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Dynamics of Soft Robots

Configuration space using Lie group theory

In contrast to a rigid robot, whose mechanical structure consists of static links and joints, a soft robot lacks the physical notion of joints and therefore cannot be viewed as an ordinary multi-body. From a mechanical perspective, a soft robotic system is more closely related to a continuous deformable medium with infinite degrees-of-freedom rather. Given this notion, a soft robotic system can be modeled as a one-dimensional Cosserat beam together with the geometrically exact beam theories proposed by [4].

To define a spatial coordinate frame, let us introduce a parameter $\sigma \in \mathcal{X}$ that lies on a bounded domain $\mathcal{X} \in [0, l] \subset \mathbb{R}$ (with $l \in \mathbb{R}_{>0}$ the extensible length of the soft robot). Given this description, we can represent the position $p(\sigma, t) \in \mathbb{R}^3$ and orientation matrix $R(\sigma, t) \in \mathbb{SO}(3)$ for any point σ on the smooth backbone of the soft robot by a functional curve $g : \mathcal{X} \times \mathbb{R} \mapsto \mathbb{SE}(3)$, that is,

$$g(\sigma, t) := \begin{pmatrix} R(\sigma, t) & p(\sigma, t) \\ 0_3^\top & 1 \end{pmatrix} \in \mathbb{SE}(3), \tag{1.1}$$

where $\mathbb{SE}(3)$ is the Lie group of rigid body transformations in $\mathbb{R}^3$ [2, 7].

Since the backbone curve g is space-time variant, the variations in space and time can be characterized by two vector field in the Lie algebra $\mathfrak{se}(3)$. Throughout this work, we denote the partial derivatives $\partial(\cdot)/\partial\sigma$ and $\partial(\cdot)/\partial t$ by a 'prime' and 'dot', respectively. By regarding the partial derivative with respect to time of (1.1), the time-twist field can be defined as follows

$$\dot{g} = g\hat{\eta} \quad \Longrightarrow \quad \hat{\eta} := g^{-1}\dot{g} = \begin{pmatrix} \Omega_\times & V \\ 0_3^\top & 0 \end{pmatrix} \in \mathfrak{se}(3), \tag{1.2}$$

where $\Omega = (\omega_1, \omega_2, \omega_3)^\top$ and $V = (v_1, v_2, v_3)^\top$ denote the angular velocity vector and the linear velocity vector, respectively. Note that in (1.2) we used a property of the Lie algebra $\mathfrak{so}(3) \cong \mathbb{R}^3$ with the isomorphism $\Omega \mapsto \Omega_\times$ [2]. To be more specific on geometric interpretation, the vector field $\eta(\sigma, t)$ defines the infinitesimal local transformation undergone by a frame at position σ between two infinitesimally close instances t and $t + dt$. Second, by regarding the partial derivative with respect to space of (1.1), the space-twist field can be defined as follows

$$g' = g\hat{\xi} \implies \hat{\xi} := g^{-1}g' = \begin{pmatrix} K_\times & E \\ 0_3^\top & 0 \end{pmatrix} \in \mathfrak{se}(3), \tag{1.3}$$

where $K = (k_1, k_2, k_3)^\top$ and $E = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^\top$ denote the curvature-torsion strain vector and the stretch-shear strain vector, respectively. Similar to its geometric counterpart, the vector field $\xi(\sigma, t)$ defines the infinitesimal local transformation undergone by a frame at an instance t between two infinitesimally close positions σ and $\sigma + d\sigma$. Since $\mathfrak{se}(3) \cong \mathbb{R}^6$, we can express(1.2) and (1.3) as a column vector in $\mathbb{R}^6$ as follows

$$\eta(\sigma, t) = \begin{pmatrix} \Omega \\ V \end{pmatrix}; \qquad \xi(\sigma, t) = \begin{pmatrix} K \\ E \end{pmatrix}. \tag{1.4}$$

Continuous kinematics for soft robots

By using the equality of mixed partials, we may invoke that $\frac{\partial}{\partial t}(g') = \frac{\partial}{\partial \sigma}(\dot{g})$ holds for any instance in space and time. Accordingly, substitution of relations (1.2) and (1.3) into this commutative relation leads to

$$\dot{g}\xi + g\dot{\hat{\xi}} = g'\hat{\eta} + g\hat{\eta}', \tag{1.5}$$

which implies

$$g\hat{\eta}\hat{\xi} + g\dot{\hat{\xi}} = g\hat{\xi}\hat{\eta} + g\hat{\eta}'. \tag{1.6}$$

