HW7 Collaborative

CS40 Spring '22

Due: Wednesday, June 1, 2022 at 11:59PM on Gradescope

In this assignment,

You will work with recursively defined sets and functions and prove properties about them, practicing induction, contradiction, and other proof strategies. You will also practice applying functions to counting problems and related proofs.

For all HW assignments:

Please see the instructions and policies for assignments on the class website and on the writeup for HW1. In particular, these policies address

- Collaboration policy
- Where to get help
- Typing your solutions
- Expectations for full credit

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW7-Collaborative".

In your proofs and disproofs of statements below, justify each step by reference to the proof strategies we have discussed so far, and/or to relevant definitions and calculations. We include only induction-related strategies here; you can and should refer to past material to identify others.

Assigned Questions

1. Prove the product rule: If A and B are finite sets $|A \times B| = |A||B|$. Hint: Find a witness bijective function $f : A \times B \to \{1, 2, 3, ...|A| \cdot |B|\}$.

   Answer:
   let i(x) define the index of x within a set. Indices begin from 0.
   We define f as f(a,b)=(i(a)*$|B|$ + i(b))
   Proof: f is a bijective function
   Goal 1: Show that f is well-defined
   Any arbitrary tuple of a and b maps to exactly one value y $\in N$
   $i(a) \geq 0 \wedge i(b) \geq 0$
   $f(a,b) \geq 0$
   (i(a)*$|B|$ + i(b)) ≥ 0
   By property of modular expressions, we know the LHS of this equation must result in a unique value for unique value pairs of i(a) and i(b) since i(b)$< |B|$. So we have proven for arbitrary tuple (a,b), f(a,b) = unique value$\in N_{<|A||B|}$
   Goal 2: Show that f is one to one
   $\forall (a,b) \in A \times B, \forall (c,d) \in A \times B, (f(a,b) = f(c,d)) \to ((a,b) = (c,d))$
   $f(a,b) = f(c,d)$
   $(i(a) * |B| + i(b) = i(c) * |B| + i(d))$
   We know that b and d are both from set B from our definitions. This means that
   $(i(b) < |B|) \wedge (i(d) < |B|)$.
   Since b and d are both smaller than $|B|$, in order for the equality to hold, i(b)=i(d).
   $(i(a) * |B| + i(b) = i(c) * |B| + i(d))$ (by property of modular expression)
   $(i(b) = i(d))$
   Assuming the above,
   $(i(a) * |B| = i(c) * |B|)$
   $(i(a) = i(c))$
   This proves the function is one-to-one.
   Goal 3: Show that f is onto
   $\forall m \in N_{<|A||B|} \exists (n,o) \in A \times B, m = f(n,o)$
   For arbitrary integer m, there exists an ordered pair (n,o) where f((n,o))=m. By the property of the modular operation, any integer can be expressed in the form a=qx+r where q is fixed and x and r correspond to the number's value. In this case, q=$|B|$, x=n, r=o.

2. Given the definition of the set of all linked lists of natural numbers L and the function $toNum : L \to \mathbb{N}$, both defined recursively below, **prove that $toNum$ is one-to-one**.

The set of linked lists of natural numbers L is defined by:

Basis Step:	$[\,] \in L$
Recursive Step:	If $l \in L$ and $n \in \mathbb{N}$, then $(n, l) \in L$

$toNum : L \to \mathbb{N}$ is defined recursively as follows:

Basis Step:	$toNum([]) = 0$
Recursive Step:	If $n \in \mathbb{N}$ and $l \in L$, then $toNum((n, l)) = 2^n \cdot 3^{toNum(l)}$

Solutions: Towards direct proof.
By definition of one-to-one, $\forall l \in L\ \forall l' \in L$ ($(toNum(l) = toNum(l')) \to (l = l')$)
Choose arbitrary l and l'.
Assume $toNum(l) = toNum(l') \ \equiv\ 2^a \cdot 3^{2^b \cdot 3^c} = 2^{a'} \cdot 3^{2^{b'} \cdot 3^{c'}}$ where c and c' continue the recursive steps.
Dividing both sides by the RHS, we get $2^{a-a'} \cdot 3^{2^b \cdot 3^c - 2^{b'} \cdot 3^{c'}} = 1$
For the product of two relatively prime positive integers each raised to a power to be 1, both must be raised to the 0^{th} power. Therefore,
$a - a' = 0 \to a = a' \qquad b - b' = 0 \to b = b' \qquad c - c' = 0 \to c = c'$
Since each individual node in l is equivalent to the node in the same position of l', $l = l'$ and $toNum$ is one-to-one.

