Linear Algebra

Matrix: A formal table in the form of

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{m\times n}$$

whose elements can be numbers, variables or functions, is called a "matrix" with m rows and n columns.

$m\times n$ is called the order of the matrix.

we can also write the matrix as $A=[a_{ij}]$

ex

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} \sqrt{5}+i & 0 \\ \sqrt{2}-i & 1 \end{bmatrix}_{2\times 2}$$

reel matrix complex matrix

Some basic matrix

1) If $m=1$, then it's called "row matrix"
2) If $n=1$, then it's called "column matrix".
3) If all $a_{ij}=0$, then it's called "zero matrix".
4) If $m=n$, then it's called "square matrix.
5) Let $m=n$, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the "prime diagonal elements."

If except the prime diagonal elements, all the other elements are zero, then it's called "diagonal matrix".

6) In a diagonal matrix, if the prime diagonal elements are equal to each other, then it's called "scalar matrix".

7) In a scalar matrix, if $a_{11} = \dots = a_{nn} = 1$, then it's called "identity matrix".

Examples

1)

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}_{1\times 4} \Rightarrow \text{row matrix}$$

2) $$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}_{5\times 1} \Rightarrow \text{column matrix}$$

3) $$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} \Rightarrow \text{zero matrix}$$

4) $$D = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}_{2\times 2} \quad E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}_{3\times 3}$$ square matrix

5)

$$F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3} \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}_{4\times 4}$$ diagonal matrix

6) $H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$ $\quad K = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ scalar matrix

7) $m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = I_2$ $\quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ $\quad I_1 = [1]$

identity matrix

Sum and Subtract of Matrix

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrix with the same order $m\times n$. Then

$$A \mp B = [a_{ij} \mp b_{ij}]$$

ex Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}_{3\times 3}$

Then

$A + B = \begin{bmatrix} 3 & 4 & -2 \\ 0 & 1 & 4 \\ 3 & 0 & -2 \end{bmatrix}_{3\times 3}$ $\quad A - B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ -1 & -2 & 2 \end{bmatrix}_{3\times 3}$

ex Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix}_{2\times 3}$ $\quad B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$

$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$ $\quad A - B = \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix}_{2\times 3}$

Properties of the matrix

Let A, B and C be matrix with the same order and λ_1 and λ_2 be two scalars. Then

1) $A+B=B+A$ (Commutative property)

ex

$A=[2 \ \ 1 \ \ 2]$, $B=[1 \ \ 0 \ \ 0]$

$A+B=[3 \ \ 1 \ \ 2] = B+A=[3 \ \ 1 \ \ 2]$

2) $A+(B+C)=(A+B)+C$ (associative property)
3) $A+O=A$ (additive identity)
4) $\lambda_1(A+B)=\lambda_1 A+\lambda_1 B$
5) $(\lambda_1 \lambda_2)A=\lambda_1(\lambda_2 A)$

Properties of matrix multiplication

Let $A=[a_{ij}]_{m\times r}$ and $B=[b_{ij}]_{r\times n}$ be two matrix. Then for $i=1,\dots,m$; $j:1,\dots,n$

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{ir}b_{rj}=\sum_{k=1}^{r}a_{ik}b_{kj}$$

$$AB=[c_{ij}]_{m\times n}$$

ex

$$A=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}_{2\times2} \quad B=\begin{bmatrix}0 & 2\\ 3 & 1\end{bmatrix}_{2\times2} \Rightarrow AB=\begin{bmatrix}0 & 4\\ 3 & 3\end{bmatrix}_{2\times2}$$

$$BA=\begin{bmatrix}2 & 2\\ 7 & 1\end{bmatrix}_{2\times2}$$

$AB \neq BA$

ex

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 5 & 10 \end{bmatrix} = BA = \begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 5 & 10 \end{bmatrix}$$

$$AB = BA$$

☆ So, it doesn't always possess $AB = BA$

Properties

Let A be $m \times n$; B and C be $n \times r$ matrix ; and λ be a scalar. D be a $r \times t$ matrix.

1) $A(BD) = (AB)D$
2) $A(B+C) = AB + AC$
3) $\lambda(AB) = (\lambda A)B = A(\lambda B)$
4) $A0 = 0A = 0$
5) $AI = IA = A$

ex

$\underset{1\times 3}{A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}$ $\quad \underset{3\times 2}{B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}$ $\quad \underset{2\times 1}{D = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$

$$A(BD) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix} = \underset{1\times 1}{\begin{bmatrix} 19 \end{bmatrix}}$$

$$(AB)D = \left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underset{1\times 1}{\begin{bmatrix} 19 \end{bmatrix}}$$

ex

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = A = IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Transpose of a matrix

The transpose of a matrix is represented by A^t, A^T or A' and obtained by changing its rows into columns and its columns into rows. So if A is $m \times n$, then A^t is $n \times m$.

ex

$$A = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 4 & 2 \\ 1 & 0 \end{bmatrix}} \Rightarrow A^t = \underset{2\times 3}{\begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 0 \end{bmatrix}}$$

$$B = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}} \Rightarrow B^t = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix}}$$

Properties of transpose

Let A and B be $m\times n$ matrix and λ be a scalar.