Multiplying both sides with g^{-1} and rearranging the equality, we find

$$\hat{\eta}' = -(\hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi}) + \dot{\hat{\xi}}, \tag{1.7}$$

where we can recognize, in the parenthesis, the Lie bracket of ξ and η. The Lie bracket $[\hat{\xi}, \hat{\eta}]$ is also an element of Lie algebra $\mathfrak{se}(3)$, and thus it may be alternatively expressed in $\mathbb{R}^6$ as the adjoint action between ξ onto η, namely $\mathrm{ad}_\xi \eta : \mathbb{R}^6 \mapsto \mathbb{R}^6$ (see [7] and [9]). Therefore, the velocity kinematics in (1.7) can be written in vector representation as

$$\eta' = -\mathrm{ad}_\xi \eta + \dot{\xi}. \tag{1.8}$$

By taking the time derivative of (1.8) and combining the previous results, the continuous kinematic model for the configuration, velocity, and acceleration can be written as system of first-order partial differential equation (PDE) of the form

$$\frac{\partial}{\partial\sigma}\begin{pmatrix} g \\ \eta \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} g\hat{\xi} \\ -\mathrm{ad}_{\xi}\eta + \dot{\xi} \\ -\mathrm{ad}_{\dot{\xi}}\eta - \mathrm{ad}_{\xi}\dot{\eta} + \ddot{\xi} \end{pmatrix}. \tag{1.9}$$

For a general case, the boundary conditions of PDE in (1.9) should satisfy $g(0,t) = g_0$, $\eta(0,t) = \eta_0$ and $\dot{\eta}(0,t) = \dot{\eta}_0$. However, in case of a manipulator whose base is spatially fixed, the boundary conditions should satisfy $g(0,t) = g_0$, and $\eta(0,t) = \dot{\eta}(0,t) = 0_6$. Notice that if the strain fields ξ, $\dot{\xi}$, and $\ddot{\xi}$ are known, the partial differential equation in (1.9) simply becomes a first-order ordinary differential equation (ODE), which can be easily solved using numerical approaches.

Continuous dynamics for soft robots

In this section, we derive the dynamical model of the soft robot through Hamilton's variational principle. Given an interval $[t_0, t_1]$, the variational principle states that the evolution of a state $q(t)$ between $q(t_0)$ and $q(t_1)$ is a stationary point regarding an action functional, $\mathcal{S} = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t)\, dt$ in which $\mathcal{L}(q, \dot{q}) := \mathcal{T}(q, \dot{q}) - \mathcal{V}(q)$ is the Lagrangian. The generalization of Hamilton's principle [1] includes an external potential contributions, and it can be formally written as

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \left[\delta\mathcal{T} - \delta\mathcal{V} + \delta\mathcal{W}_{ex}\right]\, dt = 0, \tag{1.10}$$

where the operator δ denotes the variation of functional that are fixed at the boundaries $[t_0, t_1]$, and $\mathcal{W}_{ex}$ is the external virtual work produced by nonconservative external forces acting on the system.

First, let us regard the functional variation of kinetic energy. The kinetic energy of the soft robot is defined by

$$\mathcal{T} := \frac{1}{2}\int_{\mathcal{X}} \eta^\top \mathcal{M}\eta\, d\sigma, \tag{1.11}$$

where $\mathcal{M} \in \mathfrak{se}(3) \times \mathfrak{se}(3)^*$ is the inertia tensor whose components denote the inertial properties of an infinitesimal slice of the continuous mechanical body. More specifically, the inertia tensor is $\mathcal{M} = \mathrm{blkdiag}\,\{mI_3, \mathcal{J}\}$ with $m \in \mathbb{R}_{>0}$ the line-density and $\mathcal{J} \in$

$\mathfrak{so}(3) \times \mathfrak{so}(3)^*$ the moment of inertia tensor. From the isomorphism $\mathfrak{se}(3) \cong \mathbb{R}^6$, the inertia tensor $\mathcal{M}$ may be equivalently represented as a symmetric matrix of $\mathbb{R}^{6\times 6}$. Given (1.11), the variation of the kinetic energy function is given by

$$\begin{aligned}
\delta\mathcal{T} &= \frac{\partial}{\partial a}\mathcal{T}(\eta + a\delta\eta)\bigg|_{a=0}, \\
&= \frac{1}{2}\int_{\mathcal{X}} \delta\eta^\top \mathcal{M}\eta + \eta^\top \mathcal{M}\delta\eta \; d\sigma, \\
&= \int_{\mathcal{X}} \delta\eta^\top \mathcal{M}\eta \; d\sigma.
\end{aligned} \tag{1.12}$$