3. Determine whether the following functions are well defined, and if they are injective (one-to-one) and/or surjective (onto) as well:

   (a) $f : \mathbb{N} \to \{0,1\}^*$ such that $n \mapsto (n)_2$, where $\{0,1\}^*$ is the set of all finite length bit strings, and $(n)_2$ is the binary representation of n. *(Hint: use a result you proved in the previous homework.)*
   Answer:
   We will use strong induction to prove that every integer can be represented by distinct powers of 2.
   Base Step: B(1) = 1
   Inductive Step: We can express any decimal number n+1 as 2m or 2m+1 for some integer m<n. By strong induction, we know that all decimal numbers 1 through n can be represented by distinct powers of 2, therefore we know that m (which is less than n) can be represented in binary. To form 2m with the collection of powers of 2 equal to m, we can multiply every power of 2 by 2. To form 2m+1, we can do this and then add 2^0.
   Thus if we can represent every integer from 1 through n as distinct powers of 2, we can also represent n+1 as a distinct power of 2.
   Since the powers of 2 are all distinct, we know binary representations of integers consist of only 1s and 0s.
   Goal 1: Prove f is well defined.
   Since binary representations of integers are just collections of 0s and 1s and binary strings are just collections of 0s and 1s, we know the function f is well defined because for every binary string, there will always be only one corresponding binary representation(the binary with the same number and order of 0s and 1s). Corollarily, we know that every every binary representation must also map to a unique binary string. We will use this to prove f is injective.
   Goal 2: Prove f is injective
   We must prove that if two binary strings map to the same binary representation, they are the same binary string. We know that f and f^{-1} are both well defined. This means every mapping from binary strings to representations and vice versa are unique. This is the guarantees injectivity since if f(a)=f(b), a=b must be true(since every mapping is unique).
   Goal 3: Prove f is surjective
   Earlier, we proved every binary representation consisted of only 1s and 0s. This means every representation can be mapped from its corresponding binary string that just contains the same number and order of 0s and 1s. We have proved f is well-defined, injective, and surjective. ■

   (b) $f : \mathbb{Q} \to \mathbb{Z}$ given by $\frac{m}{n} \mapsto m^n$;
   Answer: This function is not well defined, so it can not be one to one or onto.
   This function is not well defined because any rational number inputted can be expressed as any fraction that shares the correct ratio between the numerator and denominator. For example, 1/2 and 2/4 will map to different elements in the codomain even though they both represent the same element in the domain.

   (c) $f : \{0,1\}^3 \to \{0,1\}^4$ where $(b_3, b_2, b_1) \mapsto (0, b_3, b_2, b_1)$ for $b_i \in \{0,1\}$;
   Answer: This function is well defined and injective but not surjective.
   The function is well defined. Every tuple in $f : \{0,1\}^3$ maps to some single tuple in $\{0,1\}^4$ as the function f only prepends 0 to the given tuple of (b_3, b_2, b_1)
   For the arbitrary tuple $s = \{m_3, m_2, m_1\}, f(s) = \{0, s\}$
   The function is one to one(injective). Every tuple in the codomain has a unique mapping from the domain.
   $\forall s \in \{0,1\}^3 \forall t \in \{0,1\}^3, (f(s) = f(t) \to s = t)$
   Towards a direct proof, assume f(s)=f(t)

$f(s) = f(t)$
$\{0, s\} = \{0, t\}$
$s = t$
The function is not onto(surjective). We will disprove the claim that it is onto through counterexample.
$\{1, 0, 0, 0\}$ is in the codomain but has no mapping from the domain.

