AUTOMATED CONJECTURE-MAKING III: CHOMP!

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Abstract. This article makes a novel attempt at constructing winning strategies in the combinatorial game Chomp.

1. Chomp Basics

In 1974, David Gale formulated a "curious nim-type game" named Chomp.[1] Chomp is a two player impartial combinatorial game played on a rectangular array of objects called a Chomp board. While the nature of these objects is inconsequential, we use cookies. Players alternate choosing one of these cookies to remove from the board. Once a cookie is selected, it is eliminated from the board along with all cookies to the right and below. The opposing player then moves in kind. The goal of Chomp is to force the opponent to take the top left cookie on the board, referred to as the *poison cookie* (see Figure **??** for a sample game). As a combinatorial game, Chomp research makes reference to several terms unique to that field. This includes regular reference to P-positions and N-Positions.[?]

Definition 1. *A P-position is any Chomp board in which the previous player has a winning strategy. This is equivalent to the statement "all reachable positions are N-Positions".*

Definition 2. *A N-Position is any Chomp board where the next player has a winning strategy. This is equivalent to the statement "at least one reachable position is a P-Position".*

It is important to consider a few special cases of P-Positions and N-Positions encountered during Chomp game play.

Definition 3. *A Balanced L Board is a $M \times M$ dimension Chomp board such that the first row and first column have M cookies and all other rows and columns have one cookie.***??**

Remark 1. A Balanced L Board is a P-Position.

Proof. When given a Balanced L Board, a player can choose some cookie (1,j) or (i,1). The opposing player then responds by choosing (j,1) or (1,i), respectively. This process continues and it is easy to see that it ends in a win for the opposing player. Therefore, Balanced L's are P-Positions. □

Definition 4. *An Unbalanced L Board is a $M \times N$ dimension Chomp board, where $M \neq N$, such that the first row and first column have N cookies and all other rows and columns have one cookie.*??

Remark 2. An Unbalanced L Board is a N-Position.

Proof. Suppose a player is given an Unbalanced L Board of dimension M × N.
Case 1: M > N.
The player chooses the $(1, n)$ cookie giving the next player a P-Position and the game follows the balanced L strategy above.
Case 2: M < N.
The player chooses the $(m, 1)$ cookie giving the next player a P-Position and the game follows the Balanced L strategy above. □

Definition 5. *A Square Chomp board is a $M \times M$ board such that each row has M cookies.*

Remark 3. A Square Chomp board is a N-Position

Proof. Suppose that a player is given a square Chomp board. The player can take the $(2, 2)$ cookie giving the opposing player a balanced L position, which is a P-Position. □

Definition 6. *Two-rowed Chomp and Three-Rowed Chomp are Chomp Boards of the form $2 \times N$ and $3 \times N$, respectively.*

Remark 4 (Two row chomp)*.* A Chomp board in Two-rowed Chomp is a P-position if, and only if, the number of cookies in the first row is one greater than the number of cookies in the second row.

Proof. As shown in Gale's 1974 paper.[1] □

In Gale's original 1974 paper, he presents a particularly beautiful proof that, regardless of size, an initial rectangular board is an N-Position.

Theorem 5. *Given any Chomp board with dimension $M \times N$, the player who moves first has a winning strategy.*

Proof.
Case 1: Taking the (m, n) cookie is a winning strategy
The first player simply selects (m, n) and has a winning strategy.
Case 2: Taking the (m, n) cookie is not a winning strategy

If (m, n) is not a winning strategy, then there must be some other cookie (i, j) such that taking (i, j) is a winning strategy in response to the first player taking (m, n). However, the first player could have also chosen this cookie (i, j) and then would have a winning strategy. □

It is important to note that this represents a example of a nonconstructive existence proof. That is, while the proof conclusively shows that a winning strategy exists for the first player, it does no work showing the process by which this winning strategy would be formulated. This goal of constructing a winning strategy given some arbitrary Chomp board has been the goal of much research since the game's initial formulation. Doron Zeilberger's collection of Chomp research papers have this project as their main goal. Zeilberger has focused mainly on Three Rowed Chomp and has constructed several theorems regarding those boards.

