

Connectivity and Robustness in Networks

Network robustness: the ability of a network to maintain its general structural properties when it faces failures or attacks.

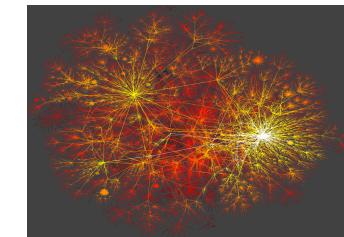
Type of attacks: removal of nodes or edges.

Structural properties: connectivity.

Examples: airport closures, internet router failures, power line failures.



Network of direct flights around the world
[Bio.Diaspora]



Internet Connectivity [K. C. Claffy]

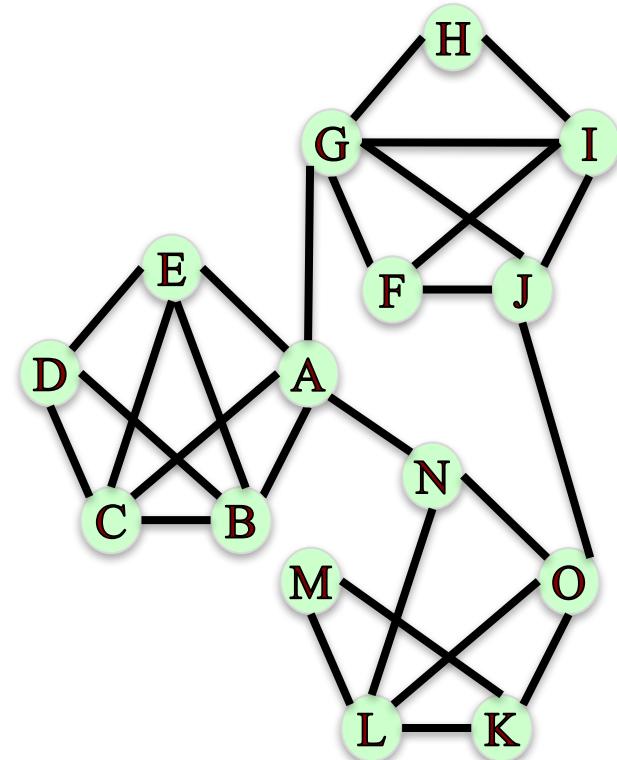
Disconnecting a Graph

What is the smallest number of nodes that can be removed from this graph in order to disconnect it?

```
In: nx.node_connectivity(G_un)  
Out: 1
```

