

Doğal Logaritma ve Doğal Üstel Fonksiyon

F1

★ $\ln x = \int_1^x \frac{dt}{t}$ ($x > 0$) denklemi ile verilen $f(x) = \ln x$ fonksiyonuna "Doğal Logaritma" denir.

★ $\ln x = e^x$ ile tanımlı ($\ln x$ fonksiyonunun ters fonksiyonu) $f(x) = e^x$ fonksiyonuna "Doğal Üstel Fonksiyon" denir.

★ $f(x) = a^x$ ($a > 0, a \neq 1$) ile tanımlı fonksiyona "Genel Üstel Fonksiyon" denir.

★ $f(x) = \log_a x$ fonksiyonuna (a^x fonksiyonunun tersidir) "Genel logaritmik fonksiyon" denir. ($\ln x = \log_e x$ dir)

Özellikleri

$$\textcircled{1} \ln e = 1 \quad \left\{ \begin{array}{l} \textcircled{2} \ln xy = \ln x + \ln y \\ \textcircled{3} \ln \frac{x}{y} = \ln x - \ln y \end{array} \right.$$

$$\textcircled{4} \ln x^r = r \ln x \quad \left\{ \begin{array}{l} \textcircled{5} \log_a a^x = x \\ \ln e^x = x \end{array} \right. \quad \textcircled{6} \quad \begin{array}{l} e^{\ln x} = x \\ a^{\log_a x} = x \end{array}$$

$$\textcircled{7} \quad \begin{array}{l} a^x = e^{x \ln a} \\ x^n = e^{n \ln x} \end{array} \quad \left\{ \begin{array}{l} \textcircled{8} \log_a x = \frac{\ln x}{\ln a} \\ \textcircled{9} \ln 1 = 0 \\ \log_a 1 = 0 \end{array} \right.$$

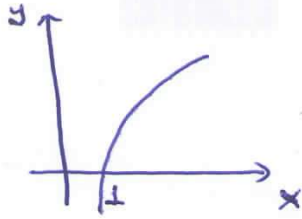
$$\textcircled{10} \log_a xy = \log_a x + \log_a y \quad \left\{ \begin{array}{l} \textcircled{11} \log_a \frac{x}{y} = \log_a x - \log_a y \end{array} \right.$$

$$\textcircled{12} \ln \frac{1}{x} = -\ln x \quad \left\{ \begin{array}{l} \textcircled{13} \log_a a = 1 \\ \textcircled{14} \log_a x = y \Leftrightarrow x = a^y \\ \ln x = y \Leftrightarrow x = e^y \end{array} \right. \\ \log_a \frac{1}{x} = -\log_a x$$

Grafikleri

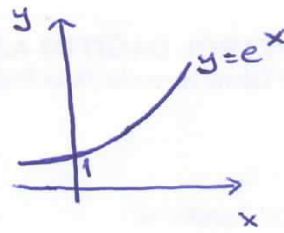
[F2]

