Report on Coupled Linear Simple Pendulums

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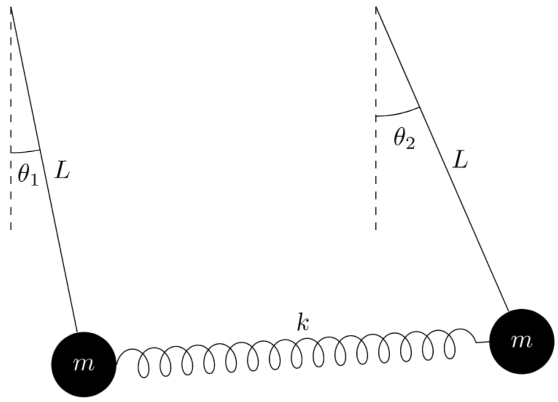
Introduction

The Coupled Simple Pendulums system consists of two pendula suspended from the same or different supports connected by a weightless structure with a weak spring nature. The spring-like structure is what causes the coupling as it allows energy transfer between the pendula which results in their motions no longer being independent of each other which must be taken into account when trying to understand the system's dynamics.



Since the way one pendulum moves affects the other, we expect this kind of coupling to be reflected in the equations of motion of the pendula as well in the form of some kind of mathematical coupling.

Equations of Motion

The total kinetic energy of the system T is

$$T = \frac{1}{2} m L^2 \left(\dot{\theta_1}^2 + \dot{\theta_2}^2 \right)$$

And the total potential energy of the system (gravitational and vibrational) is

$$V = -mgL \left(\cos\theta_1 + \cos\theta_2 \right) + \frac{1}{2} k L^2 \left(\sin\theta_1 - \sin\theta_2 \right)^2$$

Thus, the Lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m L^2 \left(\dot{\theta_1}^2 + \dot{\theta_2}^2 \right) + mgL \left(\cos\theta_1 + \cos\theta_2 \right) - \frac{1}{2} k L^2 \left(\sin\theta_1 - \sin\theta_2 \right)^2 \end{aligned}$$

Applying Hamilton's principle (The Principle of Least Action), we attempt to extremize the action S where

$$S = \int L \, dt$$

In order to do so, we make use of the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

Where in our case we have two sets of position and velocity coordinates: $\left(\theta_1, \dot{\theta_1} \right)$ and $\left(\theta_2, \dot{\theta_2} \right)$

Extremizing the system's action using the Euler-Lagrange for both sets of coordinates,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= mL^2\dot{\theta}_1 \\
\frac{\partial L}{\partial \theta_1} &= -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{d}{dt}\left(mL^2\dot{\theta}_1\right) &= -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1 \\
mL^2\ddot{\theta}_1 &= -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1
\end{aligned}
$$

Simplifying, we arrive at the θ_1 equation of motion

$$
\ddot{\theta}_1 = -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1
$$

Similarly for θ_2,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= mL^2\dot{\theta}_2 \\
\frac{\partial L}{\partial \theta_2} &= -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{d}{dt}\left(mL^2\dot{\theta}_2\right) &= -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2 \\
mL^2\ddot{\theta}_2 &= -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2
\end{aligned}
$$

$$
\ddot{\theta}_2 = -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2
$$

Thus we finally have the following **system of 2nd order coupled nonlinear differential equations** as the equations of motion for the pendula:

$$
\begin{aligned}
\ddot{\theta}_1 &= -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1 \\
\ddot{\theta}_2 &= -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2
\end{aligned}
$$

Linearization via the Small Angle Approximation

For small oscillations where the angular displacements are small $(\theta_1, \theta_2 \ll 1)$, we can linearize the system of differential equations by applying the **small angle approximations**

$$
\begin{aligned}
\sin\theta &\approx \theta \\
\cos\theta &\approx 1
\end{aligned}
$$

to obtain the following **system of 2nd order coupled linear differential equations**

$$
\begin{aligned}
\ddot{\theta}_1 &= -\frac{g}{L}\theta_1 - \frac{k}{m}(\theta_1 - \theta_2) \\
\ddot{\theta}_2 &= -\frac{g}{L}\theta_2 + \frac{k}{m}(\theta_1 - \theta_2)
\end{aligned}
$$