(d) $f : \{0, 1\}^4 \to \{0, 1\}^3$ where $(b_4, b_3, b_2, b_1) \mapsto (b_3, b_2, b_1)$ for $b_i \in \{0, 1\}$;
Answer: This function is well defined and surjective but is it not injective.
The function is well defined. Every tuple in $f : \{0, 1\}^4$ maps to some single tuple in $\{0, 1\}^3$ as the function f only removes the initial element of the given tuple s $\in \{0, 1\}^4$
For the arbitrary tuple $s = \{m_4, m_3, m_2, m_1\}, f(s) = \{m_3, m_2, m_1\}$
The function is not one to one(injective). We will disprove the claim that it is one to one through counterexample.
f being one to one claims:
$\forall u \in \{0, 1\}^4, \forall v \in \{0, 1\}^4, (f(u) = f(v) \to u = v)$
We take values u_1 and v_1 as counterexample:
$u_1 \in \{0, 1\}^4, v_1 \in \{0, 1\}^4$
$u_1 = \{0, 0, 0, 0\}, v_1 = \{1, 0, 0, 0\}$
$f(u_1) = f(v_1), u_1 \neq v_1$
Values u_1 and v_1 disprove the universal statement that proves f is one to one. Therefore f is not one to one.
The function is onto(surjective). Every element in the codomain has a mapping from some element in the domain. $\forall g \in \{0, 1\}^3, \exists h \in \{0, 1\}^4, f(h) = g$
For arbitrary $g = (i_3, i_2, i_1)$, witness b_4=0, $(f(h) = g) \to (h = (0, i_3, i_2, i_1))$
This proves the function is onto(surjective) through universal generalization.

(e) $f : \mathbb{Z} \to \mathbb{Z}$ where $k \mapsto k \bmod 19$.
Answer: This function is well defined but not one to one or onto.
This function is well defined.
$\forall k \in Z, \exists l \in Z, f(k) = l$
Since k is an integer, l must also be an integer. Therefore f is will defined
This function is not one to one.
We will disprove the claim that f is one to one through counterexample.
f being one to one claims:
$\forall m \in Z, \forall n \in Z, (f(m) = f(n) \to m = n)$
We take values m_1 and n_1 as counterexample:
$m_1 \in Z, n_1 \in Z$
$m_1 = 40, n_1 = 59$
$f(m_1) = f(n_1), m_1 \neq n_1$
Values m_1 and n_1 disprove the claim(the predicate) that proves f is one to one. Therefore f is not one to one.
The function is not onto. Not every element in the codomain is mapped to from some element in the domain.
Take 20 as a counterexample from the codomain. By the property of the modular operation, there can be no remainder that is greater or equal to the divisor. Since 20≥19, 20 can not be a remainder and has no mapping from the domain.

4. Let $W = \mathcal{P}(\{1, 2, 3, 4, 5\})$.

Sample response that can be used as reference for the detail expected in your answer for parts (a) and (b) below:

To give an example element in the set $\{X \in W \ : \ 1 \in X\} \cap \{X \in W \ : \ 2 \in X\}$, consider $\{1, 2\}$. To prove that this is in the set, by definition of intersection, we need to show that $\{1, 2\} \in \{X \in W \ : \ 1 \in X\}$ and that $\{1, 2\} \in \{X \in W \ : \ 2 \in X\}$.

- By set builder notation, elements in $\{X \in W \ : \ 1 \in X\}$ have to be elements of W which have 1 as an element. By definition of power set, elements of W are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of W. Also, by roster method, $1 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \ : \ 1 \in X\}$.
- Similarly, by set builder notation, elements in $\{X \in W \ : \ 2 \in X\}$ have to be elements of W which have 2 as an element. By definition of power set, elements of W are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of W. Also, by roster method, $2 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \ : \ 2 \in X\}$.

(a) Give two example elements in

$$\mathcal{P}(W)$$

Justify your examples by explanations that include references to the relevant definitions.

