Homework 19

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Problem 7.8 from the text: Show that if $\overline{3SAT} \in BPNP$, then PH collapses to Σ_3^p.
Hint: Recall the proof that BPP is in Σ_2^p.

Proof:
Assume that $\overline{3SAT} \in BPNP$. Then we know that there exists a probablistic Turing machine M which outputs a reduction in poly-time such that $P(x \in \overline{3SAT} \Rightarrow M(x) \in 3SAT) > 2/3$. Let M get its decisions for random transitions from an advice tape a. Thus for 2/3 of all possible a, $x \in \overline{3SAT} \Rightarrow M(x,a) \in 3SAT$.

The following proof is in a similar vein to the proof of $BPP \subseteq \Sigma_2^p$:
If 2/3 of all possible values of a produce a correct reduction and the other 1/3 do not, then it follows that there exists some grouping of values $(a_1, a_2, ...a_k)$ such that for all groups the majority of advices $a_1...a_k$ cause M reduce correctly.

Now construct a deterministic TM N with tapes x, y, z, w such that:
On input x, where x is an $\overline{SAT}$ problem:
$\exists y$ where y is grouping, such that
$\forall z$ where z is groups of advice, $M(x,z)$ is an instance of a SAT formula, such that
$\exists w$ where w is a correct assignment of variables to satisfy the formula.

N is clearly a TM with a language in Σ_3^p. Since $L(M) = L(N) = \overline{3SAT}$, $\overline{3SAT} \in \Sigma_3^p$.

Since $\overline{3SAT}$ is $coNP$-complete, and $coNP = \Pi_1^p \subseteq \Sigma_2^p \subseteq \Pi_3^p$, then $\Pi_3^p = \Sigma_3^p$ since $\overline{3SAT}$ reduces to Σ_3^p.

We can group variables from similar adjacent quantifiers to reduce the complexity of the problem to a smaller n. And we can repeat these two steps (changing quantifiers, and grouping variables into tuples) to reduce any Σ_n^p or Π_n^p problem into a Σ_3^p, collapsing the polynomial hierarchy.

In other words, since $\Sigma_3^p = \Pi_3^p$,
then $PH \subseteq \Sigma_3^p$, since $\exists ... \forall \exists \forall \exists P = \exists ... \forall \Sigma_3^p = \exists ... \forall \Pi_3^p = \exists ... \Pi_3^p ... = \Pi_3^p = \Sigma_3^p$.