★ $y = \ln x$



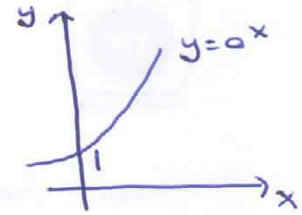
Fonk. Artandır

★ $y = e^x$



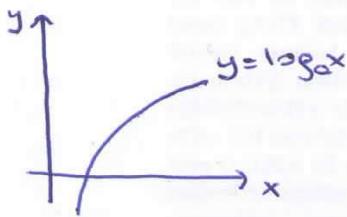
Fonk. Artandır

★ $y = a^x \quad (a > 1)$



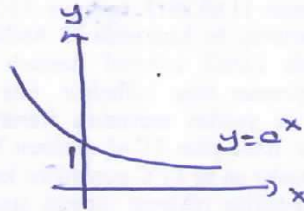
Fonk. Artandır

★ $y = \log_a x \quad (a > 1)$



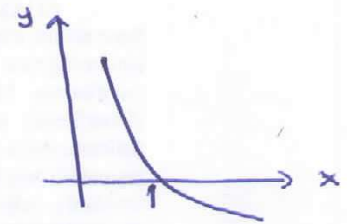
Fonk. Artandır

★ $y = a^x \quad (0 < a < 1)$



Fonk. Azalandır

★ $y = \log_a x \quad (0 < a < 1)$



Fonk. Azalandır

Limitleri

① $\lim_{x \rightarrow \infty} \ln x = \infty$

② $\lim_{x \rightarrow 0^+} \ln x = -\infty$

③ $\lim_{x \rightarrow \infty} e^x = \infty$

④ $\lim_{x \rightarrow -\infty} e^x = 0$

⑤ $\lim_{x \rightarrow \infty} a^x = 0 \quad (0 < a < 1)$

⑥ $\lim_{x \rightarrow \infty} a^x = \infty \quad (a > 1)$

⑦ $\lim_{x \rightarrow -\infty} a^x = \infty \quad (0 < a < 1)$

⑧ $\lim_{x \rightarrow -\infty} a^x = 0 \quad (a > 1)$

⑨ $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

⑩ $\lim_{x \rightarrow 0^+} (1+ax)^{1/x} = e^a$

⑪ $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

Türevleri

① $(\ln x)' = \frac{1}{x}$

② $(\ln f(x))' = \frac{f'(x)}{f(x)}$

③ $(e^x)' = e^x$

④ $(e^{f(x)})' = f'(x) \cdot e^{f(x)}$

⑤ $(a^{f(x)})' = f'(x) \cdot a^{f(x)} \ln a$

⑥ $(\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}$

Ters Trigonometrik Fonksiyonlar

Altı temel trigonometrik fonksiyon bire-bir değildir; fakat tanım kümelerini bire-bir oldukları aralıklara kısıtlayabiliriz. Bu kısıtlanmış fonksiyonlar artık bire-bir oldukları için tersleri vardır ve aşağıdaki şekilde gösterilirler:

<u>Ters Trig. Fonksiyon</u>	<u>Tanım Kümesi</u>	<u>Görüntü K.</u>
① $f(x) = \text{ArcSin } x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
② $f(x) = \text{ArcCos } x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
③ $f(x) = \text{ArcTan } x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
④ $f(x) = \text{ArcCot } x$	$-\infty < x < \infty$	$0 < y < \pi$
⑤ $f(x) = \text{ArcSec } x$	$x \leq -1$ veya $x \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
⑥ $f(x) = \text{ArcCosec } x$	$x \leq -1$ veya $x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

$$\star y = \text{ArcSin } x \Rightarrow x = \text{Sin } y$$

$$\star y = \text{ArcCos } x \Rightarrow x = \text{Cos } y$$

$$\star y = \text{ArcTan } x \Rightarrow x = \text{Tan } y$$

$$\star y = \text{ArcCot } x \Rightarrow x = \text{Cot } y$$

$$\star y = \text{ArcSec } x \Rightarrow x = \text{Sec } y$$

$$\star y = \text{ArcCosec } x \Rightarrow x = \text{Cosec } y$$

Örnek:

x	$y = \text{ArcSin } x$	$y = \text{ArcCos } x$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
0	0	$\frac{\pi}{2}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$

x	$y = \text{ArcSin } x$	$y = \text{ArcCos } x$
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$

Örnek:

x	y = Arc Tan x
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
0	0
$\frac{\sqrt{3}}{3}$	$\pi/6$

x	y = Arc Tan x
$-\sqrt{3}$	$-\pi/3$
-1	$-\pi/4$
$-\frac{\sqrt{3}}{3}$	$-\pi/6$

F4

Özellikler

① $\text{Arc Cos } x + \text{Arc Cos } (-x) = \pi$

② $\text{Arc Sin } x + \text{Arc Cos } x = \frac{\pi}{2}$

③ $\text{Arc Tan } x + \text{Arc Cot } x = \frac{\pi}{2}$

④ $\text{Arc Cosec } x + \text{Arc Sec } x = \frac{\pi}{2}$

Türevleri

① $y = \text{Arc Sin } x \Rightarrow y' = \frac{1}{\sqrt{1-x^2}}$

$y = \text{Arc Sin } f(x) \Rightarrow y' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

② $y = \text{Arc Cos } x \Rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$

$y = \text{Arc Cos } f(x) \Rightarrow y' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$

③ $y = \text{Arc Tan } x \Rightarrow y' = \frac{1}{1+x^2}$

$y = \text{Arc Tan } f(x) \Rightarrow y' = \frac{f'(x)}{1+(f(x))^2}$

④ $y = \text{Arc Cot } x \Rightarrow y' = \frac{-1}{1+x^2}$

$y = \text{Arc Cot } f(x) \Rightarrow y' = \frac{-f'(x)}{1+(f(x))^2}$

⑤ $y = \text{Arc Sec } x \Rightarrow y' = \frac{1}{|x| \cdot \sqrt{x^2-1}}$

$y = \text{Arc Sec } f(x) \Rightarrow y' = \frac{f'(x)}{|f(x)| \sqrt{(f(x))^2-1}}$