Theorem 6. *Given a Three-Rowed Chomp board with one cookie in the third row, the board is a P-Position iff the board is a balanced L with three cookies in the first row and column or the board is the transpose of a two row solution.*

Proof. As shown in Zeilberger's Three-Rowed CHOMP[?] □

Theorem 7. *Given a Three-Rowed Chomp board with two cookies in the third row, the board is a P-Position iff the number of cookies in the first row is two greater than the number of cookies in the second row*

Proof. As Shown in Zeilberger's Three-Rowed Chomp[?] □

While a winning strategy is known for these special boards, no general form of a winning strategy for some arbitrary board has been constructed. Furthermore, discovering additional P-Positions has required extensive computer assisted evaluation. All *reachable* boards must be evaluated for reachable P-Positions and this must be iterated through a full game play for each possible board to determine if the original position is a P-Position.

Definition 7. *A Chomp board, B, is reachable from another Chomp board, B' if, and only if, given B a player can make one legal move and leaving their opponent with B'.*

The goal of our research is two pronged: to develop an intelligent game player for Chomp and to discover and prove Chomp Theorems. These goals are pursued in a novel way by means of automated conjecture making. Section 2 describes what conjecture making is and how exactly it is used to accomplish our goals. Section 4 illustrates the success of this method at accomplishing the first goal of developing an intelligence game player and Section 5 explores conjecturing's effectiveness at producing viable Chomp Theorems.

The conjecture-making program makes its conjectures about P-positions. This is beneficial in two ways. The conjectures help advance the theory of P-positions and may be used when deciding between potential moves. However, it is worth noting that the conjectures may not be true for all Chomp boards. But they will be true for the finite amount of boards the program knows of. As we build the database of boards the computer is able to fine tune its conjectures to a broader class of Chomp boards.

2. Conjecturing program & Conjecture Examples

Conjecturing is a program that, for our purposes, may be described as a black box. The inputs of the program are Chomp boards and invariants. It outputs conjectures applicable to the Chomp board inputs. A full description of how the program runs may be found in the forthcoming paper *Automated Conjecturing II.*

It is important to determine what Chomp boards should be added to the objects list. We start with small examples of boards whose P-ness has already been ascertained. Chomp boards such as [1], [2,1], [5,4], [4,2,2], [5,3,2] and [10,8,2] we know to be P-positions by two[1] and three-row[3] Chomp theory.

Next we supply Conjecturing with invariants that it uses to produce conjectures. An *invariant* is a number that can be associated with any Chomp board. Examples include the number of rows, the amount of cookies, and the smallest column size. Table 1 below shows the initial list of invariants used to generate conjectures.

Conjecturing feeds off of invariants. That is, the number of conjectures output is directly correlated with the number of invariants input. In order to generate several useful conjectures, it often takes a large number of both invariants and Chomp boards.

Table 2 shows some examples of an early conjecture-making output taking only simple boards and our initial invariants. After generating conjectures for these Chomp boards, individual conjectures are considered for their truth value for all boards and for P-Positions specifically. This application of conjecture making furthers the second goal of our project: to prove Chomp Theorems. However, it is possible that a conjecture made by the program will be true for all the P-Positions input and not be true for all P-Positions generally. If a P-Position counterexample is found for one of the conjectures, then this position is added to the list of Chomp board inputs thereby eliminating this conjecture from the output list and producing a different set of conjectures.

In the case of the conjectures in Table 2, the counterexamples [10,2,2,1,1,1,1,1,1] and [2,1] would be input as objects generating new conjectures. Iterations of this process are used to produce conjectures that hold true for more and more P-Position Chomp boards.

