Collision Thoughts

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We have two objects, A and B which collide. We assume the two objects are uniform density, rigid bodies, and convex polygons.

1 Two balls collide in 1D space

$$p_A^0 = m_A v_A^0 \qquad \text{(Momentum of object A)}$$
$$p_B^0 = m_B v_B^0 \qquad \text{(Momentum of object B)}$$
$$c_r = \frac{v_B^1 - v_B^0}{v_A^1 - v_A^0} \qquad \text{(Coefficient of restitution)}.$$

The coefficient of restitution, $c_r \in [0, 1]$, gives a measure of elasticity of the collision. $c_r = 0$ denotes a completely inelastic collision (no change in velocity occurs) and $c_r = 1$ denotes a completely elastic collision (kinetic energy is conserved).

Given *a priori* values for $m_A, m_B, v_A^0, v_B^0, c_r$, we are interested in determining v_A^1 and v_B^1. To do this, we solve the linear system

$$m_A v_A^0 + m_B v_B^0 = m_A v_A^1 + m_B v_B^1$$
$$c_r = \frac{v_B^1 - v_B^0}{v_A^1 - v_A^0}.$$

This yields

$$v_A^1 = \frac{m_A v_A^0 + m_B v_B^0 + c_r m_B (v_B^0 - v_A^0)}{m_A + m_B}$$
$$v_B^1 = \frac{m_B v_B^0 + m_A v_A^0 + c_r m_A (v_A^0 - v_B^0)}{m_B + m_A}.$$

Intuitively, we hope the outcome does not explicitly depend on the velocities of the objects, instead in terms of momentum and mass only. We can refomulate the results as follows. (With $m_A + m_B = m_A + m_B$.)

$$p_A^1 = \big((1-c_r)p_A^0 + (1+c_r)p_B^0\big)\left(\frac{m_A}{m_A+m_B}\right)$$
$$p_B^1 = \big((1+c_r)p_A^0 + (1-c_r)p_B^0\big)\left(\frac{m_B}{m_A+m_B}\right).$$

1.1 Sanity Checks on some example cases:

1. $m_A = m_B =: m, \quad v_A^0 = -v_B^0 =: v^0$

$$p_A^1 = -c_r p^0$$
$$p_B^1 = c_r p^0.$$

   Here, the interpretation of c_r is upheld as the linear damping of the final momenta of two identical objects with equi-opposite initial momenta.

2. $m_A = m_B =: m$

$$p_A^1 = \frac{1-c_r}{2}p_A^0 + \frac{1+c_r}{2}p_B^0$$
$$p_B^1 = \frac{1+c_r}{2}p_A^0 + \frac{1-c_r}{2}p_B^0$$

   The resulting momenta are weighted averages of the initial momenta. When $c_r = 1$, the two momenta swap so that $p_B^1 = p_A^0$ and vice versa. When $c_r = 0$, the resulting momenta are equal since the two objects are same mass and are stuck together.

3. $p_A^0 = -p_B^0 =: p^0$

$$p_A^1 = -2c_r p^0\left(\frac{m_A}{m_A+m_B}\right)$$
$$p_B^1 = 2c_r p^0\left(\frac{m_B}{m_A+m_B}\right)$$

   Both objects entered the collision with equi-opposite momentum entering the collision, and so both objects leave the collision with equi-opposite momentum, whose magnitude is linearly damped by the coefficient of restitution, and scaled by what proportion of the total mass that object accounted for.

4. $c_r = 1$ (Perfectly elastic collision)

$$p_A^1 = 2p_B^0 \left(\frac{m_A}{m_A + m_B} \right)$$
$$p_B^1 = 2p_A^0 \left(\frac{m_B}{m_A + m_B} \right).$$

Recall since this is a 1D collision, the two objects' momenta had opposite directions initially. This result shows in a perfectly elastic collision, the objects reverse directions and the magnitude of the resulting momentum is linearly proportional to the proportion of the total mass that object held.

5. $c_r = 0$ (Perfectly inelastic collision)

$$p_A^1 = (p_A^0 + p_B^0) \left(\frac{m_A}{m_A + m_B} \right)$$
$$p_B^1 = (p_A^0 + p_B^0) \left(\frac{m_B}{m_A + m_B} \right).$$

Here, the two objects have the same velocity since they are stuck together, so their final momenta vary only by what their difference in mass is.