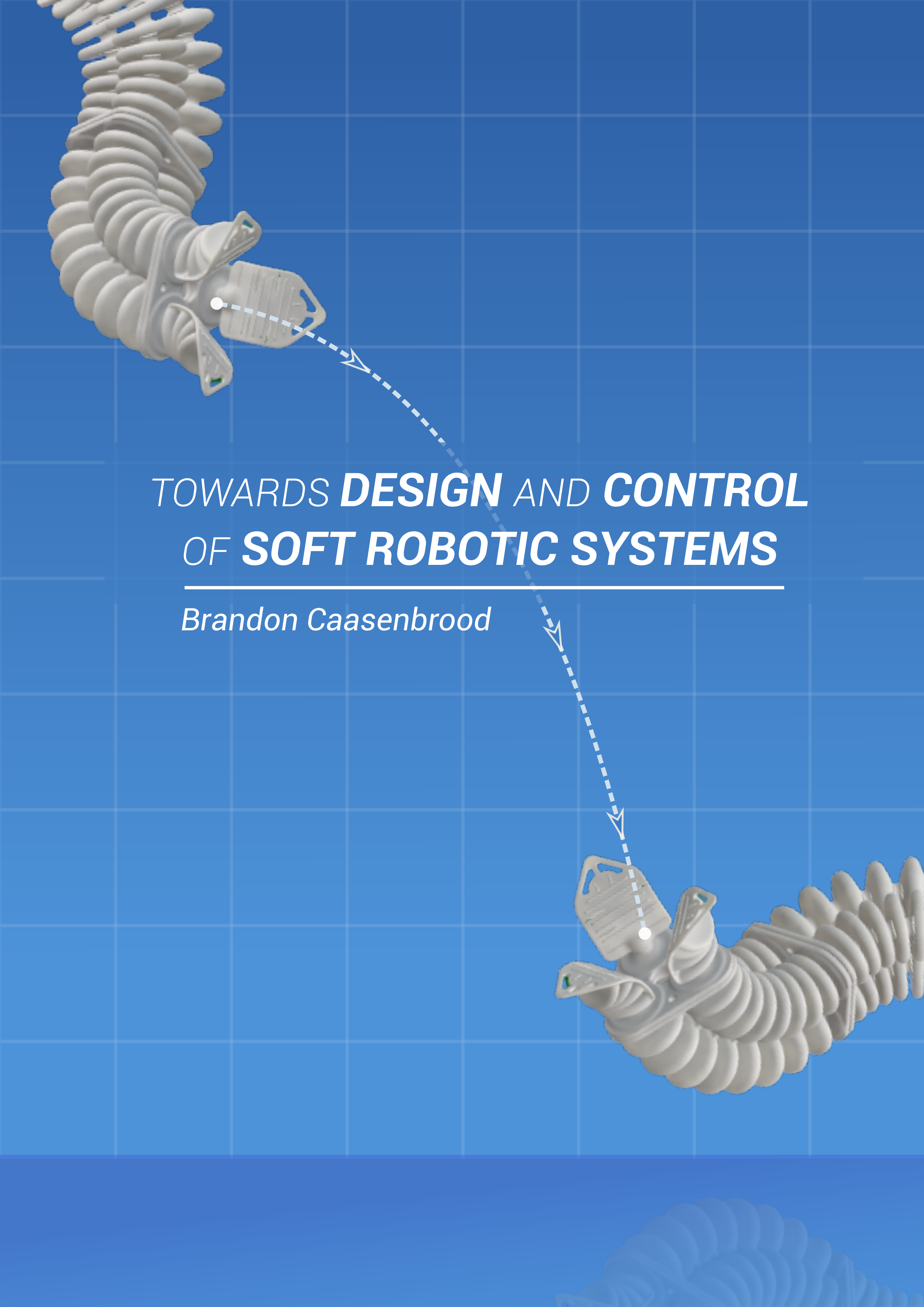

TOWARDS **DESIGN** AND **CONTROL** OF **SOFT ROBOTIC SYSTEMS**

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Dynamics of Soft Robots

1.1 Configuration space using Lie group theory

In contrast to a rigid robot, whose mechanical structure consists of static links and joints, a soft robot lacks the physical notion of joints and therefore cannot be viewed as an ordinary multi-body system. From a mechanical perspective, a soft robotic system is more closely related to a continuous deformable medium with infinite degrees-of-freedom rather than a traditional serial-chain rigid robot. Given this description, a soft robotic system can be modeled as a one-dimensional Cosserat beam together with the geometrically exact beam theories proposed by Simo et al. (1986, [1]) whose modeling approach is grounded in the field of group theory.

