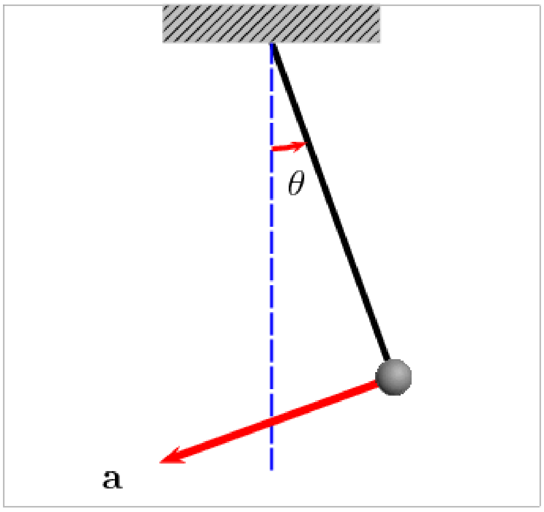
Report on the Nonlinear Simple Pendulum

Introduction

The Simple Pendulum system consists of a bob tied to a string that is suspended from a support. Upon supplying an initial acceleration to the pendulum, it starts to exhibit oscillatory behavior due to the opposing natures of the bob's momentum that tends to push the bob away from the mean position (middle) and the force of gravity acting as a restoring force that tends to pull the bob towards the mean position.



Equation of Motion

The equation of motion for the simple pendulum system is a **2nd order nonlinear differential equation**:

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0$$

Analytical Solution

Linearization via the Small Angle Approximation

The most common approach is to linearize the differential equation by consider only small oscillations, i.e., ones where the angular displacement is small enough $(\theta \ll 1)$ that the **small angle approximation** $\sin \theta \approx \theta$ can be used to reduce the differential equation to a very well known and studied **2nd order linear differential equation**, namely that of the **harmonic oscillator**

$$\ddot{\theta} + \frac{g}{L} \theta = 0$$

Common techniques to solve 2nd order linear DEs can solve this pendulum harmonic oscillator to produce the following closed-form, analytic solution

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right)$$

Nonlinear case

If one does not linearize, the nonlinear differential equation of motion does not have a closed form solution that involves elementary functions, but instead must be solved numerically using a computer.

Numerical Solution

To solve the equation of motion numerically, we first need to convert the 2nd order differential equation

$$\ddot{\theta} + \frac{g}{L}\sin(\theta(t)) = 0$$

into a 1st order system of differential equations upon which standard numerical methods like Euler and Runge-Kutta that exist for solving first order differential equations can be applied.

We do that by defining a new variable ω

$$\omega(t) = \dot{\theta}$$

Physically, $\omega(t)$ corresponds to the angular velocity of the system, i.e., the rate of change of angular displacement with time.

With this new variable introduced, the 2nd order DE can be written as the following first order system of DEs

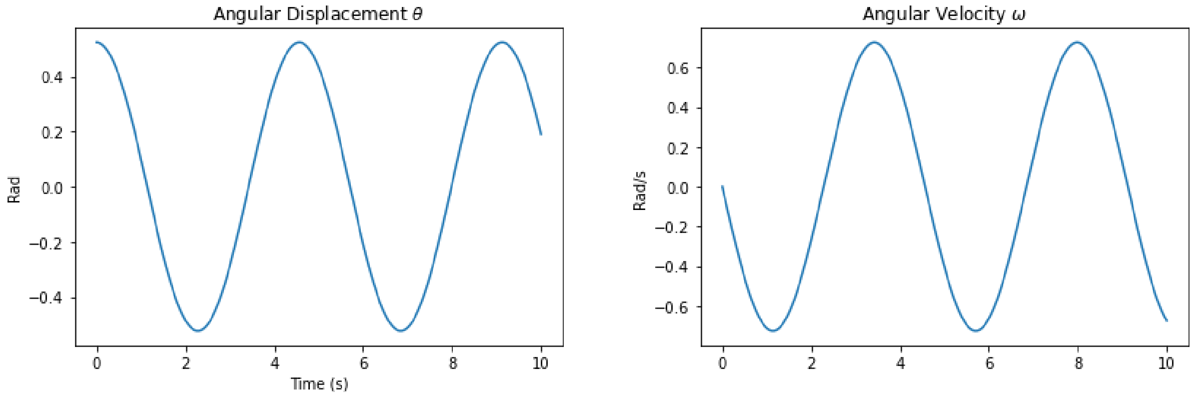
$$\begin{aligned}\dot{\theta} &= \omega(t)\\ \dot{\omega} &= -\frac{g}{L}\sin(\theta(t))\end{aligned}$$

Now, we employ the Runge-Kutta 4 method to solve this system numerically.

In Python 3.10, we implemented RK4 to solve the DE system for a pendulum of length 5 meters, iterating from 0 to 10 seconds for 10000 time steps with the initial conditions:

$$\begin{aligned}\theta(0) &= 30^\circ\\ \omega(0) &= 0\end{aligned}$$

Upon solving, we obtained the following plots for $\theta(t)$ and $\omega(t)$,



As expected, the solutions are sinusoidal in nature which is consistent with the results we obtain when employing the small angle approximation. However, these solutions are more precise.