Report on Coupled Nonlinear Simple Pendulums

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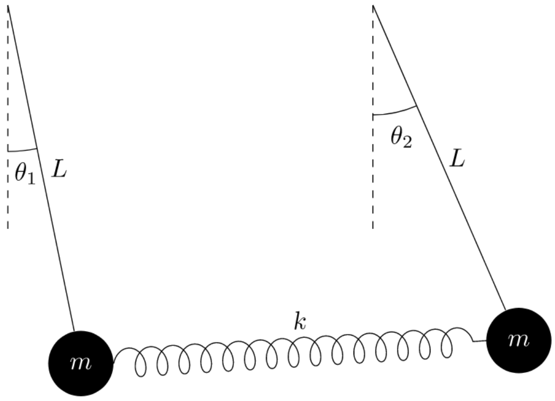
Introduction

The Coupled Simple Pendulums system consists of two pendula suspended from the same or different supports connected by a weightless structure with a weak spring nature. The spring-like structure is what causes the coupling as it allows energy transfer between the pendula which results in their motions no longer being independent of each other which must be taken into account when trying to understand the system's dynamics.



Since the way one pendulum moves affects the other, we expect this kind of coupling to be reflected in the equations of motion of the pendula as well in the form of some kind of mathematical coupling.

Equations of Motion

The total kinetic energy of the system T is

$$T = \frac{1}{2} m L^2 \left(\dot{\theta_1}^2 + \dot{\theta_2}^2 \right)$$

And the total potential energy of the system (gravitational and vibrational) is

$$V = -mgL \left(\cos\theta_1 + \cos\theta_2 \right) + \frac{1}{2} k L^2 \left(\sin\theta_1 - \sin\theta_2 \right)^2$$

Thus, the Lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m L^2 \left(\dot{\theta_1}^2 + \dot{\theta_2}^2 \right) + mgL \left(\cos\theta_1 + \cos\theta_2 \right) - \frac{1}{2} k L^2 \left(\sin\theta_1 - \sin\theta_2 \right)^2 \end{aligned}$$

Applying Hamilton's principle (The Principle of Least Action), we attempt to extremize the action S where

$$S = \int L \, dt$$

In order to do so, we make use of the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

Where in our case we have two sets of position and velocity coordinates: $\left(\theta_1, \dot{\theta_1}\right)$ and $\left(\theta_2, \dot{\theta_2}\right)$

Extremizing the system's action using the Euler-Lagrange for both sets of coordinates,

$$\frac{\partial L}{\partial \dot{\theta}_1} = mL^2\dot{\theta}_1$$
$$\frac{\partial L}{\partial \theta_1} = -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$

$$\therefore \frac{d}{dt}\left(mL^2\dot{\theta}_1\right) = -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$
$$mL^2\ddot{\theta}_1 = -mgL\sin\theta_1 - kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$

Simplifying, we arrive at the θ_1 equation of motion

$$\ddot{\theta}_1 = -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$

Similarly for θ_2,

$$\frac{\partial L}{\partial \dot{\theta}_2} = mL^2\dot{\theta}_2$$
$$\frac{\partial L}{\partial \theta_2} = -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$

$$\therefore \frac{d}{dt}\left(mL^2\dot{\theta}_2\right) = -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$
$$mL^2\ddot{\theta}_2 = -mgL\sin\theta_2 + kL^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$

$$\ddot{\theta}_2 = -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$

Thus we finally have the following **system of 2nd order coupled nonlinear differential equations** as the equations of motion for the pendula:

$$\ddot{\theta}_1 = -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$
$$\ddot{\theta}_2 = -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$

Numerical Solution

Converting each 2nd order differential equation into a set of differential equations

To solve these equations of motion numerically, we first need to convert each 2nd order differential equation

$$\ddot{\theta}_1 = -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1$$
$$\ddot{\theta}_2 = -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2$$

into a 1st order system of differential equations upon which standard numerical methods like Euler and Runge-Kutta that exist for solving first order differential equations can be applied.

