Homework 19

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Problem 7.8 from the text: Show that if $\overline{3SAT} \in BPNP$, then PH collapses to Σ_3^p.
Hint: Recall the proof that BPP is in Σ_2^p.

Proof:
Assume that $\overline{3SAT} \in BPNP$. Then we know that $\exists M$ where M is a Turing machine which does the reduction $\overline{3SAT} \leq_p 3SAT$ in polynomial time such that $P(\overline{3SAT}(M(x)) = 3SAT(x)) > 2/3$.

If $\overline{3SAT} \leq_p 3SAT$ with, then there is a bijective relationsihp between problems in NP and co-NP since $3SAT$ and $\overline{3SAT}$ are complete in each complexity class, respectively. Given that bijective relationship, any NP problem represented as $\exists y$ such that $F(x, y)$ iff $x \in L$ can be rewritten as $\forall y$ then $!F(x, y)$ iff $x \notin L$.

Furthermore, if we can change a quantifier for any NP or co-NP problem, we could use the same argument for any Σ_n^p or Π_n^p problem to swap quantifiers. Then we could group variables from similar adjacent quantifiers to reduce the complexity of the problem to a smaller n. Finally we could repeat these two steps (changing quantifiers, and grouping variables into tuples) to reduce any Σ_n^p or Π_n^p problem into a NP or co-NP problem with a single quantifier, collapsing the polynomial hierarchy.

The last issue to be addressed is that our reduction M only has a greater than 2/3 success rate, but our previous argument requires a reduction with a perfect bijective relationship. So it remains to be shown that we can tack on a few more quantifiers to our original problem collapsed to NP or co-NP, to guarantee the reduction to change quantifiers is always correct.

(proof with "weddings")

Old Stuff:
$\overline{3SAT} \in BPNP \Rightarrow PH \subseteq \Sigma_3^p$
Assume $\overline{3SAT} \in BPNP$.

End of the proof:
$\Sigma_3^p = \Pi_3^p$
Therefore $PH \subseteq \Sigma_3^p$, since $\exists ... \forall \exists \forall \exists P = \exists ... \forall \Sigma_3^p = \exists ... \forall \Pi_3^p = \exists ... \Pi_3^p ... = \Pi_3^p = \Sigma_3^p$.

Brett's note: Proving that $\overline{3SAT} \leq_p 3SAT$ would imply that $NP = coNP$, which implies that $PH = NP = \Sigma_1^p$, which is a stronger conclusion than the homework. Since it's stronger, it is probably harder to prove, so I would avoid this route.