BLM2502 Theory of Computation

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Course Outline

Week	Content
1	Introduction to Course
2	Computability Theory, Complexity Theory, Automata Theory, Set Theory, Relations, Proofs, Pigeonhole Principle
3	Regular Expressions
4	Finite Automata
5	**Deterministic and Nondeterministic Finite Automata**
6	**Epsilon Transition, Equivalence of Automata**
7	Pumping Theorem
8	April 10 - 14 week is the first midterm week
9	Context Free Grammars
10	Parse Tree, Ambiguity,
11	Pumping Theorem
12	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
13	Turing Machines, Recognition and Computation, Church-Turing Hypothesis
14	May 22 – 27 week is the second midterm week
15	Review
16	Final Exam date will be announced

Non-regular languages

(Pumping Lemma)

Non-regular languages

$$\{a^n b^n : \ n \geq 0\}$$

$$\{vv^R : \ v \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c + a$$

$$b + c(a+b)^*$$

$$etc...$$

How can we prove that a language L is not regular?

Prove that there is no DFA or NFA or RE that accepts L

Difficulty: this is not easy to prove

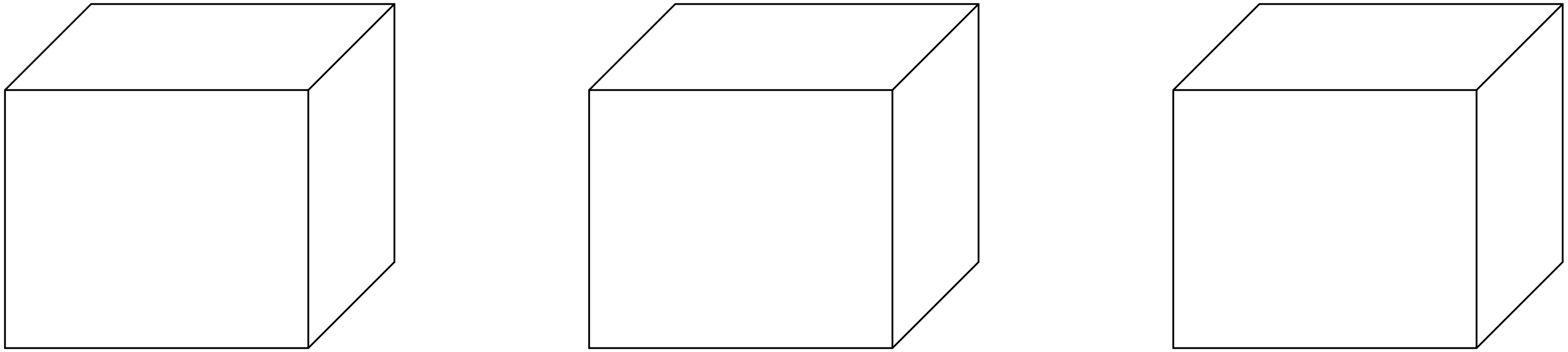
(since there is an infinite number of them)

Solution: use the Pumping Lemma !!!

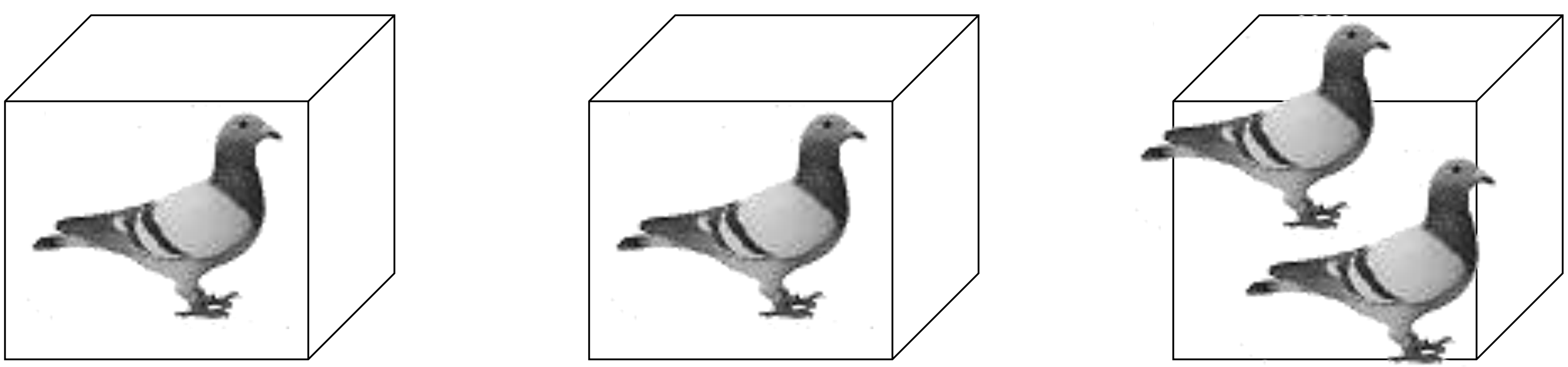
The Pigeonhole Principle

4 pigeons

3 pigeonholes

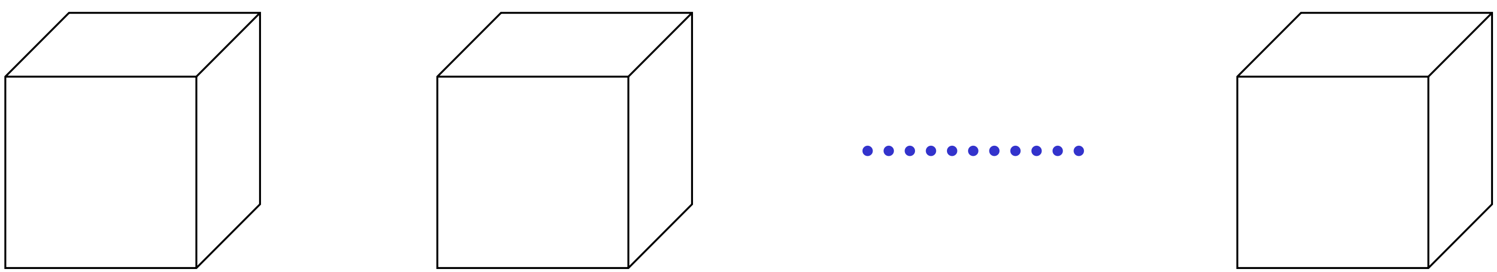


A pigeonhole must
contain at least two pigeons



n pigeons

m pigeonholes $\quad n > m$



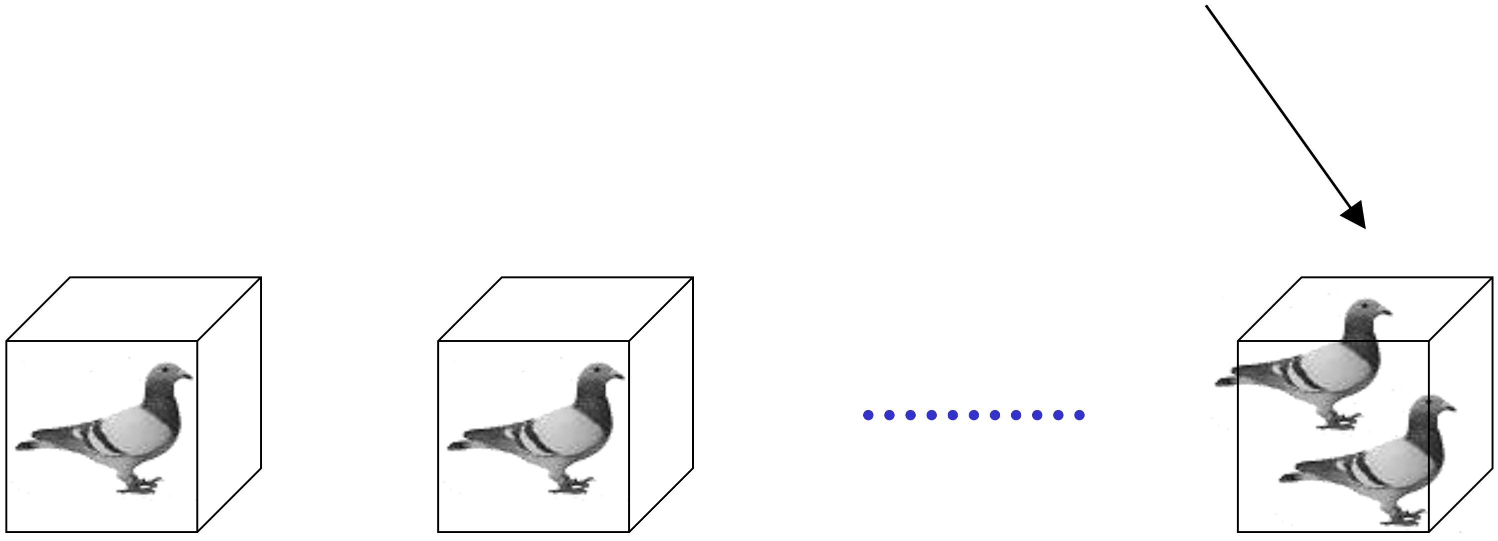
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

