1.5. An experiment consists of tossing two dice.

(*a*) Find the sample space S.

(*b*) Find the event A that the sum of the dots on the dice equals 7.

(*c*) Find the event B that the sum of the dots on the dice is greater than 10.

(*d*) Find the event C that the sum of the dots on the dice is greater than 12.

(*a*) For this experiment, the sample space S consists of 36 points (Fig. 1-3):

$$S = \{(i, j): i, j = 1, 2, 3, 4, 5, 6\}$$

where i represents the number of dots appearing on one die and j represents the number of dots appearing on the other die.

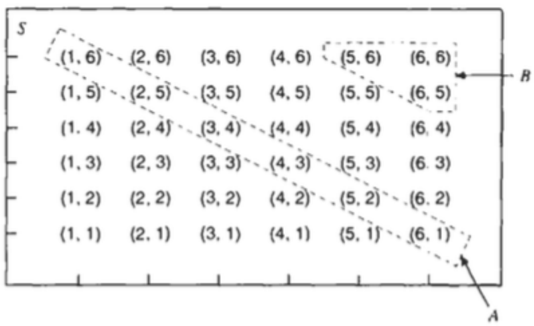
(*b*) The event A consists of 6 points (see Fig. 1-3):

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

(*c*) The event B consists of 3 points (see Fig. 1-3):

$$B = \{(5, 6), (6, 5), (6, 6)\}$$

(*d*) The event C is an impossible event, that is, $C = \varnothing$.



1.30. The sample space S of a random experiment is given by

$$S = \{a, b, c, d\}$$

with probabilities $P(a) = 0.2$, $P(b) = 0.3$, $P(c) = 0.4$, and $P(d) = 0.1$. Let A denote the event $\{a, b\}$, and B the event $\{b, c, d\}$. Determine the following probabilities: (*a*) $P(A)$; (*b*) $P(B)$; (*c*) $P(\bar{A})$; (*d*) $P(A \cup B)$; and (*e*) $P(A \cap B)$.

Using Eq. (*1.36*), we obtain

(*a*) $P(A) = P(a) + P(b) = 0.2 + 0.3 = 0.5$

(*b*) $P(B) = P(b) + P(c) + P(d) = 0.3 + 0.4 + 0.1 = 0.8$

(*c*) $\bar{A} = \{c, d\}$; $P(\bar{A}) = P(c) + P(d) = 0.4 + 0.1 = 0.5$

(*d*) $A \cup B = \{a, b, c, d\} = S$; $P(A \cup B) = P(S) = 1$

(*e*) $A \cap B = \{b\}$; $P(A \cap B) = P(b) = 0.3$

1.39. Show that if $P(A \mid B) > P(A)$, then $P(B \mid A) > P(B)$.

If $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} > P(A)$, then $P(A \cap B) > P(A)P(B)$. Thus,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} > \frac{P(A)P(B)}{P(A)} = P(B) \qquad \text{or} \qquad P(B \mid A) > P(B)$$

1.11. Consider the switching networks shown in Fig. 1-5. Let A_1, A_2, and A_3 denote the events that the switches s_1, s_2, and s_3 are closed, respectively. Let A_{ab} denote the event that there is a closed path between terminals a and b. Express A_{ab} in terms of A_1, A_2, and A_3 for each of the networks shown.

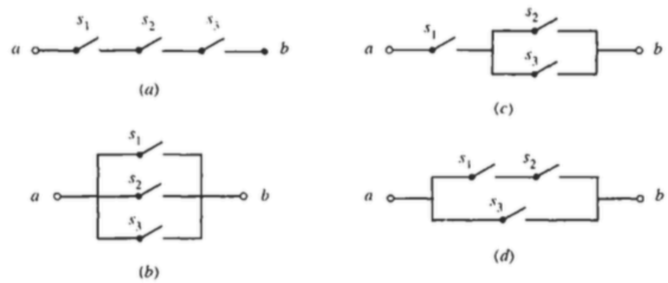


Fig. 1-5

(a) From Fig. 1-5(a), we see that there is a closed path between a and b only if all switches s_1, s_2, and s_3 are closed. Thus,

$$A_{ab} = A_1 \cap A_2 \cap A_3$$

(b) From Fig. 1-5(b), we see that there is a closed path between a and b if at least one switch is closed. Thus,

$$A_{ab} = A_1 \cup A_2 \cup A_3$$

(c) From Fig. 1-5(c), we see that there is a closed path between a and b if s_1 and either s_2 or s_3 are closed. Thus,

$$A_{ab} = A_1 \cap (A_2 \cup A_3)$$

Using the distributive law (1.12), we have

$$A_{ab} = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

which indicates that there is a closed path between a and b if s_1 and s_2 or s_1 and s_3 are closed.