Solutions: $\{\{1\}\}$ and $\{\{2\}\}$

- By definition of power set, all subsets of $\{1, 2, 3, 4, 5\}$ are elements of W. Since $1 \in \{1, 2, 3, 4, 5\}$, $\{1\} \subseteq \{1, 2, 3, 4, 5\}$ and $\{1\}$ is is an element of $P(\{1, 2, 3, 4, 5\})$, or $\{1\} \in W$. By definition of power set, all subsets of W are in $\mathcal{P}(W)$. Since $\{1\} \in W$, $\{\{1\}\} \subseteq W$. Thus, $\{\{1\}\}$ is a subset of W and hence $\{\{1\}\} \in \mathcal{P}(W)$.
- By definition of power set, all subsets of $\{1, 2, 3, 4, 5\}$ are elements of W. Since $2 \in \{1, 2, 3, 4, 5\}$, $\{1\} \subseteq \{1, 2, 3, 4, 5\}$ and $\{2\}$ is is an element of $P(\{1, 2, 3, 4, 5\})$, or $\{2\} \in W$. By definition of power set, all subsets of W are in $\mathcal{P}(W)$. Since $\{2\} \in W$, $\{\{2\}\} \subseteq W$. Thus, $\{\{2\}\}$ is a subset of W and hence $\{\{2\}\} \in \mathcal{P}(W)$.

(b) Give one example element in

$$\mathcal{P}(W) \times \mathcal{P}(W)$$

that is **not** equal to $(\emptyset, \emptyset)$ or to (W, W). Justify your example by an explanation that includes references to the relevant definitions.

Solutions: $\{\ \{\{1\}\}, \{\{2\}\}\ \}$

Continuing from part (a), $\{\{1\}\}$ and $\{\{2\}\}$ are elements in the set $\mathcal{P}(W)$. Taking the intersection of $\{\{1\}\}$ and $\{\{2\}\}$, we get $\{\ \{\{1\}\}, \{\{2\}\}\ \}$. Since both elements in this intersection are elements of set $\mathcal{P}(W$, we know this intersection is an element in $\mathcal{P}(W) \times \mathcal{P}(W)$.

5. Let W be a finite set with n elements. How many pairs (X, Y) are there such that $X \subseteq W$, $Y \subseteq W$, and $X \subseteq Y$? Provide an expression to compute the answer that shows your reasoning. You don't need to express your answer in its most reduced form.
Answer: $|W| = n, |Y| = P(W), |X| = P(Y)$
$|W| = n, |Y| = 2^{|W|}, |X| = 2^{|Y|}$
$|W| = n, |Y| = 2^n, |X| = 2^{2^n}$
The number of pairs (X, Y) is equal to $|X| * |Y|$
$|X| * |Y| = 2^{2^n} * 2^n$

6. Write a recursive algorithm to compute the arithmetic mean of a sequence of integers. Then, use induction on the length of the sequence to prove that your algorithm outputs the correct value for every non-empty input sequence.

Algorithm: Computing the arithmetic mean of a sequence recursively

```
procedure mean_recursive( (a_0,...a_n): a sequence of integers)
if arr.size = 1
    return arr[0]
prevmean = mean_recursive(arr.pop)
diff = (arr[last] - prevmean)/arr.size
m = prevmean + diff

return (m){m is the arithmetic mean of (a_0,...a_n)}
```

Now we prove this algorithm works through induction
Base Case: For an arbitrary array of size 1, the mean is arr[0].
Inductive Case: If we know meanrecursive(arr.pop) is accurate, we know meanrecursive(arr) is accurate.
meanrecursive(arr.pop) is accurate. meanrecursive(arr.pop)=$\frac{(a_0+a_1+...+a_{n-1})}{n}$
We must prove $meanrecursive(arr) = \frac{(a_0+a_1+...+a_n)}{n+1}$
For arbitrary element arr[last],
meanrecursive(arr) = meanrecursive(arr.pop) + (arr[last] - meanrecursive(arr.pop))/arr.size
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})}{n}$ + (arr[last] - meanrecursive(arr.pop))/arr.size
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})}{n} + \frac{a_n - \frac{(a_0+a_1+...+a_{n-1})}{n}}{n+1}$
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})}{n} + \frac{\frac{(na_n-a_0-a_1+...-a_{n-1})}{n}}{n+1}$
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})}{n} + \frac{(na_n-a_0-a_1+...-a_{n-1})}{n(n+1)}$
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})(n+1)}{n(n+1)} + \frac{(na_n-a_0-a_1+...-a_{n-1})}{n(n+1)}$
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1})(n+1)+(na_n-a_0-a_1+...-a_{n-1})}{n(n+1)}$
meanrecursive(arr) = $\frac{(na_0+a_0+na_1+a_1+...+na_{n-1}+a_{n-1})+(na_n-a_0-a_1+...-a_{n-1})}{n(n+1)}$
meanrecursive(arr) = $\frac{(na_0+na_1+...+na_{n-1})+(na_n)}{n(n+1)}$
meanrecursive(arr) = $\frac{n(a_0+a_1+...+a_{n-1}+a_n)}{n(n+1)}$
meanrecursive(arr) = $\frac{(a_0+a_1+...+a_{n-1}+a_n)}{n+1}$
Thus we have proven the algorithm works through induction ■