Which node?

```
In: nx.minimum_node_cut(G_un)  
Out: {'A'}
```



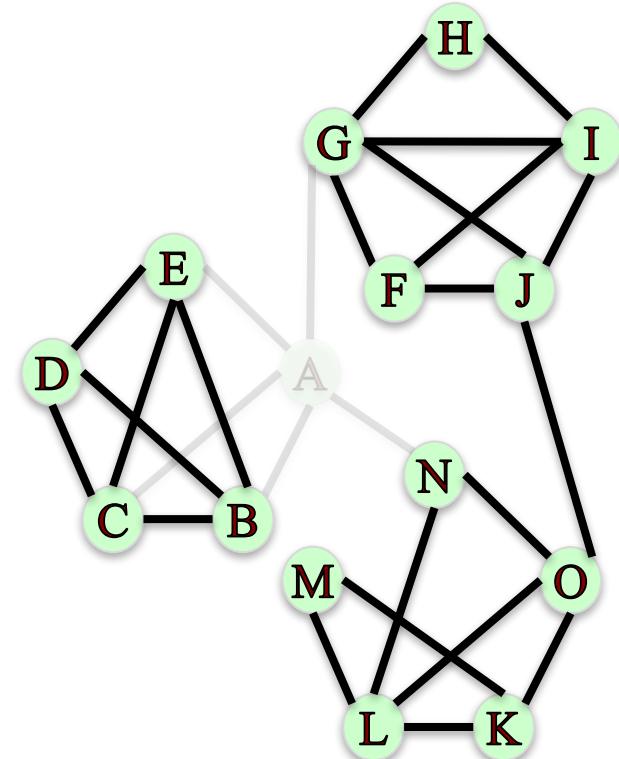
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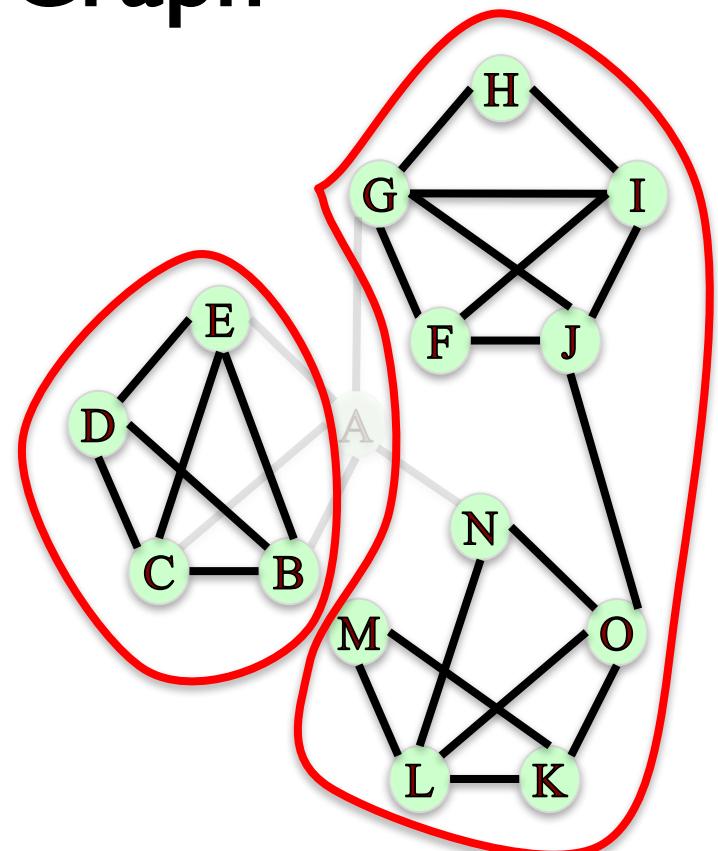
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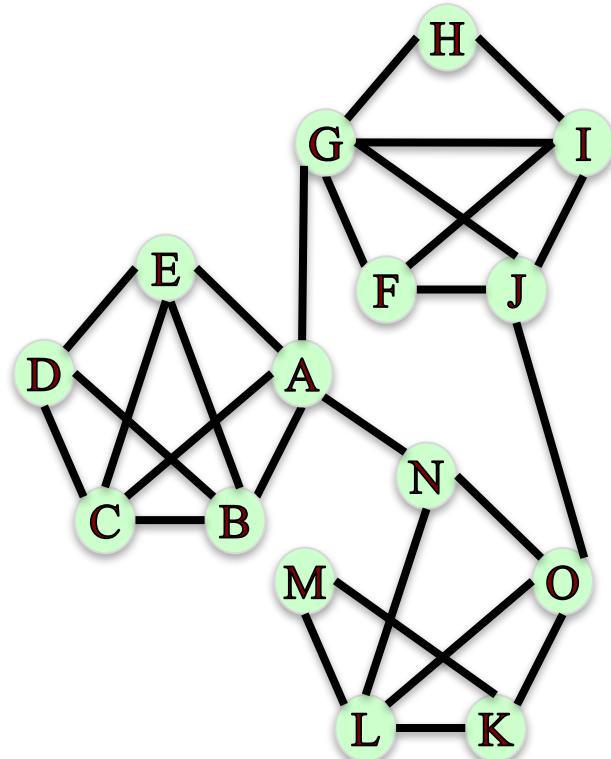
Disconnecting a Graph

What is the smallest number of **edges** that can be removed from this graph in order to disconnect it?

```
In: nx.edge_connectivity(G_un)  
Out: 2
```

Which edges?

