## CS B553 Homework 1: Math Review and Univariate Optimization

Due date: 1/24/2012

1. Use L’Hospital’s rule to find the limit of sin(x)/x as x→0.
2. Derive the expression of the 1st and 2nd order Taylor expansion of the function f(x) = c exp(-d x2) around the point x0. Find an expression for the k’th order Taylor expression that works for all k.
3. Consider modeling a dataset D={(xi,yi),i=1,…,n} with a line through the origin , where m is a parameter. The error between the line and the i’th datapoint is . Let the objective function be the *sum of squared errors* . Find the critical points of . Show that there is one unique point and that it is indeed a local minimum.
4. Prove that if a function f(x) is differentiable, and its derivative f’(x) satisfies |f’(x)|≤K for all x, then f satisfies the Lipschitz condition with constant K. Use Rolle’s theorem, which states that if a differentiable function g(x) attains the same values at points *a* and *b*, then g’(x)=0 at some point x∈(*a*,*b*). Give an example of a function that is Lipschitz but is not differentiable.
5. Give pseudocode for a constrained Newton's method that limits the step in order to prevent divergence. The iterates should satisfy .
6. First, verify that L’Hospital’s rule applies. Both the numerator and the denominator approach 0 as x→0, so yes it indeed applies. So lim[x→0] sin(x)/x = lim[x→0] sin’(x)/x’ = lim[x→0] cos(x)/1 = 1/1 = 1.
7. f’(x) = -2dcx exp(-dx2), and f’’(x) = -2dc exp(-dx2) + 4dcx2exp(-dx2). The first-order Taylor expansion is f(x) = f(x0) + f’(x0)(x-x0) + O((x-x0)2), and the second-order Taylor expansion is f(x) = f(x0) + f’(x0)(x-x0) + ½ f’’(x0) (x-x0)2 + O((x-x0)2).  
     
   The k’th order Taylor expansion is given by . We simply need to find an expression for the k’th derivative of f(x) and plug it into the Taylor expansion. By inspection, note that the k’th derivative of f(x) has the form for some constants ci,k. Taking the derivative of this function we have . Equating powers of x, we must solve the recurrence:  
     
   c0,0=1 (base case)  
     
   c0,k+1 = c1,k; ci,k+1 = (i+1)ci+1,k -2dci-1,k  for all k>=0, i<k; ck+1,k+1 = -2dck,k (inductive case)
8. Write out the expression for f(m): and take the derivative . To find critical points, find the value of m that sets the derivative to zero: . This value exists and is unique as long as the denominator is nonzero. Finally, note that the second derivative which is nonnegative for all m (and is positive as long as the denominator is nonzero). So, the unique critical point is also a minimum.
9. We wish to show that for any x-values a and b, |f(a)-f(b)| <= K|a-b|. Fix the values of a and b, and then consider a line L(x) with slope equal to the secant line between b and a: L(x) = c+x\*(f(b)-f(a))/(b-a) for some constant c. Finally consider the function g(x) = f(x)-L(x). We can see that g(a)=g(b), which allows us to apply Rolle’s theorem. This says that g’(x)=0 for some x in (a,b). So, f’(x)-L’(x) = 0 for some x, and therefore |f’(x)| = |L’(x)|. Since |f’(x)| ≤ K, then |L’(x)| = |(f(b)-f(a))/(b-a)| ≤ K, and hence |f(b)-f(a)| ≤ K|b-a| as desired.
10. Because the Newton step is derived by fitting a tangent line to g(x), if you move a tiny amount in the direction of the Newton step, then you are guaranteed to make a reduction in the absolute value of g. Our approach is to scale the step by half until a function reduction is achieved. Replace the inner loop of the standard Newton step with the following procedure:  
    a. Let .

b. While (or the termination criteria are not met):  
 c.

c. Let

(There are many other strategies that would be acceptable, but all of them must take progressively smaller steps in order to ensure a reduction in |g|.)