We do that by defining two new variables ω_1, ω_2

$$\omega_1(t) = \dot{\theta}_1$$
$$\omega_2(t) = \dot{\theta}_2$$

Physically, ω_1, ω_2 correspond to the angular velocities of the pendula, i.e., the rate of change of angular displacements with time. With these variables introduced, the 2nd order DE system can be written as the following first order system of DEs

$$
\begin{aligned}
\dot{\theta_1} &= \omega_1 \\
\dot{\theta_2} &= \omega_2 \\
\dot{\omega_1} &= -\frac{g}{L}\sin\theta_1 - \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_1 \\
\dot{\omega_2} &= -\frac{g}{L}\sin\theta_2 + \frac{k}{m}(\sin\theta_1 - \sin\theta_2)\cos\theta_2
\end{aligned}
$$

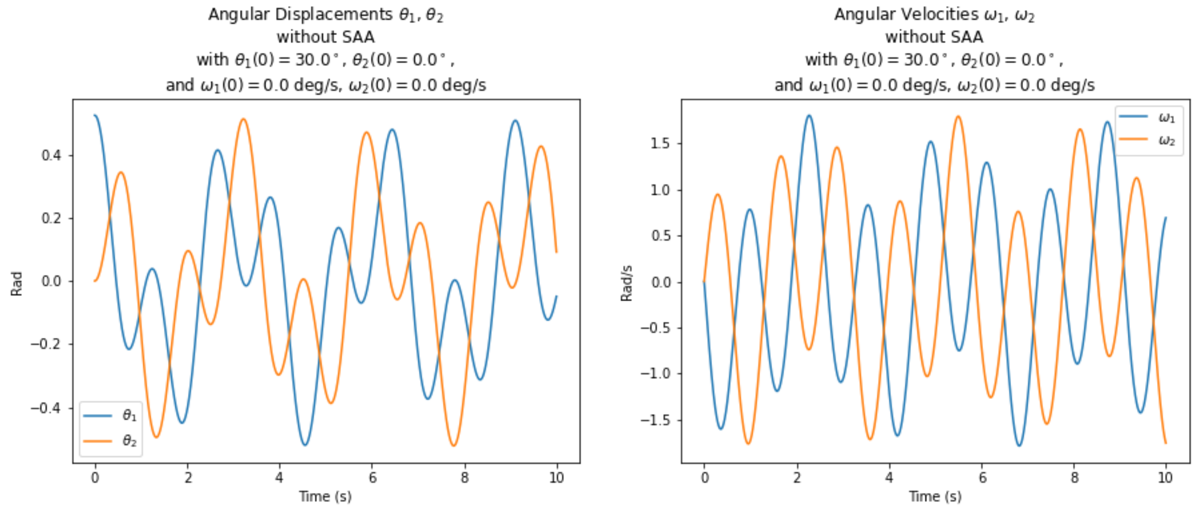
Now, we can employ the Runge-Kutta 4 method to solve this system numerically.