1) $(A+B)^t = A^t + B^t$
2) $(A^t)^t = A$
3) $(\lambda A)^t = \lambda A^t$
4) If A and B be multiplyable matrix, then $(m=n)$
$(AB)^t = B^t A^t$

ex

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix}^t = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\parallel$$

$$A^t + B^t = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$$

ex

$$(AB)^t = \left(\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right)^t = \begin{bmatrix} 2 & 2 \\ -4 & -1 \end{bmatrix}^t = \begin{bmatrix} 2 & -4 \\ 2 & -1 \end{bmatrix}$$

$$\parallel$$

$$B^t A^t = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & -1 \end{bmatrix}$$

Def: Let A be a square matrix.

If $A^t = A$, it's called "symmetric matrix".

If $A^t = -A$, " " inverse symmetric matrix".

If $A^2 = A$, " " idempotat matrix.

If $A^2 = I$, " " involutarymatrix.

ex

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 0 \end{bmatrix} = A \quad \text{sym. matrix}$$

ex

$$A=\begin{bmatrix}0&2&1\\-2&0&-1\\-1&1&0\end{bmatrix} \quad A^t=\begin{bmatrix}0&-2&-1\\2&0&1\\1&-1&0\end{bmatrix}=-A$$

inverse sym. m.

ex

$$A=\begin{bmatrix}-1&3&5\\1&-3&-5\\-1&3&5\end{bmatrix}$$

idemp. matrix

$$A^2=\begin{bmatrix}-1&3&5\\1&-3&-5\\-1&3&5\end{bmatrix}\begin{bmatrix}-1&3&5\\1&-3&-5\\-1&3&5\end{bmatrix}=\begin{bmatrix}-1&3&5\\1&-3&-5\\-1&3&5\end{bmatrix}=A$$

ex

$$A=\begin{bmatrix}4&-1\\15&-4\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}4&-1\\15&-4\end{bmatrix}\begin{bmatrix}4&-1\\15&-4\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$$

inv. m.

ex

If $A=\begin{bmatrix}4&3&k\\-1&0&-1\\m&-4&-3\end{bmatrix}$ is an involutory matrix, then what is $k+m=?$

$$\begin{bmatrix}4&3&k\\-1&0&-1\\m&-4&-3\end{bmatrix}\begin{bmatrix}4&3&k\\-1&0&-1\\m&-4&-3\end{bmatrix}=\begin{bmatrix}13+km&12-4k&k-3\\-4-m&1&-k+3\\m+4&3m+12&km+13\end{bmatrix}=I_3$$

$$\begin{array}{r}k=3\\ +\quad m=-4\\ \hline -1\end{array}$$

Conjugate of a matrix

In a A matrix whose elements are complex numbers, the matrix obtained by replacing each element with its conjugate is called Conjugate of the matrix. It's shown by $\bar{A}$.

ex

$$A=\begin{bmatrix}0&-i&2+3i\\\sqrt{2}&-\sqrt{2}i&1\\0&3&1-\sqrt{3}i\end{bmatrix} \Rightarrow \bar{A}=\begin{bmatrix}0&i&2-3i\\\sqrt{2}&\sqrt{2}i&1\\0&3&1+\sqrt{3}i\end{bmatrix}$$

Properties of $\bar{A}$

Let A and B be $m\times n$ matrix and k be a scalar.

1) $(\bar{\bar{A}}) = A$
2) $(\overline{kA}) = \bar{k}\,\bar{A}$
3) $(\overline{A+B}) = \bar{A} + \bar{B}$
4) If A and B be multiplycable matrix, then $(\overline{AB}) = \bar{A}\,\bar{B}$

ex

$$A = \begin{bmatrix} 1-2i & 3i & 1 \\ 0 & -i & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1+2i & -3i & 1 \\ 0 & i & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow (\bar{\bar{A}}) = \begin{bmatrix} 1-2i & 3i & 1 \\ 0 & -i & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} = A$$

ex

$$A = \begin{bmatrix} i & -2i \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1-2i \\ 2 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} i & -2i \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1-2i \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3i & 3i+2 \\ 1 & 1-2i \end{bmatrix}, \quad (\overline{AB}) = \begin{bmatrix} 3i & -3i+2 \\ 1 & 1+2i \end{bmatrix}$$

$$\bar{A}\,\bar{B} = \begin{bmatrix} -i & 2i \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1+2i \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3i & 2-3i \\ 1 & 1+2i \end{bmatrix}$$