To express a space-time variant coordinate frame, lets start by introducing a spatial coordinate $\sigma \in \mathbb{X}$ that lies on a bounded domain $\mathbb{X} \in [0, l] \subset \mathbb{R}$, and a temporal coordinate $t \in \mathbb{T}$ with $\mathbb{T} \subseteq \mathbb{R}$. Given these notations, we can represent the position $p(\sigma, t) \in \mathbb{R}^3$ and orientation matrix $R(\sigma, t) \in \mathbb{SO}(3)$ for any point σ and instance t on the smooth backbone of the soft robot by a functional curve $g : \mathbb{X} \times \mathbb{T} \mapsto \mathbb{SE}(3)$, that is,

$$g(\sigma, t) := \begin{pmatrix} R(\sigma, t) & p(\sigma, t) \\ 0_3^\top & 1 \end{pmatrix} \in \mathbb{SE}(3), \tag{1.1}$$

where $\mathbb{SE}(3)$ is the Lie group of rigid body transformations in $\mathbb{R}^3$ [2, 3].

Since the backbone curve g is space-time variant, the variations in space and time can be characterized by two vector field in the Lie algebra $\mathfrak{se}(3)$. Throughout this work, we denote the partial derivatives $\partial(\cdot)/\partial\sigma$ and $\partial(\cdot)/\partial t$ by a 'prime' and 'dot', respectively. By regarding the partial derivative with respect to time of (1.1), the time-twist field can be defined as follows

$$\dot{g} = g\hat{\eta} \implies \hat{\eta} := g^{-1}\dot{g} = \begin{pmatrix} \Omega_\times & V \\ 0_3^\top & 0 \end{pmatrix} \in \mathfrak{se}(3), \tag{1.2}$$

where $\Omega = (\omega_1, \omega_2, \omega_3)^\top$ and $V = (v_1, v_2, v_3)^\top$ denote the angular velocity vector and the linear velocity vector, respectively. The skew-symmetric matrix $\Omega_\times$ can be related to $\mathfrak{so}(2) \cong \mathbb{R}^3$ given the isomorphism $\Omega \mapsto \Omega_\times$ [2, 4, 5]. To be more specific on geometric interpretation of the time-twist field, the vector field $\eta(\sigma, t)$ defines an infinitesimal local transformation undergone by a frame at position σ between two infinitesimally close instances t and $t+dt$. Second, by regarding the partial derivative with respect to space of (1.1), the space-twist field can be defined as follows

$$g' = g\hat{\xi} \implies \hat{\xi} := g^{-1}g' = \begin{pmatrix} K_\times & E \\ 0_3^\top & 0 \end{pmatrix} \in \mathfrak{se}(3), \tag{1.3}$$

where $K = (k_1, k_2, k_3)^\top$ and $E = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^\top$ denote the curvature-torsion strain vector and the stretch-shear strain vector, respectively. Similar to its geometric counterpart, the vector field $\xi(\sigma, t)$ defines an infinitesimal local transformation undergone by a frame at an instance t between two infinitesimally close positions σ and $\sigma + d\sigma$. Since $\mathfrak{se}(3) \cong \mathbb{R}^6$ with isomorphism $\eta \mapsto \hat{\eta}$, we can express (1.2) and (1.3) as column vectors in $\mathbb{R}^6$ as follow

$$\eta(\sigma, t) = \begin{pmatrix} \Omega \\ V \end{pmatrix}; \qquad \xi(\sigma, t) = \begin{pmatrix} K \\ E \end{pmatrix}. \tag{1.4}$$

1.2 Continuous kinematics for soft robots

By using the equality of mixed partials, we may invoke that $\frac{\partial}{\partial t}(g') = \frac{\partial}{\partial \sigma}(\dot{g})$ holds for any instance in space and time. Accordingly, substitution of relations (1.2) and (1.3) into this commutative relation leads to

$$\dot{g}\xi + g\dot{\hat{\xi}} = g'\hat{\eta} + g\hat{\eta}', \tag{1.5}$$

which implies

$$g\hat{\eta}\hat{\xi} + g\dot{\hat{\xi}} = g\hat{\xi}\hat{\eta} + g\hat{\eta}'. \tag{1.6}$$