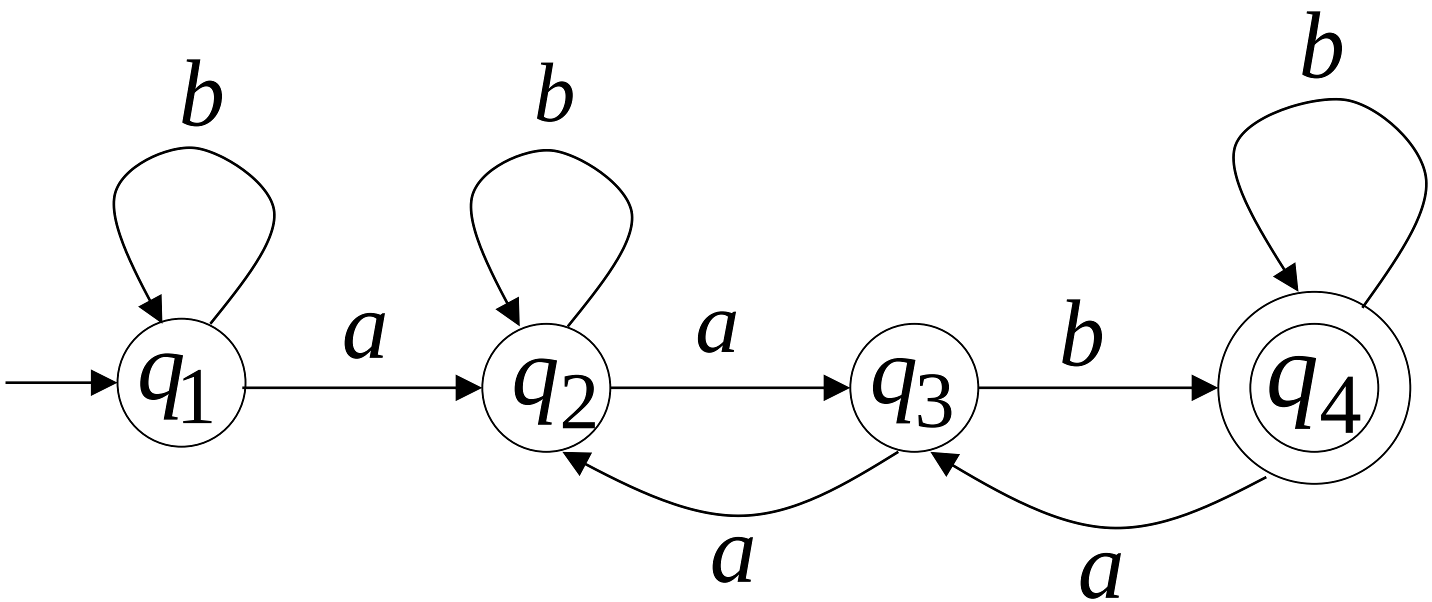


The Pigeonhole Principle

and

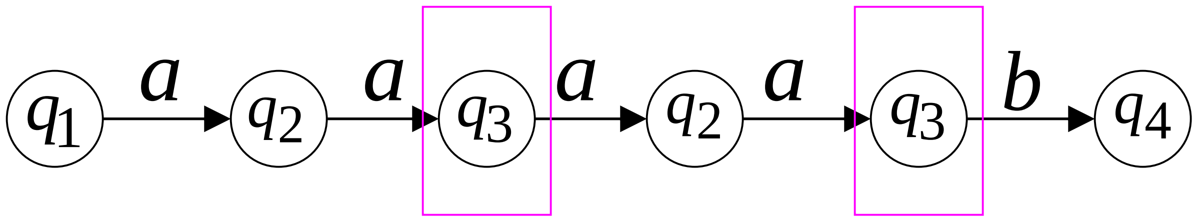
DFAs

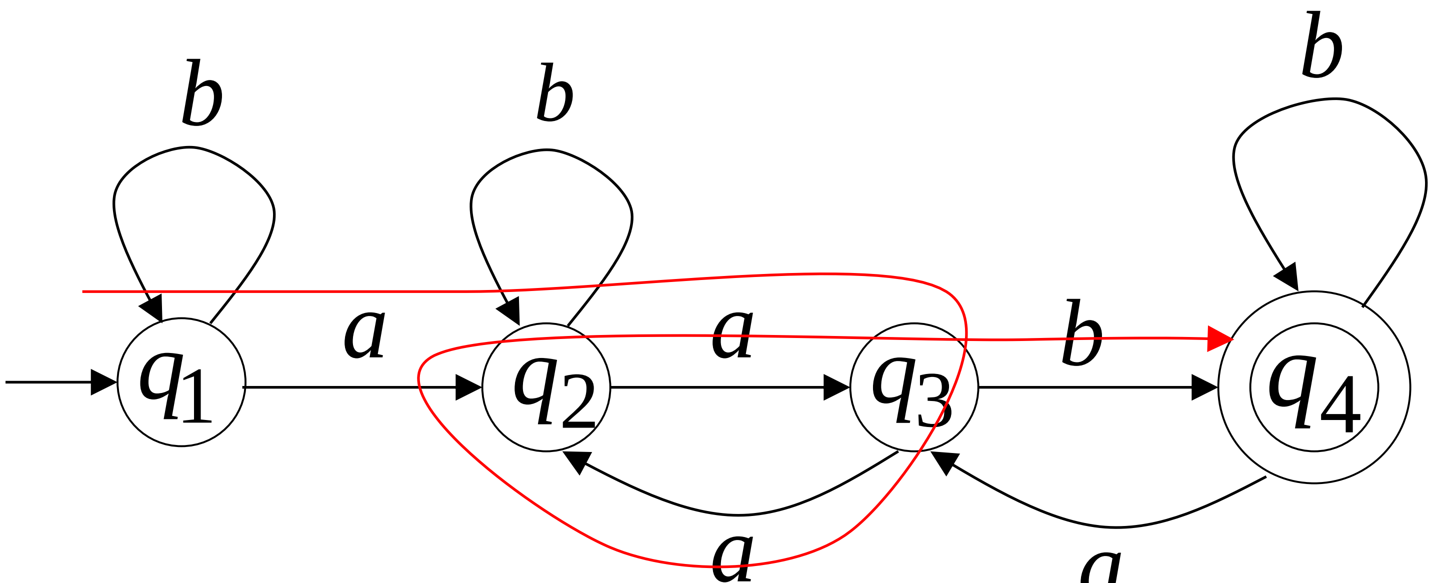
Consider a DFA with 4 states



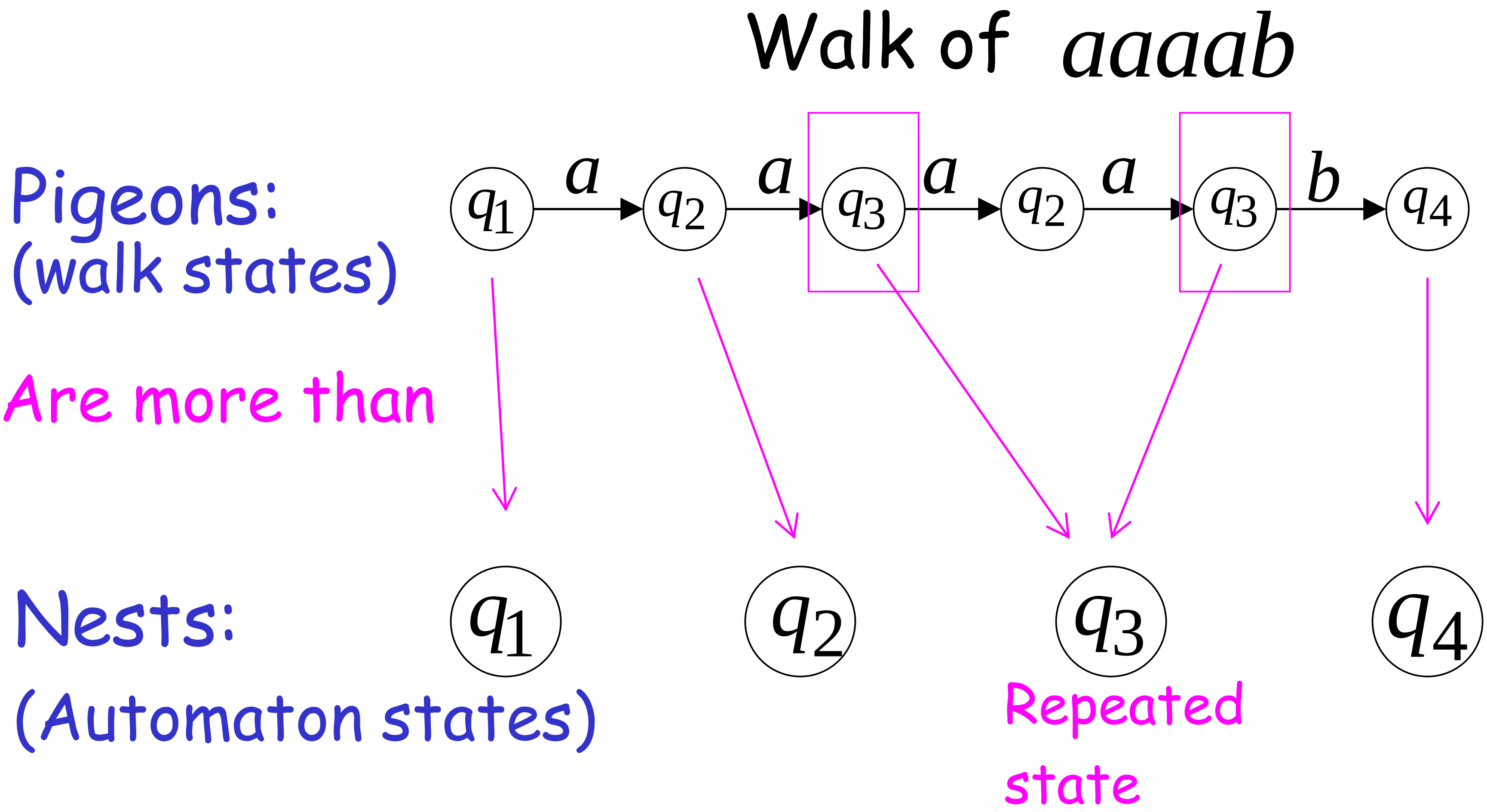
Consider the walk of a “long’’ string: $aaaab$
(length at least 4)

A state is repeated in the walk of $aaaab$





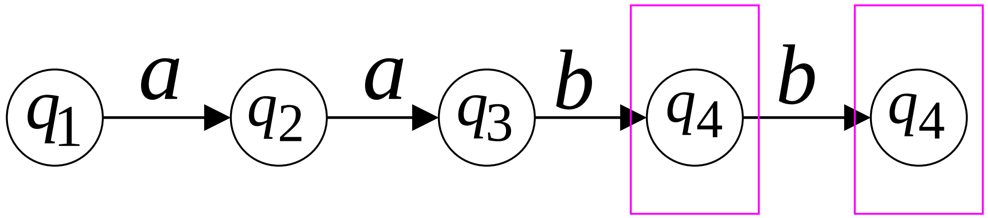
The state is repeated as a result of the pigeonhole principle