Numerical Solution

Converting each 2nd order differential equation into a set of differential equations

To solve these equations of motion numerically, we first need to convert each 2nd order differential equation

$$
\begin{aligned}
\ddot{\theta_1} &= -\frac{g}{L}\theta_1 - \frac{k}{m}(\theta_1 - \theta_2) \\
\ddot{\theta_2} &= -\frac{g}{L}\theta_2 + \frac{k}{m}(\theta_1 - \theta_2)
\end{aligned}
$$

into a 1st order system of differential equations upon which standard numerical methods like Euler and Runge-Kutta that exist for solving first order differential equations can be applied.

We do that by defining two new variables ω_1, ω_2

$$
\begin{aligned}
\omega_1(t) = \dot{\theta_1} \\
\omega_2(t) = \dot{\theta_2}
\end{aligned}
$$

Physically, ω_1, ω_2 correspond to the angular velocities of the pendula, i.e., the rate of change of angular displacements with time. With these variables introduced, the 2nd order DE system can be written as the following first order system of DEs

$$
\begin{aligned}
\dot{\theta_1} &= \omega_1 \\
\dot{\theta_2} &= \omega_2 \\
\dot{\omega_1} &= -\frac{g}{L}\theta_1 - \frac{k}{m}(\theta_1 - \theta_2) \\
\dot{\omega_2} &= -\frac{g}{L}\theta_2 + \frac{k}{m}(\theta_1 - \theta_2)
\end{aligned}
$$

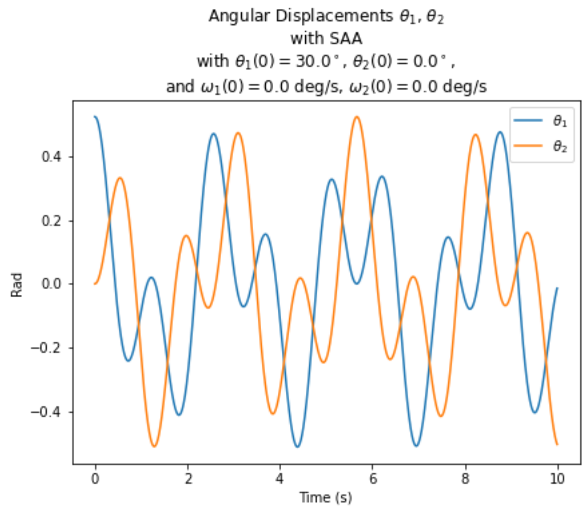
Now, we can employ the Runge-Kutta 4 method to solve this system numerically.

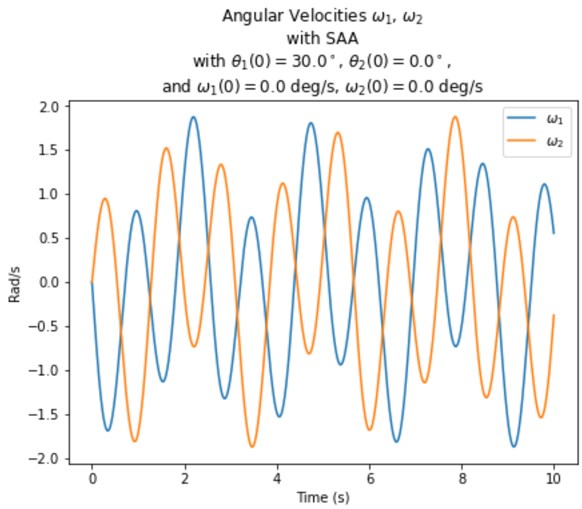
Results

In Python 3.10, we implemented RK4 to solve the DE system for pendula of equal length 2 meters and equal bob masses 1 kg with a spring constant of 10 N/m, iterating from 0 to 10 seconds for 10000 time steps with the initial conditions

$$
\begin{aligned}
\theta_1(0) = 30^\circ \\
\theta_2(0) = 0 \\
\omega_1(0) = 0 \\
\omega_2(0) = 0
\end{aligned}
$$