In: `nx.minimum_edge_cut(G_un)`
Out: `{('A', 'G'), ('O', 'J')}`



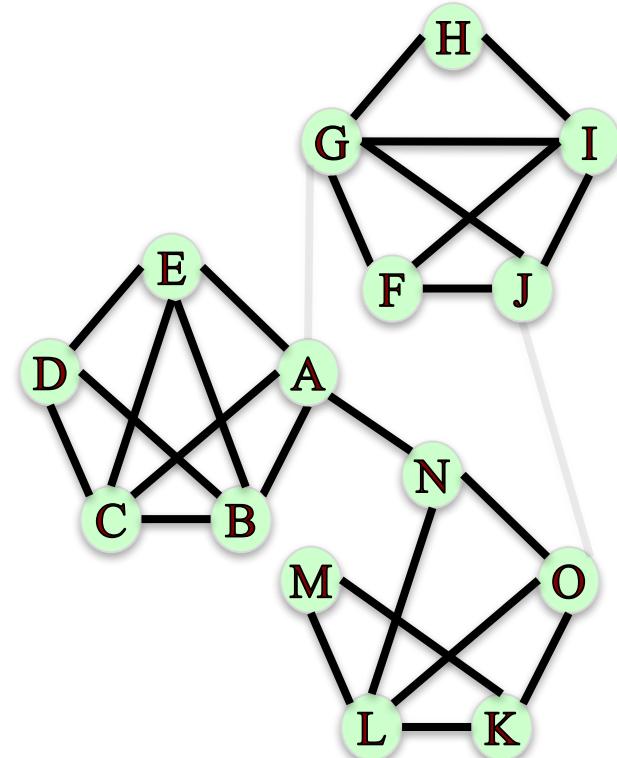
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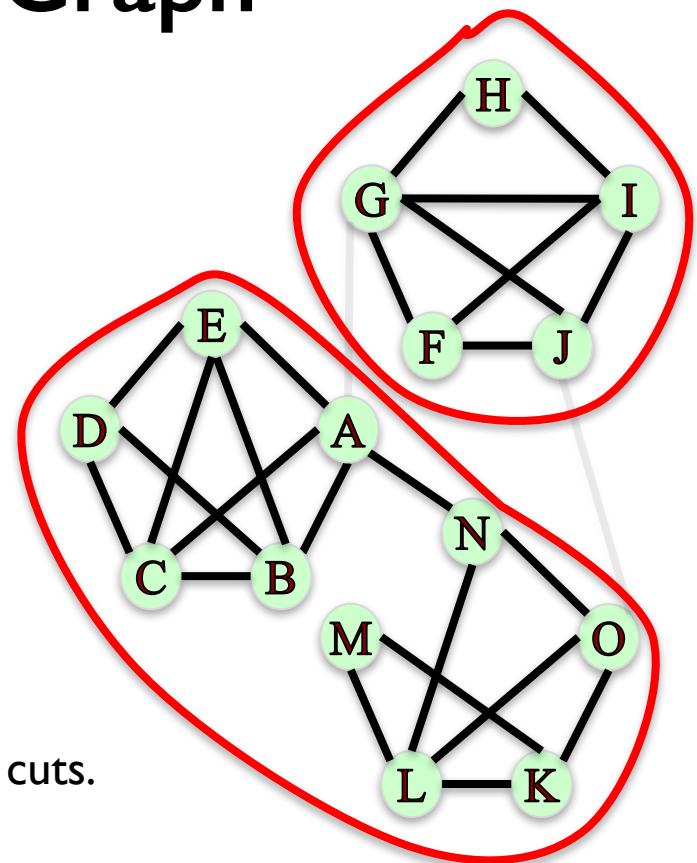
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Robust networks have large minimum node and edge cuts.



Simple Paths

Imagine node G wants to send a message to node L by passing it along to other nodes in this network.

What options does G have to deliver the message?

```
In: sorted(nx.all_simple_paths(G, 'G', 'L'))
```