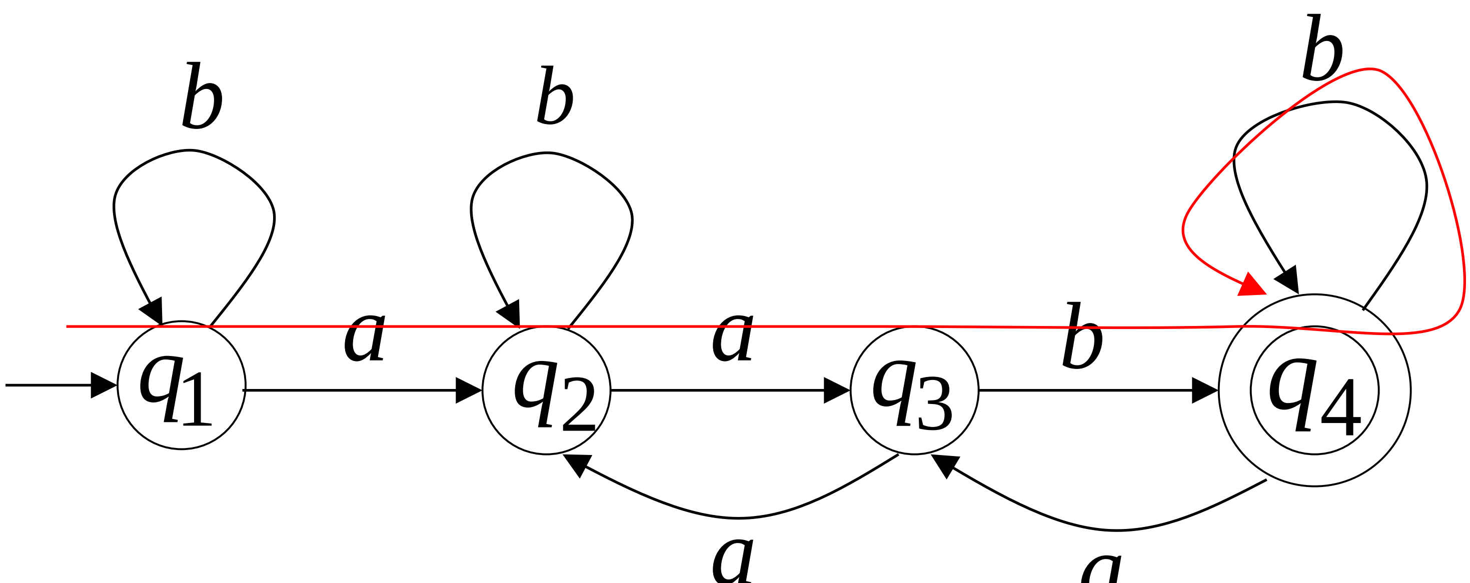


Consider the walk of a "long'' string: $aabb$
(length at least 4)

Due to the pigeonhole principle:

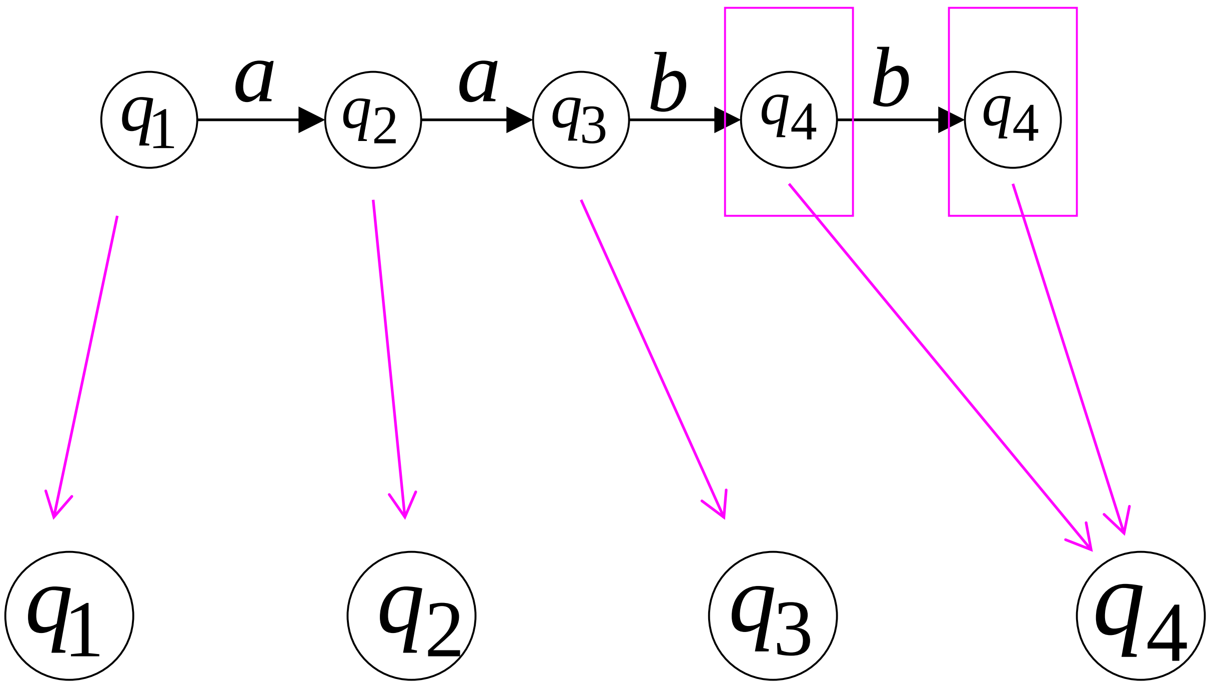
A state is repeated in the walk of $aabb$





The state is repeated as a result of the pigeonhole principle

Walk of $aabb$



Pigeons:
(walk states)

Are more than

Nests:
(Automaton states)

Automaton States

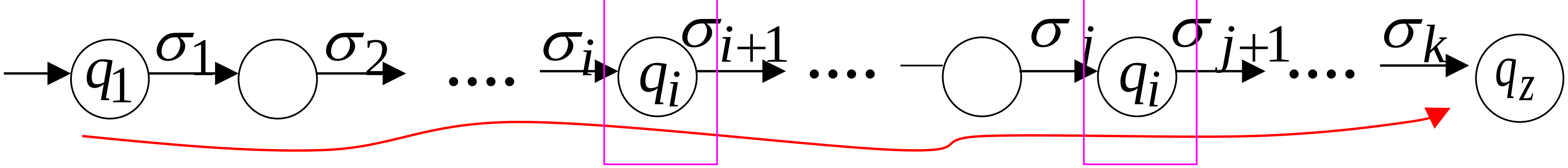
Repeated state

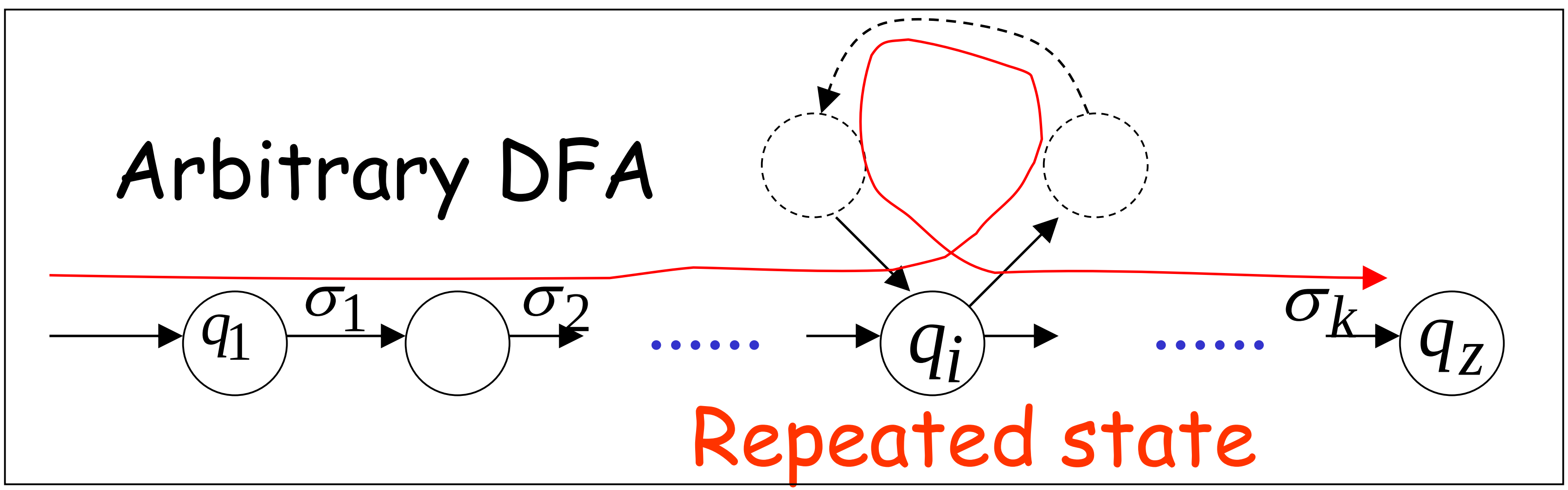
In General: If $|w| \geq$ #states of DFA,

