# HW8 Collaborative

CS40 Fall'21

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Due: Monday, Nov 29, 2021 at 10:00PM on Gradescope

**In this assignment,**

You will have more practice with induction and other proof strategies.

**For all HW assignments:**

Please see the instructions and policies for assignments on the class website and on the writeup for HW1. In particular, these policies address

- Collaboration policy
- Where to get help
- Typing your solutions
- Expectations for full credit

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW8".

In your proofs and disproofs of statements below, justify each step by reference to the proof strategies we have discussed so far, and/or to relevant definitions and calculations. We include only induction-related strategies here; you can and should refer to past material to identify others.

**Proof by Mathematical Induction**: To prove a universal quantification over the set of all integers greater than or equal to some base integer $b$:

| | |
|---|---|
| Basis Step: | Show the statement holds for $b$. |
| Recursive Step: | Consider an arbitrary integer $n$ greater than or equal to $b$, assume (as the **induction hypothesis**) that the property holds for $n$, and use this and other facts to prove that the property holds for $n + 1$. |

**Proof by Strong Induction** To prove that a universal quantification over the set of all integers greater than or equal to some base integer $b$ holds, pick a fixed nonnegative integer $j$ and then:

| | |
|---|---|
| Basis Step: | Show the statement holds for $b$, $b+1$, $\ldots$, $b+j$. |
| Recursive Step: | Consider an arbitrary integer $n$ greater than or equal to $b+j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** $b$, $b+1$, $\ldots$, $n$, and use this and other facts to prove that the property holds for $n+1$. |

## Assigned questions

1. Assume $L$ is the set of linked lists defined recursively in HW7 and $toNum$ is a function from $L \to \mathbb{N}$ defined recursively as follows (also covered in Week 9 Monday's lecture):

   Basis Step: $toNum([]) = 0$
   Recursive Step: If $n \in \mathbb{N}$ and $l \in L$, then $toNum((n, l)) = 2^n \cdot 3^{toNum(l)}$

   Prove that $toNum$ is one-to-one.

   **Answer:** .
   $\forall l1 \in L \forall l2 \in L(tonum(l1) = tonum(l2) \to (l1 = l2))$
   Direct proof:
   Assume tonum(l1) = tonum(l2)
   Then: $2^n * 3^{tonum(l1')} = 2^n * 3^{tonum(l2')}$ // Divide both sides by $2^n$
   $3^{tonum(l1')} = 3^{tonum(l2')}$

   $N1 = n2$
   $tonum(l1') = tonum(l2')$

   Induction: on l1 choose arbitrary l2
   Base case: l1 = []
   WTS: tonum([]) = tonum(l2) → [] = l2
   Direct proof:
   Assume tonum([]) = tonum(l2)
   0 = tonum(l2) by basis step
   L2 = [] by basis step

   Inductive Case: l1 = (n1,l1'). Choose arbitrary (n2,l2)
   Assume as inductive hypothesis: $tonum(l1') = tonum(l2) \to (l1' = l2)$
   WTS: $tonum((n1, l1')) = tonum(n2, l2) -> ((n1, l1') = (n2, l2))$
   Direct proof:
   Assume tonum((n1,l1')) = tonum(n2,l2)
   $2^{n1} * 3^{tonum(l1')} = 2^{n2} * 3^{tonum(l2)}$ // by recursive step
   $2^{n1-n2} * 3^{tonum(l1')-tonum(l2)} = 1$
   n1=n2
   tonum(l1') = tonum(l2)
   ⇒ (l1' = l2) // by inductive hypothesis
   ⇒((n1,l1) = (n2,l2)) // by adding n1 and n2 to each side since n1 = n2

2. Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$$

   Answer the questions that follow:

   (a) Verify that $P(3)$ is true, and then express $P(k)$ and $P(k+1)$

**Answer:** .

Verify P(3) $LHS = \sum_{j=1}^{3} j^2 = 3^2 + 2^2 + 1^2 = 9 + 4 + 1 = 14$
$RHS = 3(3+1)(2(3)+1)/6 = 3*4*7/6 = 14$
LHS = RHS verifying P(3)
$P(k) : \sum_{j=1}^{k} k^2 = k(k+1)(2(k)+1)/6$
$P(k) : \sum_{j=1}^{k+1}(k+1)^2 = (k+1)(k+1+1)(2(k+1)+1)/6$

(b) What is the basis step for an inductive proof of $\forall n \in \mathbb{Z}^+(P(n))$

**Answer:** .

Basis Step, n = 1 so P(1) is the assertion that
$P(k) : \sum_{j=1}^{1} j^2 = 1(1+1)(2(1)+1)/6$
Evaluating: $LHS = 1^2 = 1$
RHS = 1(2)(3)/6 = 6/6 = 1
LHS = RHS so n = 1 is true

(c) What would be the inductive hypothesis? What must be proven in the inductive step?

**Answer:** .

Inductive hypothesis would be that the assertion is true for n = k
So IH is P(k): $\sum_{j=1}^{k+1} k^2 = (k+1)(k+1+1)(2(k+1)+1)/6$
And what must be prove is n = k + 1
So WTS is P(k+1): $\sum_{j=1}^{k+1}(k+1)^2 = (k+1)(k+1+1)(2(k+1)+1)/6$
$= k^3/3 + 3k^2/2 + 13k/6 + 1$

(d) Write the complete inductive proof for the provided assertion by combining all your answers from the previous parts

**Answer:** .

So by inductive proof over n
Basis Step, n = 1 so P(1) is the assertion that
$\sum_{j=1}^{1} j^2 = 1(1+1)(2(1)+1)/6$
Evaluating: LHS = $1^2 = 1$
RHS = 1(2)(3)/6 = 6/6 = 1
LHS = RHS so n = 1 is true

Inductive step: Now with that being true we form the IH of n = k being true,
So IH is P(k): $\sum_{j=1}^{k}(k)^2 = k(k+1)(2(k)+1)/6 = k^3/3 + k^2/2 + k/6$
We WTS that its true for n = k + 1 which is
P(k+1): $\sum_{j=1}^{k+1}(k+1)^2 = (k+1)(k+1+1)(2(k+1)+1)/6$
Evaluating LHS = $\sum_{j=1}^{k+1}(k+1)^2$
$= (k+1)^2 + \sum_{j=1}^{k} k^2$
$= k^2 + 2k + 1 + \sum_{j=1}^{k} k^2$
RHS = (k+1)(k + 1 + 1)(2(k+ 1) + 1)/6
$= k^3/3 + 3k^2/2 + 13k/6 + 1$
$= k^3/3 + k^2/2 + k^2 + k/6 + 2k + 1$
$= k^2 + 2k + 1 + \sum_{j=1}^{k} k^2$
LHS = RHS so proving it

3. (*Graded for correctness in evaluating statement and for fair effort completeness in the justification*)

Consider the functions $f_a : \mathbb{N} \to \mathbb{N}$ and $f_b : \mathbb{N} \to \mathbb{N}$ defined recursively by

$$f_a(0) = 0 \qquad \text{and for each } n \in \mathbb{N}, \qquad f_a(n+1) = f_a(n) + 2n + 1$$

$$f_b(0) = 0 \qquad \text{and for each } n \in \mathbb{N}, \qquad f_b(n+1) = 2f_b(n)$$

Which of these two functions (if any) equals $2^n$ and which of these functions (if any) equals $n^2$? Use induction to prove the equality or use counterexamples to disprove it.

4. (*Graded for correctness*) Prove the following statement:

$$\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{Z}^{\geq n_0}\, (n^2 < 2^n)$$

In your proof, you may use the following lemma:

$$\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{Z}^{\geq n_0}\, (1 + 2n \ < n^2)$$

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*Proof of lemma. This proof can also be used as reference for a possible approach for the statement you are trying to prove:*

To prove the existential claim, consider the witness $n_0 = 3$. We will prove that

$$\forall n \in \mathbb{Z}^{\geq 3} \, (1 + 2n \ < n^2)$$

using mathematical induction.