Multiplying both sides with g^{-1} and rearranging the equality, we find

$$\hat{\eta}' = -(\hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi}) + \dot{\hat{\xi}}, \tag{1.7}$$

where we can recognize, in the parenthesis, the Lie bracket of ξ and η. The Lie bracket $[\hat{\xi}, \hat{\eta}]$ is also an element of Lie algebra $\mathfrak{se}(3)$, and thus it may be alternatively expressed in $\mathbb{R}^6$ as the adjoint action between ξ onto η, namely $\mathrm{ad}_\xi \eta : \mathbb{R}^6 \mapsto \mathbb{R}^6$ (see [3] and [5]). Therefore, the velocity kinematics in (1.7) can be written in vector representation as

$$\eta' = -\mathrm{ad}_\xi \eta + \dot{\xi}. \tag{1.8}$$

By taking the time derivative of (1.8) and combining the previous results, the continuous kinematic model for the configuration, velocity, and acceleration can be written as system of first-order partial differential equation (PDE) of the form

$$\frac{\partial}{\partial \sigma}\begin{pmatrix} g \\ \eta \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} g\hat{\xi} \\ -\mathrm{ad}_{\xi}\eta + \dot{\xi} \\ -\mathrm{ad}_{\dot{\xi}}\eta - \mathrm{ad}_{\xi}\dot{\eta} + \ddot{\xi} \end{pmatrix}. \tag{1.9}$$

For a general case, the boundary conditions of PDE in (1.9) should satisfy $g(0,t) = g_0$, $\eta(0,t) = \eta_0$ and $\dot{\eta}(0,t) = \dot{\eta}_0$. However, in case of a manipulator whose base is spatially fixed, the boundary conditions should satisfy $g(0,t) = g_0$, and $\eta(0,t) = \dot{\eta}(0,t) = 0_6$. Notice that if the strain fields ξ, $\dot{\xi}$, and $\ddot{\xi}$ are known, the partial differential equation in (1.9) simply becomes a first-order ordinary differential equation (ODE), which can be easily solved using numerical methods.

1.3 Continuous dynamics for soft robots

In this section, we derive the dynamical model of the soft robot through Hamilton's variational principle. Given an interval $[t_0, t_1]$, the variational principle states that the evolution of a state $q(t)$ between $q(t_0)$ and $q(t_1)$ is a stationary point regarding an action functional, $\mathcal{S} = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t)\, dt$ in which $\mathcal{L}(q, \dot{q}) := \mathcal{T}(q, \dot{q}) - \mathcal{V}(q)$ is the Lagrangian. The generalization of Hamilton's principle [4] includes an external potential contributions, and it can be formally written as

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \left[\delta\mathcal{T} - \delta\mathcal{V} + \delta\mathcal{W}_{ex}\right]\, dt = 0, \tag{1.10}$$

where the operator δ denotes the variation of functional that are fixed at the boundaries $[t_0, t_1]$, and $\mathcal{W}_{ex}$ is the external virtual work produced by nonconservative external forces acting on the system.

First, let us regard the functional variation of kinetic energy. The kinetic energy of the soft robot is defined by

$$\mathcal{T} := \frac{1}{2}\int_{\mathbb{X}} \eta^{\top}\mathcal{M}\eta\, d\sigma, \tag{1.11}$$

where $\mathcal{M} \in \mathfrak{se}(3) \times \mathfrak{se}(3)^*$ is the inertia tensor whose components denote the inertial properties of an infinitesimal slice of the continuous mechanical body. More specifically, the inertia tensor is $\mathcal{M} = \mathrm{blkdiag}\{mI_3, \mathcal{J}\}$ with $m \in \mathbb{R}_{>0}$ the line-density and $\mathcal{J} \in \mathfrak{so}(2) \times \mathfrak{so}(2)^*$ the moment of inertia tensor. From the isomorphism

$\mathfrak{se}(3) \cong \mathbb{R}^6$, the inertia tensor $\mathcal{M}$ may be equivalently represented as a symmetric matrix of $\mathbb{R}^{6\times 6}$. Given (1.11), the variation of the kinetic energy function is given by

$$\begin{aligned}\delta\mathcal{T} &= \frac{\partial}{\partial a}\mathcal{T}(\eta + a\delta\eta)\Big|_{a=0},\\ &= \frac{1}{2}\int_{\mathbb{X}} \delta\eta^\top \mathcal{M}\eta + \eta^\top\mathcal{M}\delta\eta \; d\sigma,\\ &= \int_{\mathbb{X}} \delta\eta^\top\mathcal{M}\eta \; d\sigma. \end{aligned} \tag{1.12}$$

