B^C \vec{F}\cdot d\vec{l}_1 + \int_C^A \vec{F}\cdot d\vec{l}_2 + = -10 + 24 - 18 = -4$$

之前练习： 对于矢量场 $\vec{F} = 3y\hat{x} + (3x - y^2)\hat{y}$

$$\int_A^B \vec{F}\cdot d\vec{l} = 0 \qquad \int_B^C \vec{F}\cdot d\vec{l}_2 = 18 \qquad \int_C^A \vec{F}\cdot d\vec{l}_1 = -18$$

$$\oint \vec{F}\cdot d\vec{l} = \int_A^B \vec{F}\cdot d\vec{l} + \int_B^C \vec{F}\cdot d\vec{l}_1 + \int_C^A \vec{F}\cdot d\vec{l}_2 = 0$$



验算： $\nabla\times\left(xy\hat{x} + (3x - y^2)\hat{y}\right) = (3 - x)\hat{z}$ $\qquad$ $\nabla\times\left(3y\hat{x} + (3x - y^2)\hat{y}\right) = 0$

三角型ABC内有旋！ $\qquad$ 三角型ABC内无旋！

进一步验算一下：

对于矢量场 $\vec{F} = xy\hat{x} + (3x - y^2)\hat{y}$

$$\nabla \times \vec{F} = (3 - x)\hat{z}$$

$$\int_{\Delta ABC} \nabla \times \vec{F} \cdot d\vec{s} = \oint \vec{F} \cdot d\vec{l}$$

$$\int_{\Delta ABC} \nabla \times \vec{F} \cdot d\vec{s} = \iint (3 - x)\hat{z} \cdot \color{red}{\hat{z}dxdy}$$

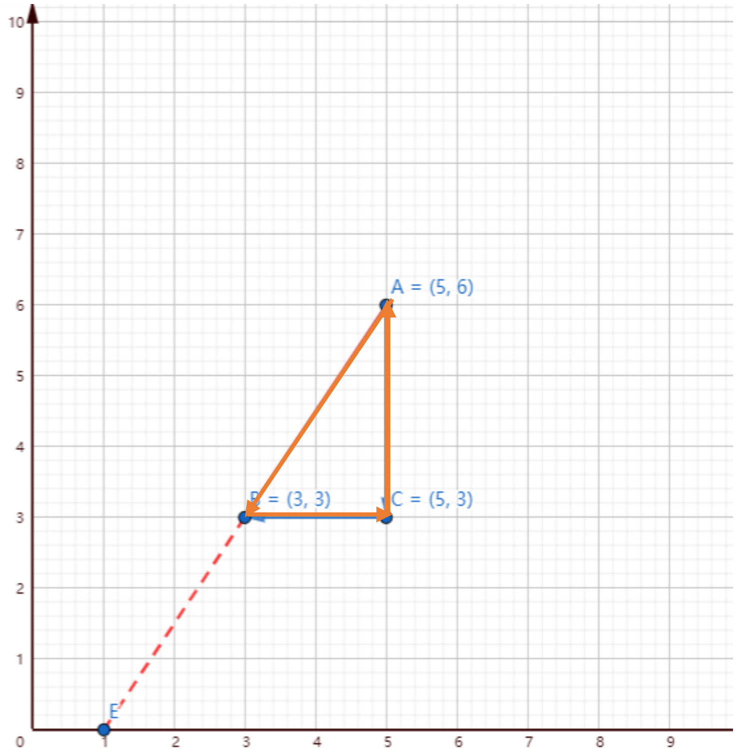
$$= \int_3^5 \left[\int_3^{\frac{3}{2}(x-1)} (3 - x)dy \right] dx$$

