CS 1510 Homework 2

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a

- Let A be an algorithm which produces a list fill amounts a motorist takes at each stop.
- For sake of contradiction assume A is not correct.
- Therefore there exists an input I where A produces a non-optimal output.
- Let $opt(I)$ be the optimal output that differs the least from $A(I)$.
- Since $A(I) \neq opt(I)$, there exists a first item in the list at stop x_i where $A(I)$'s fill amount f differs from $opt(I)$'s fill amount g.
- Every prior fill amount before x_i must be identical between $A(I)$ and $opt(I)$.
- Since $f \neq g$, then $f > g$ or $f < g$.
- If $f > g$:
  - Since $opt(I)$ is correct, g was enough fuel to get to the next stop.
  - Therefore f was not the minimum amount of gas needed.
  - A guarantees only filling up the minimum amount of gas needed.
  - There is a contradiction.
- So it can only be the case that $f < g$.
- $A(I)$ spends less time ($\frac{g-f}{r}$ seconds) filling up at x_i.
- Since $A(I)$ is correct (yet suboptimal), f was enough fuel to get to the next stop.
- Consider an alternative optimal solution $opt'(I)$ identical to $opt(I)$ except fill amount g is replaced with f at x_i.
- The time is still optimal, since less time is spent fueling at x_i. ($\frac{g-f}{r}$) seconds less.
- $opt'(I)$ cannot use less time than $opt(I)$, otherwise $opt(I)$ would not be optimal. So $opt'(I)$ utilizes $\frac{g-f}{r}$ seconds somewhere else.
- This extra time can be used to ensure that $opt'(I)$ is still a correct algorithm.

b

This algorithm A does not provide an optimal output for the following input I:

- Let A be the first gas station at kilometer 0
- Let x be the second gas station at kilometer 2
- Let B be the destination at kilometer 4

- Let C be the capacity 3 liters
- Let F be the consumption rate of 1 liter per kilometer
- Let r be the fill rate of 1 liter per minute

$A(I)$ produces the following output (a list of actions):

1. Fuel tank starts at 0/3.
2. Not enough gas to make it to x, so fill the tank up all the way with 3 liters at A, taking 3 minutes. (Fuel tank at 3/3).
3. Travel to gas station x, 2 kilometers away. (Fuel tank at 1/3).
4. Not enough gas to make it to B, so fill the tank up all the way with 2 liters at x taking 2 minutes. (Fuel tank at 3/3).
5. Arrive at destination B, 2 kilometers away. (Fuel tank at 1/3).

$A(I)$ requires 3 + 2 = 5 minutes of fueling.

A better output $O(I)$ is:

1. Fuel tank starts at 0/3.
2. Fill 3 liters at A, taking 3 minutes. (Fuel tank at 3/3).
3. Travel to gas station x, 2 kilometers away. (Fuel tank at 1/3).
4. Fill 1 liters at x, taking 1 minute. (Fuel tank at 1/3).
5. Arrive at destination B, 2 kilometers away. (Fuel tank at 0/3).

$O(I)$ requires 3 + 1 = 4 minutes of fueling.

Since there exists a more optimal output than $A(I)$, A is not a correct algorithm.

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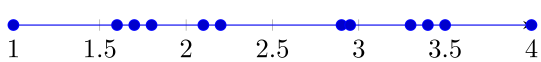
a

The presented algorithm (hereafter referred to as A) is not correct as it does not provide an optimal output for the presented problem for the following input (hereafter referred to as P). For the sake of simplicity I will be presenting the input (P) as a list of points on the real line and the unit intervals as intervals of length 1 on the real line. Additionally, S shall be the solution set of intervals.

- Let $P = \{1, 1.6, 1.7, 1.8, 2.1, 2.2, 2.9, 2.95, 3.3, 3.4, 3.5, 4\}$



- A always selects the interval that covers the most points; clearly, the intervals from from 1.6 to 2.2 and from 2.9 to 3.5 cover the most points as they both cover 5 points each. For the sake of argument, let's say A selects the interval from 1.6 to 2.2 first and adds it to S.
- A will next select the interval from 2.9 and 3.5 and add it to S as it also covers 5 points.
- A is now left with points 1 and 4 to cover with intervals. They will have to be covered separately, as they are more than 1 unit away from each other.
- After covering 1 and 4, A's optimal output is: $P = \{$1, 1.6-2.2, 2.9-3.5 , 4$\}$
- However, a more optimal output is the obvious three intervals from 1-2, 2-3, and 3-4.
- Since there exists a more optimal output than $A(P)$, A is not a correct algorithm.

b

This algorithm A solves the problem. Because I am using A to refer to the algorithm, I will refer to the set of points listed in the problem as P.

- Assume the algorithm A is incorrect, and has some input I that causes it to give suboptimal output.
- Let $O(I)$ be the optimal algorithm on I that differs the least from $A(I)$.
- Let j be the first point that the algorithms differ. That is, before we've reached p_j, the $j'th$ item in P, $A(I) = O(I)$.
- Because of our assumption, $O(I)$ chose another way of covering point p_j with a unit interval. Because we know that $A(I)$ will have a unit interval beginning at point p_j, by definition of the algorithm, that leaves 2 possibilities for how $O(I)$ placed the unit interval.
- Either $O(I)$ has a unit interval ending at point p_j, or $O(I)$ placed a unit interval, where p_j appears somewhere in between.
- Neither of these situations are more optimal than the one A chose.
- In the case that the interval ends at p_j, we know that it is not more optimal, as if it covered more points before p_j, A would have already selected one of those earlier points, placing a unit interval beginning at one of those points and covering p_j.
- Similarly, in the case that p_j appears somewhere in the middle of the interval placed by O, no points appearing earlier would be captured (for the reason stated above), and any point after p_j that would be captured by this interval would also be captured by the interval that A placed.
- Thus, we can construct $O'(I)$ by removing the interval it placed at step j, and adding an interval beginning at p_j.
- $O'(I)$ agrees with $A(I)$ for one more step without sacrificing correctness, so we have reached a contradiction.