Def: Let A be a square matrix. If

$(\bar{A})^t = A$, then it's Hermitian matrix

$(\bar{A})^t = -A$, " " inverse " "

ex

$$A = \begin{bmatrix} 5 & 2-i & 3 \\ 2+i & 0 & -i \\ 3 & i & 7 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 5 & 2+i & 3 \\ 2-i & 0 & i \\ 3 & -i & 7 \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} 5 & 2-i & 3 \\ 2+i & 0 & -i \\ 3 & i & 7 \end{bmatrix} = A$$

Hermitian

ex

$$A = \begin{bmatrix} i & 1-i & 2 \\ 1-i & 3i & i \\ -2 & i & 0 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} -i & 1+i & 2 \\ -1+i & -3i & -i \\ -2 & -i & 0 \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} -i & -1+i & -2 \\ 1+i & -3i & -i \\ 2 & -i & 0 \end{bmatrix} = -A$$

inverse Hermition

Theorem:

Let A be a $n \times n$ square matrix. If there exists a matrix B provides

$$AB = BA = I_n$$

then B is called the "inverse of A" and given by A^{-1}. A is called (regular) "invertible matrix"

ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

★ If A has no inverse, then it's called "singular matrix."

Echelon form of a matrix

An $m \times n$ matrix A is said to be in reduced row echelon form if it satisfies the following properties:

a) All zero rows, if there are any, appear at the bottom of the matrix.

b) The first nonzero entry from the left of a nonzero row is a 1. This entry is called a leading one of its row.

c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

d) If a column contains a leading one, then all other entries in that column are zero.

(+) If a $m \times n$ matrix satisfying a, b, c is said to be in row echelon matrix.

(+) If (a) ... (d) are provided, it's normal form

ex

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{echelon matrix}$$

Def: In the row echelon form, the row number which is non-zero is called the rank of the matrix.

ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{rank}(A) = 3$$

ex

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 3 & 1 \\ 2 & -2 & 1 & 0 & 2 \\ 1 & 1 & -1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 2 & -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}(A) = 3$$

hom

$$A=\begin{bmatrix}1&2&-2&3&1\\1&3&-2&3&0\\2&4&-3&6&4\\1&1&-1&4&6\end{bmatrix}\qquad \text{rank}(A)=?$$

$\rightarrow$ ★

Inverse of a matrix

$[A : I_n] \sim [I_n : A^{-1}]$ augumented matrix

ex

$$A=\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&3&1&0&1&0\\0&4&3&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&-1&-2&0&1&-1\\0&4&3&0&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&2&0&-1&1\\0&0&-5&0&4&-3\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&2&0&-1&1\\0&0&1&0&-\frac{4}{5}&\frac{3}{5}\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&0&0&1&\frac{8}{5}&-\frac{6}{5}\\0&1&0&0&\frac{3}{5}&-\frac{1}{5}\\0&0&1&0&-\frac{4}{5}&\frac{3}{5}\end{array}\right]\Rightarrow A^{-1}=\begin{bmatrix}1&8/5&-6/5\\0&3/5&-1/5\\0&-\frac{4}{5}&3/5\end{bmatrix}$$

ex

$$A=\begin{bmatrix}1&1&1\\0&2&3\\5&5&1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&3&0&1&0\\5&5&1&0&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&3&0&1&0\\0&0&-4&-5&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&3/2&0&1/2&0\\0&0&1&\frac{5}{4}&0&-\frac{1}{4}\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-\frac{15}{8}&\frac{1}{2}&\frac{3}{8}\\0&0&1&\frac{5}{4}&0&-\frac{1}{4}\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&\frac{13}{8}&-\frac{1}{2}&-\frac{1}{8}\\0&1&0&-\frac{15}{8}&\frac{1}{2}&\frac{3}{8}\\0&0&1&\frac{5}{4}&0&-\frac{1}{4}\end{array}\right]$$

A^{-1}

4

Def: A matrix can be reduced to both row and column echelon form by applying elementary row and elementary column operations to the matrix. This matrix can be equal to

$$[I_k \ \ 0], \quad \begin{bmatrix} I_k \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \quad I_k$$

writing the matrix as any of them is called "normalizing the matrix."

ex

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right]$$

The third row has all zeros

So, A is a singular matrix.