By applying variational calculus on the Lie group, we can express the variation of the velocity field as $\delta\eta = \delta\dot{\epsilon} + \mathrm{ad}_\eta \delta\epsilon$ in which $\delta\epsilon = g^{-1}\delta g \in \mathfrak{se}(3)$ with $\delta\epsilon(t_0) = \delta\epsilon(t_1) = 0$. Therefore, substitution of the variation into (1.12) and followed by integration by parts leads to derivation

$$\begin{aligned}
\int_{t_1}^{t_2} \delta\mathcal{T}\; dt &= \left[\int_{\mathcal{X}} \delta\epsilon^\top M\eta\right]_{t_0}^{t_1} + \int_{t_1}^{t_2}\int_{\mathcal{X}} \delta\epsilon^\top \left(M\dot{\eta} - \mathrm{ad}_\eta^\top M\eta\right)\; d\sigma dt \\
&= \int_{t_1}^{t_2}\int_{\mathcal{X}} \delta\epsilon^\top \left(M\dot{\eta} - \mathrm{ad}_\eta^\top M\eta\right)\; d\sigma dt.
\end{aligned} \tag{1.13}$$

Note that since the variations are fixed at the boundaries of $[t_0, t_1]$, the first right hand part in (1.13) vanished. Since the variations are fixed at the boundaries of $[t_0, t_1]$, the first right hand part in (1.13) vanishes. The expression in (1.13) will be recalled later, but first, let us describe the functional variation of potential energy.

The internal potential energy of the soft robot is defined as

$$\mathcal{V} := \int_{\mathcal{X}} \xi^\top \Lambda \; d\sigma. \tag{1.14}$$

where $\Lambda \in \mathfrak{se}(3)^*$ is the field of internal wrenches along the continuum elastic body. Notice that vector field of internal wrenches is an element of the dual space of $\mathfrak{se}(3)$. This field and the strains vector field are related through a material constitutive law. In general concerning soft robotic applications, the use of linear constitutive relations for an isotropic elastic material are not sufficient, since large deformations introduce nonlinear material behavior. However, for the sake of simplicity, we consider the simplest viscoelastic constitutive model - the Kelvin-Voigt model. The Kelvin-Voigt model is a linear elasticity model with a linear viscous contribution that is proportional to the rate of strain ξ,

$$\Lambda = K\xi + \Gamma\dot{\xi} \tag{1.15}$$

where K and Γ are the elasticity and viscosity material tensor, respectively. Similar to the kinetic energy, the variation of the potential energy function V is given by

$$\delta\mathcal{V} = \frac{\partial}{\partial a}\mathcal{V}(\xi + a(\delta\xi))\bigg|_{a=0} = \int_{\mathcal{X}} \delta\xi^\top \Lambda \, d\sigma, \tag{1.16}$$

Recalling the commutativity of the Lie algebra, we can express the variation as $\delta\xi = \epsilon' + \mathrm{ad}_\xi \epsilon$. Therefore, substitution into (1.16) and using integration by parts leads to

$$\begin{aligned}\int_{t_1}^{t_2} \delta\mathcal{V}\, dt &= \left[\int_{\mathcal{X}} \delta\epsilon^\top \Lambda\right]_{t_0}^{t_1} + \int_{t_1}^{t_2}\int_{\mathcal{X}} \epsilon^\top \left(\Lambda' - \mathrm{ad}_\xi^\top \Lambda\right)\, d\sigma dt \\ &= \int_{t_1}^{t_2}\int_{\mathcal{X}} \delta\epsilon^\top \left(\Lambda' - \mathrm{ad}_\xi^\top \Lambda\right)\, d\sigma dt. \end{aligned}\tag{1.17}$$

Now, assuming that $\delta\mathcal{W}_{ex} = \int \delta\epsilon^\top \mathcal{F}\, d\sigma$ with $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}^6$ an external wrench acting on the continuous body, we can substitute the kinetic and potential energy variations (1.13) and (1.17) into the Hamilton's variational principle (1.10) which leads to the weak formulation of the continuous dynamics

$$\delta\mathcal{S} = \int_{t_1}^{t_2}\int_{\mathcal{X}} \delta\epsilon^\top \left(\mathcal{M}\dot{\eta} - \mathrm{ad}_\eta^\top \mathcal{M}\eta - \Lambda' + \mathrm{ad}_\xi^\top \Lambda + \mathcal{F}\right)\, d\sigma dt = 0, \tag{1.18}$$

which holds for all variations $\delta\epsilon \in \mathfrak{se}(3)$. Given the weak formulation in (1.18), the strong form of the continuous dynamics is represented by a first-order partial differential equation of the following form

$$\frac{\partial \Lambda}{\partial \sigma} = \mathcal{M}\dot{\eta} - \mathrm{ad}_\eta^\top \mathcal{M}\eta + \mathrm{ad}_\xi^\top \Lambda + \mathcal{F} \tag{1.19}$$

with boundary conditions $\Lambda(0,t) = -(\mathcal{F}_-)$ and $\Lambda(l,t) = \mathcal{F}_+$, i.e., the external reaction forces acting on the boundaries of the material domain $\mathcal{X} \in [0,l]$. In case of a manipulator whose base is spatially fixed with a free end-effector, the boundary conditions should satisfy $\Lambda(0,t) = -(\mathcal{F}_-)$.

