HW9 Collaborative

CS40 Fall'21

Benjamin Cruttenden (4672440)
Bharat Kathi (5938444)
Sean Oh (4824231)
Marco Wong (4589198)

Due: Thursday, Dec 02, 2021 at 10:00PM on Gradescope

In this assignment,

You will have more practice with proof strategies learned so far and using functions to compare the sizes of sets.

For all HW assignments:

Please see the instructions and policies for assignments on the class website and on the writeup for HW1. In particular, these policies address

- Collaboration policy
- Where to get help
- Typing your solutions
- Expectations for full credit

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW9-Collabrative".

Assigned questions

1. Let $B = \{0,1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

   (a) Find a bijection between B^9 and E_{10} and prove that your function is a bijection.

   Answer: .

   $B9 : 2^9 = 512$ different combinations
   $E10 : (2^10)/2 = 1024/2 = 512$ different combinations
   $|B^9| = |E_10|$
   $|B^9| = 2^9 = 512$
   $|E_10| = 1024/2 = 512$
   $512 = 512$ so there is a bijection

   $000000000 \rightarrow 0000000000$
   $000000001 \rightarrow 0000000011$
   $000000010 \rightarrow 0000000101$
   $000000011 \rightarrow 0000000110$
   $f : E_10- > B^9$
   $f(es) = e$, where es in E_{10} and s in $\{0,1\}$

   Proof:

   f is well defined:
   Consider arbitrary es in E_{10}. WTS it maps to exactly one element of B^9.
   Since es in E_{10}, it is a binary string with 10 bits.
   Since s in $\{0,1\}$ it is a binary string with 1 bit.
   Removing s from es gives a binary string with 9 bits, so e in B^9.
   Thus, each input for f maps to exactly 1 output.

   f is one-to-one:
   For a, b in domain E_{10}, if $f(a) = f(b)$, then $a = b$
   We want to prove that if $f(a) = f(b)$, then $a = b$
   Towards a direct proof, assume f(a) = f(b)
   Since $f(a) = f(b)$, $e_a = e_b$, by definition of f.
   Add arbitrary s in $\{0,1\}$ to both sides: $e_{as} = e_{bs}$
   Then, a = b

   (b) What is $|E_{10}|$?

   Answer: .

   $|E_{10}| = (2^{10})/2 = 2^9 = 512$

2. Prove that $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$ by showing that the witness function $f : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ with $f(x) = \{x^2\}$ is one-to-one.

3. Apply the pigeonhole principle/generalized pigeonhole principle to answer the following question. If the pigeonhole principle can not be applied, give a specific counterexample.

(a) A team of four high jumpers all have a personal record that is at least 6 feet and less than 7 feet. Is it necessarily true that two of the team members must have personal records that are within (less than or equal to) four inches of each other? Heights are measured to within a precision of $\frac{1}{4}$ inch.

(b) There are 121.4 million people in the United States who earn an annual income that is at least $\$10,000$ and less than $\$1000,000$ dollars. Annual income is rounded to the nearest dollar. Show that there are 123 people who earn the same annual income in dollars.

4. For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. Justify your choice using an appropriate proof strategy

   (a) $h : \mathbb{Z} \to \mathbb{Z}.h(x) = x^3$

   (b) $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}.g(x, y) = (1 - y, 1 - x)$

5. Consider the binary relation R on the set of integers define as $R_m = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 3 \cdot m\}$ for some positive integer m. Prove or disprove that R_m is a equivalence relation.

6. Extra credit: Recall the definitions of the greatest common divisor, gcd: **Greatest common divisor** Let a and b be integers, not both zero. The largest integer d such that d is a factor of a and d is a factor of b is called the greatest common divisor of a and b and is denoted by $gcd(a, b)$.

   Lemma 2: For every two integers a and b, not both zero, with $gcd(a, b) = 1$, it is not the case that both a is even and b is even.

   Proof of Lemma 2:

   Towards a universal generalization, let a and b be integers, not both zero, with $gcd(a, b) = 1$. We will proceed in a **proof by contradiction** to show that it is not the case that both a is even and b is even.

   ... *Proof would continue here* ...

   Since the goal in a proof by contradiction is to prove $\neg p \to (r \wedge \neg r)$ for some propositions p and r, what would be **assumed** in the next step of the proof?

   Write this assumption both symbolically and in English.

   You do not need to complete the proof for credit.