Invariant	**Explanation**
ChompBoard.number_of_cookies	
ChompBoard.number_of_rows	
ChompBoard.number_of_columns	
ChompBoard.full_rectangle	Number of rows times the number of columns.
ChompBoard.rows_of_different_length	
ChompBoard.trace_square	Counts the number of cookies on the main diagonal of a squared Chomp Board.
ChompBoard.squareness	The absolute value of the number of rows minus the number of columns.
ChompBoard.row_product	The product of the row lengths.
ChompBoard.column_product	The product of the column lengths.
ChompBoard.average_cookies_per_column	The number of cookies divided by columns
ChompBoard.average_cookies_per_row	The number of cookies divided by rows

Table 1. Initial Chomp board invariants

3. Using Conjectures in Robot player

The computer selects the best position to move to based on the generated conjectures.

4. Intelligent Game Player Example & Simulation Data

The first aforementioned goal of our Chomp research is to produce an intelligent game player. The immediately obvious hurdle comes in constructing a functional definition of what it means to be intelligent. It is impossible to explicate a functional definition of artificial intelligence without strong consideration of the groundbreaking work of Alan Turing. The Turing Test serves as a foundational motivation for how intelligence is defined for the Chomp computer player. That is testing an artificial Chomp player against players of differing levels of intelligence. According to Theorem 5, We have written a simple game player that allows a human to play Chomp against the computer. The human player determines their moves on their own accord while the robot player follows a given position evaluation function.

5. Chomp Theorems and P-position Theory

Lemma 8. *For any arbitrary Chomp board B, B is a P position if, and only if, B^T is a P position.*

Proof. Suppose some board B is a P position. By definition, every reachable board is then an N position. Every reachable board is a result of some move (i, j) where $1 \leq i \leq m$ and $1 \leq j \leq n$ where m is the number of rows of the board and n is

Conjecture	**Truth Value**
number_of_cookies(x) $\geq$ number_of_rows(x)	True as a product of One-Rowed Chomp[1]
number_of_cookies(x) $\geq$ 2 $\times$ number_of_columns(x) - 1	Proof in Section 5
number_of_cookies(x) $\geq$ average_cookies_per_column(x) $\times$ rows_of_different_length(x)	Vacuously True
number_of_cookies(x) $\geq$ average_cookies_per_row(x) $\times$ number_of_rows(x)	Vacuously True
number_of_cookies(x) $\geq$ 2 $\times$ number_of_rows(x) - 1	Proof in Section 5
number_of_cookies(x) $\geq$ -(average_cookies_per_row(x) -number_of_columns(x))^2 + full_rectangle(x)	[2,1] is a counterexample
number_of_cookies(x) $\geq$ minimum(row_product(x), -1/2*average_cookies_per_row(x) + 1/2 $\times$ full_rectangle(x))	[10,2,2,1,1,1,1,1,1] is a counterexample. See Section 5

TABLE 2. Initial Chomp board conjectures

the number of cookies in the ith row of the board. This move takes the cookies denoted C_{ij} along with every cookie such that removing any cookie $C_{i_o j_o}$ removes all cookies $C_i j$ such that $i_o \leq i \leq m$ and $j_o \leq j \leq n$. B^T is defined as the matrix of cookies such that each cookie $C_i j$ in B is mapped to $C_j i$ in B^T. So, removing any cookie $C_{j_o i_o}$ in B^T removes all cookies $C_j i$ such that $j_o \leq j \leq n$ and $i_o \leq i \leq m$. Since every move in B corresponds to a moves in B^T, a player could counteract any possible move in B by removing the corresponding cookie in B^T. This implies every reachable position in B^T is an N position, so B^T is a P position. Without loss of generality, it can be shown that B^T being a P position implies that B is a P position. Since these two statements imply each other, we can conclude that B is a P position if, and only if, B^T is a P position. $\square$

Definition 8. *An "$L+1$ Position" is a chomp board which has x cookies in its first row, two cookies in its second row, and one cookie in all other rows. These positions can be characterized by their width, x, and their height, y.*