**Basis step** For the basis step, we need to show that $1 + 2 \cdot 3 \ < \ 3^2$. Evaluating: $1 + 2 \cdot 3 = 1 + 6 = 7$ and $3^2 = 9$. Since $7 < 9$, the basis step is complete.

**Recursive step** Consider arbitrary integer $n$ that is greater than or equal to 3. Assume, as the induction hypothesis, that $1 + 2n \ < \ n^2$. We need to show that $1 + 2(n+1) \ < (n+1)^2$. Calculating:

$$\begin{aligned}
(n+1)^2 &= (n+1)(n+1) = n^2 + 2n + 1 \\
&> (1 + 2n) + 2n + 1 && \text{by the induction hypothesis} \\
&> 2n + 2n + 1 && \text{since } 1 > 0 \\
&> 2n + 2 \cdot 1 + 1 && \text{since } n > 1 \text{ by assumption that } n \geq 3 \\
&= 2(n+1) + 1 = 1 + 2(n+1) && \text{as required to complete the recursive step.}
\end{aligned}$$

Thus, the universal quantification was proved using mathematical induction and so the witness $n_0 = 3$ proves the existential. ■

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5. (*Graded for fair effort completeness*) Can the statement you proved above be used to prove or disprove the following statement? Why or why not?

$$\exists n_0 \in \mathbb{N} \, \forall n \in \mathbb{Z}^{\geq n_0} \, (2^n < n^2)$$

6. Prove the following upper bound for the given recurrence relation using strong induction.

   Define the sequence $\{a_n\}$ as follows:

   $a_1 = a_2 = a_3 = 1$
   $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$
   Prove that

$$\forall n \in \mathbb{Z}^{\geq 1} \, ( \ a_n \leq 2^n \ )$$

7. Use induction to prove that the following algorithm is correct. Binary strings are the set of all strings of length 0 or more made up of characters from the set $\{0, 1\}$.

Recursively computing the set of all binary strings of a fixed length $n$

```
procedure StringSet(n: a non-negative integer)

S := ∅
if n = 0
    Add λ to S
    return (S)
end if
T := StringSet(n − 1)
for every x ∈ T
    y := 0x
    Add y to S
    y := 1x
    Add y to S
end for
return (S) {output is S}
```

8. Let $W = \mathcal{P}(\{1, 2, 3, 4, 5\})$.

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*Sample response that can be used as reference for the detail expected in your answer for parts (a) and (b) below:*

To give an example element in the set $\{X \in W \mid 1 \in X\} \cap \{X \in W \mid 2 \in X\}$, consider $\{1, 2\}$. To prove that this is in the set, by definition of intersection, we need to show that $\{1, 2\} \in \{X \in W \mid 1 \in X\}$ and that $\{1, 2\} \in \{X \in W \mid 2 \in X\}$.

- By set builder notation, elements in $\{X \in W \mid 1 \in X\}$ have to be elements of $W$ which have 1 as an element. By definition of power set, elements of $W$ are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of $W$. Also, by roster method, $1 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \mid 1 \in X\}$.
- Similarly, by set builder notation, elements in $\{X \in W \mid 2 \in X\}$ have to be elements of $W$ which have 2 as an element. By definition of power set, elements of $W$ are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of $W$. Also, by roster method, $2 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \mid 2 \in X\}$.

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(a) Give two example elements in

$$W \times W.$$

Justify your examples by explanations that include references to the relevant definitions.

(b) Give one example element in

$$\mathcal{P}(W)$$

that is **not** equal to $\emptyset$ or to $W$. Justify your example by an explanation that includes references to the relevant definitions.

9. We define the following function:

$$f : \{0, 1\}^4 \to \{0, 1\}^3,$$

where the output of $f$ is obtained by taking the input string and dropping the first bit. For example $f(1011) = 011$. Indicate whether the $f$ is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

# Attributions