HW3 Collaborative

CS40 Spring '22

Due: Monday, April 26, 2022 at 11:59PM on Gradescope

For all Collaborative HW assignments:

Collaborative homeworks may be done individually or in groups of up to 4 students. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission.

All submitted homework for this class must be typed. Diagrams may be hand-drawn and scanned and included in the typed document. You can use a word processing editor if you like (Microsoft Word, Open Office, Notepad, Vim, Google Docs, etc.) but you might find it useful to take this opportunity to learn LaTeX. LaTeX is a markup language used widely in computer science and mathematics. The homework assignments are typed using LaTeX and you can use the source files as templates for typesetting your solutions.[1]

Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.
- You may not collaborate on homework with anyone other than your group members. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza. You *cannot* use any online resources about the course content other than the text book and class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions we will use this quarter.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW3-Collaborative".

[1] To use this template, you will need to copy both the source file (extension `.tex`) you'll be editing and the file containing all the "shortcut" commands we've defined for this class)

Summary of Proof Strategies (so far)

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion**: To prove that $\forall x\, P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization**: To prove that $\forall x\, P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.
- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Proof of Conditional by Direct Proof**: To prove that the implication $p \to q$ is true, we can assume p is true and use that assumption to show q is true.

Assigned Questions

1. For each of these arguments, identify what rule of inference is used.[2]

   (a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
   Simplification

   (b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
   Disjunctive syllogism

   (c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
   Modus ponens

   (d) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.
   Hypothetical syllogism

2. For each of these arguments, determine whether the argument is correct or incorrect and explain why. If the argument is correct, state which rule of inference is used, and if it is incorrect, give a reasonable explanation for why it is incorrect.

   (a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory.Therefore, Mia is not enrolled in the university.
   Correct. It uses Modus tollens

   (b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
   Incorrect. A convertible car is fun to drive. This does not imply that all fun cars must be convertible. It is possible that Isaac's car is not a convertable but is nonetheless fun to drive.

   (c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, Eight Men Out is an action movie.
   Incorrect. Quincy liking a movie does not imply it must be action. Quincy is able to like movie besides action movies. Therefore we can not infer that Eight Men Out is an action movie

   (d) They allow you into the dining commons only if you have a green badge. Olivia is inside the dining commons. Therefore, Olivia has a green badge.
   Correct. It uses modus ponens

[2] See zyBooks 3.5 for list of rules of inference

3. Is the following argument valid or invalid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid using truth tables.

   Solutions: p = get a promotion, s = buy stereo, g = buy sunglasses

$$\begin{array}{ll} s \wedge g \rightarrow p & \\ \neg p & \\ g & \\ \hline \therefore & \neg s \end{array}$$

p	s	g	$s \wedge g \rightarrow p$	$\neg p$	$(s \wedge g \rightarrow p) \wedge (\neg p) \wedge g$	$\neg s$
T	T	T	T	F	F	F
T	T	F	T	F	F	F
T	F	T	T	F	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

   Conclusion: The form is valid, as the resolution is true when the hypotheses are all true (second to last row in the table above).

4. The domain for variables x and y is a group of people P. The predicate $F(x, y)$ is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, $\neg F(x, y)$ means that x is an enemy of y.

   Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

   (a) Everyone has an enemy.
   $\forall x \exists y F(x, y)$ Logical Expression
   $\neg \forall x \exists y F(x, y)$ Negation
   $\exists x \forall y \neg F(x, y)$ Simplified
   There is someone with no enemies. English

   (b) Everyone is their own friend.
   $\forall x F(x, x)$ Logical Expression
   $\neg \forall x F(x, x)$ Negation
   $\exists x \neg F(x, y)$ Simplified
   There is someone who is not friends with themselves. English

   (c) At least two different people are friends.
   $\exists x \exists y (F(x, y) \wedge x \neq y)$ Logical Expression
   $\neg \exists x \exists y (F(x, y) \wedge x \neq y)$ Negation
   $\forall x \forall y \neg (F(x, y) \wedge x \neq y)$ Simplification step
   $\forall x \forall y (\neg F(x, y) \vee \neg (x \neq y))$ Simplification step
   $\forall x \forall y (\neg F(x, y) \vee x = y)$ Simplified
   Everyone is friends with nobody but themselves. English

   (d) "The enemy of my enemy is my friend"[3].

   Solutions:

$$
\begin{aligned}
\text{Logical expression: } & \forall x \forall y \forall z \ \ (\neg F(x, y) \wedge \neg F(y, z)) \rightarrow F(x, z) \\
\text{Negation: } & \neg \forall x \forall y \forall z \ \ (\neg F(x, y) \wedge \neg F(y, z)) \rightarrow F(x, z) \\
\equiv \ & \exists x \exists y \exists z \ \ \neg((\neg F(x, y) \wedge \neg F(y, z)) \rightarrow F(x, z)) \\
\equiv \ & \exists x \exists y \exists z \ \ \neg((F(x, y) \vee F(y, z)) \vee F(x, z))) \\
\equiv \ & \exists x \exists y \exists z \ \ (\neg F(x, y) \wedge \neg(F(y, z) \wedge \neg F(x, z))) \\
\text{Translation to English: } & \text{Someone is enemies with their enemy's enemy.}
\end{aligned}
$$

