HW4 Individual

CS40 Spring '22

Bharat Kathi (5938444)

Due: Friday, May 6, 2022 at 11:59PM on Gradescope

Integrity reminders for individual homeworks

- "Individual homeworks" must be solely your own work.
- You may not collaborate on individual homeworks with anyone or seek help from online tutors or entities outside the class.
- You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza. However, the staff will only answer clarifying questions on these homeworks. You *cannot* use any online resources about the course content other than the text book and class material from this quarter.
- Do not share written solutions or partial solutions for homework with other students. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW4-Individual".

Summary of Proof Strategies (so far)

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion**: To prove that $\forall x\, P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization**: To prove that $\forall x\, P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.

- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Strategies for conjunction**: To prove that $p \wedge q$ is true, have two subgoals: subgoal (1) prove p is true; and, subgoal (2) prove q is true. To prove that $p \wedge q$ is false, it's enough to prove that p is false. To prove that $p \wedge q$ is false, it's enough to prove that q is false.
- **Proof of Conditional by Direct Proof**: To prove that the implication $p \rightarrow q$ is true, we can assume p is true and use that assumption to show q is true.
- **Proof of Conditional by Contrapositive Proof**: To prove that the implication $p \rightarrow q$ is true, we can assume $\neg q$ is true and use that assumption to show $\neg p$ is true.

Assigned Questions

1. **Theorem**: If n and m are odd integers, then $n \cdot m$ is odd.

   For each of the following proof attempts of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

   (a) Let n and m be odd integers. Then $n = 2k + 1$ and $m = 2j + 1$. Plugging into the expression $n \cdot m$ gives
   $$n \cdot m = (2k + 1)(2j + 1) = 2(j \cdot k + j + k) + 1$$
   Since k and j are integers, $j \cdot k + j + k$ is also an integer. Then $n \cdot m$ equals to two times an integer plus one, therefore $n \cdot m$ is odd.

   Answer: The proof should state that the definition of an odd number is being used to show that n = 2k + 1 and m = 2j + 1. The proof should also state that k and j are some arbitrary integers. The steps in multiplying (2k+1) and (2j+1) need to be shown. The result is missing a coeffcient of 2 in front of $j * k$.

   (b) Let n and m be odd integers. Since n is an odd integer, then $n = 2k + 1$ for some integer k. Since m is an odd integer, then $m = 2j + 1$ for some integer j. Plugging in $2k + 1$ for n and $2j + 1$ for m into the expression $n \cdot m$ gives
   $$n \cdot m = (2k + 1)(2j + 1)$$
   Since $n \cdot m$ is equal to two times an integer plus one, then $n \cdot m$ is an odd integer.

   Answer: The proof needs to show the steps in multiplying (2k+1) and (2j+1).

   (c) Let n and m be odd integers. Since n is an odd integer, then $n = 2k + 1$ for some integer k. Since m is an odd integer, then $m = 2k + 1$ for some integer k. Plugging into the expression $n \cdot m$ gives
   $$n \cdot m = (2k + 1)(2k + 1) = 2(2k^2 + 2k) + 1$$
   Since k is an integer, $2k^2 + 2k + 1$ is also an integer. Since $n \cdot m$ is equal to two times an integer plus one, then $n \cdot m$ is an odd integer.

   Answer: Since m and n are two distinct integers, they need sepereate defintions for being odd numbers. They cannot both use the variable k. In the current setup, m and n are both equal to 2k+1 which would mean that m and n are equal.

2. **Theorem** For all non-zero integers, x, y, z, if x does not divide yz, then x does not divide y

   For each of the following proof attempts of the theorem, explain where the proof uses invalid reasoning or skips essential steps.

   (a) We prove this by contrapositive. Let x, y, z be non-zero integers. We assume that x divides y. Then $y = kx$. So,
   $$zy = z(kx) = (kz)x$$
   Since z, k are integers, zk is also an integer. So, x divides yz.

   Answer: The proof should explicitly state that the definition of being divisible is used, and that k is some arbitrary integer. It also needs to show some steps and that z is being multiplied to both sides of the equation. It should also explicitly state that yz is of the form mx, where m = zk is some integer.