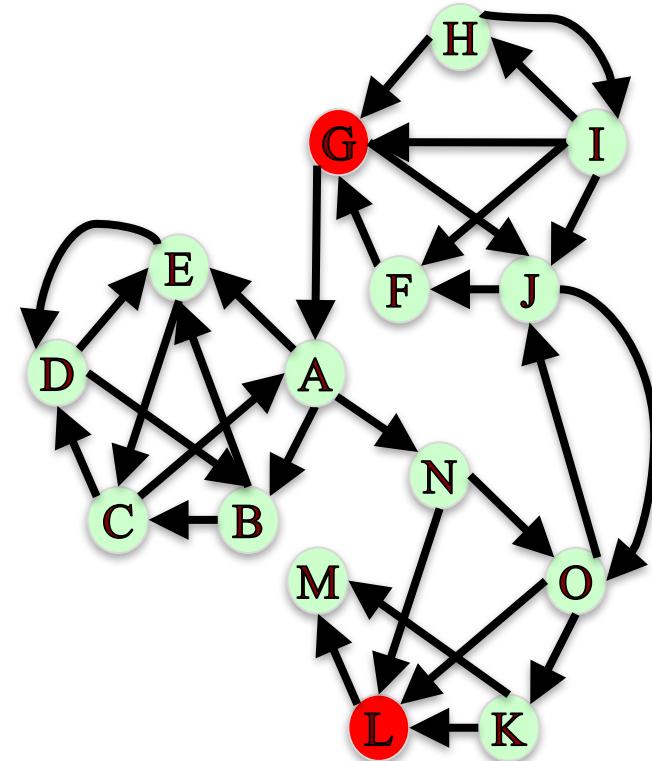
Out: `['G', 'A', 'N', 'L']`

['G', 'A', 'N', 'O', 'K', 'L'],

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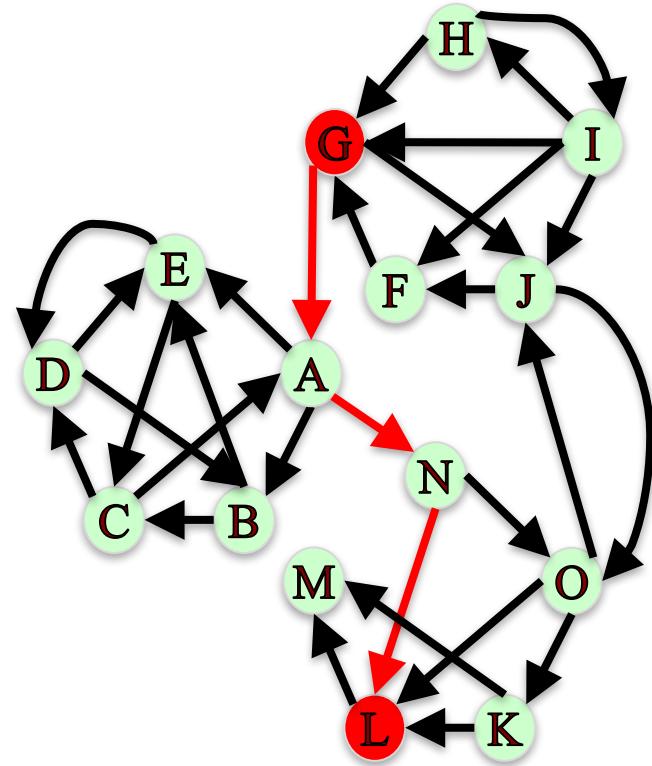
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Out: [['G', 'A', 'N', 'L'],
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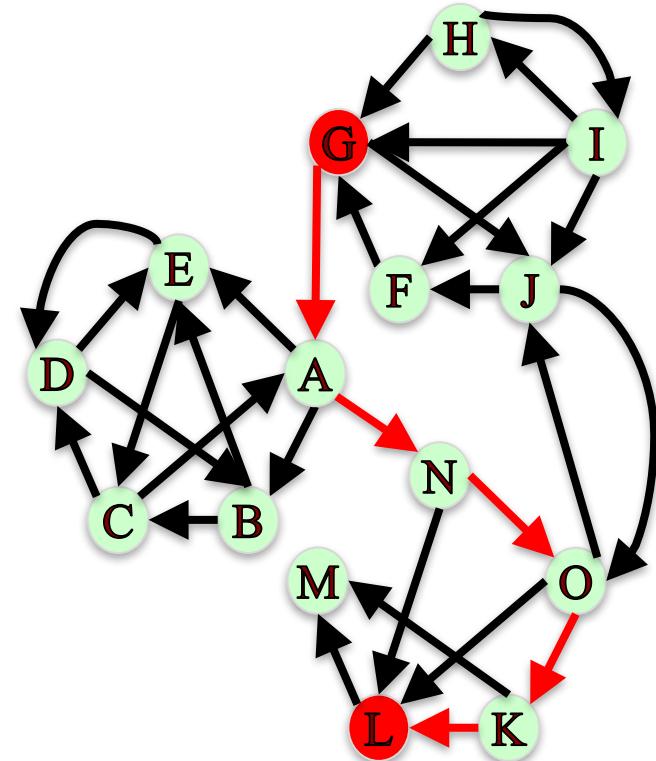
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Simple Paths

