Application

1) If $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow adj(A) = ?$

$\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$1 + 8 = 9$

$$adj\,A = \begin{bmatrix} \overset{+}{9} & \overset{-}{-1} & \overset{+}{4} \\ \overset{-}{+12} & \overset{+}{1} & \overset{-}{-4} \\ \overset{+}{-3} & \overset{-}{+5} & \overset{+}{1} \end{bmatrix}^t$$

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$$adj\,A = \begin{bmatrix} 9 & -1 & 4 \\ +12 & 1 & -4 \\ -3 & +5 & 1 \end{bmatrix}^t = \begin{bmatrix} 9 & 12 & -3 \\ -1 & 1 & 5 \\ 4 & -4 & 1 \end{bmatrix}$$

Application

1) If $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow adj(A) = ?$

$\Rightarrow A^{-1} = ?$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$adj\,A = \begin{bmatrix} 9 & -1 & 4 \\ +12 & 1 & -4 \\ -3 & +5 & 1 \end{bmatrix}^t = \begin{bmatrix} 9 & 12 & -3 \\ -1 & 1 & 5 \\ 4 & -4 & 1 \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} 9 & -1 & 4 \\ +12 & 1 & -4 \\ -3 & +5 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 12 & -3 \\ -1 & 1 & 5 \\ 4 & -4 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{21} & \frac{1}{21} & \frac{5}{21} \\ \frac{4}{21} & -\frac{4}{21} & \frac{1}{21} \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 0 & 4 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & -2 \end{vmatrix} = 1 + 12 + 0 - 0 + 8 - 0 = 21$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

2) $$A = \begin{bmatrix} 4 & 1 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$|A| = 4 \underbrace{\begin{vmatrix} -1 & 2 & 0 \\ 1 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix}} + \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

$$\underbrace{-2 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \cdot 4}_{(-1-4)(-8)} + \underbrace{(-1) \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}}_{-(-1-4)} = 40 + 5 = 45$$

ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = (-1)(0-2) = 2 \neq 0$$

$\Rightarrow$

$$\mathrm{Adj}\,A = \begin{bmatrix} 2 & 0 \\ & \\ & \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{matrix}$$

ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

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$$\text{Adj } A = \begin{bmatrix} 2 & 0 & 0 & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{matrix}$$

ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = (-1)(0-2) = 2 \neq 0$$

$$\text{Adj } A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}^t \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & 0 & -1/2 & -1 \\ 0 & 0 & -1/2 & 0 \end{bmatrix}$$

ex $A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

$A_{13} = ? \Rightarrow (-1)^4 \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1$

$M_{23} = ? \Rightarrow M_{23} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

ex If $|A| = \frac{3}{4}$ with 4×4, then $|4A|, \left|\frac{A}{2}\right|, |A^{-1}|, |A^3| = ?$

$|4A| = 4^4 \cdot |A| = 4^{\cancel{4}\,3} \cdot \frac{3}{\cancel{4}} = 64 \cdot 3 = 192$ $\quad \left|\frac{A}{2}\right| = \left(\frac{1}{2}\right)^4 |A| = \frac{1}{16} \cdot \frac{3}{4} = \frac{3}{64}$

$|A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{4}{3}$ $\quad |A^3| = |A||A||A| = \frac{27}{64}$

Break

ex If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$ is a singular matrix, then what is c? $(c \geq 0)$ $\rightarrow |A| = 0$

a) 0 b) 2 c) 3 d) 5 e) 6

ex If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$ is a singular matrix, then what is c? $(c \geq 0)$ $\rightarrow |A| = 0$

a) 0 b) 2 c) 3 d) 5 e) 6

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = 27 - c^2 - (3 - c) + (c - 9) = 0$

$= 27 - c^2 - 3 + c + c - 9 = 0 \Rightarrow +c^2 - 2c - 15 = 0 \rightarrow c_1 = 5, \; c_2 = -3$

$-5 \quad +3$

ex Let $A=\begin{bmatrix}1&2\\1&0\end{bmatrix}$ and $f(x)=2x^2-3x+7$. $f(A)=?$

$$f(A)=2A^2-3A+7I$$

$$A^2=\begin{bmatrix}1&2\\1&0\end{bmatrix}\begin{bmatrix}1&2\\1&0\end{bmatrix}=\begin{bmatrix}3&2\\1&2\end{bmatrix}$$

$$f(A)=\begin{bmatrix}6&4\\2&4\end{bmatrix}-\begin{bmatrix}3&6\\3&0\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}=\begin{bmatrix}10&-2\\-1&11\end{bmatrix}$$

$$f(A)=2A^2-3A+7I$$

$$A^2=\begin{bmatrix}1&2\\1&0\end{bmatrix}\begin{bmatrix}1&2\\1&0\end{bmatrix}=\begin{bmatrix}3&2\\1&2\end{bmatrix}$$