(b) We prove this by contrapositive. Let x, y, z be non-zero integers. We assume that x divides y. So, $y = kx$ for some integer k. Since zk is also an integer, we conclude that x divides zy, thus proving the theorem.

Answer: The proof is missing all the steps to show that yz is of the form mx, where m = zk is an integer and x is nonzero.

(c) We prove this by contrapositive. Let x, y, z be non-zero integers. Assume that x divides y. So, $x = ky$ for some integer k. We further assume that $ky = ayz$ for some integer a, and show that x divides yz. We divide $ky = ayz$ out to get $a = \frac{k}{z}$. Since k, z are both integers, kz is also an integer, thus proving that x divides yz.

Answer: The proof has an incorrect definition of x — y, and also assumes that x — yz.

3. Consider the statement: The sum of any two integers is odd if and only if at least one of them is odd.

(a) Define predicates as necessary and write the symbolic form of the statement using quantifiers.

Answer: .

Let O(n) be the predicate "n is odd".
$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}(O(x+y) \leftrightarrow (O(x) \vee O(y)))$

(b) Prove or disprove the statement. Specify which proof strategy is used.

Answer: .

Disproving by counterexample.
Let x = 1 and y = 1.
The proposition is now $(O(1+1) \leftrightarrow (O(1) \vee O(1)))$
Since $1+1 = 2$ and 2 is not even, the left side of our conditional is false, while the right side is true.
Our conditional is equivalent to $F \leftrightarrow T$ which evaluates to F.
Therefore, the statement is disproven.

4. Consider the statement: If x and y are integers such that $x + y \geq 5$, then $x > 2$ or $y > 2$.

(a) Write the symbolic form of the statement using quantifiers.

Answer: $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}((x + y \geq 5) \rightarrow (x > 2 \vee y > 2))$

(b) Prove or disprove the statement. Specify which proof strategy is used.

Answer: .

Proof by contrapositive.
Let x and y be arbitrary integers.
Assume that $x \leq 2 \wedge y \leq 2$.
We will prove that $x + y < 5$.
Since $x \leq 2$ and $y \leq 2$, adding the two results in $x + y \leq 4$. Since $4 < 5$, we can say that $x + y < 5$.
Therefore, our original statement is proved by the contrapositive.

5. Consider the statement: The average of two odd integers is an integer.

(a) Write the symbolic form of the statement using quantifiers.

Answer: $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}(((\exists k \in \mathbb{Z}(x = 2k+1)) \wedge (\exists j \in \mathbb{Z}(y = 2j+1))) \rightarrow (\frac{x+y}{2} \in \mathbb{Z}))$

(b) Prove or disprove the statement. Specify which proof strategy is used.

Answer: .

Proof by direct proof.
Let x and y be arbitrary integers.
Assume that x and y are odd. We will prove that $\frac{x+y}{2}$ is an integer.
Since x is odd, that means that $x = 2k + 1$ for some intejer k.
Since y is odd, that means that $y = 2j + 1$ for some intejer j.
Substitute $x = 2k + 1$ and $y = 2j + 1$ into $\frac{x+y}{2}$.
$= \frac{(2k+1)+(2j+1)}{2}$
$= \frac{(2k+2j+2)}{2}$
$= \frac{2(k+j+1)}{2}$
$= k + j + 1$
Since we know that k and j are integers, $k + j + 1$ must also be an integer.
Therefore, $\frac{x+y}{2}$ is an integer.

6. Consider the statement: For any three consecutive integers, their product is divisible by 6.

(a) Write the symbolic form of the statement using quantifiers.

Answer: $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \forall z \in \mathbb{Z}(((y = x+1) \wedge (z = x+2)) \rightarrow (6|xyz))$

(b) Prove or disprove the statement. Specify which proof strategy is used.

Answer: .

Proof by cases.
Since x, y, and z are all consecutive, we can write them as the following:
$x = x$
$y = x + 1$
$z = x + 3$
Our 3 consecutive numbers are now x, x+1, and x+2.
So $xyz = x(x + 1)(x + 2)$
Since 6 = 2 x 3, an integer that is divisble by both 2 and 3 must also be divisible by 6.
Therefore, we can prove $6|x(x + 1)(x + 2)$ is divisible by 6 by proving $(2|x(x + 1)(x + 2) \wedge 3|x(x + 1)(x + 2))$.