[3] For all people in the domain, enemies of their enemies are their friends

5. Assuming that the domains of all quantifiers are the same, use rules of inference to show that vxcz

$$\text{if } \forall x(P(x) \vee Q(x)) \wedge \forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x)),$$

$$\text{then } \forall x(\neg R(x) \rightarrow P(x)).$$

$$\forall x(P(x) \vee Q(x)) \wedge \forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$$
$$\forall x((P(x) \vee Q(x)) \wedge \neg P(x) \wedge Q(x)) \rightarrow R(x))$$
$$\forall x(\neg P(x) \rightarrow R(x))$$
$$\forall x(\neg R(x) \rightarrow P(x))$$

Solutions: Show that $\forall x(P(x) \vee Q(x)) \wedge \forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x)) \rightarrow \forall x(\neg R(x) \rightarrow P(x))$:

$$\begin{array}{lll}
c \text{ is an arbitrary element} & \forall x(P(x) \vee Q(x)) \wedge \forall x((\neg P(x) \wedge Q(x) \rightarrow R(x)) & \\
& \Longrightarrow (P(c) \vee Q(c)) \wedge ((\neg P(c) \wedge Q(c)) \rightarrow R(c)) & \text{Universal Instantiation} \\
& \equiv (P(c) \vee Q(c)) \wedge (\neg(\neg P(c) \wedge Q(c)) \vee R(c)) & \text{Conditional Rule} \\
& \equiv (P(c) \vee Q(c)) \wedge ((P(c) \vee \neg Q(c)) \vee R(c)) & \text{De Morgan's Law} \\
& \equiv (P(c) \vee Q(c)) \wedge (P(c) \vee [\neg Q(c) \vee R(c)]) & \text{Associative Law} \\
& \equiv P(c) \vee (Q(c) \wedge (\neg Q(c) \vee R(c))) & \text{Distributive Law} \\
& \equiv P(c) \vee [(Q(c) \wedge \neg Q(c)) \vee (Q(c) \wedge R(c))] & \text{Distributive Law} \\
& \equiv P(c) \vee [F \vee (Q(c) \wedge R(c))] & \text{Complement Law} \\
& \equiv [P(c) \vee Q(c)] \wedge [P(c) \vee R(c)] & \text{Distributive Law} \\
& \Longrightarrow P(c) \vee R(c) & \text{Simplification} \\
& \equiv R(c) \vee P(c) & \text{Commutative Law} \\
& \equiv \neg R(c) \rightarrow P(c) & \text{Conditional Identity} \\
& \therefore \forall x \;\; \neg R(x) \rightarrow P(x) & \text{Universal Generalization}
\end{array}$$

6. Consider the predicate $F(a, b) =$ "a is a factor of b" over the domain $\mathbb{Z}^{\neq 0} \times \mathbb{Z}$ that was introduced in lecture. Consider the following quantified statements

(i) $\forall x \in \mathbb{Z}\ (F(1, x))$

(ii) $\forall x \in \mathbb{Z}^{\neq 0}\ (F(x, 1))$

(iii) $\exists x \in \mathbb{Z}\ (F(1, x))$

(iv) $\forall x \in \mathbb{Z}^{\neq 0}(\neg F(x, 1))$

(v) $\forall x \in \mathbb{Z}^{\neq 0}\ \exists y \in \mathbb{Z}\ (F(x, y))$

(vi) $\exists x \in \mathbb{Z}^{\neq 0}\ \forall y \in \mathbb{Z}\ (F(x, y))$

(vii) $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}^{\neq 0}\ (F(x, y))$

(viii) $\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}^{\neq 0}\ (F(x, y))$

(a) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **proved** by the following proof?

> By universal generalization, **choose** e to be an **arbitrary** integer. We need to show $\exists y \in \mathbb{Z}^{\neq 0}(F(y,e))$. By definition of the predicate F, we can rewrite this goal as $\exists y \in \mathbb{Z}^{\neq 0} \exists c \in \mathbb{Z}\ (e = c \cdot y)$. We pick the **witnesses** $y = 1$ and $c = e$. y is a non-zero integer and therefore in the domain. Similarly, c is an integer and therefore in the domain. Plugging the value of the witnesses y and c, we get $c \cdot y = e \cdot 1 = e$, as required. Since the predicate $\exists y \in \mathbb{Z}^{\neq 0}(F(y,e))$ evaluates to true for the arbitrary integer e, the claim has been proved.
>
> ■

Hint: It may be useful to identify the key words in the proof that indicate proof strategies.

Solutions: (vii) Universal generalization eliminates the $\forall y$ and arbitrary element e replaces y to reach $\exists y \in \mathbb{Z}^{\neq 0}(F(y,e))$, matching the proof.

(b) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **disproved** by the following proof?