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Note

$(A^{-1})^{-1}=A$

ex If $A=\begin{bmatrix}1&3\\2&5\end{bmatrix}$ and $\underbrace{[X^t+2I_2]^{-1}}=\underbrace{A} \Rightarrow X=?$

$X^t+2I=A^{-1}$

$$f(A)=\begin{bmatrix}6&4\\2&4\end{bmatrix}-\begin{bmatrix}3&6\\3&0\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}=\begin{bmatrix}10&-2\\-1&11\end{bmatrix}$$

$\underbrace{(X^t)^t}_{X}=A^t$

ex If $A=\begin{bmatrix}1&3\\2&5\end{bmatrix}$ and $\underbrace{[X^t+2I_2]^{-1}}=\underbrace{A} \Rightarrow X=?$

$X^t+2I=A^{-1}$

$A^{-1} \Rightarrow |A|=5-6=-1$

$\text{Adj}A=\begin{bmatrix}5&-2\\-3&1\end{bmatrix}^t \Rightarrow \begin{bmatrix}5&-3\\-2&1\end{bmatrix}$

$A^{-1}=\begin{bmatrix}-5&3\\2&-1\end{bmatrix}$

$X^t=\begin{bmatrix}-5&3\\2&-1\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}-7&3\\2&-3\end{bmatrix}$

$X=\begin{bmatrix}-7&2\\3&-3\end{bmatrix}$

+1) $\det(AB) = \det(A)\det(B)$ → $|A| = |A^t|$

$|AB| = |A||B|$

+2) $\boxed{\det(AB)} = \det(BA)$ → $|AB| = |A||B| = |B||A| = |BA|$

✓3) $\det(A^t B) = \det(B^t A)$

4) If we change 2. row and 3. row, then $|A|$ is not changing.

→ $|A^t B| = |(A^t B)^t| = |B^t A|$

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✓3) $\det(A^t B) = \det(B^t A)$

X 4) If we change 2. row and 3. row, then $|A|$ is not changing.

→ $|A^t B| = |(A^t B)^t| = |B^t A|$

$$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

ex

$$\begin{vmatrix} x & x & x & x \\ y & y & y & -y \\ z & z & -z & -z \\ t & -t & -t & -t \end{vmatrix} = \begin{vmatrix} 2x & 2x & 2x & x \\ 0 & 0 & 0 & -y \\ 0 & 0 & -2z & -z \\ 0 & -2t & -2t & -t \end{vmatrix} = 2x\cdot(-y)\left[0(-2t) - 4zt\right]$$

$$= (-2xy)(-4zt) = \underline{8xyzt}$$

ex

$$|A| = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & ac-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix} = \Big[\overbrace{(b^2-a^2)}^{(a+b)(b-a)}\underbrace{(ab-bc)}_{b(a-c)} - \overbrace{(c^2-a^2)}^{(c-a)(c+a)}\underbrace{(ac-bc)}_{c(a-b)}\Big]$$

$$= (b-a)(a-c)\left[\underbrace{(a+b)\cdot b-(a+c)\cdot c}\right] = (b-a)(a-c)(a+b+c)(b-c)$$

$$ab+b^2-ac-c^2$$

$$a(b-c)+(b-c)(b+c)$$

$$(a+b+c)(b-c)$$

ex

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & & & & \\ b & b & b & \cdots & a \end{vmatrix}_{n\times n} = \begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & \cdots & b \\ \vdots & \vdots & & \\ a+(n-1)b & b & & a \end{vmatrix}$$

$$[a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & & & \\ 1 & b & \cdots & a \end{vmatrix}$$

$$[a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & & & \\ 1 & b & \cdots & a \end{vmatrix} = [a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ 0 & a-b & \cdots & 0 \\ 0 & 0 & a-b \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a-b \end{vmatrix}$$

$$\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a-b \end{vmatrix}_{n-1} = (a-b)\, I_{n-1}$$

$$\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & a-b \end{vmatrix}_{n-1} = (a-b)^{n-1} I_{n-1}$$

$$= [a+(n-1)b](a-b)^{n-1} I_{n-1}$$

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$$A = \begin{bmatrix} 1 & 0 & 4 & 1 & 7 \\ 0 & 1 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{4\times 5} \qquad rank(A) = 1$$

$$\begin{vmatrix} 1 & 4 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1.(-1)^2 \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 2\cdot 3 = -4 \neq 0$$