By applying variational calculus on the Lie group, we can express the variation of the velocity field as $\delta\eta = \delta\dot{\epsilon} + \mathrm{ad}_\eta\delta\epsilon$ in which $\delta\epsilon = g^{-1}\delta g \in \mathfrak{se}(3)$ with $\delta\epsilon(t_0) = \delta\epsilon(t_1) = 0$. Therefore, substitution of the variation into (1.12) and followed by integration by parts leads to derivation

$$\begin{aligned}\int_{t_1}^{t_2} \delta\mathcal{T}\; dt &= \left[\int_{\mathbb{X}} \delta\epsilon^\top M\eta\right]_{t_0}^{t_1} + \int_{t_1}^{t_2}\int_{\mathbb{X}} \delta\epsilon^\top\Big(M\dot{\eta} - \mathrm{ad}_\eta^\top M\eta\Big)\; d\sigma dt\\ &= \int_{t_1}^{t_2}\int_{\mathbb{X}} \delta\epsilon^\top\left(M\dot{\eta} - \mathrm{ad}_\eta^\top M\eta\right)\; d\sigma dt. \end{aligned} \tag{1.13}$$

Note that since the variations are fixed at the boundaries of $[t_0, t_1]$, the first right hand part in (1.13) vanished. Since the variations are fixed at the boundaries of $[t_0, t_1]$, the first right hand part in (1.13) vanishes. The expression in (1.13) will be recalled later, but first, let us describe the functional variation of potential energy.

The internal potential energy of the soft robot is defined as

$$\mathcal{V} := \int_{\mathbb{X}} \xi^\top\Lambda\; d\sigma. \tag{1.14}$$

where $\Lambda \in \mathfrak{se}(3)^*$ is the field of internal wrenches along the continuum elastic body. Notice that vector field of internal wrenches is an element of the dual space of $\mathfrak{se}(3)$. This field and the strains vector field are related through a material constitutive law. In general concerning soft robotic applications, the use of linear constitutive relations for an isotropic elastic material are not sufficient, since large deformations introduce nonlinear material behavior. However, for the sake of simplicity, we consider the simplest viscoelastic constitutive model - the Kelvin-Voigt model. The Kelvin-Voigt model is a linear elasticity model with a linear viscous contribution that is proportional to the rate of strain ξ,

$$\Lambda = K\xi + \Gamma\dot{\xi} \tag{1.15}$$

where K and Γ are the elasticity and viscosity material tensor, respectively. Similar to the kinetic energy, the variation of the potential energy function V is given by

$$\delta\mathcal{V} = \frac{\partial}{\partial a}\mathcal{V}(\xi + a(\delta\xi))\Big|_{a=0} = \int_{\mathbb{X}} \delta\xi^\top\Lambda\; d\sigma, \tag{1.16}$$

Recalling the commutativity of the Lie algebra, we can express the variation as $\delta\xi = \epsilon' + \mathrm{ad}_\xi \epsilon$. Therefore, substitution into (1.16) and using integration by parts leads to

$$\begin{aligned} \int_{t_1}^{t_2} \delta\mathcal{V}\, dt &= \left[\int_{\mathbb{X}} \delta\epsilon^\top \Lambda\right]_{t_0}^{t_1} + \int_{t_1}^{t_2}\int_{\mathbb{X}} \epsilon^\top \left(\Lambda' - \mathrm{ad}_\xi^\top \Lambda\right)\, d\sigma dt \\ &= \int_{t_1}^{t_2}\int_{\mathbb{X}} \delta\epsilon^\top \left(\Lambda' - \mathrm{ad}_\xi^\top \Lambda\right)\, d\sigma dt. \end{aligned} \tag{1.17}$$