> To disprove the statement, we need to find a counterexample. We choose -1, which is a nonzero integer so in the domain. We need to show $F(-1,1)$. By definition of the predicate F, we can rewrite this goal as $\exists c \in \mathbb{Z}\ (1 = c \cdot -1)$. We pick the **witness** $c = -1$, which is an integer and therefore in the domain. Plugging the value of the witness c, we get $c \cdot -1 = -1 \cdot -1 = 1$, as required. So the counterexample works to disprove the original statement. ■

Hint: It may be useful to identify the key words in the proof that indicate proof strategies.

Solutions: (iv) Option 4 is the only one negating the function F (expecting all *false*), and the proof is attempting to disprove a statement with a counterexample evaluating to true (implying that all inputs should not evaluate to *false*).

(c) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and proof*) Translate the following statement to English and then prove or disprove it:

$$\forall x \in \mathbb{Z}^{\neq 0}\ \forall y \in \mathbb{Z}^{\neq 0}\ (\ F(x,y) \to F(x, x+y)\ )$$

Proof: Assume a and b to be arbitrary non-zero integers. We need to show $\forall x \in \mathbb{Z}^{\neq 0}\ \forall y \in \mathbb{Z}^{\neq 0}\ (\ F(x,y) \to F(x,x+y)\ )$ Towards proof by contrapositive, assume $\neg F(a, a+b)$. We need to show $\neg F(a,b)$

$\neg F(a,a+b) \equiv$	$a + b = c \cdot a + d,\ 0 < d < a$	Division Theorem
$\equiv$	$a + b - a = c \cdot a + d - a,\ 0 < d < a$	Subtracting a from both sides
$\equiv$	$b = (c-1) \cdot a + d,\ 0 < d < a$	Distributive Property
$\equiv$	$b = k \cdot a + d,\ k = c - 1,\ 0 < d < a$	Define k as $c - 1$
$\equiv$	$\neg F(a,b)$	■

7. Let $P = \{a : a \in Z^+, a \leq 50\}$ and $F(a, b)$ be the predicate defined in the previous question. We want to list all pairs (a, b) of integers in P such that a is a factor of b. In other words, list all the elements of the set $\{(a, b) \in P \times P \mid F(a, b)\}$

   Listing all such pairs is time-consuming, so we have decided to start with an arbitrarily created initial set $K \subseteq P \times P$, defined as $K = \{(1, 2), (1, 4), (2, 4), (3, 9)\}$.

   We know that the set K does not contain all possible pairs of factors – that is, we know that there exist other pairs $(a, b) \notin K$ such that $F(a, b)$, but we do not want to have to list them all out.

   To guess, we make the following assumption about integer pairs that are related through factoring:
   (i) for all a, $b \in P$, if a is a factor of b then a is a factor of ab,
   (ii) For all a, b, c in P such that $a \neq c$, $a \neq b$ and $b \neq c$, if a is a factor of b and b is a factor of c then a is a factor of c, and
   (iii) For all a, b, c in P such that $a \neq c$, $a \neq b$ and $b \neq c$, if a is a factor of b and a is a factor of c then a is a factor of $b + c$

   (a) Make a recursive definition for a set $L \subseteq P \times P$ that captures the following notion: $(a, b) \in L$ if and only if we can deduce from K and our assumptions (i), (ii), (iii) that a is a factor of b.
   $K \subseteq L$
   $\forall (a, b) \in P \times P, (a, b) \in L \rightarrow (a, ab) \in$ L
   $\forall (a, b) \in P \times P, (a, b) \in L \wedge (b, c) \in L \wedge a \neq c \wedge a \neq b \wedge b \neq c \rightarrow (a, c) \in L$
   $\forall (a, b) \in P \times P, (a, b) \in L \wedge (a, c) \in L \wedge a \neq c \wedge a \neq b \wedge b \neq c \rightarrow (a, b + c) \in L$

   (b) List five example elements in the set $\{(a, b) \in P \times P \mid F(a, b)\}$ that are each not in the set $L \subseteq P \times P$ as per your recursive definition in (7a). Write your answer in roster notation.
   {(5,10), (7,14), (11,22), (13,26), (17,34)}

   (c) Modify the recursive definition from (7a) so that that the resulting set contains all pairs of numbers in $P \times P$, where the first number is a factor of the second number. The same constraints as part (7a) apply here with the exception that in this question you may use a different set of assumptions than the ones provided as a starting guess. Write your solution in a way so that it generalizes well to any different choice of the sets P and K. Note that although non-recursive definitions are possible, you must provide a recursive definition to receive credit for this question.
   $\forall b \in P : (b, b) \in J$
   $J \subseteq L$
   $\forall (a, b) \in P \times P, \forall c \in P, (((a, b) \in L) \wedge ac \in P) \rightarrow (a, ac) \in$ L

   (d) Critique your solution for part(c) by discussing whether it generalizes to any choice of the sets $P \subseteq \mathbb{Z}^+$ and $K \subseteq P \times P$, and if so under what assumptions.
   This recursive definition generalizes to set K. The recursive definition defines all pairs (a,b) that fall within P × P, which is set K.

Attributions

Some of the problems on this homework are based on questions originally created by Mia Minnes, Joe Politz, and Daniel Lokshtanov.