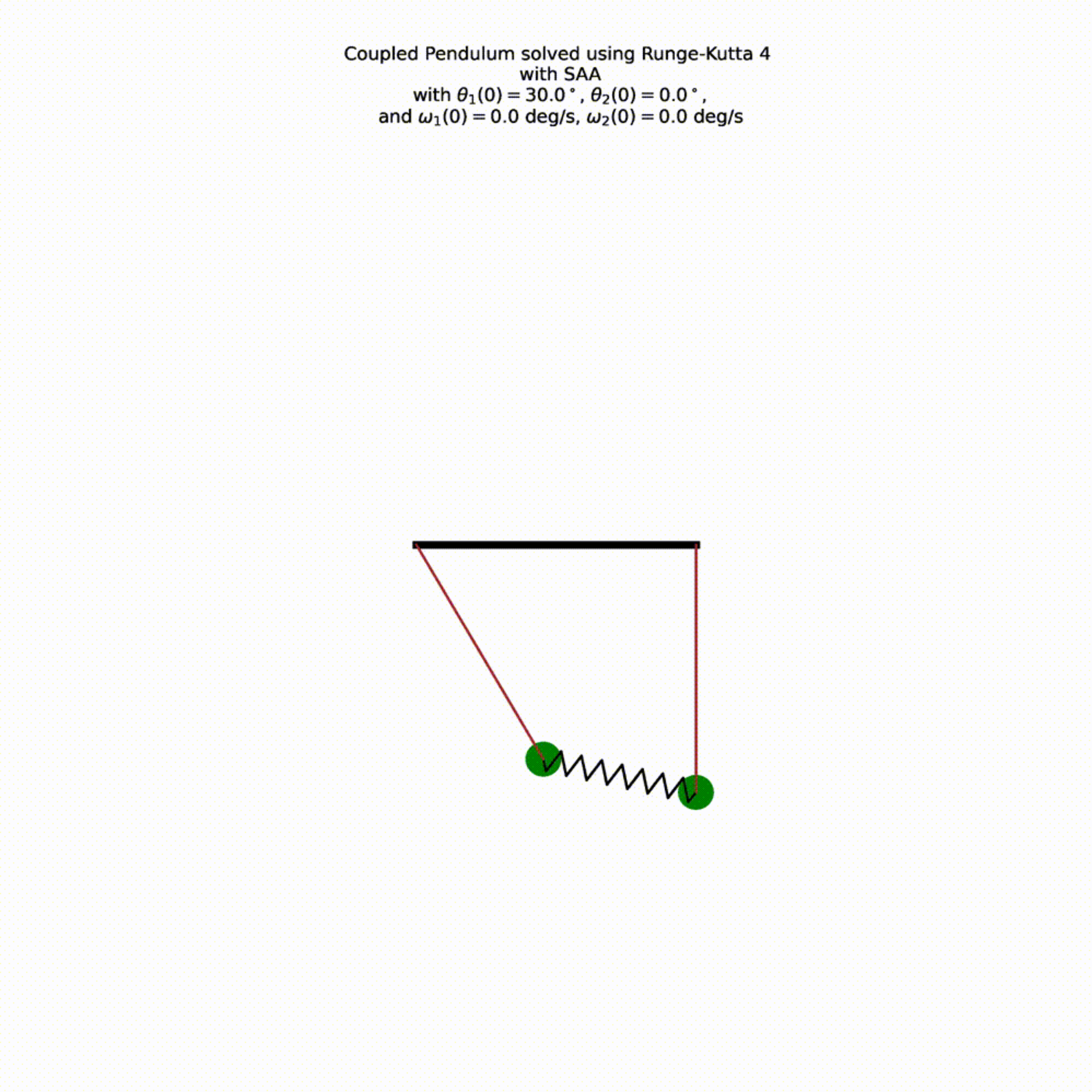
Upon solving, we obtain the following solutions for $\theta_1(t), \theta_2(t)$ and $\omega_1(t), \omega_2(t)$,





Observe that apart from the individual pendula themselves, there seems to be an overall oscillation in the state/configuration of the system in that after a certain time period, the system seems to return to the same state, similar to the simple pendulum system.