Theorem 9. *Let B be an $L+1$ Position chomp board of width w and height h.*

(1) $(w = h+1) \wedge (w \text{ is odd }) \wedge (h \geq 2) \implies (B \text{ is a P-Position})$
(2) $(w = h-1) \wedge (w \text{ is even }) \wedge (h \geq 3) \implies (B \text{ is a P-Position})$
(3) *All other $L+1$ Positions are N-Positions.*

Proof. This is a proof by induction on n.
Let B be an arbitrary $L+1$ Position with height of $2n$ and width of $2n+1$. Let $n = 1$, then B has 2 rows and 3 columns with 3 cookies in the first row and 2 cookies in the second row. This is a P-Position by remark 4. Thus, the base case holds. Let k be an arbitrary integer such that $2 \leq k \leq n$ is true. Assume that all $L+1$ Positions of width $2k$ and height $2k-1$ are P-Positions. Then, given a board where $k = n+1$, its width is $2(n+1) = 2n+2$ and its height is $2(n+1)-1 = 2n+1$. Using the definition of P-Position in term 1, in order for this board to be a P-Position, there must be no reachable P-Positions from this board. Examining the possible cases, the following results are gathered using the notation that "$P1(x, y)$" means that player one removes the xth cookie in row y.

$P1(2,1)$ This move leaves a board with exactly one row with more than one cookies. This is an N-Position using definition **??**.

$P1(1,3)$

□

Definition 9. *An "$L+2$ Position" is a chomp board which has x cookies in its first row, two cookies in its second row, two cookies in its third row, and one cookie in all other rows. These positions can be characterized by their width, x, and their height, y.*

Theorem 10. *Let B be an $L+2$ Position chomp board of width w and height h.*

(1) $(w = h+1) \wedge (w \text{ is even }) \wedge (h \geq 3) \implies (B \text{ is a P-Position})$
(2) $(w = h-1) \wedge (h \text{ is even }) \wedge (h \geq 6) \implies (B \text{ is a P-Position})$
(3) *All other $L+2$ Positions are N-Positions.*

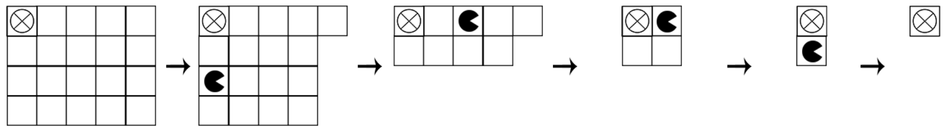

Figure 1. A Chomp Game

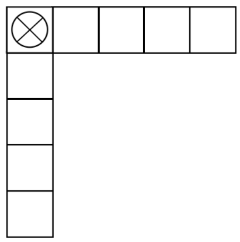

Figure 2. Balanced L Position

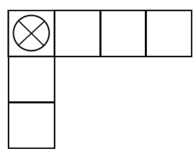

Figure 3. Unbalanced L Position

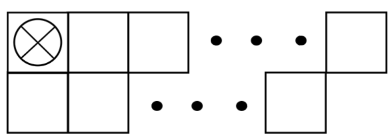

Figure 4. The P-postion form in two-row Chomp

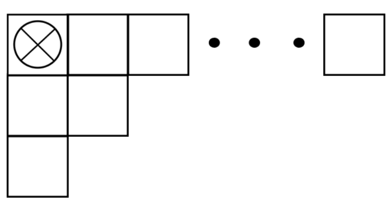

Figure 5. L + 2 Position

Proof. This is a proof by induction on n.

Let B_a be an $L+2$ Position chomp board of width $2n$ and height $2n-1$ for some $n \geq 2$. Let B_b be an $L+2$ Position chomp board of width $2m-1$ and height $2m$ for some $m \geq 3$. Let $n=2$. Then, B_a is an $L+2$ Position board of width 4 and height 3. Using **??**, this is a P board, so the base case for B_a holds. Now let $m=3$. Then B_b is an $L+2$ Position board of width 5 and height 6. This can (through the process of elimination) seen to be a P board. Let k be an integer such that $2 \leq k \leq n$ and l be an integer such that $3 \leq l \leq m$. We will assume that all L+2 Position boards of width $2k$ and height $2k-1$ are P boards (Inductive assumption A) and that all L+2 Position boards of width $2l-1$ and height $2l$ are also P positions (Inductive Assumption B).