7. Recall the definition of the set of rational numbers, $Q = \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$. We define the set of **irrational** numbers $\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$. Fill in the blank in the following argument.

   Claimed statement: $-\sqrt{2} \in \overline{\mathbb{Q}}$

   > **Proof**: Towards a proof by contradiction, we will prove that $\underline{\quad BLANK \quad}$ guarantees $\sqrt{2} \in \overline{\mathbb{Q}} \wedge \sqrt{2} \notin \overline{\mathbb{Q}}$. We proceed by direct proof and assume the hypothesis of the conditional. To prove the conclusion of the conditional, we have two subgoals. Subgoal (1): We need to prove the first conjunct, that $\sqrt{2} \in \overline{\mathbb{Q}}$. This is proved in Chapter 7.2 (Theorem 7.2.1 in Zybook). Subgoal (2): It remains to prove that $\sqrt{2} \notin \overline{\mathbb{Q}}$, in other words (by double negation), that $\sqrt{2} \in \mathbb{Q}$. By our assumption that $-\sqrt{2} \notin \overline{\mathbb{Q}}$ and double negation, we have $-\sqrt{2} \in \mathbb{Q}$. By definition of the set of rational numbers, there are integers p' and q' (with q' nonzero) such that $-\sqrt{2} = \frac{p'}{q'}$. Consider the witnesses $p = -p'$ and $q = q'$. These are integers (since -1 times an integer is an integer) and q is nonzero, so they are candidate witness for the fraction in the definition of rational numbers. Moreover,
   >
   > $$\frac{p}{q} = \frac{-p'}{q'} = (-1)\frac{p'}{q'} \overset{\text{by def of } p', q'}{=} (-1)(-\sqrt{2}) = \sqrt{2}$$
   >
   > Thus, we have proved (under our assumption) that $\sqrt{2} \in \mathbb{Q}$, and subgoal (2) is complete. Since the direct proof is complete, we have proved that assuming the negation of the claimed statement leads to a contradiction, and therefore the original statement must be true. QED

   Answer: BLANK = "-$\sqrt{2} \notin \mathbb{Q}$"

8. Consider the binary relation R on the set of integers define as $R_m = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 3 \cdot m\}$ for some positive integer m. Prove or disprove that R_m is a equivalence relation.
   If R_m is symmetric, reflexive, and transitive, it is an equivalence relation.
   R_m is not symmetric.
   $\neg\forall(a, b) \in \mathbb{Z} \times \mathbb{Z},\ aR_m b \rightarrow bR_m a$
   Towards a proof by contradiction, we assume $a - b = 3m_1$ and $b - a = 3m_2$. By multiplying the first expression by -1, we find $b - a = -3m_1$. $m_1 = m_2$ but both $m_1 and m_2$ must be positive, leading to a contradiction.
   R_m is not reflexive.
   $\neg\forall a \in Z, aR_m a$
   Assuming $aR_m a$ means
   $a - a = 3m$
   $0 = 3m$
   $m = 0$
   However m must be a positive integer, meaning this assumption leads to a contradiction.
   R_m is transitive
   $(aR_m b \wedge bR_m c) \rightarrow aR_m c$
   Towards a direct proof, we assume:
   $\exists m \in Z^+, \exists n \in Z^+, a - b = 3m \wedge b - c = 3n$
   $a - b = 3m, b - c = 3n$
   $a - b = 3m, b = c + 3n$
   $a - (c + 3n) = 3m$
   $a - c = 3m + 3n$
   $a - c = 3(m + n)$
   m and n are both positive integers so m+n must be a positive integer. This proves $aR_m c$ and concludes the direct proof.
   Because R_m is not reflexive, it is not an equivalence relation.