by the pigeonhole principle,

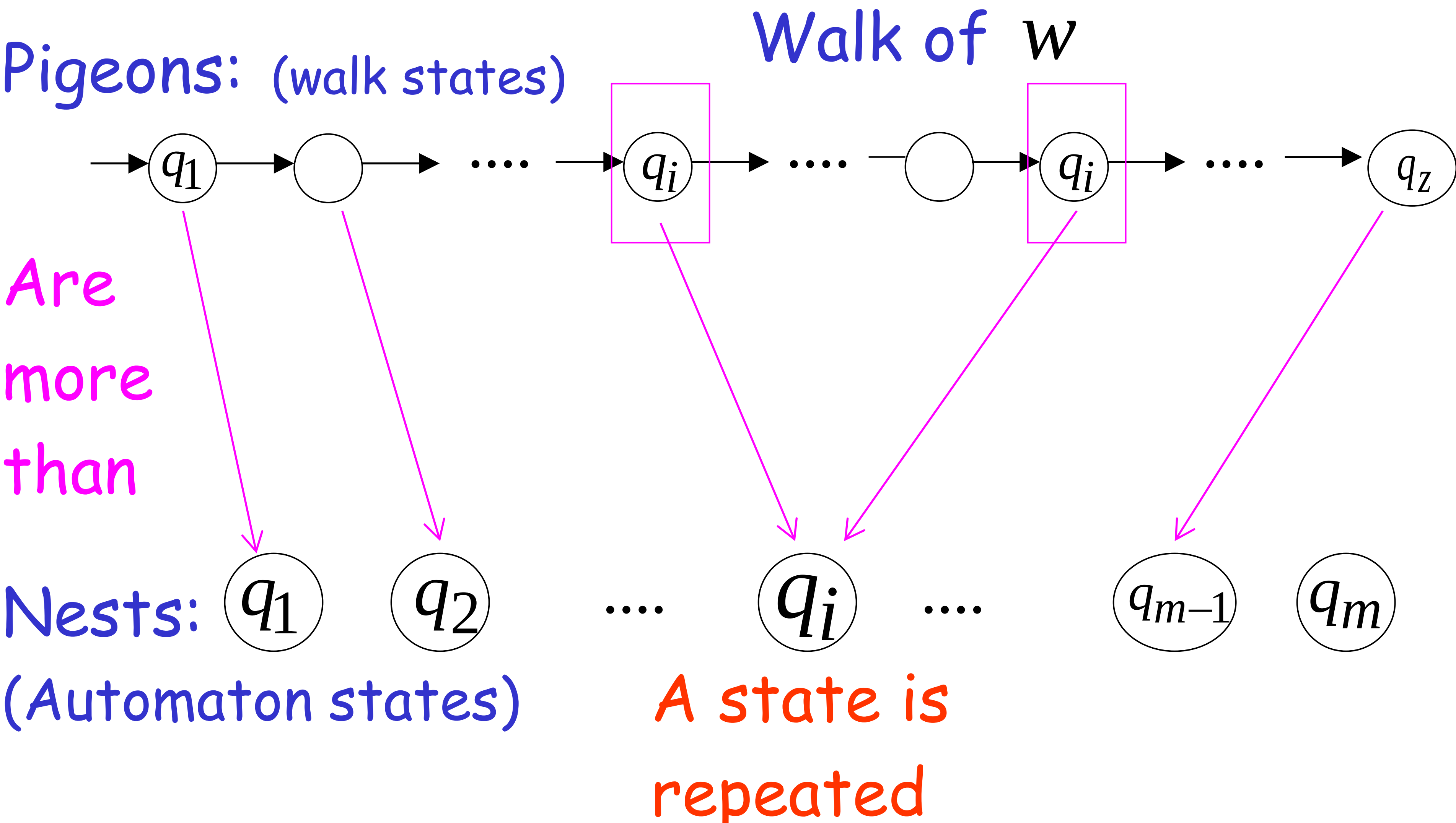
a state is repeated in the walk w

Walk of $w = \sigma_1 \sigma_2 \cdots \sigma_k$





$|w| \geq \#\text{states of DFA} = m$

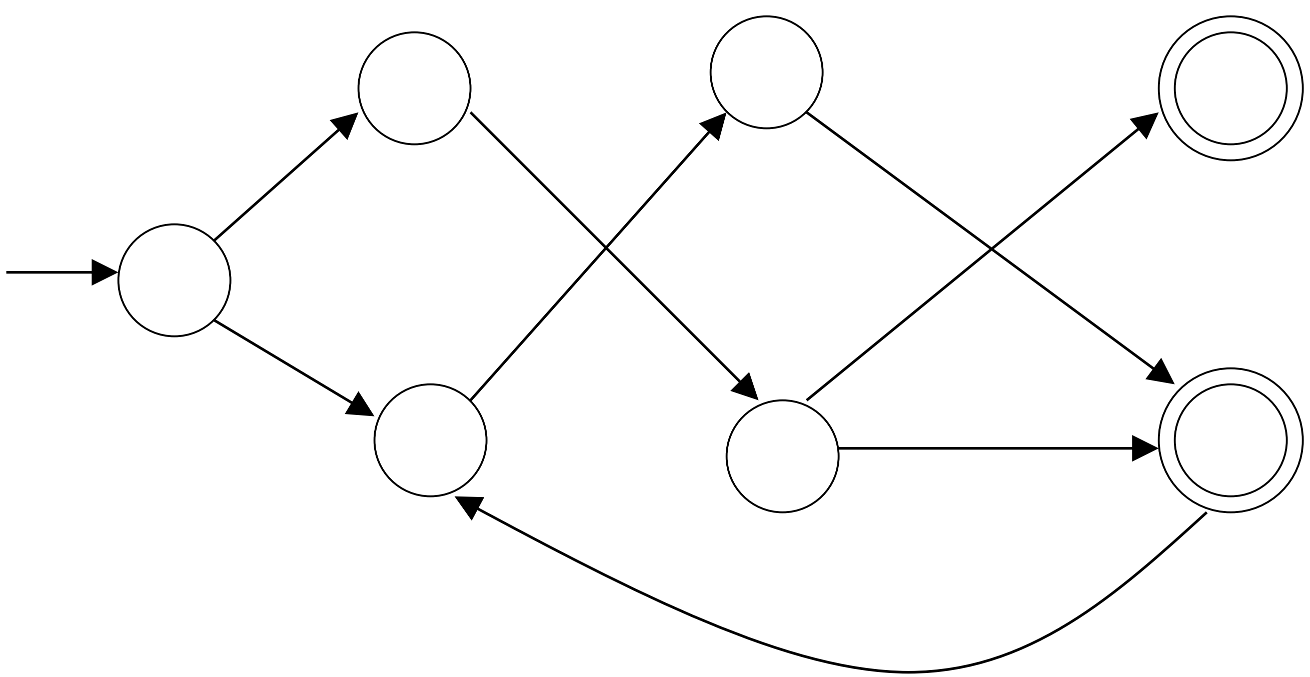


The Pumping Lemma

Take an infinite regular language L

(contains an infinite number of strings)

There exists a DFA that accepts L



m states

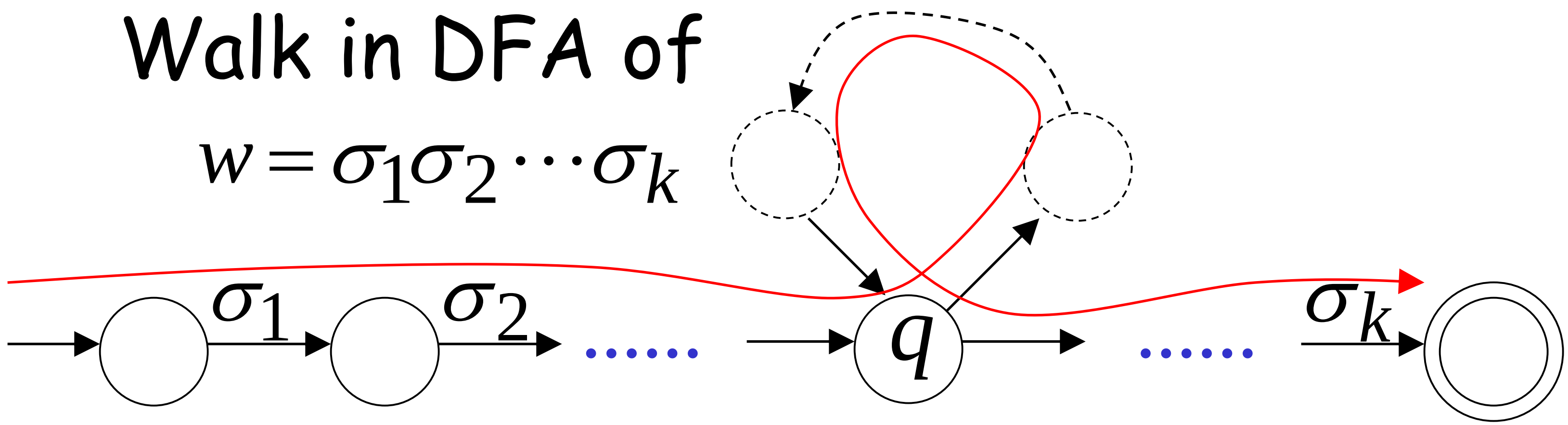
Take string $w \in L$ with $|w| \geq m$

(number of states of DFA)

then, at least one state is repeated in the walk of w

Walk in DFA of

$w = \sigma_1 \sigma_2 \cdots \sigma_k$

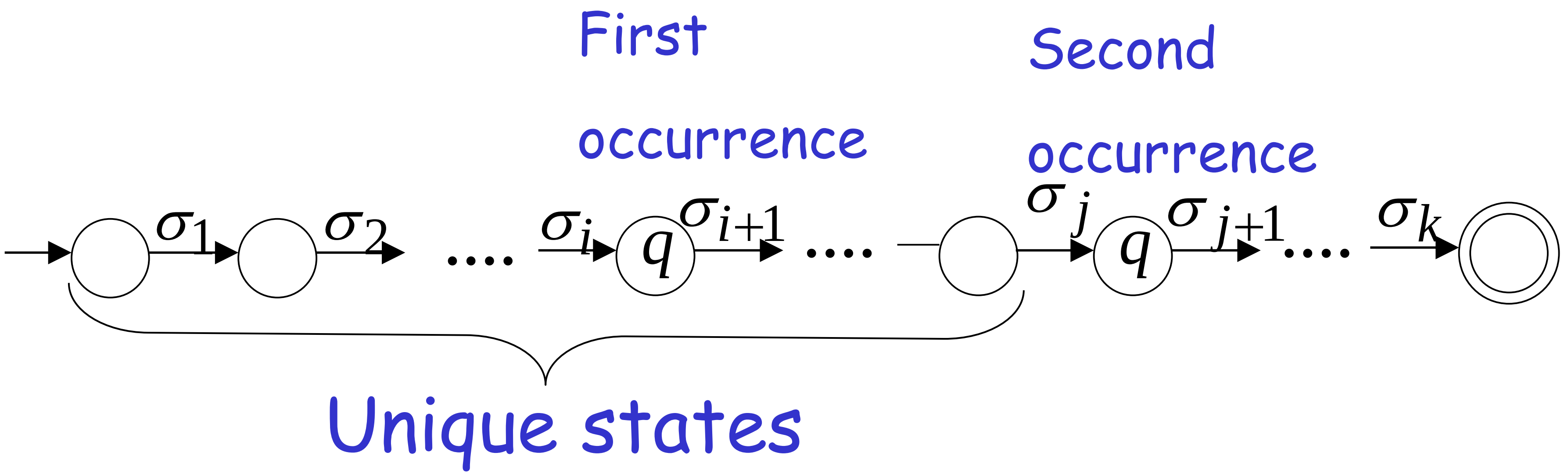


Repeated state in DFA

There could be many states repeated

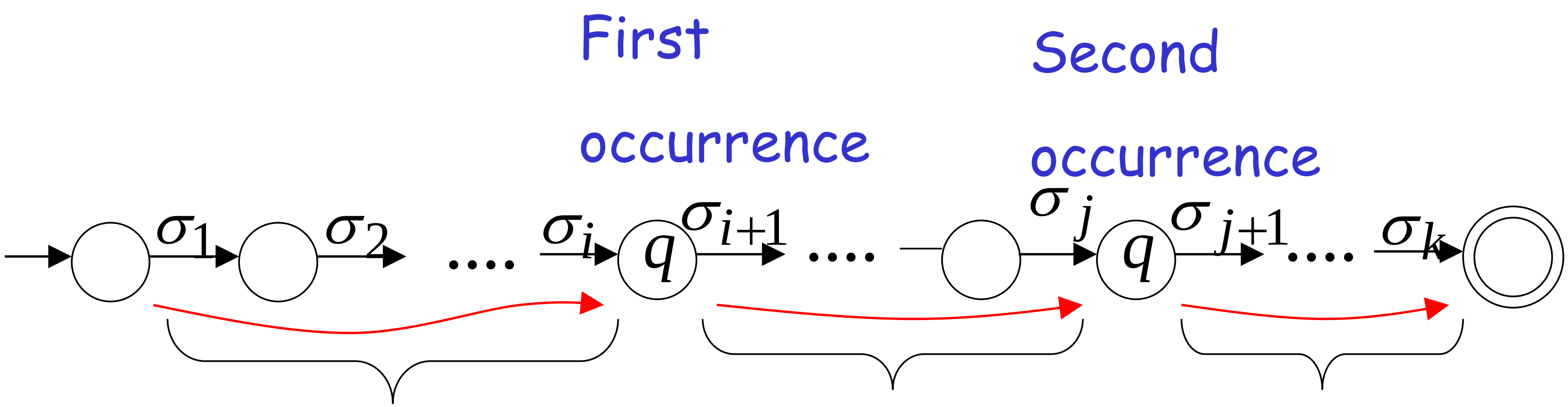
Take q to be the first state repeated

One dimensional projection of walk w:



We can write $w = xyz$

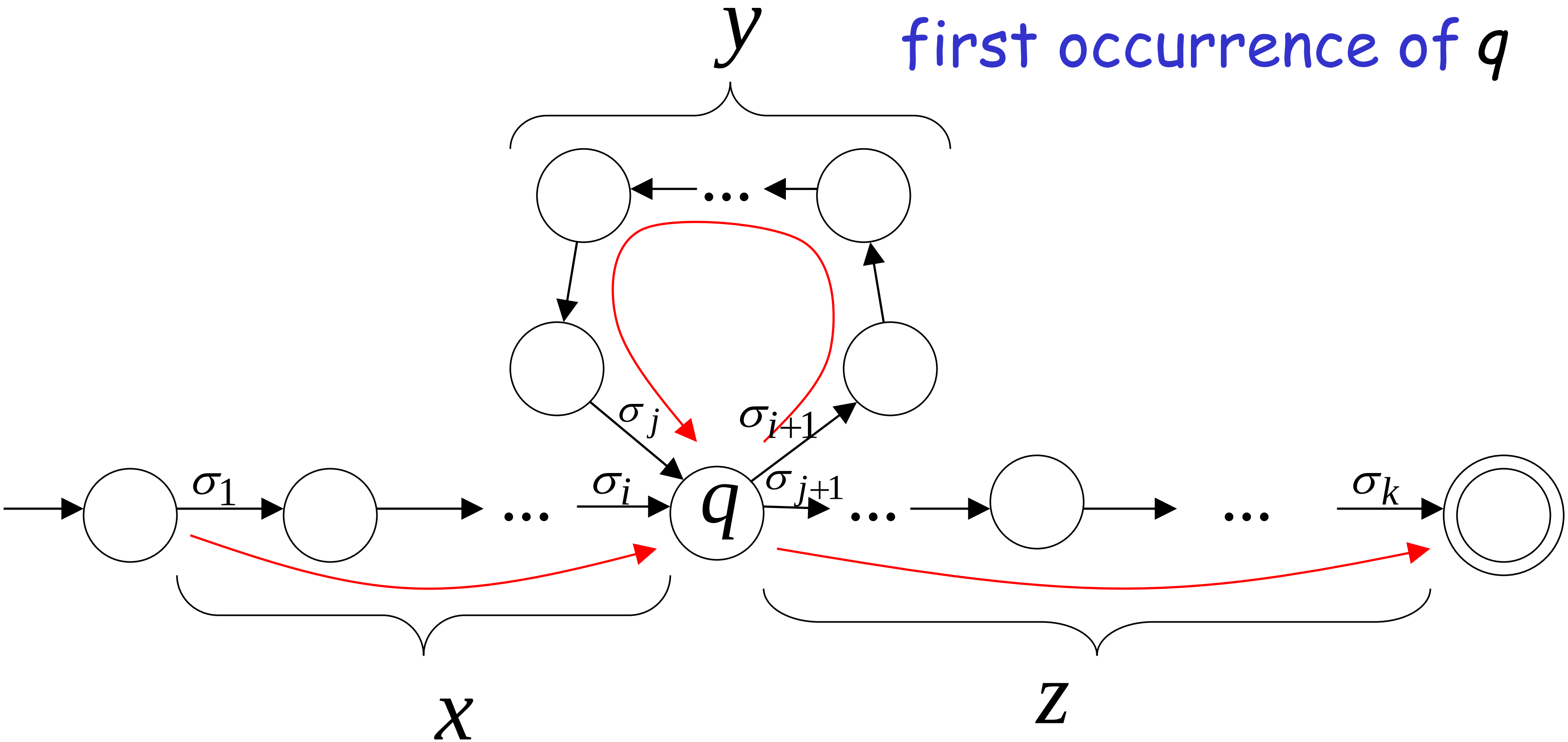
One dimensional projection of walk w:



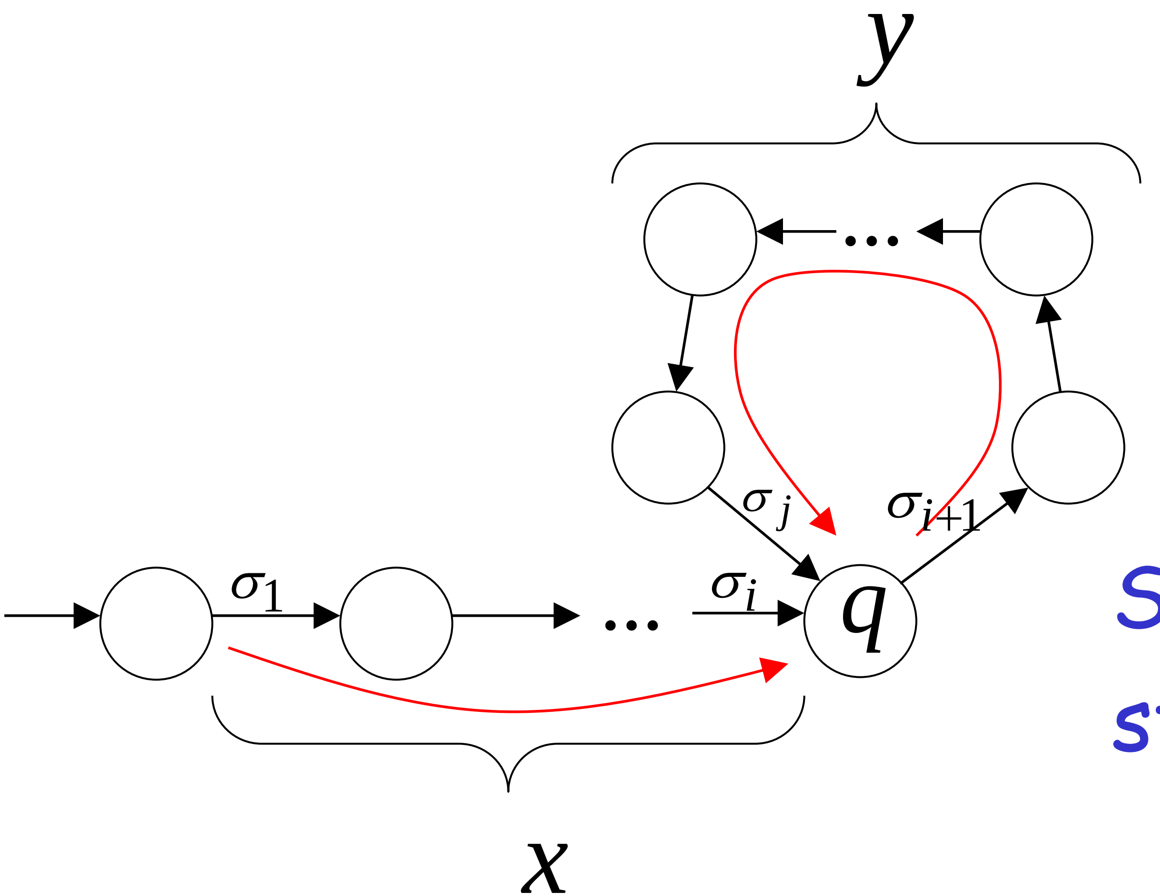
$x = \sigma_1 \cdots \sigma_i$ $\quad y = \sigma_{i+1} \cdots \sigma_j$ $\quad z = \sigma_{j+1} \cdots \sigma_k$

In DFA: $w = x\ y\ z$

contains only
first occurrence of q



Observation: length $|x\,y| \leq m$ number of states of DFA

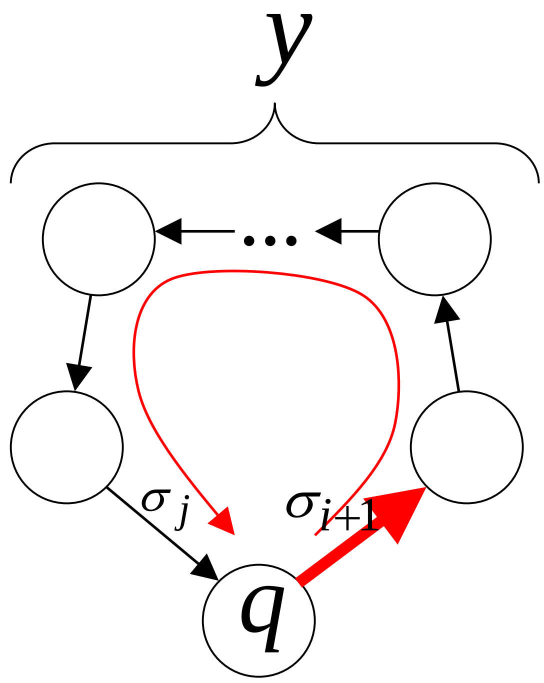


Unique States

Since, in xy no state is repeated (except q)