$$= \int_3^5 \left[(3 - x)\left(\frac{3}{2}(x - 1) - 3 \right) \right] dx$$

$$= -\frac{3}{2}\int_3^5 \left[(x - 3)^2 \right] dx$$

$$= -4$$

$$= \oint \vec{F} \cdot d\vec{l}$$

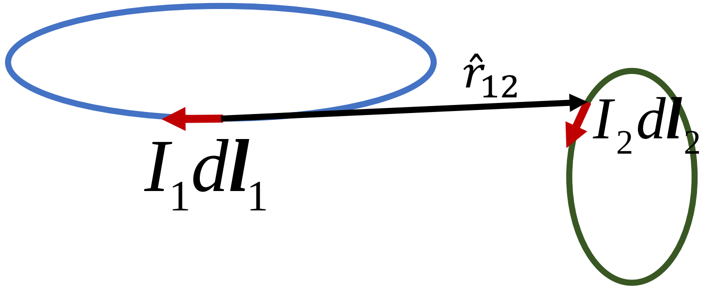


$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

例题1.22：根据Biot-Savart 定律，两个闭合通电线圈之间的作用力可以表示为

$$\boldsymbol{F}_{12}=\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2d\boldsymbol{l}_2\times\left(d\boldsymbol{l}_1\times\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2},$$

$$\boldsymbol{F}_{21}=\frac{\mu_0}{4\pi}\oint_{(L_2)}\oint_{(L_1)}\frac{I_2I_1d\boldsymbol{l}_1\times\left(d\boldsymbol{l}_2\times\hat{\boldsymbol{r}}_{21}\right)}{r_{21}^2}\qquad \vec{r}_{12}=-\vec{r}_{21}$$



试证明：$\boldsymbol{F}_{12}=-\boldsymbol{F}_{21}$

$$d\boldsymbol{l}_2\times\left(d\boldsymbol{l}_1\times\hat{\boldsymbol{r}}_{12}\right)=d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\hat{\boldsymbol{r}}_{12}\right)-\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)$$

$$d\boldsymbol{l}_1\times\left(d\boldsymbol{l}_2\times\hat{\boldsymbol{r}}_{21}\right)=d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)-\hat{\boldsymbol{r}}_{21}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)=d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)+\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)$$

$$\boldsymbol{F}_{12}=\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2\left[d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\hat{\boldsymbol{r}}_{12}\right)-\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)\right]}{r_{12}^2}=\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2}-\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)}{r_{12}^2}$$

$$\boldsymbol{F}_{21}=\frac{\mu_0}{4\pi}\oint_{(L_2)}\oint_{(L_1)}\frac{I_2I_1\left[d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)+\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)\right]}{r_{21}^2}=\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2}+\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2\hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)}{r_{12}^2}$$

令: $\boldsymbol{P}_1 = \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1 I_2 \hat{\boldsymbol{r}}_{12}\left(d\boldsymbol{l}_1\cdot d\boldsymbol{l}_2\right)}{r_{12}^2}$

则: $\boldsymbol{F}_{12} = -\boldsymbol{P}_1 + \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1 I_2 d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2}$ $\quad \boldsymbol{F}_{21} = \boldsymbol{P}_1 + \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1 I_2 d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2}$

$$\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1 I_2 d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2} = \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)} I_1 I_2 d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\frac{\hat{\boldsymbol{r}}_{12}}{r_{12}^2}\right) = -\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)} I_1 I_2 d\boldsymbol{l}_1\left(d\boldsymbol{l}_2\cdot\nabla\left(\frac{1}{r_{12}}\right)\right)$$

$$= -\frac{\mu_0}{4\pi}\oint_{(L_1)} I_1 I_2 d\boldsymbol{l}_1 \oint_{(L_2)}\nabla\left(\frac{1}{r_{12}}\right)\cdot d\boldsymbol{l}_2 = 0$$

$\oint_C \nabla u\cdot \mathrm{d}\vec{l} = \oint_C du = 0$

$$or = -\frac{\mu_0}{4\pi}\oint_{(L_1)} I_1 I_2 d\boldsymbol{l}_1 \int_{S_2}\nabla\times\nabla\left(\frac{1}{r_{12}}\right)\cdot d\boldsymbol{s} = 0$$

$\int_s (\nabla\times\vec{F})\cdot d\vec{s} = \oint_l \vec{F}\cdot d\vec{l}$ $\quad \nabla\times(\nabla u)\equiv 0$

同理: $\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1 I_2 d\boldsymbol{l}_2\left(d\boldsymbol{l}_1\cdot\hat{\boldsymbol{r}}_{12}\right)}{r_{12}^2} = 0$

所以: $\boldsymbol{F}_{12} = -\boldsymbol{P}_1; \boldsymbol{F}_{21} = \boldsymbol{P}_1 \Rightarrow \boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$