Results

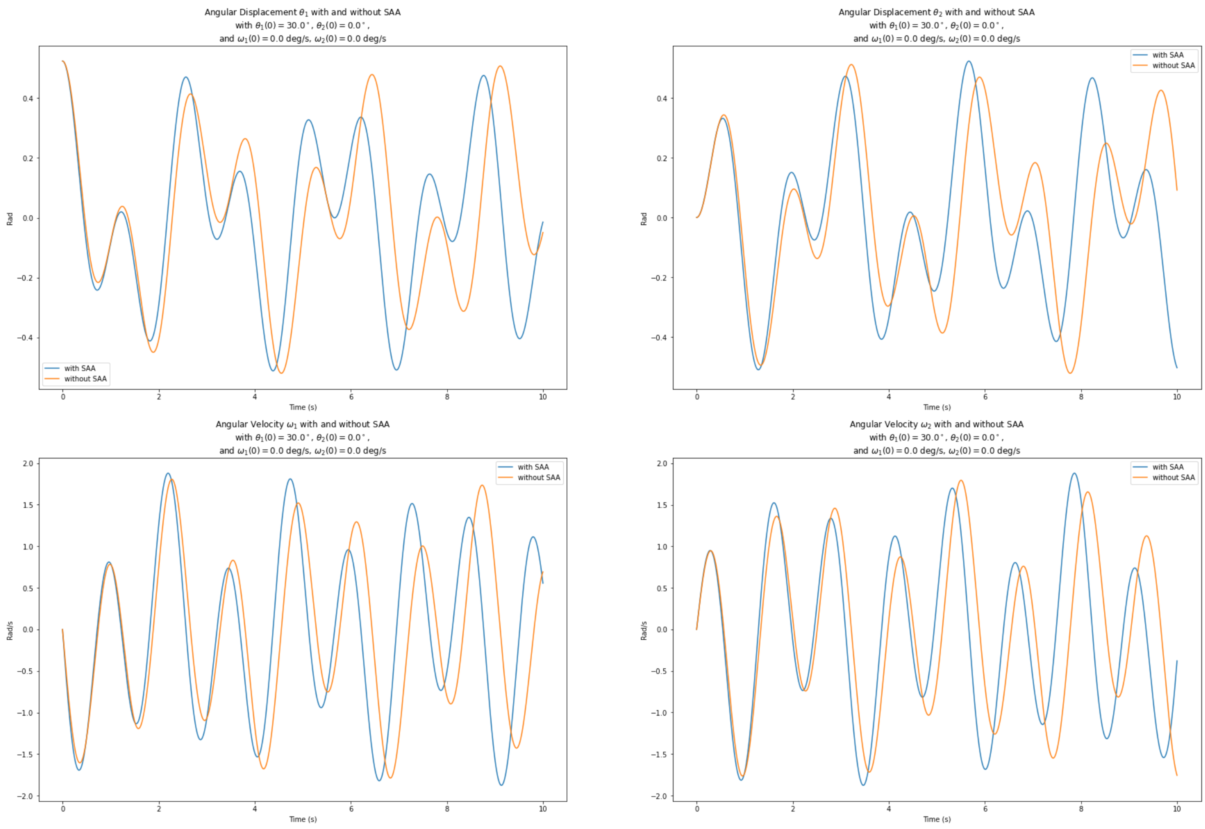
In Python 3.10, we implemented RK4 to solve the DE system for pendula of equal length 2 meters and equal bob masses 1 kg with a spring constant of 10 N/m, iterating from 0 to 10 seconds for 10000 time steps with the initial conditions

$$
\begin{aligned}
\theta_1(0) &= 30^\circ \\
\theta_2(0) &= 0 \\
\omega_1(0) &= 0 \\
\omega_2(0) &= 0
\end{aligned}
$$

Upon solving, we obtain the following solutions for $\theta_1(t), \theta_2(t)$ and $\omega_1(t), \omega_2(t)$,