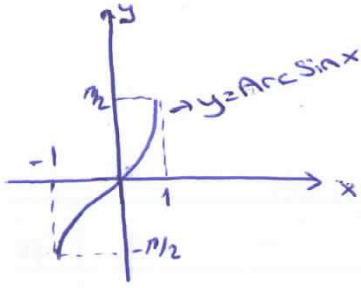
⑥ $y = \text{Arc Cosec } x \Rightarrow y' = \frac{-1}{|x| \sqrt{x^2-1}}$

$y = \text{Arc Cosec } f(x) \Rightarrow y' = \frac{-f'(x)}{|f(x)| \sqrt{f^2-1}}$

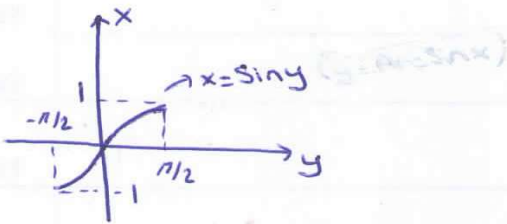
Grafikleri

F5

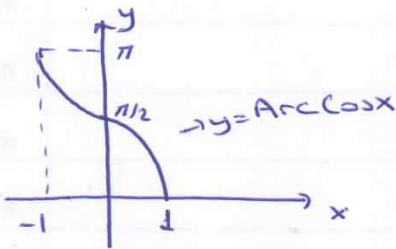
① $y = \text{ArcSin } x$ T.K: $[-1, 1]$ G.K: $[-\frac{\pi}{2}, \frac{\pi}{2}]$



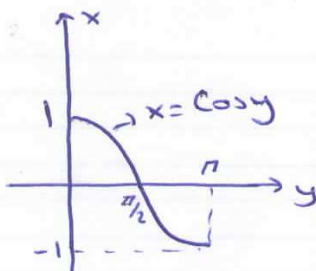
II. Yol $y = \text{ArcSin } x \Rightarrow x = \text{Sin } y$ olusunu kullanirsak:



② $y = \text{ArcCos } x$ T.K: $[-1, 1]$ G.K: $[0, \pi]$



veya: $y = \text{ArcCos } x \Rightarrow x = \text{Cos } y$ olusunu kullanirsak:



Hiperbolik fonksiyonlar

[F6]

*İki üstel fonksiyon e^x ve e^{-x} in birleşimi ile oluşan fonksiyonlardır.

$$\text{Sinüs hiperbolik fonksiyonu: } \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{Cosinüs " " : } \cosh x = \frac{e^x + e^{-x}}{2}$$

*Bu temel çiftten hareketle Tanjant, Cotanjan, Secant ve Cosecant Hiperbolik fonk. tanımlanır.

*Hiperbolik Fonksiyonlar isimlerini aldıkları Trigonometrik fonksiyonlar ile birçok benzerlik gösterirler.

Özellikler

$$\textcircled{1} \tanh x = \frac{\sinh x}{\cosh x}$$

$$\textcircled{2} \coth x = \frac{\cosh x}{\sinh x}$$

$$\textcircled{3} \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\textcircled{4} \operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\textcircled{5} \sinh 2x = 2 \sinh x \cosh x$$

$$\textcircled{6} \cosh^2 x - \sinh^2 x = 1$$

$\textcircled{7}$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1$$

$$\textcircled{8} \tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\textcircled{9} \coth^2 x = 1 + \operatorname{cosech}^2 x$$

