1

Control of Soft Robots

Before addressing the controller synthesis, we briefly introduce the notion of under-actuated dynamical systems [4, 1, 3, 2].

Definition 1.1 (Under-actuated system)**.** *A second-order dynamical system described by the partial differential equation*

$$\ddot{q} = f(q, \dot{q}, u, t) \tag{1.1}$$

is considered fully-actuated in a state $(q, \dot{q})$ *at time* t *if and only if the map* f *is surjective, i.e, for every* $\ddot{q}(t)$ *there exists a control input* $u(t)$ *such that the instantaneous acceleration is realizable. Otherwise, the dynamical system is said to be under-actuated. Regarding under-actuated systems in the control-affine form, that is,*

$$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u(t), \tag{1.2}$$

a sufficient condition is $\operatorname{rank}(f_2(q, \dot{q}, t)) < \dim(q)$.

In other words, an under actuated dynamical system cannot steer its states in any arbitrary direction. As a consequence, under actuated systems are generally more difficult to control. By definition, a soft robotic system is an under-actuated system since they theoretically pose infinitely many degrees-of-freedom. Including the distributed control inputs to the Lagrangian model, we write the dynamics for a soft robotic system as follows

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = S^\top(q)u, \tag{1.3}$$

where $M(q)$ is the positive definite mass matrix, $c(q, \dot{q}) \in \mathbb{R}^n$ is a vector of Coriolis forces, $g(q) \in \mathbb{R}^n$ a vector of conservative potential forces, $S(q) \in \mathbb{R}^{n\times m}$ is a (nonlinear) mapping that projects the active control inputs onto the acceleration pace of q, and $u(t) \in \mathbb{R}^m$ is the lower-dimensional control input. Since the system is under-actuated, it shall be clear that $\dim(u) < \dim(q)$. In literature [1], the matrix S is referred to as the synergy matrix whose columns describe actuation patterns of the soft robot's input space. Without loss of generality, let the synergy matrix S be defined by a set of linearly independent column vectors of actuation patterns $s_i : \mathbb{R}^n \to \mathbb{R}^n$,

$$S(q) := [\, s_1(q),\, s_2(q),\, ...,\, s_m(q) \,], \tag{1.4}$$

which implies that the matrix has $\operatorname{rank}(S) = m$.

A

Fundamentals on Lie Group Theory

In this chapter, we will discuss the fundamentals on Lie groups and their associated Lie algebras.

A.1 Lie group

A Lie group encompasses the concepts of 'group' and 'smooth manifold' in a unique embodiment. To be more specific, the Lie Group $\mathcal{G}$ is a smooth manifold whose elements satisfy the group axioms:

1. Closure: if $g_1, g_2 \in \mathcal{G}$, then $g_1 g_2$ is also an element of $\mathcal{G}$,
2. Identity: there exists an element e such that $ge = eg = g$ for any $g \in G$,
3. Inversion: For any $g \in \mathcal{G}$, there exists an element $g^{-1} \in \mathcal{G}$ such that $gg^{-1} = g^{-1}g = e$,
4. Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for any $g_1, g_2, g_3 \in \mathcal{G}$.

The smoothness of the Lie groups intuitively suggests the existence of useful differential geometries. For any elements g on the smooth manifold $\mathcal{G}$, there exists a tangent linear space denoted by $T_g\mathcal{G}$. The tangent space of the Lie group at the identity element e is referred to as the associative Lie algebra $\mathfrak{g}$ of the group. it allows us to perform algebra computation concerning the Lie group.

A.2 Geometric Jacobian for Lie groups: $\mathbb{SE}(3)$

The geometric Jacobian can be computed through finding the solutions of the configuration space g and an unknown integrand Ξ over the domain $[0, \sigma]$:

$$\frac{\partial}{\partial \sigma}\begin{pmatrix} g \\ \Xi \end{pmatrix} = \begin{pmatrix} g\hat{\xi} \\ \mathrm{Ad}_g \Phi(\sigma) \end{pmatrix}, \tag{A.1}$$

with initial conditions $g(0) = g_0$. Recall that the strain field denoted by $\hat{\xi} \in \mathfrak{se}(3)$ is approximated using a finite set of shape functions $\hat{\xi} = (\Phi(\sigma)q + \xi_0)^\wedge$. Then, given the solution to the ordinary differential equation (A.1), the geometric Jacobian associated with a point σ can be evaluated using $J(\sigma, q) = \mathrm{Ad}_g^{-1}\Xi(\sigma, q)$. Taking the partial derivative with respect to time of the Jacobian matrix, we may express the change in the geometric Jacobian as

$$\begin{aligned} \dot{J} &= \dot{\mathrm{Ad}}_g^{-1}\Xi + \mathrm{Ad}_g^{-1}\dot{\Xi}, \\ &= -\mathrm{ad}_\eta \mathrm{Ad}_g^{-1}\Xi + \mathrm{Ad}_g^{-1}\int_0^\sigma \mathrm{ad}_{\eta(s)}\mathrm{Ad}_{g(s)}\Phi(s)\, ds. \end{aligned} \tag{A.2}$$

Combining the formulations in (A.1) and (A.2), the geometric Jacobian and its time derivative can be computed through finding the solutions of the following ODE

$$\frac{\partial}{\partial \sigma}\begin{pmatrix} g \\ \eta \\ \Xi \\ \dot{\Xi} \end{pmatrix} = \begin{pmatrix} g\hat{\xi} \\ -\mathrm{ad}_\xi \eta + \dot{\xi} \\ \mathrm{Ad}_g \Phi(\sigma) \\ ad_\eta \mathrm{Ad}_g \Phi(\sigma) \end{pmatrix}, \tag{A.3}$$

given the initial conditions $g(0) = g_0$ and $\eta(0) = \eta_0$.

Bibliography

[1] Cosimo Della Santina, Lucia Pallottino, Daniela Rus, and Antonio Bicchi. Exact Task Execution in Highly Under-Actuated Soft Limbs: An Operational Space Based Approach. *IEEE Robotics and Automation Letters*, 4(3):2508–2515, 2019.

[2] Mark W. Spong. Energy based control of a class of underactuated mechanical systems. *IFAC Proceedings Volumes*, 29(1):2828 – 2832, 1996.

[3] Mark W. Spong. Underactuated mechanical systems. In *Control Problems in Robotics and Automation*, pages 135–150, Berlin, Heidelberg, 1998. Springer Berlin Heidelberg.

[4] Russ Tedrake. Underactuated robotics: Learning, planning, and control for ecient and agile machines course notes for mit 6.832. 2009.