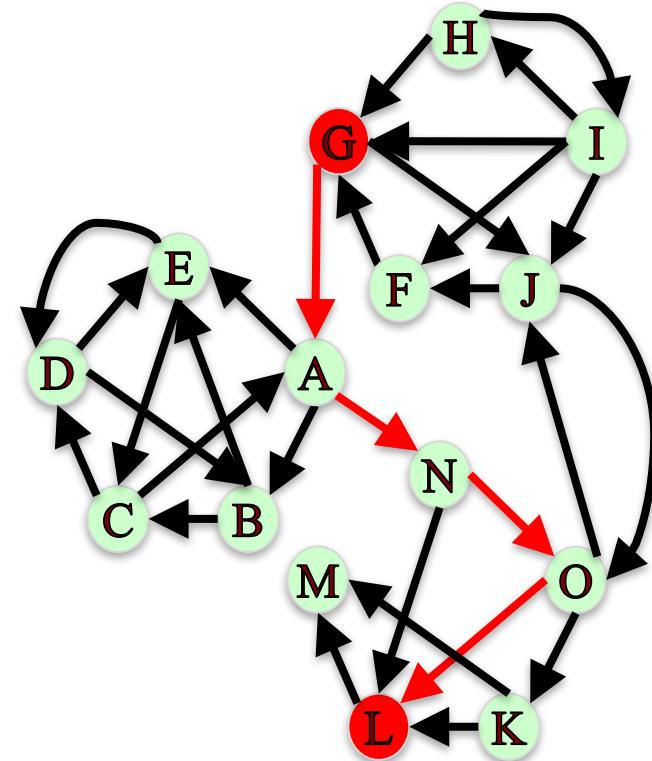
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```



Simple Paths

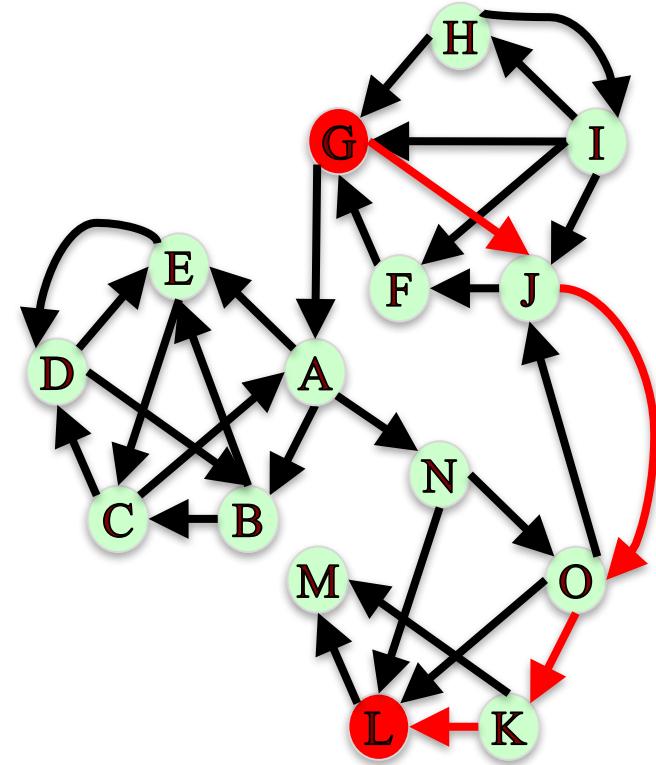
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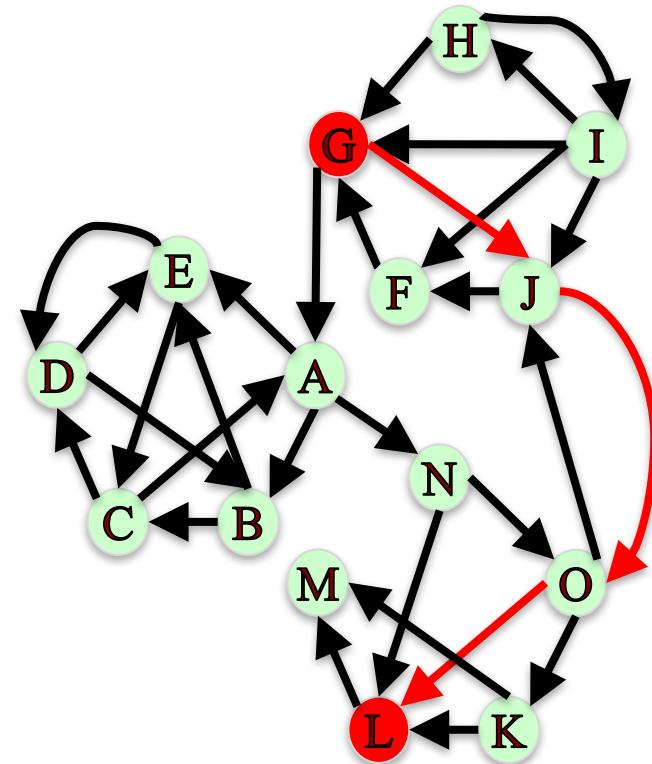
Out: `['G', 'A', 'N', 'L']`