Türevleri

$$\textcircled{1} y = \sinh x \rightarrow y' = \cosh x$$

$$y = \sinh f(x) \rightarrow y' = f'(x) \cdot \cosh f(x)$$

$$\textcircled{2} y = \cosh x \rightarrow y' = \sinh x$$

$$y = \cosh f(x) \rightarrow y' = f'(x) \sinh f(x)$$

$$\textcircled{3} y = \tanh x \rightarrow y' = \operatorname{sech}^2 x$$

$$y = \tanh f(x) \rightarrow y' = f'(x) \cdot \operatorname{sech}^2 f(x)$$

$$\textcircled{4} y = \coth x \rightarrow y' = -\operatorname{cosech}^2 x$$

$$y = \coth f(x) \rightarrow y' = -f'(x) \operatorname{cosech}^2 f(x)$$

$$\textcircled{5} y = \operatorname{sech} x \rightarrow y' = -\operatorname{sech} x \tanh x$$

$$y = \operatorname{sech} f(x) \rightarrow y' = -f'(x) \operatorname{sech} f(x) \tanh f(x)$$

$$\textcircled{6} y = \operatorname{cosech} x \rightarrow y' = -\coth x \cdot \operatorname{cosech} x$$

Ters Hiperbolik Fonksiyonlar

$$\star y = \sinh^{-1} x \Rightarrow x = \sinh y$$

$$\star y = \cosh^{-1} x \Rightarrow x = \cosh y$$

$$\star y = \tanh^{-1} x \Rightarrow x = \tanh y$$

$$\star y = \coth^{-1} x \Rightarrow x = \coth y$$

$$\star y = \operatorname{cosech}^{-1} x \Rightarrow x = \operatorname{cosech} y$$

$$\star y = \operatorname{sech}^{-1} x \Rightarrow x = \operatorname{sech} y$$

Özellikler

$$\textcircled{1} \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \quad \textcircled{2} \operatorname{cosech}^{-1} x = \sinh^{-1} \frac{1}{x} \quad \textcircled{3} \coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Türevleri

$$\textcircled{1} y = \sinh^{-1} x \Rightarrow y' = \frac{1}{\sqrt{1+x^2}}$$

$$\textcircled{2} y = \cosh^{-1} x \Rightarrow y' = \frac{1}{\sqrt{x^2-1}}$$

$$\textcircled{3} y = \tanh^{-1} x \Rightarrow y' = \frac{1}{1-x^2}$$

$$\textcircled{4} y = \coth^{-1} x \Rightarrow y' = \frac{1}{1-x^2}$$

$$\textcircled{5} y = \operatorname{sech}^{-1} x \Rightarrow y' = \frac{-1}{x \cdot \sqrt{1-x^2}}$$

$$\textcircled{6} y = \operatorname{cosech}^{-1} x \Rightarrow y' = \frac{-1}{1 \times 1 \cdot \sqrt{1+x^2}}$$

Çözümli Sorular

(F8)

① $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$ olduğunu gösteriniz.

$$\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} \Rightarrow$$

$$2\cosh^2 x - 1 = \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

\Downarrow

$$2\sinh^2 x + 1 = \frac{e^{2x} - 2 + e^{-2x}}{2} + 1 = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x \checkmark$$

② $2\cosh(\ln x) = ?$

$\cosh x = \frac{e^x + e^{-x}}{2}$ olduğundan

$$2\cosh(\ln x) = 2 \cdot \frac{e^{\ln x} + e^{-\ln x}}{2}$$

$$\left(\begin{array}{l} e^{\ln x} = x \\ e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x} \end{array} \right)$$

$$= 2 \cdot \frac{x + \frac{1}{x}}{2} = x + \frac{1}{x}$$

③ $y = \ln(\sinh x) \Rightarrow y' = ?$

$$y' = \frac{(\sinh x)'}{\sinh x} = \frac{\cosh x}{\sinh x} = \coth x$$

④ $y = \sinh(e^{\cosh x}) \Rightarrow y' = ?$

$$y = (\sinh x) \cdot (e^{\cosh x})$$

$$y' = \sinh x \cdot e^{\cosh x} \cdot \cosh(e^{\cosh x})$$

⑤ $y = \operatorname{sech}(\ln(\cos x)) \Rightarrow y' = ?$

$$y' = \frac{-\sin x}{\cos x} \cdot (1 - \operatorname{sech}(\ln \cos x) \cdot \tanh(\ln \cos x))$$

$$\textcircled{6} \quad \sin\left(\arccos \frac{\sqrt{2}}{2}\right) = ? \quad \Rightarrow \quad \sin\left(\arccos \frac{\sqrt{2}}{2}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\textcircled{7} \quad \sec\left(\arccos \frac{1}{2}\right) = ? \quad \Rightarrow \quad \sec\left(\arccos \frac{1}{2}\right) = \sec \frac{\pi}{3} = 2$$