Lets look at $2|x(x + 1)(x + 2)$ first.

Case 1: x is even.
This means that $x = 2k$, for some integer k since x is even.
Substitute $x = 2k$ into $x(x + 1)(x + 2)$.
$x(x + 1)(x + 2) = 2k(2k + 1)(2k + 2)$
$x(x + 1)(x + 2)$ can be written as $2m$ where $m = k(2k + 1)(2k + 2)$.
Since k is an integer, $k(2k + 1)(2k + 2)$ is also an integer, making m an integer as well.
Therefore, $x(x + 1)(x + 2)$ is divisible by 2.

Case 2: x is odd.
This means that $x = 2k + 1$, for some integer k since x is odd.

Substitute $x = 2k + 1$ into $x(x + 1)(x + 2)$.
$x(x + 1)(x + 2) = (2k + 1)(2k + 2)(2k + 3)$
$= (2k + 1)(2(k + 1))(2k + 3)$
$= 2(2k + 1)(k + 1)(2k + 3)$
$x(x + 1)(x + 2)$ can be written as $2m$ where $m = (2k + 1)(k + 1)(2k + 3)$.
Since k is an integer, $(2k + 1)(k + 1)(2k + 3)$ is also an integer, making m an integer as well.
Therefore, $x(x + 1)(x + 2)$ is divisible by 2.

Now that we have proven $2|x(x + 1)(x + 2)$, lets look at $3|x(x + 1)(x + 2)$.

Case 1: x is a multiple of 3.
This means that $x = 3k$ for some integer k.
Substitute $x = 3k$ into $x(x + 1)(x + 2)$.
$x(x + 1)(x + 2) = 3k(3k + 1)(3k + 2)$
$x(x + 1)(x + 2)$ can be written as $3m$ where $m = k(3k + 1)(3k + 2)$.
Since k is an integer, $k(3k + 1)(3k + 2)$ is also an integer, making m an integer as well.
Therefore, $x(x + 1)(x + 2)$ is divisible by 3.

Case 2: x is one number greater than a multiple of 3.
This means that $x = 3k + 1$ for some integer k.
Substitute $x = 3k + 1$ into $x(x + 1)(x + 2)$.
$x(x + 1)(x + 2) = (3k + 1)(3k + 2)(3k + 3)$
$= (3k + 1)(3k + 2)(3(k + 1))$
$= 3(3k + 1)(3k + 2)(k + 1)$
$x(x + 1)(x + 2)$ can be written as $3m$ where $m = (3k + 1)(3k + 2)(k + 1)$.
Since k is an integer, $(3k + 1)(3k + 2)(k + 1)$ is also an integer, making m an integer as well.
Therefore, $x(x + 1)(x + 2)$ is divisible by 3.

Case 3: x is two numbers greater than a multiple of 3.
This means that $x = 3k + 2$ for some integer k.
Substitute $x = 3k + 2$ into $x(x + 1)(x + 2)$.
$x(x + 1)(x + 2) = (3k + 2)(3k + 3)(3k + 4)$
$= (3k + 1)(3(k + 1))(3k + 4)$
$= 3(3k + 1)(k + 1)(3k + 4)$
$x(x + 1)(x + 2)$ can be written as $3m$ where $m = (3k + 1)(k + 1)(3k + 4)$.
Since k is an integer, $(3k + 1)(k + 1)(3k + 4)$ is also an integer, making m an integer as well.
Therefore, $x(x + 1)(x + 2)$ is divisible by 3.

Now that we have proven both $2|x(x + 1)(x + 2)$ and $3|x(x + 1)(x + 2)$, we have proven that $6|x(x + 1)(x + 2)$.
Therefore, we have proven that $6|xyz$.

7. Consider the following statements:

(i) If x and y are even integers, then $x + y$ is an even integer.

(ii) If $x + y$ is an even integer, then and x and y are both even integers.

(iii) If x and y are integers and $x^2 = y^2$, then $x = y$.

(iv) If x and y are real numbers and $x < y$, then $x^2 < y^2$.

(v) If x and y are positive real numbers and $x < y$, then $x^2 < y^2$.

(a) (*Graded for correctness*[1]) Express the statements (i) - (v) as quantified statements. Define any predicates as needed.