9. Recall that in a movie recommendation system, each user's ratings of movies is represented as a n-tuple (with the positive integer n being the number of movies in the database), and each component of the n-tuple is an element of the collection $\{-1, 0, 1\}$.

   Assume there are five movies in the database, so that each user's ratings can be represented as a 5-tuple. Let R be the set of all ratings, that is, the set of all 5-tuples where each component of the 5-tuple is an element of the collection $\{-1, 0, 1\}$.

   Consider the following two binary relations on R:

$$A_1 = \{(u, v) \in R \times R \mid \text{users } u \text{ and } v \text{ agree about the first movie in the database}\}$$

$$A = \{(u, v) \in R \times R \mid \text{users } u \text{ and } v \text{ don't care or haven't seen the same number of movies}\}$$

   Binary relations that satisfy certain properties (namely, are reflexive, symmetric, and transitive) can help us group elements in a set into categories.

   (a) **True** or **False**: The relation A_1 holds of $u = (1, 1, 1, 1, 1)$ and $v = (-1, -1, -1, -1, -1)$.
   False, u liked the first movie while v disliked the first movie.

   (b) **True** or **False**: The relation A holds of $u = (1, 0, 1, 0, -1)$ and $v = (-1, 0, 1, -1, -1)$.
   False, u doesn't care or hasn't see 2 movies but v doesn't care or hasn't seen 1 movie. $2 \neq 1$

   (c) **True** or **False**: A_1 is reflexive; namely, $\forall u \in R\ (\ (u, u) \in A_1\ )$
   True, A_1 is reflexive. A user will agree about the first movie with themself because they are comparing themself to themself.

(d) **True** or **False**: A_1 is symmetric; namely, $\forall u \in R\ \forall v \in R\ (\ (u, v) \in A_1 \to (v, u) \in A_1\ )$
True, A_1 is symmetric. A user will agree about the first movie with someone who agreed on the same first movie as them.

(e) **True** or **False**: A_1 is transitive; namely, $\forall u \in R\ \forall v \in R\ \forall w \in R(\ ((u, v) \in A_1 \wedge (v, w) \in A_1) \to (u, w) \in A_1\ )$
True, A_1 is transitive. If user u and v agree about the first movie and users v and arbitrary user w also agree on the first movie, they must have all agreed on the same conclusion, meaning user u and w also agree.

(f) **True** or **False**: A is reflexive; namely, $\forall u \in R\ (\ (u, u) \in A\ )$
True, A is reflexive. Any user will not care or has not seen the same number of movies as themself because they are themself.

(g) **True** or **False**: A is anti-symmetric; namely, $\forall u \in R\ \forall v \in R\ (\ (\ (u, v) \in A \wedge (v, u) \in A\ ) \to (u = v)\ )$
False, A is not anti-symmetric (there exists a pair of relations). If user u doesn't care or hasn't seen the same number of movies as user v, it is impossible for user v to not have seen the same number or not care about the same number of movies as user u.

(h) **True** or **False**: A is transitive; namely, $\forall u \in R\ \forall v \in R\ \forall w \in R(\ ((u, v) \in A \wedge (v, w) \in A) \to (u, w) \in A\ )$
True, A is transitive. If user u and v haven't seen or don't care about the same number of movies and users v and arbitrary user w hasn't seen or don't care about the same number of movies, then all of them must have not seen or don't care about the same number of movies, meaning users u and w have not seen or don't care about the same number of movies.

10. In the previous question select any one of parts (c) to (h) that evaluated to true and provide a formal proof using the strategies you have learned in CS40
Solutions: Proof by universal generalization on part (c) of Question 9.
Need to show $\forall u \in R\ (\ (u, u) \in A_1\ )$.
Take arbitrary $u \in R$ and expand to $u = (u_1, u_2, u_3, u_4, u_5)$.
By definition of A_1, $(u, v) \in A_1$ if users u and v agree about the first movie in the database, or in other words $u_1 = v_1$.
Since $u_1 = u_1$ (the rating for the first movie doesn't change), $\forall u \in R\ (\ (u, u) \in A_1\ )$. (i.e. a person agrees with himself about the first movie in the database).