Consider a board of width $2(n+1)$, or $2n+2$ and height of $2(n+1)-1$, or $2n+1$. There are a several cases for which move that player one (P1) can make. Applying proof by contradiction, assume these boards are N boards. Then, there exists at least one reachable board which will be an P board. Examining the possible cases, the following results are gathered using the notation that "$P1(x,y)$" means that player one removes the xth cookie in row y.

$P1(1,3)$ This move converts the board to a 2-row chomp board, which has been solved for all cases. Since the first row has an even number of cookies, it will never have exactly one more cookies than the second row, so this must be an N board.

$P1(1,5)$ P2 can respond by playing $P2(3,1)$. By 8, each board is identical to its transpose, so this 2-column board is identical to the transposed 2-row board. Then, since the first row is one greater than the second, it must be a P board. Then, that means P2 was handed an N board, so after $P1(1,5)$ this board must be an N board.

$P1(1,2k)$ Then, player 2 (P2) can respond to P1 by playing $P2(2k+1,1)$, which, by Inductive Assumption A will convert this board to a P board. Then, that means P2 was handed an N board, so after $P1(1,2k)$ this board must be an N board.

$P1(1,2k-1)$ Then, player 2 can respond to P1 by playing $P2(2k,1)$, which, by Inductive Assumption B will convert this board to a P board. Then, that means P2 was handed an N board, so after $P1(1,2k-1)$ this board must be an N board.

$P1(2,2)$ This move converts the board into an unbalanced L position, which is known to be an N board.

$P1(2,3)$ $L+1$ positions have been completely classified, and this position is known to be an N board.

$P1(1,2n+1)$ If P2 responds with $P2(2n+1,1)$, which is a board of equal height and width that is not a balanced L position. This is known to be a P board, so that means P2 was handed an N board, so after $P1(1, 2n+1)$ the board must be an N board.

$P1(3,1)$ By 8, each board is identical to its transpose, so this 2-column board is identical to the transposed 2-row board. Then, since the first row is not exactly one greater than the second, it must be an N board.

$P1(4,1)$ Each board is identical to its transpose, so this 3-column board is identical to the transposed 3-row board. There are exactly two 3-row boards that are P positions with 1 cookies in their third row, and this is not one of them, so it is an N board.

$P1(5,1)$ Then this can be followed by $P2(1,4)$ which is assumed to be a P board as it is the base case for inductive assumption A. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(6,1)$ P2 can move $P2(1,7)$ which, using Inductive Assumption B, is a P board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(2k-1,1)$ Note here that $k \geq 3$ since all cases below that have been explicitly examined above. Then, P2 can respond with $P2(1, 2k-2)$ which, by Inductive Assumption A, is a P board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(2k,1)$ Again, assume $k \geq 3$ since all lower cases have been explicitly examined above. Then, P2 can respond with $P2(1, 2k+1)$, which is a P board using Inductive Assumption B. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(2n+2,1)$ Since the width and height of the board are the same, P2 can respond $P2(2,2)$ and give P1 a balanced L position, which is a P board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

Since in all possible cases, P1 must hand P2 an N board, we can conclude that our assumption that an L+2 board with width of $2n+2$ and height of $2n+1$ is an N position was incorrect. Then, it must be a P position, assuming that Inductive Assumption B holds. To prove this, we will next examine a board of width $2(n+1)-1$, or $2n+1$ and height $2(n+1)$, or $2n+2$.