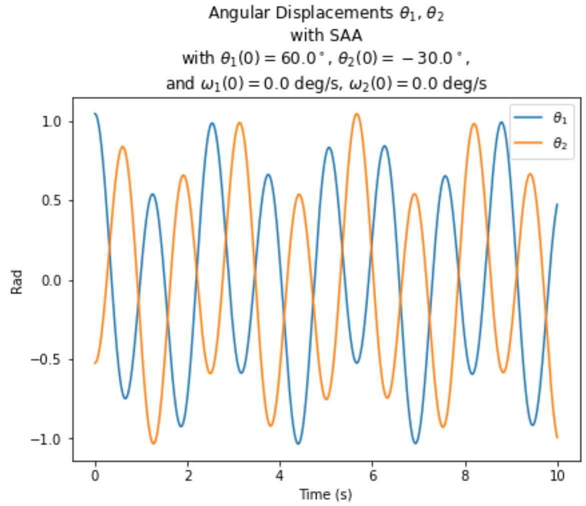
This would have been difficult to make out from just observing the physical system qualitatively. To illustrate that point further, we used Python to create a handy simulation that simulates and visualizes the pendulums' movements over time. The full simulation as a .gif file can be found here.

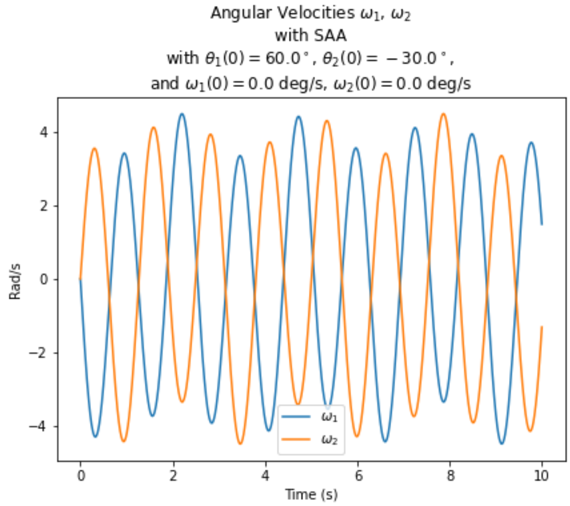


A couple more solutions for different initial conditions are presented below.

For

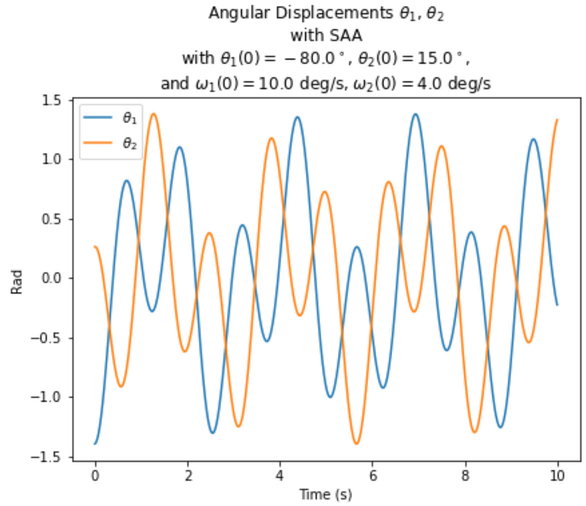
$$\theta_1(0) = 60^\circ$$
$$\theta_2(0) = -30^\circ$$
$$\omega_1(0) = 0 \text{ deg}/s$$
$$\omega_2(0) = 0 \text{ deg}/s$$