Now, assuming that $\delta\mathcal{W}_{ex} = \int \delta\epsilon^\top \mathcal{F}\, d\sigma$ with $\mathcal{F} : \mathbb{X} \mapsto \mathbb{R}^6$ an external wrench acting on the continuous body, we can substitute the kinetic and potential energy variations (1.13) and (1.17) into the Hamilton's variational principle (1.10) which leads to the weak formulation of the continuous dynamics

$$\delta\mathcal{S} = \int_{t_1}^{t_2}\int_{\mathbb{X}} \delta\epsilon^\top \left(\mathcal{M}\dot{\eta} - \mathrm{ad}_\eta^\top \mathcal{M}\eta - \Lambda' + \mathrm{ad}_\xi^\top \Lambda + \mathcal{F}\right)\, d\sigma dt = 0, \tag{1.18}$$

which holds for all variations $\delta\epsilon \in \mathfrak{se}(3)$. Given the weak formulation in (1.18), the strong form of the continuous dynamics is represented by a first-order partial differential equation of the following form

$$\Lambda' = \mathrm{ad}_\xi^\top \Lambda + \mathcal{M}\dot{\eta} - \mathrm{ad}_\eta^\top \mathcal{M}\eta + \mathcal{F} \tag{1.19}$$

subjected to the boundary conditions $\Lambda(0,t) = -(\mathcal{F}_-)$ and $\Lambda(l,t) = \mathcal{F}_+$, i.e., the external reaction forces acting on the boundaries of the material domain $\mathbb{X} \in [0,l]$. In case of a manipulator whose base is spatially fixed with a free end-effector, the boundary conditions should satisfy $\Lambda(0,t) = -(\mathcal{F}_-)$. Consequently, the infinite dimensional kinematics and dynamics for a slender soft body can be compactly represented as a system of partial differential equations of the form

$$\Sigma := \begin{cases} g' &= g\hat{\xi} \\ \eta' &= -\mathrm{ad}_\xi \eta + \dot{\xi} \\ \dot{\eta}' &= -\mathrm{ad}_{\dot{\xi}} \eta - \mathrm{ad}_\xi \dot{\eta} + \ddot{\xi} \\ \Lambda' &= \mathrm{ad}_\xi^\top \Lambda + \mathcal{M}\dot{\eta} - \mathrm{ad}_\eta^\top \mathcal{M}\eta + \mathcal{F} \end{cases} \tag{1.20}$$

1.4 Projection into Lagrangian model

Although the system of PDEs in (1.20) is useful for solving the forward kinematics or dynamics, it is generally more difficult to apply control theory for PDEs. By definition, a PDE involves a differential equation with multiple continuous variables; often subjected to a set of boundary values. These distinctions make systematic controller design more challenging as Lyapunov theorems for stability are not suited. An important tool in control theory for PDEs is model reduction, where only a finite-dimensional subsystem is controlled [6, 7]. In this approach, a infinite-dimensional dynamical system is projected onto a finite-dimensional subspace that contains the basis elements with attributes of the expected solution. A well-known example is the Galerkin projection method, commonly used in finite element methods. As a result, the PDE model can be replaced by a system of ordinary differential equations (ODEs) that does allow for traditional control theory. It is, however, important to show robustness for neglecting the remaining infinite dimensional dynamics absent in the reduced model.

Similar to finite element methods, suppose the components of the strain field $\xi := (g^{-1}g')^\vee$ can be closely approximated by a finite number of orthogonal shape functions[1] $\varphi : \mathbb{X} \to \mathbb{R}$, namely

$$\xi_i(\sigma, t) \cong \sum_{i=1}^{N} \varphi_i(\sigma) q_i(t) + \xi_{i,0}(\sigma), \quad \forall \sigma \in \mathbb{X}, t \in \mathbb{T} \tag{1.21}$$

where $\xi_0 = (g_0^{-1}g_0')^\vee$ is a vector field of zero strains (i.e., the space-twist field corresponding to the undeformed configuration of the elastic body), $\mathbb{X}$ a spatial set, and $\mathbb{T} \subseteq \mathbb{R}$ a time set. Furthermore, we refer $\{\varphi_i\}_{i\in\mathbb{N}}$ as the set of basis functions and $q = (q_1, ..., q_n)^\top$ as the modal coefficients regarding the basis $\{\varphi_i\}_{i\in\mathbb{N}}$. From a robotics perspective, we interchangeability refer to q as the joint variables or generalized coordinates of the finite-dimensional subset.