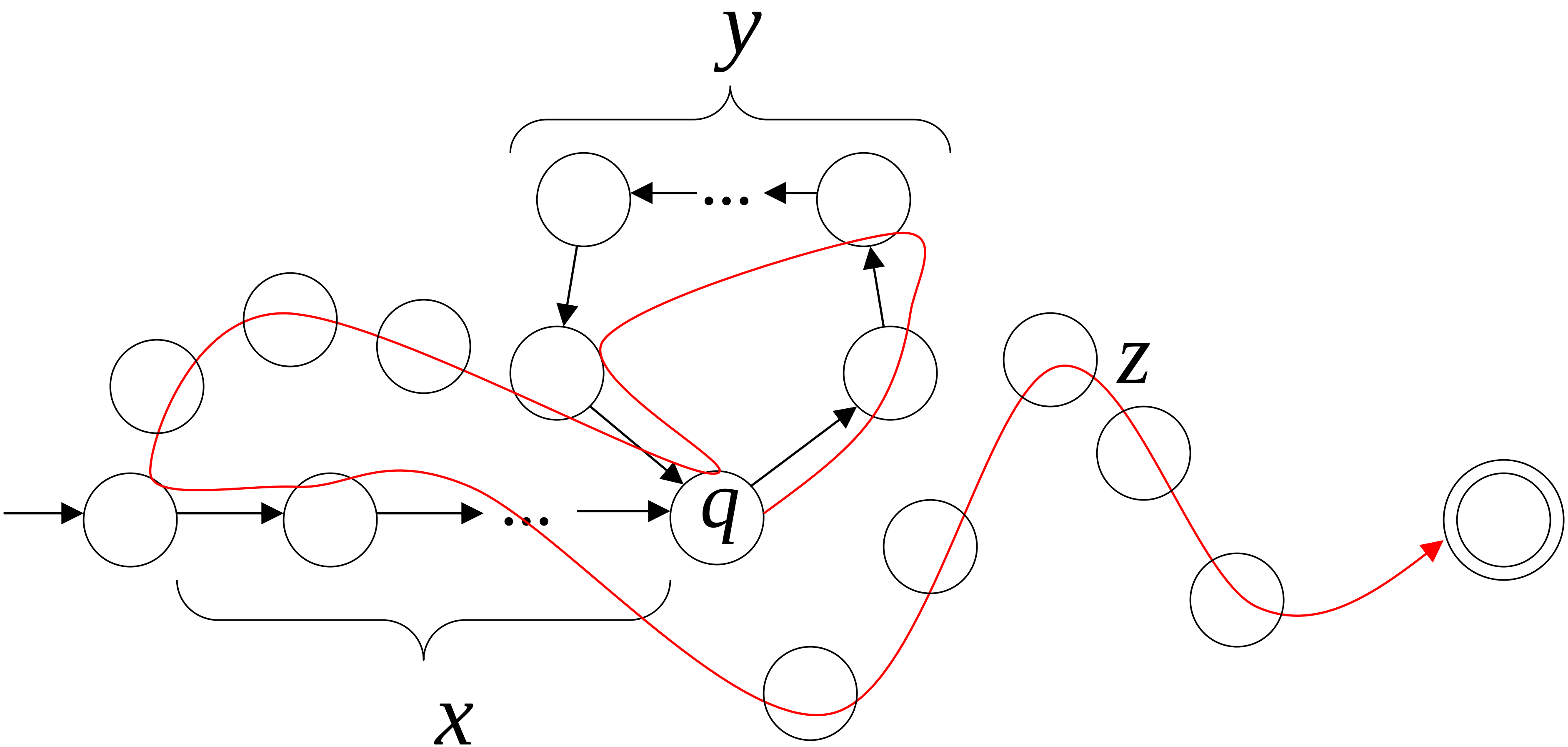
Observation: length $|y| \geq 1$

Since there is at least one transition in loop



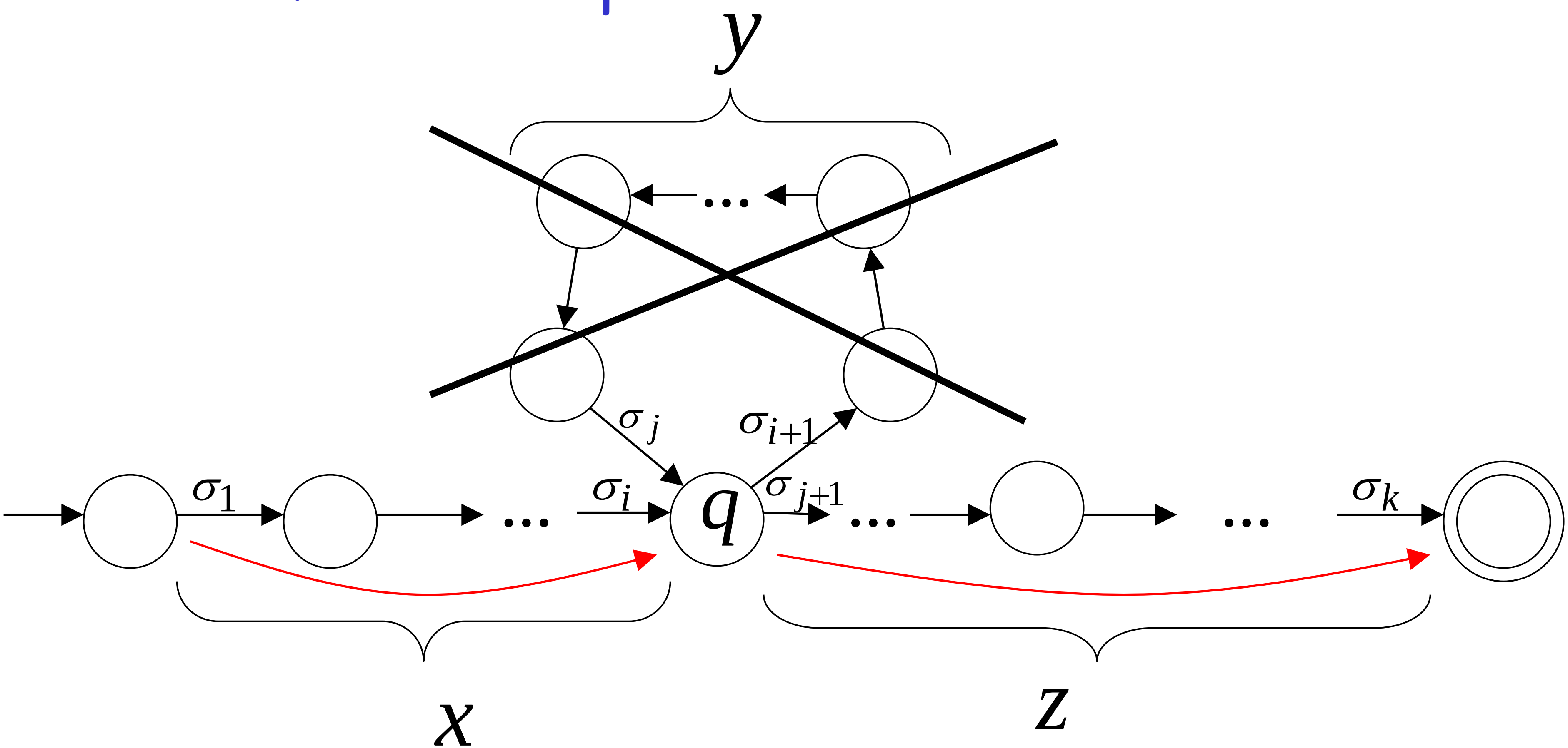
We do not care about the form of string z

z may actually overlap with the paths of x and y



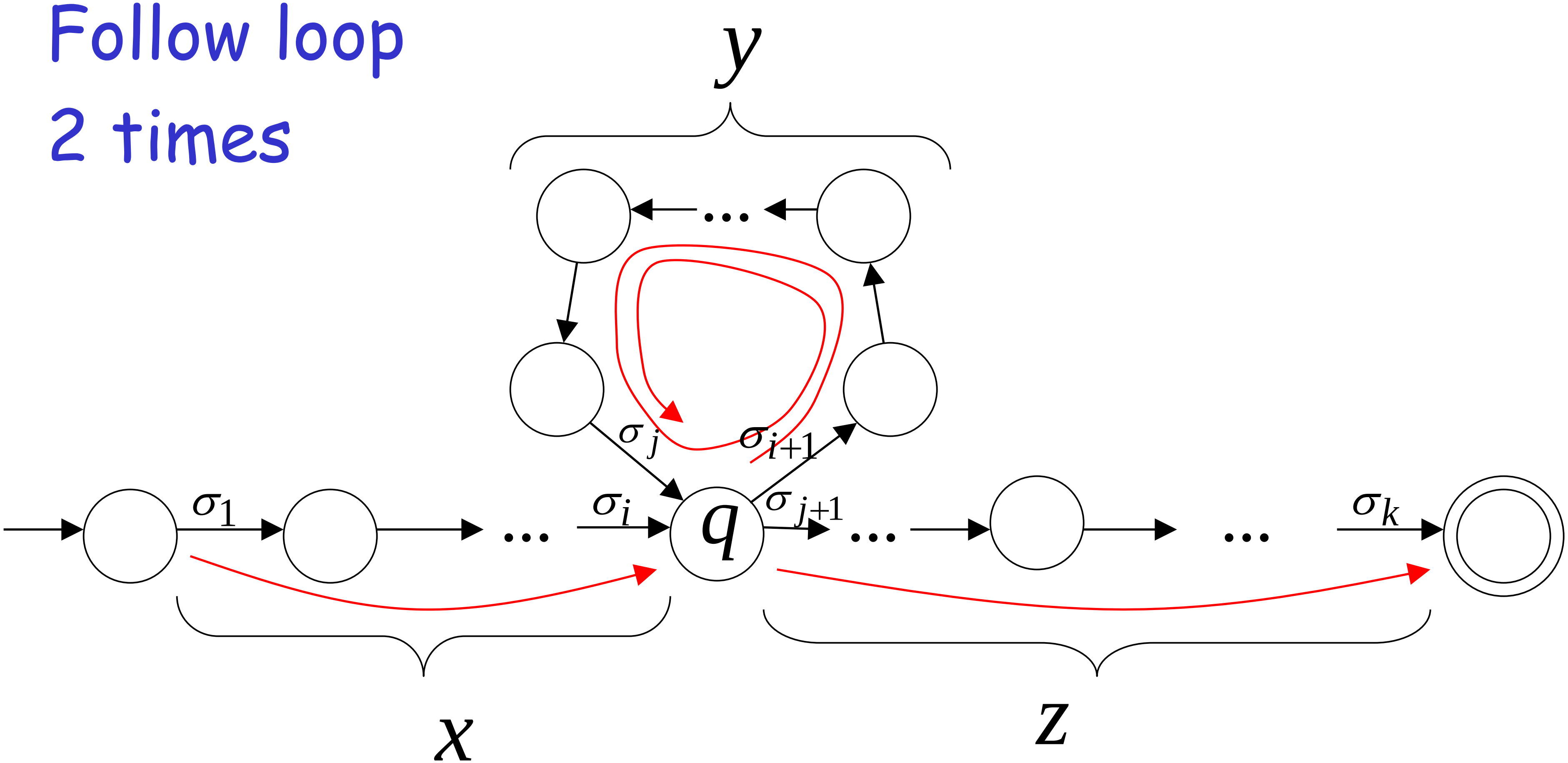
Additional string: The string $x\ z$ is accepted

