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Control of Soft Robots

1.1 Underactuated system

Before addressing the controller synthesis, we briefly introduce the notion of under-actuated dynamical systems [1, 2, 3, 4].

Definition 1.1 (Under-actuated system). *A second-order dynamical system described by the partial differential equation*

$$\ddot{q} = f(q, \dot{q}, u, t) \tag{1.1}$$

is considered fully-actuated in a state $(q, \dot{q})$ *at time* t *if and only if the map* f *is surjective, i.e, for every* $\ddot{q}(t)$ *there exists a control input* $u(t)$ *such that the instantaneous acceleration is realizable. Otherwise, the dynamical system is said to be under-actuated. Regarding under-actuated systems in the control-affine form, that is,*

$$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u(t), \tag{1.2}$$

a sufficient condition is $\operatorname{rank}(f_2(q, \dot{q}, t)) < \dim(q)$.

In other words, an under actuated dynamical system cannot steer its states in any arbitrary direction. As a consequence, under actuated systems are generally more difficult to control. By definition, a soft robotic system is an under-actuated system since they theoretically posses infinitely many degrees-of-freedom. Including the distributed control inputs to the Lagrangian model, we write the dynamics for a soft robotic system as follows

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = S_a^\top(q)\tau, \tag{1.3}$$

where $M(q)$ is the positive definite mass matrix, $c(q, \dot{q}) \in \mathbb{R}^n$ is a vector of Coriolis forces, $g(q) \in \mathbb{R}^n$ a vector of conservative potential forces, $S_a(q) \in \mathbb{R}^{n \times m}$ is a (nonlinear) mapping that projects the active control inputs onto the acceleration space of q, and $u(t) \in \mathbb{R}^m$ is the lower-dimensional control input. Since the system is under-actuated, it shall be clear that $\dim(u) < \dim(q)$. In some studies [1], the matrix S is referred to as the synergy matrix whose columns describe actuation patterns of the soft robot's input space. Without loss of generality, let the synergy matrix S be defined by a set of linearly independent column vectors of actuation patterns $s_i : \mathbb{R}^n \to \mathbb{R}^n$,

$$S_a(q) := [\, s_1(q),\, s_2(q),\, ...,\, s_m(q)\,]\,, \tag{1.4}$$

which implies that the matrix has $\mathrm{rank}(S) = n_a$.

1.2 Discontinuous shape functions

An approach to distinguish the actuator dynamics and the redundant dynamics due to mechanical flexibility is the use of discontinuous shape representation. This is especially important for soft robotics as their (theoretical) infinite number of degrees of freedom cannot be match by the same number of actuators. To recall, each component of the geometric strain field $\xi_i(\sigma,t) \in \mathfrak{se}(3)$ is approximated using a finite number of shape functions $\xi_i(\sigma,t) \cong \sum_{j=1}^{N} \Phi_j(\sigma) q_j(t)$ with $\Phi_j : \mathbb{R}[0,l] \mapsto \mathbb{R}$. First, let us consider the conventional Chebyshev polynomials (of the first kind) which are defined as

$$\Phi_n(\sigma) = \cos\left[n \arccos(\sigma)\right], \tag{1.5}$$

where n is the order of the polynomial. Now, suppose a uniformly distributed actuation field is applied to continuous body within the spatial domain $\mathcal{X}_a \in [\sigma_-, \sigma_+]$ with $0 \leq \sigma_- < \sigma_+ \leq l$. With this subset $\mathcal{X}_a$ in mind, lets modify the Chebyshev polynomial as follows

$$\Phi_n(\sigma) = w(\sigma) \cos\left[n \arccos\left(\frac{\sigma - \sigma_-}{\sigma_+ - \sigma_-}\right)\right], \tag{1.6}$$

with the weighting function

$$w(\sigma) = \begin{cases} 1 & \sigma \in \mathcal{X}_a, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Using the discontinuous variant of the Chebyshev polynomial in (1.6), the strain field of the soft robot can be decomposed into active and passive parts. This approach infers that the synergy matrix S_a containing the column vectors of actuation patterns is time-invariant.

1.3 Partial feedback linearization

Consider the permutation matrix $S = [\, S_p^\top, S_a^\top \,]^\top$ that partitions the state vector $q \in \mathbb{R}^n$ into passive states denoted by $q_p \in \mathbb{R}^{n_p}$ and active states $q_a \in \mathbb{R}^{n_a}$ such that $Sq = [\, q_p^\top,\, q_a^\top \,]^\top$. Recall from earlier that the matrix S_a is denoted as the synergy matrix of actuation patterns. Similarly, the redundant flexibility patterns of the soft body are related to the matrix S_p. As such, (1.3) can be written as

$$M(q)S^\top \ddot{q} + c(q, \dot{q}) + g(q) = S_a^\top \tau, \tag{1.8}$$

A property of the selection matrices is that their columns are mutually orthogonal which implies that $S_i S_i^\top = I$ and $S_i S_j^\top = O$ for $i, j \in \{a, p\}$. Exploiting their mutual orthogonality, we may decompose the dynamic model of the soft robotic system as follows

$$M_{11}\ddot{q}_p + M_{12}\ddot{q}_a + h_1 = 0, \tag{1.9}$$
$$M_{21}\ddot{q}_p + M_{22}\ddot{q}_a + h_2 = \tau. \tag{1.10}$$

where $h(q, \dot{q}) = c(q, \dot{q}) + g(q)$. Intuitively, the dynamics in (1.9) are related to the redundant flexibility of the hyper-elastic body, whereas (1.10) reflects the active control shapes corresponding the distributed actuation field. Using (1.9), we may solve for the hyper-redundant dynamics of the soft robot as follows

$$\ddot{q}_p = -M_{11}^{-1}\left(M_{12}\ddot{q}_a + h_1\right) \tag{1.11}$$

whose solutions are unique since $M_{11} = S_p M S_p^\top$ is positive definite $\forall q \in \mathbb{R}^n$. Substitution of the obtained dynamics into (1.10) leads to

$$\bar{M}_a \ddot{q}_a + \bar{h}_a = \tau, \tag{1.12}$$

where $\bar{M}_a = \left(M_{22} - M_{21}M_{11}^{-1}M_{12}\right)$ and $\bar{h}_a = \left(h_2 - M_{21}M_{11}^{-1}h_1\right)$. From the relation in (1.12), we propose a partial feedback linearization approach by Spong (1996) as follows

$$\tau = \bar{M}_a(-K_p q_a - K_d \dot{q}_a + K\nu) + \bar{h}_a, \tag{1.13}$$

which ensures asymptotic stability of the linear subsystem for $K_p, K_d > 0$ and the outer control law $\nu = 0$. However, as stated in Spong (1996), the proposed collocated feedback linearization above can results in the system having unstable zero dynamics.

Bibliography

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