11. *No justifications are required for credit for this question. It's a good idea to think about how you would explain how you arrived at your examples.* Given the relations A_1 and A in Q9 answer the following questions:

(a) Give two distinct examples of elements in $[\ (1, 0, 0, 0, 0)\ ]_{A_1}$
Solutions: $(\ (1, 1, 1, 1, 1), (1, 0, 0, 0, 1)\ )$ and $(\ (1, 1, 1, 1, 1), (1, 0, 0, 0, -1)\ )$

(b) Give two distinct examples of elements in $[\ (1, 0, 0, 0, 0)\ ]_A$
Solutions: $(\ (0, 0, 0, 0, 1), (0, 0, 0, 1, 0)\ )$ and $(\ (0, 0, 0, 0, 1), (0, -1, 0, 0, 0)\ )$

(c) Find examples $u, v \in R$ where $[u]_{A_1} \neq [v]_{A_1}$ but $[u]_A = [v]_A$
Solutions: $u = (1, 0, 0, 0, 1)$, $v = (0, -1, 0, -1, 0)$

(d) Find examples $u, v \in R$ (different from the previous part) where $[u]_{A_1} = [v]_{A_1}$ but $[u]_A \neq [v]_A$
Solutions: $u = (1, 0, 0, 1, 1)$, $v = (1, 0, 0, -1, 0)$

12. Consider an old game of matches. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction on the integer j to show that the first player can always win if $n = 4j, 4j + 2$, or $4j + 3$ for some non-negative integer j.

    The second player will win when the game starts with 4n+1 matches for some arbitrary non-negative integer n.

    P(x)-Your opponent(the first player) starting their turn with x sticks will give you a win.

    Base Case: $P(1)$

    If your opponent starts with 1 stick, you win the game.

    Inductive Case: $P(n) \rightarrow P(n + 4)$

    If your opponent starts with n+4 sticks, they can make you start your turn with: n+1, n+2, or n+3 sticks. But since you move second, you can react to whatever move they make. By taking the complement of 4, you can make your opponent start with n sticks no matter if they leave you with n+1, n+2, or n+3 sticks. This means if you can win whenever your opponent starts with n sticks, you can also win whenever your opponent starts with n+4 sticks.

    Thus, the second play will win whenever the game starts with 1+4n matches for some arbitrary non-negative integer n.

    There are only 4 possible cases to how many matches the game starts with. The game must start with one of these values for an arbitrary non-negative integer n: 4n,4n+1,4n+2,4n+3.

    Corollary, the first player will win whenever the game starts with 4n, 4n+2, or 4n+3 matches for some arbitrary non-negative integer n.

13. (Extra Credit) Define $\mathbb{R}^{(0,1)} = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Here we will show that $|\mathbb{R}^{(0,1)}| = |\mathbb{R}|$.

(a) Show that the function $f : \mathbb{R}^{(0,1)} \to \mathbb{R}^+$ for which $x \mapsto \frac{x}{1-x}$ is bijective.

Answer: We define f(x)=$\frac{x}{1-x}$ and $f^{-1} = \frac{x+1}{x}$. We must prove f is bijective.
To prove f is bijective, we will prove that f and f^{-1} are both one to one
Goal 1: Prove f is one to one
$f(a) = f(b)$
$\frac{a}{1-a} = \frac{b}{1-b}$
a(1-b)=b(1-a)
a-ab=b-ab
a=b
Goal 2: Prove f^{-1} is one to one
$f^{-1}(a) = f^{-1}(b)$
$\frac{a+1}{a} = \frac{b+1}{b}$
(a+1)b=(b+1)a
ab+b=ab+a
b=a
By proving that both f and f^{-1} are both one to one for f(x)=$\frac{x}{1-x}$, we have proven that the function is bijective.

(b) Show that the function $g : \mathbb{R}^{(0,1)} \to \mathbb{R}$ is bijective, where

$$g(x) = \begin{cases} -f(1-2x) & 0 < x < \frac{1}{2}, \\ 0 & x = \frac{1}{2}, \\ f(2x-1) & \frac{1}{2} < x < 1. \end{cases}$$

and conclude that $|\mathbb{R}^{(0,1)}| = |\mathbb{R}|$.