Do not follow loop



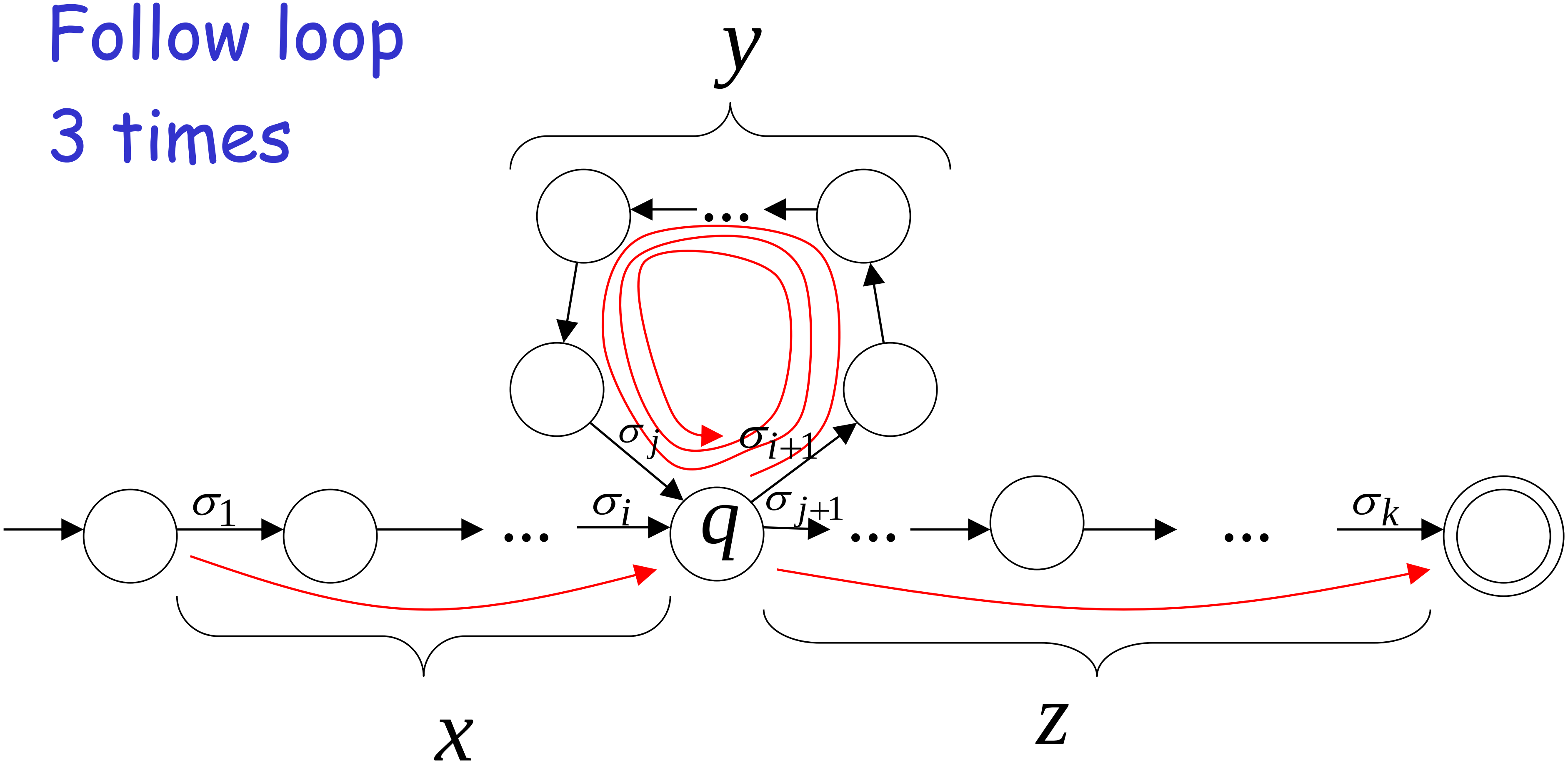
Additional string: The string $x\,y\,y\,z$ is accepted

Follow loop 2 times



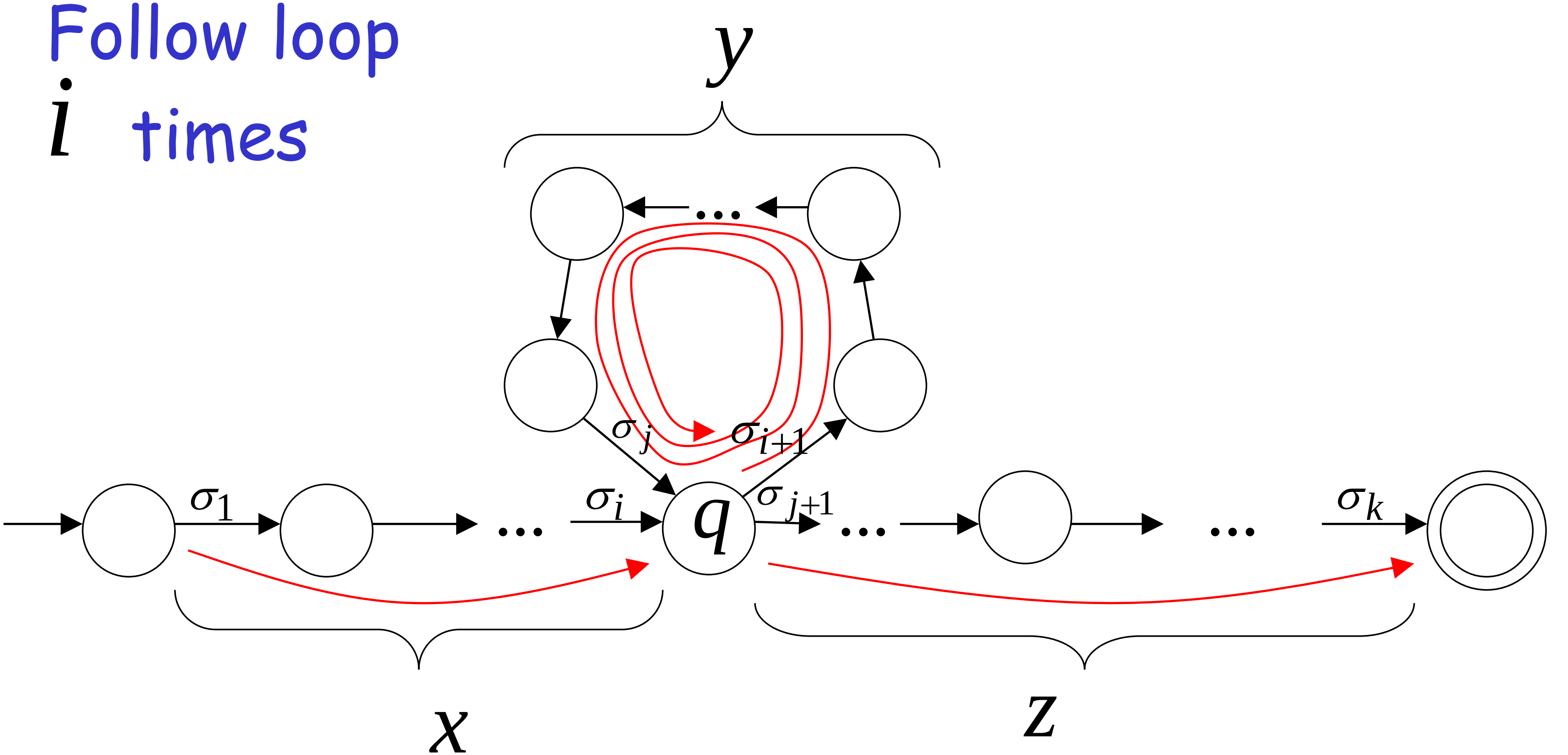
Additional string: The string $x\,y\,y\,y\,z$ is accepted

Follow loop 3 times



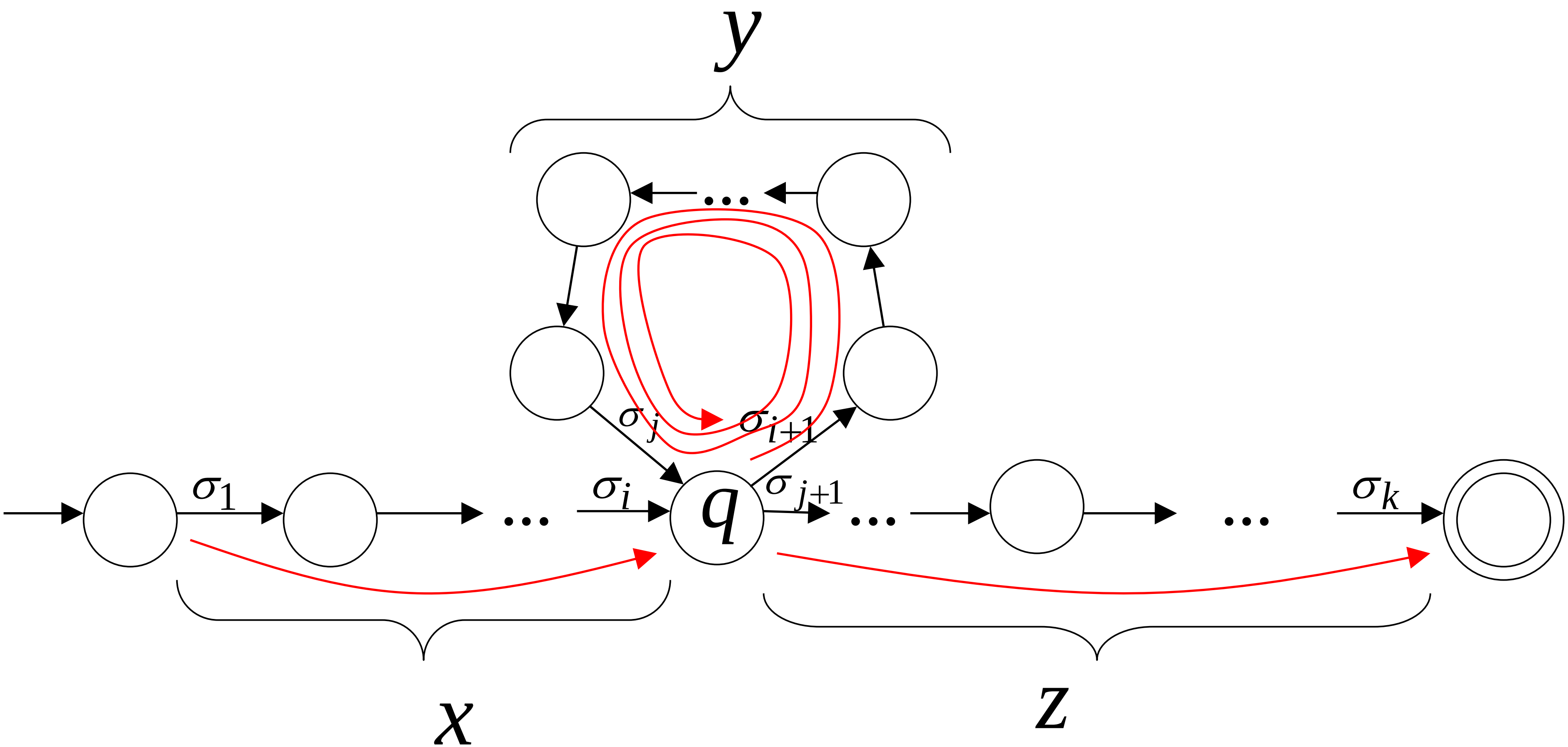
In General: The string $x\ y^i\ z$ is accepted $i = 0, 1, 2, \ldots$