For the sake of simplicity, lets assume $\xi_0 = 0_6$ for now. Accordingly, we can rewrite the n-th order expansion of the geometric strain twist as

$$\begin{aligned} \xi(\sigma, t) &\cong \left(B_a \otimes \begin{bmatrix} \varphi_1 & \dots & \varphi_N \end{bmatrix}\right) q(t), \\ &= \underbrace{\begin{pmatrix} \varphi_1 & \dots & \varphi_N & \dots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \dots & \varphi_1 & \dots & \varphi_N \end{pmatrix}}_{\Phi(\sigma)} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \end{aligned} \tag{1.22}$$

[1] Orthogonality here implies that $\int_{\mathbb{X}} \varphi_i \, \varphi_j \; d\sigma = 0$ for any $i \neq j$ and non-zero otherwise.

where $\Phi : \mathbb{R} \mapsto \mathbb{R}^{m\times n}$ is the shape function matrix whose columns are mutually-orthogonal, $B_a \subseteq \text{span}\,(\mathbb{I}_6)$ a selection matrix of unconstrained strains, and $\otimes$ denotes the Kronecker product. Here, the selection matrix B_a allows for some internal kinematic constraints by eliminating components from the strain field ξ.

Now, let us recall the PDE model related to the velocity twist $\hat{\eta} \in \mathfrak{se}(3)$

$$\eta' = -\text{ad}_\xi \eta + \dot{\xi}, \tag{1.23}$$

where again $\text{ad}_\xi : \mathbb{R}^6 \mapsto \mathbb{R}^6$ denotes the adjoint action of the algebra $\hat{\xi} \in \mathfrak{se}(3)$. Using the differential property $d\text{Ad}_g/ds = \text{Ad}_g \text{ad}_\Upsilon$ given a twist $\Upsilon = (g^{-1}dg/ds)^\vee$, it follows that $-\text{ad}_\xi = (\text{Ad}_{g^{-1}})'\text{Ad}_g$. Substitution of this geometric relation into (1.23), we can rewrite the space-variation of the velocity twist as

$$\eta' = (\text{Ad}_{g^{-1}})'\,\text{Ad}_g \eta + \dot{\xi}. \tag{1.24}$$

Since the open-chain soft robot is fixed at the ground-plane, the following boundary conditions can be imposed $\eta_0 = 0_6$ and $g_0 = e$. As such, the analytic solution to the velocity twist η can be obtained by explicit integration of (1.24) over the domain $[0, \sigma]$

$$\eta(\sigma, t) = \text{Ad}_{g^{-1}} \int_0^\sigma Ad_g \Phi(\sigma)\; d\sigma\, \dot{q} := J\dot{q}. \tag{1.25}$$

which gives rise the geometric Jacobian $J(\sigma, q) : \mathbb{X} \times \mathbb{R}^n \mapsto \mathbb{R}^{6\times n}$ that linearly maps joint velocities to the velocity twist expressed in a moving inertial frame at point σ. It is worth mentioning that the space-time variant of the Jacobian matrix requires both the joint variables and the spatial coordinate on the continuous body. Given (1.25) and the boundary values $\dot{\eta}_0 = 0_6$, we can further detail the continuous kinematics at acceleration level, that is,

$$\begin{aligned}\dot{\eta}(\sigma, t) &= \text{Ad}_{g^{-1}} \int_0^\sigma Ad_g \Phi(\sigma)\; d\sigma\, \ddot{q} + \text{Ad}_{g^{-1}} \int_0^\sigma Ad_g \text{ad}_\eta \Phi(\sigma)\; d\sigma\, \dot{q}, \\ &= J\ddot{q} + \dot{J}\dot{q}. \end{aligned} \tag{1.26}$$

Notice that on right-hand side in (1.26), we now obtain the expression for the time-derivative of the geometric Jacobian, i.e., $\dot{J}$. Given the expressions for the velocity twist and acceleration twist respectively in (1.26) and (1.25), it is now possible to express the continuous dynamics in the Lagrangian form. Recall the partial differential equation for the continuous dynamics

$$\Lambda' = \text{ad}_\xi^\top \Lambda + \mathcal{M}\dot{\eta} - \text{ad}_\eta^\top \mathcal{M}\eta + \mathcal{F}, \tag{1.27}$$

which is nothing more than the continuum description of the Newton-Euler equation of motion for slender elastic objects undergoing free motion in $\mathbb{R}^3$. Before solving the original PDE, we introduce a slight modification to the PDE in (1.27). Since $\mathrm{ad}_\eta \eta = 0_6$ for any arbitrary $\eta \in \mathbb{R}^6$, then we can introduce a null vector $\mathcal{M}\mathrm{ad}_\eta \eta$ into (1.27) without affecting the continuous dynamics [8]. The importance of this null modification will be discussed later in this section. Using the previous knowledge, we can solve (1.27) explicitly over the material domain $\mathbb{X} = [0, l]$,

$$\Lambda = \int_{\mathbb{X}} J^\top \Big[\mathcal{M}\dot{\eta} + \Big(\mathcal{M}\mathrm{ad}_\eta - \mathrm{ad}_\eta^\top \mathcal{M}\Big)\, \eta + \mathcal{F} \Big] \; d\sigma \tag{1.28}$$