(d) From Fig. 1-5(d), we see that there is a closed path between a and b if either s_1 and s_2 are closed or s_3 is closed. Thus

$$A_{ab} = (A_1 \cap A_2) \cup A_3$$

1.51. Two numbers are chosen at random from among the numbers 1 to 10 without replacement. Find the probability that the second number chosen is 5.

Let A_i, $i = 1, 2, \ldots, 10$ denote the event that the first number chosen is i. Let B be the event that the second number chosen is 5. Then by Eq. (1.44),

$$P(B) = \sum_{i=1}^{10} P(B \mid A_i)P(A_i)$$

Now $P(A_i) = \frac{1}{10}$. $P(B \mid A_i)$ is the probability that the second number chosen is 5, given that the first is i. If $i = 5$, then $P(B \mid A_i) = 0$. If $i \neq 5$, then $P(B \mid A_i) = \frac{1}{9}$. Hence,

$$P(B) = \sum_{i=1}^{10} P(B \mid A_i)P(A_i) = 9\left(\tfrac{1}{9}\right)\left(\tfrac{1}{10}\right) = \tfrac{1}{10}$$

2.7. Show that

(*a*) $P(a \le X \le b) = P(X = a) + F_X(b) - F_X(a)$ (2.64)

(*b*) $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$ (2.65)

(*c*) $P(a \le X < b) = P(X = a) + F_X(b) - F_X(a) - P(X = b)$ (2.66)

(*a*) Using Eqs. (*1.23*) and (*2.10*), we have

$$\begin{aligned} P(a \le X \le b) &= P[(X = a) \cup (a < X \le b)] \\ &= P(X = a) + P(a < X \le b) \\ &= P(X = a) + F_X(b) - F_X(a) \end{aligned}$$

(*b*) We have

$$\begin{aligned} P(a < X \le b) &= P[(a < X < b) \cup (X = b)] \\ &= P(a < X < b) + P(X = b) \end{aligned}$$

Again using Eq. (*2.10*), we obtain

$$\begin{aligned} P(a < X < b) &= P(a < X \le b) - P(X = b) \\ &= F_X(b) - F_X(a) - P(X = b) \end{aligned}$$

(*c*) Similarly,

$$\begin{aligned} P(a \le X \le b) &= P[(a \le X < b) \cup (X = b)] \\ &= P(a \le X < b) + P(X = b) \end{aligned}$$

Using Eq. (*2.64*), we obtain

$$\begin{aligned} P(a \le X < b) &= P(a \le X \le b) - P(X = b) \\ &= P(X = a) + F_X(b) - F_X(a) - P(X = b) \end{aligned}$$

2.8. Let X be the r.v. defined in Prob. 2.3.

(*a*) Sketch the cdf $F_X(x)$ of X and specify the type of X.

(*b*) Find (i) $P(X \le 1)$, (ii) $P(1 < X \le 2)$, (iii) $P(X > 1)$, and (iv) $P(1 \le X \le 2)$.

(*a*) From the result of Prob. 2.3 and Eq. (*2.18*), we have

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

which is sketched in Fig. 2-11. The r.v. X is a discrete r.v.

(*b*) (i) We see that

$$P(X \le 1) = F_X(1) = \tfrac{1}{2}$$

(ii) By Eq. (*2.10*),

$$P(1 < X \le 2) = F_X(2) - F_X(1) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

(iii) By Eq. (*2.11*),

$$P(X > 1) = 1 - F_X(1) = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

(iv) By Eq. (*2.64*),

$$P(1 \le X \le 2) = P(X = 1) + F_X(2) - F_X(1) = \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{3}{4}$$

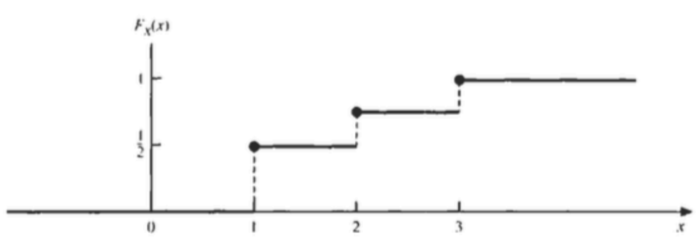


Fig. 2-11

2.16. Let X be a binomial r.v. with parameters (n, p).

(a) Show that $p_X(x)$ given by Eq. (2.36) satisfies Eq. (2.17).

(b) Find $P(X > 1)$ if $n = 6$ and $p = 0.1$.

(a) Recall that the binomial expansion formula is given by

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Thus, by Eq. (2.36),

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} = (p + 1 - p)^n = 1^n = 1$$

(b) Now

$$P(X > 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - \binom{6}{0}(0.1)^0(0.9)^6 - \binom{6}{1}(0.1)^1(0.9)^5$$

$$= 1 - (0.9)^6 - 6(0.1)(0.9)^5 \approx 0.114$$

2.17. Let X be a Poisson r.v. with parameter λ.

(a) Show that $p_X(x)$ given by Eq. (2.40) satisfies Eq. (2.17).