Follow loop i times



Therefore: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

In the book:

Critical length m = Pumping length p

Applications

of

the Pumping Lemma

Observation:

Every language of finite size has to be regular

(we can easily construct an NFA that accepts every string in the language)

Therefore, every non-regular language has to be of infinite size

(contains an infinite number of strings)

Suppose you want to prove that
An infinite language L is not regular

1. Assume the opposite: L is regular

2. The pumping lemma should hold for L

3. Use the pumping lemma to obtain a contradiction

4. Therefore, L is not regular

Explanation of Step 3: How to get a contradiction

1. Let m be the critical length for L

2. Choose a particular string $w \in L$ which satisfies the length condition $|w| \geq m$

3. Write $w = xyz$

4. Show that $w' = xy^i z \notin L$ for some $i \neq 1$

5. This gives a contradiction, since from pumping lemma $w' = xy^i z \in L$

Note: It suffices to show that only one string $w \in L$ gives a contradiction

You don't need to obtain contradiction for every $w \in L$

Example of Pumping Lemma application

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m$

From the Pumping Lemma:

we can write $w = a^m b^m = x\ y\ z$

with lengths $|x\,y| \le m, \quad |y| \ge 1$

$$w = xyz = a^m b^m = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}$$

$$\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z}$$

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^m b^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^2\ z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{b...b}^{m} \in L$$

(where z spans $a...ab...b$)

Thus: $a^{m+k} b^m \in L$

$$a^{m+k}b^m \in L \qquad\qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

$$\Downarrow$$

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular language $\{a^n b^n : \ n \geq 0\}$

Regular languages

$L(a^* b^*)$

More Applications

of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$

Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

From the Pumping Lemma:

we can write: $w = a^m b^m b^m a^m = x\ y\ z$

with lengths: $|x\,y| \le m, \quad |y| \ge 1$

$$w = xyz = \underbrace{\overbrace{a...a}^{}a...a...a}_{}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}$$

$$w = xyz = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\,...a}^{m}\ \underbrace{\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z}$$

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \ \in\ L$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\, y^2\, z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{...a}_{}}^{m+k}\underbrace{\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z} \in L$$

Thus: $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k}b^m b^m a^m \in L \qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$\Downarrow$

$$a^{m+k}b^m b^m a^m \notin L$$

CONTRADICTION!!!

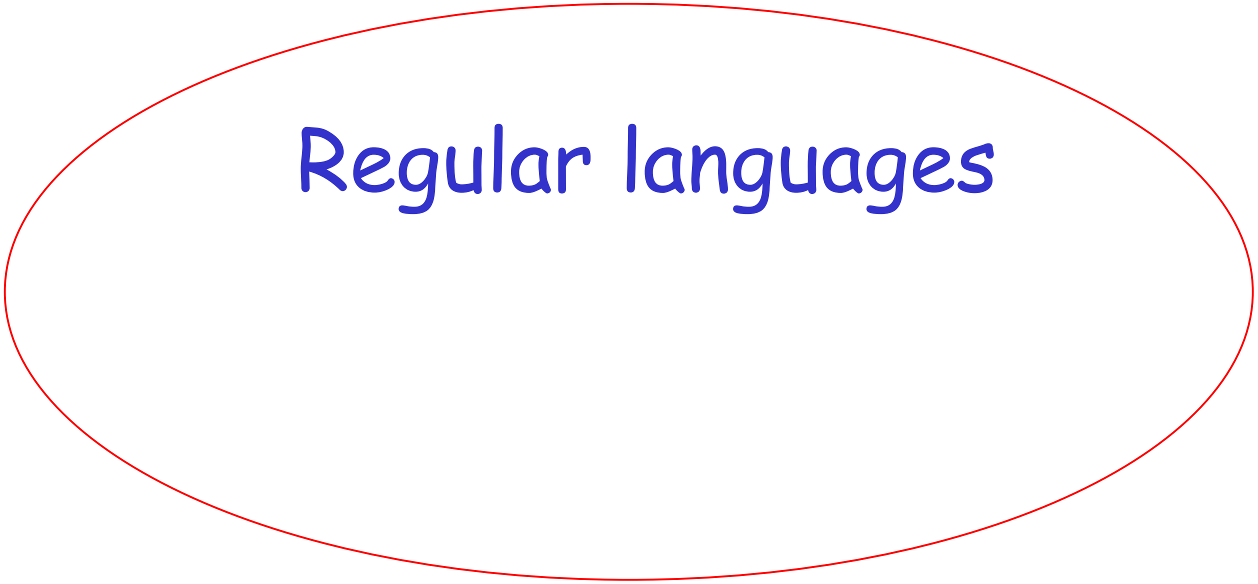
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$



Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

From the Pumping Lemma:

We can write $w = a^m b^m c^{2m} = x\ y\ z$

With lengths $|x\,y| \le m, \quad |y| \ge 1$

$$w = xyz = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}^{m}\underbrace{\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}}_{z}}$$

Thus: $y = a^k, \quad 1 \le k \le m$

$$x\ y\ z = a^m b^m c^{2m}$$ $$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $$x\ y^i\ z \in L$$

$$i = 0, 1, 2, \ldots$$

Thus: $$x\ y^0\ z = xz \in L$$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \underbrace{\overbrace{a...aa...a}^{m-k}}_{x}\ \overbrace{b...b}^{m}\ \overbrace{c...cc...c}^{2m} \in L$$

(where $x = a...a$ and $z = a...ab...bc...cc...c$)

Thus: $\quad a^{m-k} b^m c^{2m} \in L$

$$a^{m-k}b^m c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \ n,l \geq 0\}$

$\Downarrow$

$$a^{m-k}b^m c^{2m} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^{n!} : \ n \geq 0\}$$

Regular languages

Theorem: The language $L = \{a^{n!} : \; n \geq 0\}$

is not regular

$$n! \; = \; 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!}: n \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$

length $|w| \geq m$

$\boxed{\text{We pick } w = a^{m!}}$

From the Pumping Lemma:

We can write $w = a^{m!} = x\ y\ z$

With lengths $|x\ y| \leq m, \quad |y| \geq 1$

$$w = xyz = a^{m!} = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

where z spans from after y to the end.

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\,y\,z = a^{m!} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^{m!} \qquad\qquad y = a^k,\ \ 1 \le k \le m$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{a...aa...a}^{m!-m} \ \in\ L$$

(where z spans $a...aa...aa...a$)

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

Since: $L = \{a^{n!}: \; n \geq 0\}$

There must exist p such that:

$$m! + k = p!$$

However: $m!+k \leq m!+m$ for $m > 1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : n \geq 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Example 2

Theorem: $B = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof. Assume the contrary that B is regular.

Let p be the pumping length. Then, $s = 0^p 1^p$ can be decomposed as $s = xyz$ with $|y| \geq 1$, and $xy^n z \in B$ for any $n \geq 0$.

There can be three cases $y = 0^k, y = 0^k 1^l$, and $y = 1^l$, for some nonzero k, l. In each case, we can eazily see that $xy^n z \notin B$, which leads to contradiction. $(n \geq 2)$

Example 3

Theorem: $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$ *is not regular.*

Proof. Assume the contrary that C is regular.

Let p be the pumping length. Then, $s = 0^p 1^p$ can be decomposed as $s = xyz$ with $|y| \geq 1$ and $|xy| \leq p$, and $xy^n z \in C$ for any $n \geq 0$.

Then we must have $y = 0^k$ for some nonzero k.

We can eazily see that $xy^n z \notin C$, which contradicts the assumption.

Alternative proof of Example 3

- The class of regular languages is closed under the intersection operation. This is easy to prove if we run two DFAs parallelly and accept only string which are accepted by both of the DFAs.
- Now, assume C is regular.
- Then, $C \cap 0^*1^* = B$ is also regular.
- This contradicts what we proved in Example 1.38.???

Example 4

Theorem: $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

Proof. Assume the contrary that F is regular.

Let p be the pumping length and let $s = 0^p10^p1 \in F$. This s can be split into pieces like $s = xyz$ with $|y| \geq 1$ and $|xy| \leq p$, and $xy^nz \in F$ for any $n \geq 0$.

Then we must have $y = 0^k$ for some nonzero k.

We can eazily see that $xy^nz \notin F$, which contradicts the assumption.

Example 5

Theorem: $D = \{1^{n^2} \mid n \geq 0\}$ *is not regular.*

Proof. Assume the contrary that D is regular.

Let p be the pumping length and let $s = 1^{p^2} \in D$. This s can be split into pieces like $s = xyz$ with $|y| \geq 1$ and $|xy| \leq p$, and $xy^n z \in D$ for any $n \geq 0$.

The length of $xy^n z$� grows linearly with n, while the lengths of strings in D grows as $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

These two facts are incompatible as can be easily seen.

Example 6 (Pumping Down)

Theorem : $E = \{0^i 1^j \mid i > j\}$ *is not regular.*

Proof. Assume the contrary that E is regular.

Let p be the pumping length and let $s = 0^{p+1}1^p$. This s can be split

into $s = xyz$ with $|y| \geq 1$ and $|xy| \leq p$, and $xy^n z \in E$ for any $n \geq 0$.

Then we must have $y = 0^k$ for some nonzero k.

We can eazily see that $xy^0 z = xz \notin E$, which contradicts the assumption.

Example 6 (Differential Encoding)

Claim : $D = \{w \mid w \text{ contains equal number of occurrences of the substrings } 01 \text{ and } 10\}$ *is regular*.