2

Control of Soft Robots

2.1 Underactuated system

Before addressing the controller synthesis, we briefly introduce the notion of under-actuated dynamical systems [3, 5, 6, 8].

Definition 2.1 (Under-actuated system)**.** *A second-order dynamical system described by the partial differential equation*

$$\ddot{q} = f(q, \dot{q}, u, t) \tag{2.1}$$

is considered fully-actuated in a state $(q, \dot{q})$ at time t if and only if the map f is surjective, i.e, for every $\ddot{q}(t)$ there exists a control input $u(t)$ such that the instantaneous acceleration is realizable. Otherwise, the dynamical system is said to be under-actuated. Regarding the control-affine form of underactuated systems, that is,

$$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u(t), \tag{2.2}$$

a sufficient condition is $\operatorname{rank}(f_2(q, \dot{q}, t)) < \dim(q)$.

In other words, an underactuated dynamical system cannot steer its states in any arbitrary direction. As a consequence, under actuated systems are generally more difficult to control. By definition, a soft robotic system is an under-actuated system since they theoretically posses infinitely many degrees-of-freedom. Including the distributed control inputs to the Lagrangian model, we write the dynamics for a soft robotic system as follows

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = S_a^\top(q)\tau, \tag{2.3}$$

where $M(q)$ is the positive definite mass matrix, $c(q, \dot{q}) \in \mathbb{R}^n$ is a vector of Coriolis forces, $g(q) \in \mathbb{R}^n$ a vector of conservative potential forces, $S_a(q) \in \mathbb{R}^{n \times m}$ is a (nonlinear) mapping that projects the active control inputs onto the acceleration space of q, and $u(t) \in \mathbb{R}^m$ is the lower-dimensional control input. Since the system is under-actuated, it shall be clear that $\dim(u) < \dim(q)$. In some studies [3], the matrix S is referred to as the synergy matrix whose columns describe actuation patterns of the soft robot's input space. Without loss of generality, let the synergy matrix S be defined by a set of linearly independent column vectors of actuation patterns $s_i : \mathbb{R}^n \to \mathbb{R}^n$,

$$S_a(q) := [\, s_1(q),\, s_2(q),\, ...,\, s_m(q)\,]\,, \tag{2.4}$$

which implies that the matrix has $\operatorname{rank}(S) = n_a$.

2.2 Discontinuous shape functions

An approach to distinguish the actuator dynamics and the redundant dynamics due to mechanical flexibility is the use of discontinuous shape representation. This is of particular importance for soft robotics as their (theoretical) infinite number of degrees of freedom cannot be matched by the same number of actuators. To recall, each component of the geometric strain field $\xi_i(\sigma, t) \in \mathfrak{se}(3)$ is approximated using a finite set of shape functions

$$\xi_i(\sigma, t) \cong \sum_{k=1}^{N} \Phi_k(\sigma) q_k(t) \tag{2.5}$$

with $\Phi_j : \mathbb{R} \mapsto \mathbb{R}$ and the N the approximation order. First, let us consider the original Chebyshev polynomials (of the first kind) as described by

$$\Phi_n(\sigma) = \cos\left[n \arccos(\sigma)\right]. \tag{2.6}$$

Now, suppose there exists a locally uniform actuation field is applied to continuous body within the spatial domain $\mathcal{X}_a \in [\sigma_-, \sigma_+]$ with $0 \leq \sigma_- < \sigma_+ \leq l$. With this subset $\mathcal{X}$ in mind, lets propose a modification to the Chebyshev polynomial as follows

$$\Phi_n(\sigma) = w(\sigma) \cos\left[n \arccos\left(\frac{\sigma - \sigma_-}{\sigma_+ - \sigma_-}\right)\right] \tag{2.7}$$

where the weighting function defined as $w(\sigma) = 1\ \forall \sigma \in \mathcal{X}_a$ and zero otherwise. Using the discontinuous variant of the Chebyshev polynomial in (2.7), the strain field of the soft robot can be decomposed into active and passive parts. The main benefit of this approach is that the synergy matrix S_a containing the columns of actuation patterns is now invariant regarding the generalized coordinates q.

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