Homework 3

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4

a

Claim:
There is no Turing Machine that takes input TM M and determines whether the language $L(M)$ accepted by M is the empty language.

Proof by reduction:
We will reduce the Halting Problem HP to the stated Empty Language Problem ELP. Let $\langle X, w\rangle$ be the input to HP, where X is a TM and w is the input to the TM. Then create a new TM N from $\langle X, w\rangle$ where N rejects in all cases where input to N is not w, and when the input is w, N runs X on w. Finally, pass N as input to the ELP and return the opposite of ELP's output.
If X halts on w, then:

- N will accept on w
- ELP will return 0 since N accepts a non-empty input
- HP returns the opposite, which is 1

so HP returns 1 if ELP returns 0. Next, if X loops forever on w, then:

- N will not accept on w
- N rejects any other input
- N always rejects
- ELP returns 1 since N has no strings in $L(N)$ that accept
- HP returns the opposite, which is 0

so HP returns 0 if ELP returns 1.
Finally, $HP \leftrightarrow !ELP$, so $HP \leq ELP$, and thus ELP is undecidable.

b

Claim:
There is no Turing Machine that takes input TM M and determines whether the language $L(M)$ accepted by M is the language of every string over the input alphabet.

Proof by reduction:
We will reduce the Halting Problem HP to the stated Complete Language Problem CLP. Let $\langle X, w\rangle$ be the input to HP, where X is a TM and w is the input to the TM. Then create a new TM N from $\langle X, w\rangle$ where N accepts in all cases where the input to N is not w, and when the input is w, N runs X on w. Finally, pass N as input to the CLP and return the same as CLP's output.
If X halts on w, then:

- N will accept on w

- N accepts all other inputs
- N always accepts
- CLP will return 1 since N accepts all strings
- HP returns the same output, 1

so HP returns 1 if CLP returns 1. Next, if X loops forever on w, then:

- N will not accept on w
- N did not accept all inputs
- CLP returns 0 since there are strings in $L(N)$ that N does not accept
- HP returns the same, 0

so HP returns 0 if CLP returns 0.
Finally, $HP \leftrightarrow CLP$, so $HP \leq CLP$, and thus CLP is undecidable.

c

Claim:
There is no Turing Machine that takes input TM M and determines whether the language $L(M)$ accepted by M includes the string 11110.

Proof by reduction:
We will reduce the Halting Problem HP to the stated Specific Language Problem SLP. Let $\langle X, w\rangle$ be the input to HP, where X is a TM and w is the input to the TM. Then create a new TM N from $\langle X, w\rangle$ where N runs X on w, and then accepts if the input to $N = 11110$. Finally, pass N as input to the SLP and return the same as SLP's output.
If X halts on w, then:

- N will accept if the input to $N = 11110$
- SLP will return 1 since $L(N) = \{11110\}$
- HP returns the same output, 1

so HP returns 1 if SLP returns 1, that is, if $11110 \in L(N)$.
Next, if X loops forever on w, then:

- N will not accept on w
- N will not accept any strings.
- SLP returns 0 since $L(N) = \phi$ and thus $11110 \notin L(N)$.
- HP returns the same, 0

so HP returns 0 if SLP returns 0.
Finally, $HP \leftrightarrow SLP$, so $HP \leq SLP$, and thus SLP is undecidable.

d

Claim:
Given some property P of languages, there is no Turing Machine that takes input TM M and determines whether the language $L(M)$ accepted by M satisfies P.

Proof by reduction:
We will reduce the Halting Problem HP to the stated Property Language Problem PLP. Let $\langle X, w\rangle$ be the input to HP, where X is a TM and w is the input to the TM. Next, let M_1 be a TM that accepts all inputs that satisfy property P. Then create a new TM N from $\langle X, w\rangle$ that, on input s, runs X on w and then runs M_1 on s and accepts if and only if M_1 accepts. Finally, run PLP with input N and return PLP's result as the final result for the Halting Problem.
If X halts on w, then:

- N will run X on w and halt
- N will run M_1 on s and return 1 if M_1 accepts
- PLP will return 1 since N returned 1

so PLP returns 1 if X halts on w.
Next, if X loops forever on w, then:

- N will run X on w but never halt
- N will never halt
- PLP returns 0 since N never halts

so PLP returns 0 if X does not halt on w.
Finally, $X \leftrightarrow PLP$, so $HP \leq PLP$, and thus PLP is undecidable.

e

To show that 4.a, 4.b, 4.c are consequences of 4.d:

1. For problem 4.a, let P_a be the property that the language is empty
2. For problem 4.b, let P_b be the property that the language is every string over the input alphabet
3. For problem 4.c, let P_c be the property that the string 11110 is in the language

Each bullet's P is a valid property for a language, thus we can apply the result of 4.d to show 4.a,4.b,4.c are consequences of 4.d.