Answer:
Let's simplfy g(x) first:

$$g(x) = \begin{cases} \frac{2x-1}{2x} & 0 < x < \frac{1}{2}, \\ 0 & x = \frac{1}{2}, \\ \frac{2x-1}{2-2x} & \frac{1}{2} < x < 1. \end{cases}$$

We will now prove that g(x) is both one to one and onto to prove that it is bijective.
Goal 1: Prove that g(x) is one to one.
Case 1: $0 < x < \frac{1}{2}$
$\forall a \in \mathbb{R}^{(0,\frac{1}{2})} \forall b \in \mathbb{R}^{(0,\frac{1}{2})}, (g(a) = g(b)) \to (a = b)$
Towards a direct proof, we assume g(a)=g(b):
g(a)=g(b)
$\frac{1-2a}{2a} = \frac{1-2b}{2b}$
(1-2a)2b=(1-2b)2a
2b-4ab=2a-4ab
2b=2a
b=a
Case 2: $x = \frac{1}{2}$
f(1/2) = 0 is a unique mapping. Neither $\frac{1-2x}{2x}$ or $\frac{2x-1}{2-2x}$ can equal 0.
Case 3: $\frac{1}{2} < x < 1$.

$\forall a \in \mathbb{R}^{(\frac{1}{2},1)} \forall b \in \mathbb{R}^{(\frac{1}{2},1)}, (g(a) = g(b)) \rightarrow (a = b)$
Towards a direct proof, we assume g(a)=g(b):
g(a)=g(b)
$\frac{2a-1}{2-2a} = \frac{2b-1}{2-2b}$
(2a-1)(2-2b)=(2b-1)(2-2a)
4a-4ab-2+2b=4b-4ab-2+2a
4a+2b=4b+2a
2a=2b
a=b
Thus, we have proven g(x) is one to one.
Goal 2: Prove that g(x) is onto.
$\forall b \in \mathbb{R}\ \exists a \in \mathbb{R}^{(0,1)}, b = g(a)$
The problem is that the set of all real numbers is not well ordered and so it is really hard to prove that everything in it is mapped to(since the thing is infinite and not well ordered).
So I will now show g(x) is a continuous function on the two intervals and that because their bounds are 0 and -infinity,infinity, every real number must be covered.
We will prove this predicate true with 3 cases that describe every possible value of b.
Case 1: b>0
$\forall b^{>0} \in \mathbb{R}\ \exists a \in \mathbb{R}^{(\frac{1}{2},1)}, b = g(a)$
The equation for g(x) on $\mathbb{R}^{(\frac{1}{2},1)}$ is $\frac{2x-1}{2-2x}$. Since we know this function is continuous across the region and has an upper bound of infinity and a lower bound of 0, we know every positive real number is mapped by this function for a $\in \mathbb{R}^{(\frac{1}{2},1)}$. This proves the predicate
Case 2: b<0
$\forall b^{<0} \in \mathbb{R}\ \exists a \in \mathbb{R}^{(0,\frac{1}{2})}, b = g(a)$
The equation for g(x) on $\mathbb{R}^{(0,\frac{1}{2})}$ is $\frac{2x-1}{2x}$. Since we know this function is continuous across the region and has an upper bound of 0 and a lower bound of negative infinity, we know every negative real number is mapped by this function for a $\in \mathbb{R}^{(0,\frac{1}{2})}$. This proves the predicate
Case 3: b=0
$g(\frac{1}{2}) = 0$
By exhaustively proving:
$\forall b^{>0} \in \mathbb{R}\ \exists a \in \mathbb{R}^{(\frac{1}{2},1)}, b = g(a)$
$\forall b^{<0} \in \mathbb{R}\ \exists a \in \mathbb{R}^{(0,\frac{1}{2})}, b = g(a)$
$g(\frac{1}{2}) = 0$
we have proven $\forall b \in \mathbb{R}\ \exists a \in \mathbb{R}^{(0,1)}, b = g(a)$
and thus have proven the function is onto.
By proving g(x) is both one to one and onto, we have proven that g(x) is bijective ■

14. **Bonus - not for credit (but much appreciated):** Please complete the course ESCI and TA evaluations by June 3 (Friday).

Attributions