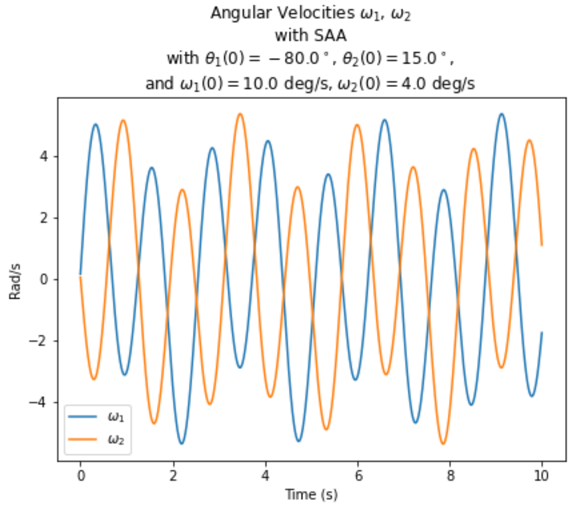




For

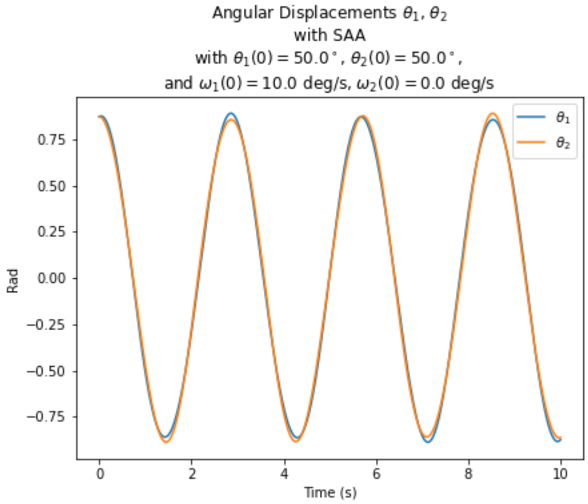
$$\theta_1(0) = -80^\circ$$
$$\theta_2(0) = 15^\circ$$
$$\omega_1(0) = 10 \text{ deg}/s$$
$$\omega_2(0) = 4 \text{ deg}/s$$

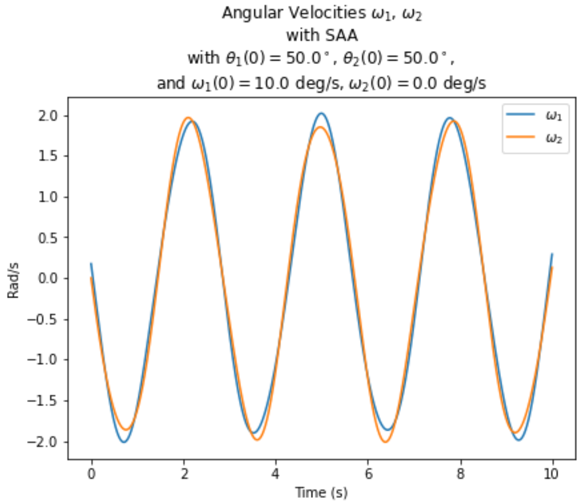




For

$$\theta_1(0) = 50^\circ$$
$$\theta_2(0) = 50^\circ$$
$$\omega_1(0) = 10 \text{ deg}/s$$
$$\omega_2(0) = 0 \text{ deg}/s$$





References

The code written for the analysis and simulations presented above is located in this Jupyter Notebook:

Simple-and-coupled-pendulums/Coupled Pendulum - Runge-Kutta.ipynb at main · BK-Modding/Simple-and-coupled-pendulums

Jupyter Notebooks written while attempting to understand and solve the Simple Pendulum and Coupled Pendulum systems. - Simple-and-coupled-pendulums/Coupled Pendulum - Runge-Kutta.ipynb at main · BK...

https://github.com/BK-Modding/Simple-and-coupled-pendulums/blob/main/Coupled%20Pendulum%20-%20Runge-Kutta.ipynb