(b) Find $P(X > 2)$ with $\lambda = 4$.

(a) By Eq. (2.40),

$$\sum_{k=0}^{\infty} p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

(b) With $\lambda = 4$, we have

$$p_X(k) = e^{-4} \frac{4^k}{k!}$$

and

$$P(X \le 2) = \sum_{k=0}^{2} p_X(k) = e^{-4}(1 + 4 + 8) \approx 0.238$$

Thus,

$$P(X > 2) = 1 - P(X \le 2) \approx 1 - 0.238 = 0.762$$

1.52. Consider the binary communication channel shown in Fig. 1-15. The channel input symbol X may assume the state 0 or the state 1, and, similarly, the channel output symbol Y may assume either the state 0 or the state 1. Because of the channel noise, an input 0 may convert to an output 1 and vice versa. The channel is characterized by the channel transition probabilities p_0, q_0, p_1, and q_1, defined by

$$p_0 = P(y_1 \mid x_0) \quad \text{and} \quad p_1 = P(y_0 \mid x_1)$$
$$q_0 = P(y_0 \mid x_0) \quad \text{and} \quad q_1 = P(y_1 \mid x_1)$$

where x_0 and x_1 denote the events $(X = 0)$ and $(X = 1)$, respectively, and y_0 and y_1 denote the events $(Y = 0)$ and $(Y = 1)$, respectively. Note that $p_0 + q_0 = 1 = p_1 + q_1$. Let $P(x_0) = 0.5$, $p_0 = 0.1$, and $p_1 = 0.2$.

(a) Find $P(y_0)$ and $P(y_1)$.

(b) If a 0 was observed at the output, what is the probability that a 0 was the input state?

(c) If a 1 was observed at the output, what is the probability that a 1 was the input state?

(d) Calculate the probability of error P_e.

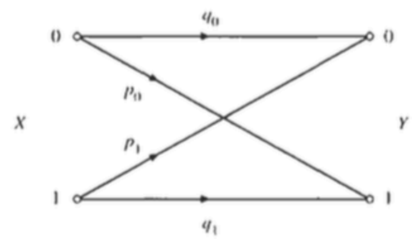


Fig. 1-15

(a) We note that

$$P(x_1) = 1 - P(x_0) = 1 - 0.5 = 0.5$$
$$P(y_0 \mid x_0) = q_0 = 1 - p_0 = 1 - 0.1 = 0.9$$
$$P(y_1 \mid x_1) = q_1 = 1 - p_1 = 1 - 0.2 = 0.8$$

Using Eq. (*1.44*), we obtain

$$P(y_0) = P(y_0 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.9(0.5) + 0.2(0.5) = 0.55$$
$$P(y_1) = P(y_1 \mid x_0)P(x_0) + P(y_1 \mid x_1)P(x_1) = 0.1(0.5) + 0.8(0.5) = 0.45$$

(b) Using Bayes' rule (*1.42*), we have

$$P(x_0 \mid y_0) = \frac{P(x_0)P(y_0 \mid x_0)}{P(y_0)} = \frac{(0.5)(0.9)}{0.55} = 0.818$$

(c) Similarly,

$$P(x_1 \mid y_1) = \frac{P(x_1)P(y_1 \mid x_1)}{P(y_1)} = \frac{(0.5)(0.8)}{0.45} = 0.889$$

(d) The probability of error is

$$P_e = P(y_1 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.1(0.5) + 0.2(0.5) = 0.15.$$

1.35. Consider the experiment of tossing a fair coin repeatedly and counting the number of tosses required until the first head appears.

(a) Find the sample space of the experiment.

(b) Find the probability that the first head appears on the kth toss.

(c) Verify that $P(S) = 1$.

(a) The sample space of this experiment is

$$S = \{e_1, e_2, e_3, \dots\} = \{e_k: k = 1, 2, 3, \dots\}$$

where e_k is the elementary event that the first head appears on the kth toss.

(b) Since a fair coin is tossed, we assume that a head and a tail are equally likely to appear. Then $P(H) = P(T) = \frac{1}{2}$. Let

$$P(e_k) = p_k \qquad k = 1, 2, 3, \dots$$

Since there are 2^k equally likely ways of tossing a fair coin k times, only one of which consists of $(k - 1)$ tails following a head we observe that

$$P(e_k) = p_k = \frac{1}{2^k} \qquad k = 1, 2, 3, \dots \tag{1.79}$$

(c) Using the power series summation formula, we have

$$P(S) = \sum_{k=1}^{\infty} P(e_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \tag{1.80}$$