With slight abuse of formulation, we may substitute Λ with the non-conservative external forces acting on the finite-dimensional system, formally denoted by the control input of the mechanical system $\tau(t)$. Furthermore, we may distinguish the conservative wrenches into a visco-elastic contribution $\mathcal{F}_e$ and a gravitational contribution $\mathcal{F}_g$. The internal wrenches due to gravitational potential field are defined by

$$\mathcal{F}_g = \mathcal{M}\mathrm{Ad}_{g^{-1}} a_z, \tag{1.29}$$

where $a_z \in \mathbb{R}^6$ is a constant gravitational acceleration vector expressed as a wrench. As for the internal wrenches due to the visco-elastic contribution, we propose a hyper-elastic model with linear dissipation (Rayleigh damping). Therefore, it follows that

$$\mathcal{F}_e = \mathcal{K}(\xi)\, \xi + \Gamma \dot{\xi} \tag{1.30}$$

where $\mathcal{K} : \mathfrak{se}(3) \mapsto \mathfrak{se}(3) \times \mathfrak{se}^*(3)$ is denoted as a hyper-elastic stiffness tensor, and $\Gamma \in \mathfrak{se}(3) \times \mathfrak{se}^*(3)$ is a constant damping tensor. Since we assume that any deformation is reversible, it follows that $\mathrm{argmin}_\xi \, \|\mathcal{K}(\xi)\|_2 = \xi_0$. By substituting the kinematic relations (1.24) and (1.26) into (1.28), we can recognize the standard Lagrangian structure as a second-order ordinary differential equation of the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + k(q) + R\dot{q} = \tau(t) \tag{1.31}$$

with

$$M(q) = \int_{\mathbb{X}} J^\top \mathcal{M} J \; d\sigma, \tag{1.32}$$

$$C(q,\dot{q}) = \int_{\mathbb{X}} J^\top \mathcal{M} \dot{J} + J^\top \Big(\mathcal{M}\mathrm{ad}_\eta - \mathrm{ad}_\eta^\top \mathcal{M}\Big)\, J \; d\sigma, \tag{1.33}$$

$$g(q) = \int_{\mathbb{X}} J^\top \mathcal{M}\mathrm{Ad}_{g^{-1}}\, a_z \; d\sigma, \tag{1.34}$$

$$k(q) = \int_{\mathbb{X}} J^\top K(\xi)\xi \; d\sigma, \tag{1.35}$$

$$R = \int_{\mathbb{X}} J^\top \Gamma\, \Phi \; d\sigma, \tag{1.36}$$

Let it be clear that it is necessary to precompute the Jacobian matrices J and $\dot{J}$ using the kinematic expressions (1.24) and (1.26), respectively.

1.5 Port-Hamiltonian model

Before deriving the dynamics into the port-Hamiltonian form, let us briefly recall the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q^{nc} \tag{1.37}$$

in which $\mathcal{L}(q, \dot{q}) := \mathcal{T}(q, \dot{q}) - \mathcal{V}(q)$ is the Lagrangian, and Q^{nc} the vector of non-conservative forces acting on the system. Note that the Euler-Lagrange equation of motion is second-order differential equation expressed in terms of the generalized coordinates $q = (q_1, ..., q_k)^\top$ and the generalized velocities $\dot{q} = (\dot{q}_1, ..., \dot{q}_k)^\top$. Now, let us define a vector of generalized momenta $p = (p_1, ..., p_n)^\top$ such that the momenta can be expressed by $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ given any Lagrangian function $\mathcal{L}(q, \dot{q})$. Recalling the expression for the kinetic energy, that is, $\mathcal{T} = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$, the generalized momenta can be described analogously as

$$p = M(q)\dot{q}. \tag{1.38}$$

Exploiting the positive definite and symmetric properties of the inertia matrix $M(q) = M^\top(q) \succ 0$, the total energy stored in the dynamical system $\mathcal{H}(q, \dot{q}) := \mathcal{T}(q, \dot{q}) + \mathcal{V}(q)$ can be rewritten as

$$\mathcal{H}(q, p) = p^\top \dot{q} - \mathcal{L} \quad \iff \quad \mathcal{H}(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + \mathcal{V}(q). \tag{1.39}$$