['G', 'A', 'N', 'O', 'K', 'L'],

['G', 'A', 'N', 'O', 'L'],

['G', 'J', 'O', 'K', 'L'],

[G, J, O, L])



Node Connectivity

If we wanted to block the message from G to L by removing **nodes** from the network, how many nodes would we need to remove?

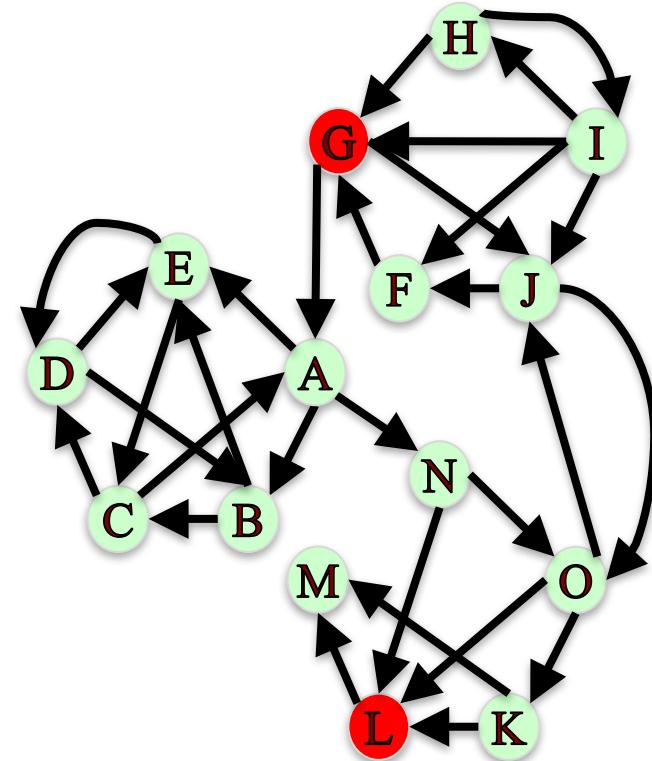
In: `nx.node_connectivity(G, 'G', 'L')`

Out: 2

Which nodes?