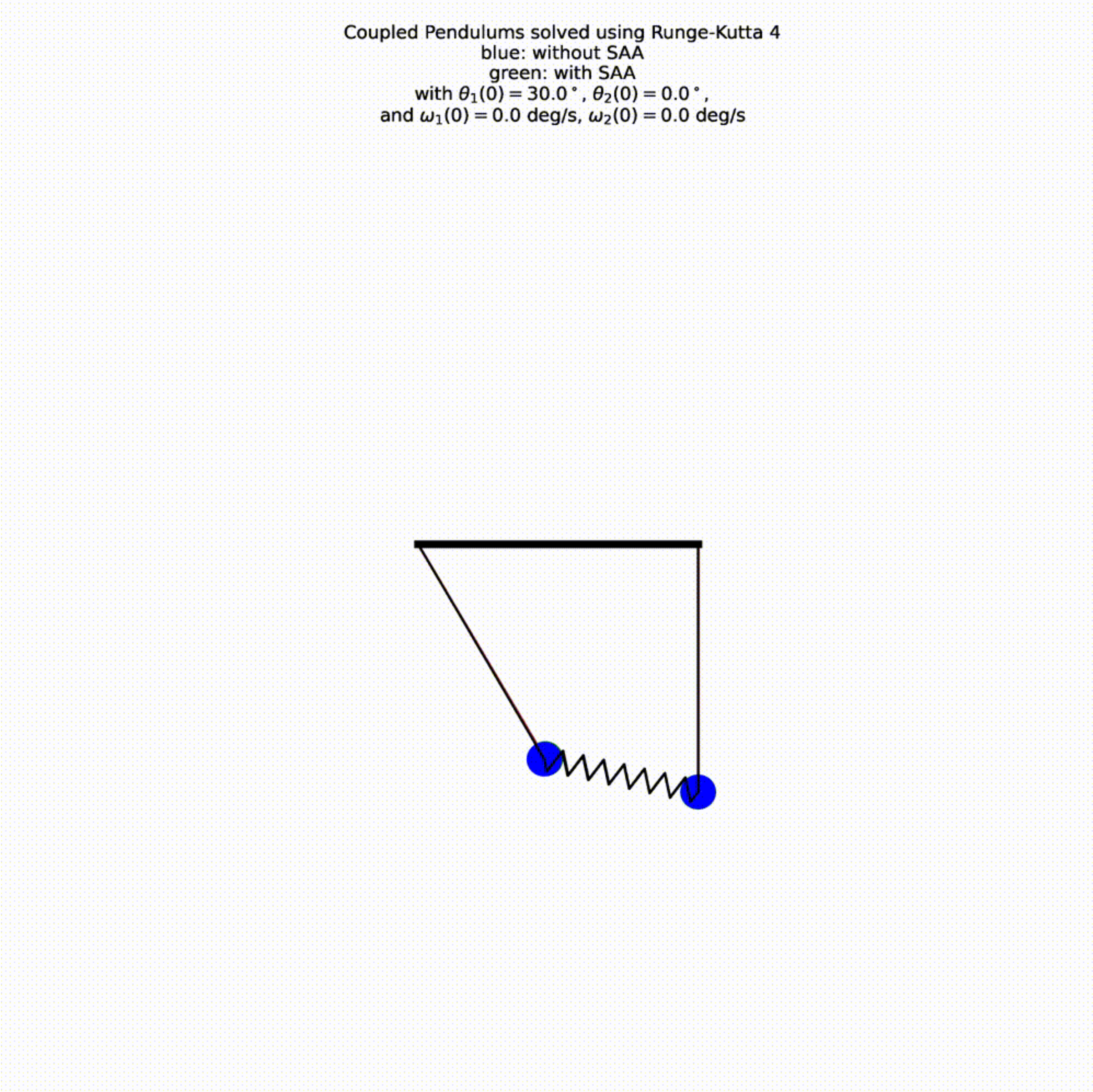


Usually, this system of DEs would have been solved analytically/numerically by linearizing via the small angle approximation (SAA). Let's compare the solutions obtained with and without linearization to see how accurate SAA is in this case.



The differences between the solutions are initially negligible but notice that as time goes on the differences get larger. This is due to the nonlinear nature of the coupled pendulums motions creeping in, because as time increases the nonlinear effects compound and become more significant, making it difficult to ignore them when predicting the system's behavior.

To illustrate this further, we used Python to create a handy simulation that details this increase of nonlinear influence over time visually by simulating the motion of two coupled pendulum systems each with and without SAA. The full simulation as a .gif file can be found here.