In literature [9–11], the total energy of the system described by $\mathcal{H}(q, p)$ is called the Hamiltonian. From (1.39), it can be easily shown that generalized velocities can be written in terms of partial derivatives of the Hamiltonian function

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = M^{-1} p. \tag{1.40}$$

Similarly, we aim to find a differential equation that relates the time evolution of p and the Hamiltonian. By applying the chain rule of differentiation to the momenta in (1.38), we find

$$\begin{aligned} \dot{p} &= \dot{M}\dot{q} + M\ddot{q} \\ &= \left(\dot{M} - C\right) M^{-1} p - \frac{\partial \mathcal{L}}{\partial q} - R\dot{q} + \tau(t), \end{aligned} \tag{1.41}$$

where the dissipative forces are modeled after Rayleigh damping $\mathcal{F}_d = R\dot{q}$ with a constant matrix $R \succeq 0$. Whereas, taking the partial derivate of the Hamiltonian in (1.39) with respect to the generalized coordinates q, we obtain

$$\frac{\partial \mathcal{H}}{\partial q} = \frac{1}{2} \frac{\partial}{\partial q} \left(\dot{q}^\top M(q) \dot{q}\right) + \frac{\partial \mathcal{L}}{\partial q}. \tag{1.42}$$

To express (1.41) in terms of (1.42), we exploit some useful structural properties in the Lagrangian model, which directly follows from the formulation of the Coriolis matrix (1.33) in the previous section. Let us consider a defined by $N(q, \dot{q}) = \dot{M} - 2C$. According to the Spong et. al (2006, [3]), if the Coriolis matrix is expressed in terms of the Christoffel symbols corresponding to the inertia matrix $M(q)$, it can be proven that post-multiplication of N with $\dot{q}$ leads to the following equality

$$N\dot{q} = -\frac{\partial}{\partial q}\left(\dot{q}^\top M(q)\dot{q}\right) - \dot{M}\dot{q}. \tag{1.43}$$

By combining (1.42), (1.41) and (1.43), and introducing the state vector $(q^\top, p^\top)^\top = (q_1, ..., q_n, p_1, ..., p_n)^\top$, we can transform the Euler-Lagrange equation of motion as a system of first-order equations of the form

$$\Sigma := \begin{cases} \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p}, \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial q} - R\dfrac{\partial \mathcal{H}}{\partial p} + \tau, \end{cases} \tag{1.44}$$

where the Hamiltonian is defined by $\mathcal{H}(q, p) = \frac{1}{2}p^\top M^{-1}(q)p + \mathcal{V}(q)$. Due to its formulation with the We refer to the system of equations above as the Hamiltonian equations of motion. The energy balance of the dynamical system follows directly from the time evolution of Hamiltonian as

$$\dot{\mathcal{H}} = \left(\frac{\partial \mathcal{H}}{\partial q}\right)^\top \dot{q} + \left(\frac{\partial \mathcal{H}}{\partial p}\right)^\top \dot{p} = -\dot{q}^\top R + \dot{q}^\top \tau \leq \dot{q}^\top \tau, \tag{1.45}$$

which states that the total energy increase of the system will always be equal or lower to the supplied energy from the environment. This also implies that the system cannot store more energy than is supplied from the environment.

Property 1.1 (Passivity)**.** If the Hamiltonian $\mathcal{H}(q, p)$ is the sum of the kinetic energy and the potential energy which is lower bounded, that is,

$$\mathcal{H} = \frac{1}{2}p^\top M^{-1}(q)p + \mathcal{V}(q), \tag{1.46}$$

with $M(q) = M^\top(q) \succ 0$ and $\exists \beta > -\infty$ such that $\mathcal{V}(q) \geq \beta$; then it follows that (1.44) with the input $u = \tau$ and the output $y = \dot{q}$ is a passive system together with the storage function $\mathcal{H}(q, p) - \beta \geq 0$.

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