In: `nx.minimum_node_cut(G, 'G', 'L')`

Out: `{'N', 'O'}`



Node Connectivity

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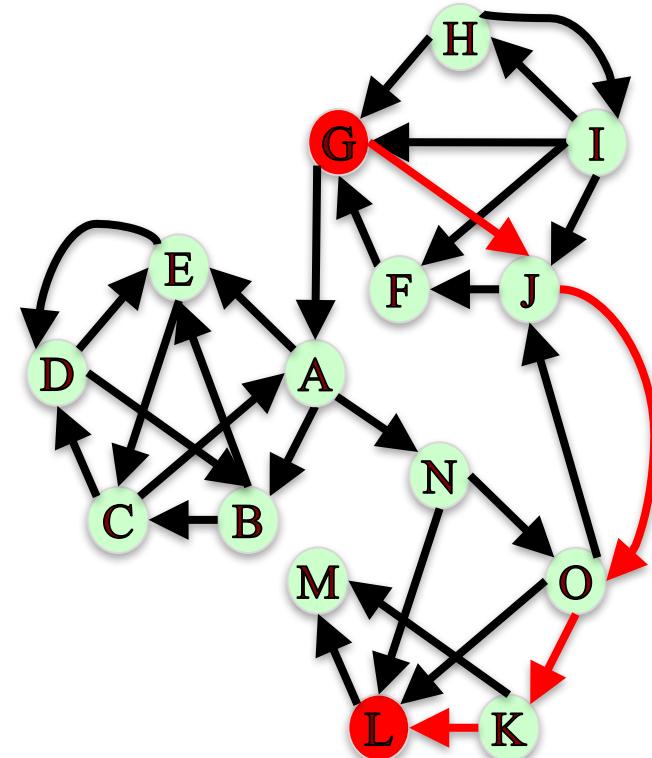
Out: 2

Which nodes?

In: `nx.minimum_node_cut(G, 'G', 'L')`

Out: `{'N', 'O'}`

If we only remove node N, message can go on path
 $G \rightarrow J \rightarrow O \rightarrow K \rightarrow L$.



Node Connectivity

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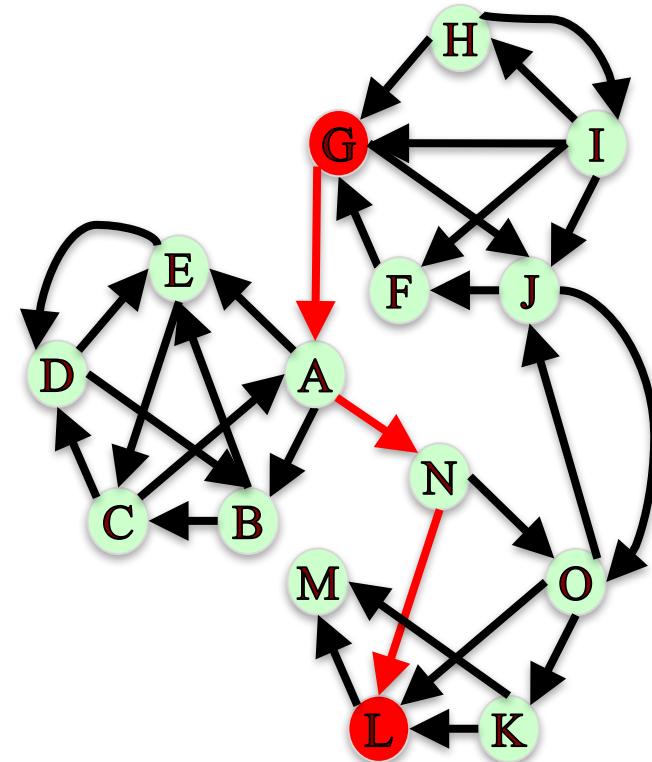
Out: 2

Which nodes?

In: `nx.minimum_node_cut(G, 'G', 'L')`

Out: `{'N', 'O'}`

If we only remove node O, message can go on path
 $G \rightarrow A \rightarrow N \rightarrow L$.



Edge Connectivity

If we wanted to block the message from G to L by removing **edges** from the network, how many edges would we need to remove?

