# Collision Thoughts

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December 23, 2017

We have two objects, $A$ and $B$ which collide. We assume the two objects are uniform density, rigid bodies, and convex polygons.

# 1 Two balls collide in 1D space

$$\begin{aligned} p_A^0 &= m_A v_A^0 && \text{(Momentum of object A)} \\ p_B^0 &= m_B v_B^0 && \text{(Momentum of object B)} \\ c_r &= \frac{v_B^1 - v_B^0}{v_A^1 - v_A^0} && \text{(Coefficient of restitution).} \end{aligned}$$

The coefficient of restitution, $c_r \in [0,1]$, gives a measure of elasticity of the collision. $c_r = 0$ denotes a completely inelastic collision (no change in velocity occurs) and $c_r = 1$ denotes a completely elastic collision (kinetic energy is conserved).

Given *a priori* values for $m_A, m_B, v_A^0, v_B^0, c_r$, we are interested in determining $v_A^1$ and $v_B^1$. To do this, we solve the linear system

$$\begin{aligned} m_A v_A^0 + m_B v_B^0 &= m_A v_A^1 + m_B v_B^1 \\ c_r &= \frac{v_B^1 - v_B^0}{v_A^1 - v_A^0}. \end{aligned}$$

This yields

$$\begin{aligned} v_A^1 &= \frac{m_A v_A^0 + m_B v_B^0 + c_r m_B (v_B^0 - v_A^0)}{m_A + m_B} \\ v_B^1 &= \frac{m_B v_B^0 + m_A v_A^0 + c_r m_A (v_A^0 - v_B^0)}{m_B + m_A}. \end{aligned}$$

Intuitively, we hope the outcome does not explicitly depend on the velocities of the objects, instead in terms of momentum and mass only. We can refomulate the results as follows. (With $m_A + m_B = m_A + m_B$.)

$$\begin{aligned} p_A^1 &= \left((1 - c_r) p_A^0 + (1 + c_r) p_B^0\right) \left(\frac{m_A}{m_A + m_B}\right) \\ p_B^1 &= \left((1 + c_r) p_A^0 + (1 - c_r) p_B^0\right) \left(\frac{m_B}{m_A + m_B}\right). \end{aligned}$$

## 1.1 Sanity Checks on some example cases:

1. $m_A = m_B =: m, \quad v_A^0 = -v_B^0 =: v^0$

$$\begin{aligned} p_A^1 &= -c_r p^0 \\ p_B^1 &= c_r p^0. \end{aligned}$$

   Here, the interpretation of $c_r$ is upheld as the linear damping of the final momenta of two identical objects with identical initial momenta.

2. $m_A = m_B =: m$

$$p_A^1 = \frac{1-c}{2} p_A^0 + \frac{1+c}{2} p_B^0$$
$$p_B^1 = \frac{1+c}{2} p_A^0 + \frac{1-c}{2} p_B^0$$

The resulting momenta are weighted averages of the initial momenta,

3. $p_A^0 = -p_B^0 =: p^0$

$$p_A^1 = -2c_r p^0 \left( \frac{m_A}{m_A + m_B} \right)$$
$$p_B^1 = 2c_r p^0 \left( \frac{m_B}{m_A + m_B} \right)$$

Both objects entered the collision with equi-opposite momentum entering the collision, and so both objects leave the collision with equi-oppositemomentum, whose magnitude is linearly damped by the coefficient of restitution.