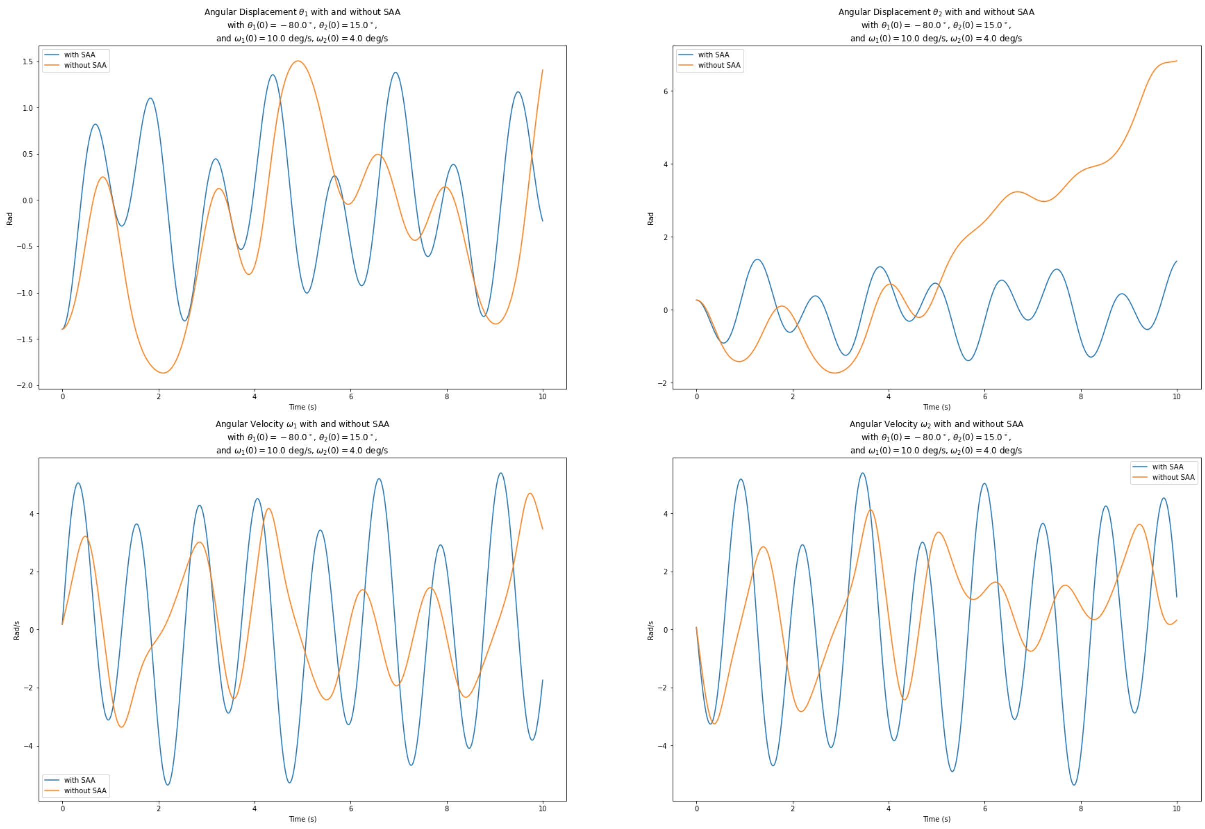


Apart from the passage of time, the nonlinear effects of the system become important when the oscillations are not small enough that SAA can be applied without large error.

As a demonstration, we solve the DE system again for all the same parameters except now we change the initial conditions to be

$$
\begin{aligned}
\theta_1(0) &= -80^\circ \\
\theta_2(0) &= 15^\circ \\
\omega_1(0) &= 10 \text{ deg}/s \\
\omega_2(0) &= 4 \text{ deg}/s
\end{aligned}
$$

Thereby increasing one of the initial displacements (and therefore the maximum amplitude of oscillations) outside the range of applicability of SAA and also supplying initial velocities. Then the solutions we obtain are



Which clearly shows that for this non-negligible set of initial conditions, the solutions diverge very quickly and also to a large degree, proving why SAA has the criteria that it does for using it.

References

The code written for the analysis and simulations presented above is located in this Jupyter Notebook:

Simple-and-coupled-pendulums/Coupled Pendulum - Runge-Kutta.ipynb at main · BK-Modding/Simple-and-coupled-pendulums

Jupyter Notebooks written while attempting to understand and solve the Simple Pendulum and Coupled Pendulum systems. - Simple-and-coupled-pendulums/Coupled Pendulum - Runge-Kutta.ipynb at main · BK...

https://github.com/BK-Modding/Simple-and-coupled-pendulums/blob/main/Coupled%20Pendulum%20-%20Runge-Kutta.ipynb

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0 Issues