$$\textcircled{8} \quad \tan\left(\arcsin\left(-\frac{1}{2}\right)\right) = ? \quad \Rightarrow \quad \tan\left(\arcsin\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

$$\textcircled{9} \quad \lim_{x \rightarrow 1^-} \arcsin x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow \infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow \infty} \operatorname{arccosec} x = 0$$

$$\textcircled{10} \quad y = \arccos \frac{1}{x} \quad \Rightarrow \quad y' = -\frac{-\frac{1}{x^2}}{\sqrt{1 - \frac{1}{x^2}}}$$

$$\textcircled{11} \quad y = \arctan(\ln x) \quad \Rightarrow \quad y' = \frac{\frac{1}{x}}{1 + \ln^2 x}$$

$$\textcircled{12} \quad y = \operatorname{arcsin} e^x \quad \Rightarrow \quad y' = \frac{e^x}{\sqrt{1 - e^{2x}}}$$

$$\textcircled{13} \quad y = \arccos(\ln \sin x) \quad \Rightarrow \quad y' = -\frac{\frac{\cos x}{\sin x}}{\sqrt{1 - (\ln \sin x)^2}} = -\frac{\cot x}{\sqrt{1 - \ln^2 \sin x}}$$

$$\textcircled{14} \quad y = \ln(\arctan x) \quad \Rightarrow \quad y' = \frac{\frac{1}{1+x^2}}{\arctan x} = \frac{1}{(1+x^2)\arctan x}$$

⑮ $\lim_{x \rightarrow 0} \frac{\text{ArcSin} x}{x} = ?$

(F10)

$$\left. \begin{array}{l} \text{ArcSin} x = y \\ x \rightarrow 0 \quad y \rightarrow 0 \\ x = \text{Sin} y \end{array} \right\} \Rightarrow \lim_{x \rightarrow 0} \frac{\text{ArcSin} x}{x} = \lim_{y \rightarrow 0} \frac{y}{\text{Sin} y} = 1$$



⑯ $\lim_{x \rightarrow 0^+} \frac{(\text{Arctan} \sqrt{x})^2}{x \sqrt{x+1}} = ?$

$$\left. \begin{array}{l} \text{Arctan} \sqrt{x} = y \\ \text{Tan } y = \sqrt{x} \\ x \rightarrow 0^+ \Rightarrow y \rightarrow 0^+ \\ 1 + \text{Tan}^2 y = \text{Sec}^2 y \end{array} \right\} \begin{aligned} \lim_{x \rightarrow 0^+} \frac{(\text{Arctan} \sqrt{x})^2}{x \sqrt{x+1}} &= \lim_{y \rightarrow 0^+} \frac{y^2}{(\text{Tan} y)^2 \cdot \sqrt{1 + \text{Tan}^2 y}} \\ &= \lim_{y \rightarrow 0^+} \underbrace{\left(\frac{y}{\text{Tan} y} \right)^2}_1 \cdot \underbrace{\frac{1}{\text{Sec} y}}_1 = 1 \end{aligned}$$

⑰ $\lim_{x \rightarrow \infty} \left(\frac{x+7}{x+3} \right)^{2x+3} = ?$

$$\lim_{x \rightarrow \infty} \left(\frac{x+7}{x+3} \right)^{2x+3} = \lim_{x \rightarrow \infty} \underbrace{\left(1 + \frac{4}{x+3} \right)^{x+3}}_{e^4}^{\frac{2x+3}{x+3}} = (e^4)^2 = e^8$$

⑱ $\lim_{x \rightarrow \infty} \left(\frac{x}{x+2} \right)^{3x} = ?$

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+2} \right)^{3x} = \lim_{x \rightarrow \infty} \frac{1}{\left(\left(1 + \frac{2}{x} \right)^x \right)^3} = \frac{1}{e^2}$$

⑲ $f(x) = \text{ArcSin} x$ 'in türevini $\frac{1}{\sqrt{1-x^2}}$ olduğunu gösteriniz.

$f(x) = \text{Sin} x$ olsun. $f^{-1}(x) = \text{ArcSin} x$ olur.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\underbrace{\text{ArcSin} x}_a)} = \frac{1}{\cos a} = \frac{1}{\sqrt{1-x^2}} \quad \leftarrow \begin{cases} \text{ArcSin} x = a \text{ olsun.} \\ \downarrow \\ x = \text{Sin} a \end{cases}$$