Application of matrix

1) what is the rank of the matrix

$$A = \begin{bmatrix} 3 & -5 & 2 & -1 \\ 4 & 5 & -2 & 6 \\ 1 & 10 & -4 & 7 \end{bmatrix} ?$$

$$\begin{bmatrix} 0 & -35 & 14 & -22 \\ 0 & -35 & 14 & -22 \\ 1 & 10 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & -4 & 7 \\ 0 & -35 & 14 & -22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$

2) If $A = \underset{3\times 3}{\begin{bmatrix} 3 & 5 & 4 \\ 2 & 4 & 1 \\ -1 & 3 & -2 \end{bmatrix}}$, $B = \underset{3\times 4}{\begin{bmatrix} 3 & -1 & -3 & 2 \\ 2 & 6 & -4 & 5 \\ 1 & -2 & 7 & -7 \end{bmatrix}}$ $AB = ?$

$$AB = \underset{3\times 4}{\begin{bmatrix} 23 & 19 & -1 & 3 \\ 15 & 20 & -15 & 3 \\ 1 & 23 & -23 & 27 \end{bmatrix}}$$

3) If $A=\begin{bmatrix}-1 & 1 & -1\\ 0 & -1 & 1\\ 1 & 1 & 1\\ 1 & 2 & 0\end{bmatrix}$, then what is A^{-1}? what is the normal form of A? rank(A)=?

A isn't a square matrix. so A^{-1} doesn't exist.

$$\begin{bmatrix}-1 & 1 & -1\\ 0 & -1 & 1\\ 1 & 1 & 1\\ 1 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}-1 & 1 & -1\\ 0 & -1 & 1\\ 0 & 2 & 0\\ 0 & 3 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 2\\ 0 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$ rank(A) = 3

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}I_3\\ 0\end{bmatrix}$$ Normal form of A

4) $D=\begin{bmatrix}1 & -1 & 0\\ 2 & 4 & 1\\ 3 & -1 & 2\\ 0 & 0 & 2\end{bmatrix}$ what is normal form of D?

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 6 & 1\\ 0 & 2 & 2\\ 0 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & -5\\ 0 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Homework I

1)

$$A=\begin{bmatrix}0&-1&3\\x&0&y\\z&-8&0\end{bmatrix} \Rightarrow A^t=\begin{bmatrix}0&x&z\\-1&0&-8\\3&y&0\end{bmatrix} \rightarrow A^t=-A$$

$$\begin{bmatrix}0&x&z\\-1&0&-8\\3&y&0\end{bmatrix}=\begin{bmatrix}0&1&-3\\-x&0&-y\\-z&8&0\end{bmatrix} \rightarrow \left.\begin{matrix}x=1\\z=-3\\y=8\end{matrix}\right\} x+y+z=6 \quad \text{E}$$

2) $$A+B=\begin{bmatrix}1&2&3\\1&3&4\\1&4&3\end{bmatrix}+\begin{bmatrix}1&1&1\\2&1&-3\\3&2&4\end{bmatrix}=\begin{bmatrix}2&3&4\\3&4&1\\4&6&7\end{bmatrix}$$

$$A-B=\begin{bmatrix}0&1&2\\-1&2&7\\-2&2&-1\end{bmatrix} \quad ; \quad B-A=\begin{bmatrix}0&-1&-2\\1&-2&-7\\2&-2&1\end{bmatrix}$$

$$A.B=\begin{bmatrix}1&2&3\\1&3&4\\1&4&3\end{bmatrix}\begin{bmatrix}1&1&1\\2&1&-3\\3&2&4\end{bmatrix}=\begin{bmatrix}14&9&7\\19&12&8\\18&11&1\end{bmatrix} \qquad B.A=\begin{bmatrix}3&9&10\\0&-5&1\\9&28&29\end{bmatrix}$$

3) I $$\left\{\left[\overline{A(\bar{A})^t}\right]\right\}^t=\left\{\bar{A}\left[\overline{(\bar{A})^t}\right]\right\}^t=\left\{\bar{A}.A^t\right\}^t=A.(\bar{A})^t \quad \checkmark$$

II. $$AB=A \Rightarrow (AB)^t=A^t \rightarrow B^t.A^t=A^t \quad \checkmark$$

$$BA=B \Rightarrow (BA)^t=B^t \rightarrow A^tB^t=B^t \quad \times$$

III $$(\bar{A})^t=-A$$

$$\left\{(\overline{i.A})\right\}^t=\left\{(-i).\bar{A}\right\}^t=(-i)\underbrace{(\bar{A})^t}_{-A}=i.A \quad \checkmark$$

$$A^t=-A \; ; \; (A^2)^t=(A.A)^t=A^t.A^t=(-A)(-A)=A^2 \quad \times$$

4) $A = \begin{bmatrix} 0 & 1 & -1 \\ 5 & k & 5 \\ 4 & -4 & 5 \end{bmatrix}$ $\quad A^2 = I$

C

$$\begin{bmatrix} 0 & 1 & -1 \\ 5 & k & 5 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 5 & k & 5 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & k+4 & 0 \\ 5k+20 & k^2-15 & 5k+20 \\ 0 & -4k-16 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$k+4=0$

$k=-4$