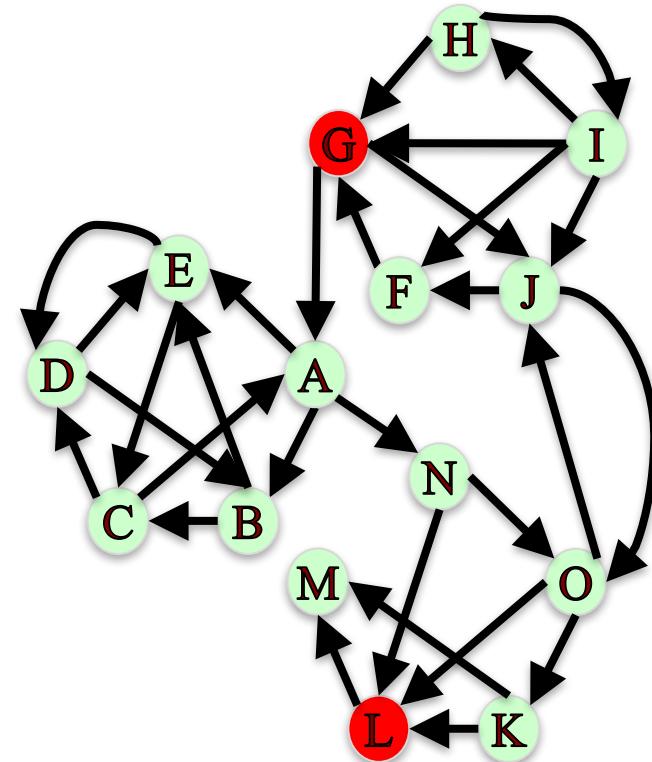
```
In: nx.edge_connectivity(G, 'G', 'L')
```

Out: 2

Which edges?

```
In: nx.minimum edge cut(G, 'G', 'L')
```

Out: $\{('A', 'N'), ('J', 'O')\}$



Edge Connectivity

If we wanted to block the message from G to L by removing **edges** from the network, how many edges would we need to remove?

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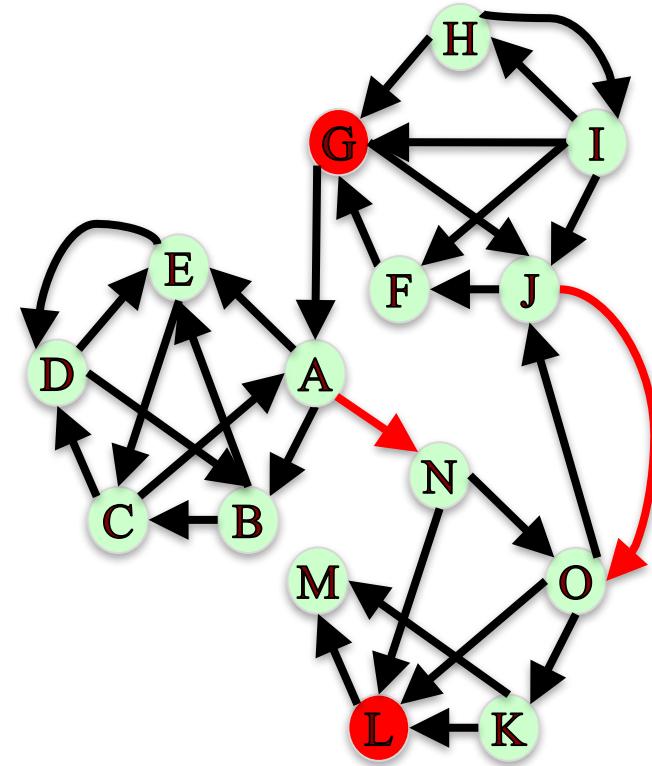
Out: 2

Which edges?

In: `nx.minimum_edge_cut(G, 'G', 'L')`

Out: `{('A', 'N'), ('J', 'O')}`

We need to remove $A \rightarrow N$ and $J \rightarrow O$ to block messages from G to L.



Summary

Node connectivity: Minimum number of nodes needed to disconnect a graph or pair of nodes.

```
nx.node_connectivity(G, 'G', 'L')  
nx.minimum_node_cut(G, 'G', 'L')
```

Edge connectivity: Minimum number of edges needed to disconnect a graph or pair of nodes.

```
nx.edge_connectivity(G, 'G', 'L')  
nx.minimum_edge_cut(G, 'G', 'L')
```

Graphs with large node and edge connectivity are more robust to the loss of nodes and edges.