Now consider a board of width $2n$, or $2n+2$ and height of $2(n+1)$, or $2n+2$. There are a several cases for which move that player one (P1) can make. Applying proof by contradiction, assume these boards are N boards. Then, there exists at least one reachable board which will be an P board. Examining the possible cases, the following results are gathered using the notation that "$P1(x,y)$" means that player one removes the xth cookie in row y.

$P1(3,1)$ Since all 2-row chomp boards for which the first row is not exactly one greater than the second row are known to be N boards, this move converts the board to an N board.

$P1(4,1)$ By theorem 6, it is known that removing this cookie creates an N board.

$P1(5,1)$ P2 can respond with $P2(1,4)$, which must return to P1 a P board using Inductive Assumption A. Since the board handed to P2 had at least one reachable P board, the board after $P1(5,1)$ must be N.

$P1(2k,1)$ Assuming $3 \leq k \leq n$, P2 can respond by playing $P2(1,2k+1)$, which by Inductive Assumption B will yield an N board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(2k+1,1)$ Assuming $2 \leq k \leq n$, P2 can respond with $P2(1,2k)$, which by Inductive Assumption A will yield an N board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(1,2k)$ Assuming $2 \leq k \leq n$, P2 can respond with $P2(2k+1,1)$, which by Inductive Assumption A will yield an N board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(1,2k+1)$ Assuming $3 \leq k \leq n$, P2 can respond with $P2(2k,1)$, which by Inductive Assumption B will yield an N board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

$P1(2,2)$ This leaves an Unbalanced L Board, which is an N board.

$P1(2,3)$ This board satisfies case 2 of theorem 9, so it is an N board.

$P1(1,2n+2)$ Since the width and height of the board are the same, P2 can respond $P2(2,2)$ and give P1 a balanced L position, which is a P board. Since P2 is able to move and make the board a P board, P2 was handed an N board.

Since in all possible cases, P1 must hand P2 an N board, we can conclude that our assumption that an L+2 board with width of $2n$ and height of $2n+2$ is an N position was incorrect. Then, it must be a P position, assuming that Inductive Assumption A holds. Since we have demonstrated that both inductive assumptions must hold, it can be concluded that To prove this, we will next examine a board of width $2(n+1)-1$, or $2n+1$ and height $2(n+1)$, or $2n+2$.

□

Theorem 11. *Let B be an arbitrary chomp board, $n(B)$ be the number of cookies in that board, and $c(B)$ be the number of columns of that board. If B is a P position, then $n(B) \geq 2 \cdot c(B) - 1$.*

Proof. Suppose $n(B) \geq 2 \cdot c(B) - 1$ for all P positions with less than k columns. Let B be an arbitrary P board with k columns. Any move yields an N position, B'. So suppose the first player takes a cookie in the last column in the first row. An arbitrary number, t cookies, are removed. Then, the resulting board is an N position, so there is at least one reachable P position. Note, however, the second player cannot reach a P position by taking any cookie in the first row, as this move could have been made by the first player, yet we assume the first player was handed a P position, and thus had no reachable P positions. Then, the second player must take a cookie on some other row to reach a P position. This move results in some arbitrary number, l cookies, removed. We will call this new P position B''.

$$n(B'') = n(B) - t - l$$
$$c(B'') = c(B) - 1$$

By the inductive hypothesis,

$$n(B'') \geq 2 \cdot c(B'') - 1$$
$$n(B) - t - l \geq 2(c(B) - 1) - 1$$
$$n(B) \geq 2 \cdot c(B) - 3 + t + l$$

t and l are both at least one (otherwise, the moves to take those cookies would not have been legal).

$$n(B) \geq 2 \cdot c(B) - 3 + 1 + 1$$
$$n(B) \geq 2 \cdot c(B) - 1$$

As desired. □

6. Open Problems

The idea of applying conjectures into a game player extends beyond Chomp boards. The game player can apply to any game given enough intelligence and theory. This includes more complicated games such as chess and Go.

7. Acquiring the Program

References

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[3] Zeilberger, Doron, Three-Rowed CHOMP, Advanced in Applied Mathematics, 26, 168–179, 2001.

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