Answer: .

Let F(x) be defined as "x is even"

(i) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}((F(x) \wedge F(y)) \rightarrow F(x+y))$
(ii) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}(F(x+y) \rightarrow (F(x) \wedge F(y)))$
(iii) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}((x^2 = y^2) \rightarrow (x = y))$
(iv) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}((x < y) \rightarrow (x^2 < y^2))$
(v) $\forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}^+((x < y) \rightarrow (x^2 < y^2))$

(b) (*Graded for correctness of choice and fair effort completeness in justification*[2]) What are the error(s) in the following attempted proof for statement (ii)?

We prove this by contrapositive. So, by **universal generalization** let x and y be arbitrary integers, such that x and y are not both even. Without loss of generality, suppose that x is even and y is odd. Then $x = 2k$ for some integer k and $y = 2j+1$ for some integer j. $x + y = 2k + 2j + 1 = 2(k + j) + 1$. Since j and k are both integers, $j + k$ is too. Therefore, $x + y$ is odd. This proves the contrapositive, thus proving the claim. ■

Answer: Universal generalization can only be used with the cases "x is even and y is odd" and "x is odd and y is even" wihtout loss of generality. The proof for these two cases does not work for the case "both x and y are odd". So a seperate proof must be used for this case.

(c) Write the correct version of the proof for statement (ii) if it is true or provide a counterexample to disprove it if it is not.

Answer: Counterexample: x = 5, y = 5

The sum $x + y = 10$ which is even.

However, both x and y are odd (not even), rendering statement (ii) false.

(d) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (v) is being **disproved** by the following proof?

To disprove the statement, we need to find a counterexample. We need to show that there exist some x and y in the domain such that $\neg$ ($(x^2 = y^2) \rightarrow (x = y)$). Rewriting this goal using the disjunctive form of the implication and applying De Morgan's Law, we need to show ($(x^2 = y^2) \wedge \neg(x = y)$).

We choose the witnesses $x = -1$ and $y = 1$, which are both in the domain and satisfy the condition $x^2 = y^2$. However, since $-1 \neq 1$, the counterexample works to disprove the original statement. ■

Answer: Statement (iii) is being disproved by this proof.

Since they are trying to find a counterexample using $\neg$ ($(x^2 = y^2) \rightarrow (x = y)$), that means they are trying to disprove the statement ($(x^2 = y^2) \rightarrow (x = y)$), which is statement (iii).

[1] This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

[2] This means that for the justification, you will get full credit so long as your submission demonstrates honest effort to answer the question. You will not be penalized for an incorrect justification.

8. We define the set of balanced parentheses S recursively as follows:

   Basis: the string () is in S

   Recursive rules:

   (a) $\forall x \in S((x) \in S)$

   (b) $\forall x \in S \forall y \in S(xy \in S)$ where xy means the concatenation of x and y

   Using structural induction, prove that for any string a in S, the number of left parentheses in x is equal to the number of right parentheses in x.

   Answer: .

   Let $L[x]$ return the number of left parentheses in x.
   Let $R[x]$ return the number of right parentheses in x.

   Theorem: $\forall s \in S(L[s] = R[s])$

   Proof by Induction.
   Base Case: s = (). L[()] = R[()] = 1.
   Inductive Step: Assuming that $x \in S$ and $y \in S$, then L[x] = R[x] and L[y] = R[y]. Then we consider the following 2 cases depending on which was the last recursive rule.

   Case 1: $s = (x)$, where $x \in S$. We assume that L[x] = R[x] and prove that L[s] = R[s].

   $L[s] = L[(x)]$, Since $s = (x)$.
   $= 1 + L[x]$, Since (x) has one more (than x.
   $= 1 + R[x]$, By the inductive hypothesis.
   $= R[(x)]$, Since (x) has one more) than x.
   $= R[s]$, Since $s = (x)$.

   Case 2: $s = xy$, where $x \in S$ and $y \in S$. We assume the inductive hypothesis and prove that L[s] = R[s].

   $L[s] = L[xy]$, Since $s = xy$.
   $= L[x] + L[y]$
   $= R[x] + R[y]$, By the inductive hypothesis.
   $= R[xy]$
   $= R[s]$, Since $s = xy$.

   Therefore, L[s] = R[s].