İntegrasyon Teknikleri

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Ör: $\int (x^5+4x^2)^3 . (5x^4+8x)\,dx = ?$

$x^5+4x^2 = u$
$(5x^4+8x)\,dx = du$

$\Rightarrow \int (x^5+4x^2)^3 . (5x^4+8x)\,dx = \int u^3\,du$

$= \frac{u^4}{4} + C$

$= \frac{(x^5+4x^2)^4}{4} + C$

Ör: $\int x^2 \sin x^3\,dx = ?$

$x^3 = u$
$3x^2\,dx = du$

$\int x^2 \sin x^3\,dx = \int \sin u . \frac{du}{3} = \frac{1}{3} \int \sin u\,du$

$= \frac{1}{3} . (-\cos u) + C$

$= -\frac{1}{3} \cos u + C$

$= -\frac{1}{3} \cos x^3 + C$.

ör: $\int \cot x . dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|u| + c.$

$= \ln|\sin x| + c.$

$\sin x = u$

$\cos x . dx = du$

ör: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|u| + c.$

$= -\ln|\cos x| + C$

$\cos x = u$

$-\sin x \, dx = du$

\#\# ör: $\int \sec x \, dx = ?$

$\sec x + \tan x = u.$

$(\sec x . \tan x + \sec^2 x) dx = du.$

$$\int \sec x . dx = \int \sec x . \frac{(\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x . \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{du}{u}$$

$= \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

سر $\displaystyle\int \operatorname{cosec}x \, dx = ?$

$\operatorname{cosec}x + \cot x = u$

$(-\operatorname{cosec}x.\cot x - \operatorname{cosec}^2 x)\,dx = du.$

$$\int \operatorname{cosec}x \, dx = \int \frac{\operatorname{cosec}x.(\operatorname{cosec}x + \cot x)}{(\operatorname{cosec}x + \cot x)} dx$$

$$= \int \frac{-du}{u}$$

$$= -\ln|u| + c$$

$$= -\ln|\operatorname{cosec}x + \cot x| + C.$$

** $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\frac{x}{a}$.

Ör: $\int \frac{dx}{9+x^2} = \int \frac{dx}{9.(1+\frac{x^2}{9})} = \frac{1}{9}\int \frac{dx}{1+(\frac{x}{3})^2}$

$= \frac{1}{9}\int \frac{3\,du}{1+u^2}$

$\frac{x}{3} = u$

$\frac{dx}{3} = du$

$= \frac{1}{3} \arctan u + c$

$= \frac{1}{3} \arctan u + c$

$= \frac{1}{3} \arctan \frac{x}{3} + c$

Ör: $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{9.(1-\frac{x^2}{9})}} = \int \frac{dx}{3\sqrt{1-(\frac{x}{3})^2}}$.

$= \frac{1}{3}\int \frac{dx}{\sqrt{1-(\frac{x}{3})^2}}$

$\frac{x}{3} = u$,

$\frac{dx}{3} = du$

$= \frac{1}{\cancel{3}} \int \frac{\cancel{3}\,du}{\sqrt{1-u^2}}$

$= \int \frac{du}{\sqrt{1-u^2}}$

** $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$

$= \arcsin u + c$.

$= \arcsin \frac{x}{3} + c$.

Ör: $\int \frac{\sin 2x}{\sqrt{1-\sin^4 x}} dx = ?$

$\sin^2 x = u$

$\underbrace{2\sin x \cos x dx}_{\sin 2x . dx} = du.$

$$\int \frac{\sin 2x}{\sqrt{1-\sin^4 x}} dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C.$$

$$= \arcsin(\sin^2 x) + C$$

Ör: $\int \sqrt{x} \sin^2 (x^{\frac{3}{2}} - 1) dx = ?$

$x^{3/2} - 1 = u.$

$\frac{3}{2} x^{1/2} dx = du$

$$\int \sqrt{x} \sin(x^{3/2} - 1) dx = \int \frac{2}{3} . \sin^2 u \, du.$$

$$= \int \frac{\cancel{2}}{3} . \left(\frac{1-\cos 2u}{\cancel{2}}\right) du.$$

$$= \frac{1}{3} \int (1-\cos 2u) du$$

$$= \frac{1}{3} . \left(u - \frac{\sin 2u}{2}\right) + C.$$

$$= \frac{1}{3} . \left(x^{3/2} - 1 - \frac{\sin 2.(x^{3/2} - 1)}{2}\right) + C.$$

$$= \frac{1}{3} . (x^{3/2} - 1) - \frac{\sin 2(x^{3/2} - 1)}{6} + C$$

İntegrasyon Teknikleri

Kısmi İntegrasyon

$$\int u\,dv = uv - \int v\,du$$

Belirli İntegralde ; $\int_a^b u\,dv = u.v\Big|_a^b - \int_a^b v\,du$.

LAPTÜ
- Ü → Üstel fonksiyon
- T → Trigonometrik fonksiyon
- P → Polinom
- A → (Arcsin, Arccos, Arctan, Arccot...) Ters Trigonometrik fonk.
- L → Logaritmik fonksiyon.

Ör: $\int x.\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$

$*\ \underbrace{\cos x\,dx = dv}$

$\Downarrow$

$\int \cos x\,dx = \int dv$

$\Downarrow$

$\sin x = v$.

• $x = u$

$\Downarrow$

$dx = du$.

Ör: $\int \ln x\,dx = x\ln x - \int x.\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + c$

$* \ln x = u \quad , \quad * dx = dv.$

$\Downarrow \qquad\qquad \Downarrow$

$\frac{1}{x}dx = du \quad , \quad \int dx = \int dv$

$x = v.$

Ör: $\int \underbrace{x^2}_{u} \underbrace{e^x dx}_{dv} = ?$

$* x^2 = u \quad , \quad * e^x dx = dv$

$2x\,dx = du \quad , \quad \int e^x dx = \int dv$

$e^x = v.$

(*) $\Rightarrow \int x^2 e^x dx = x^2 e^x - \int e^x.2x\,dx = x^2e^x - 2\underbrace{\int x e^x dx}_{\text{Tekrar Kısmi İntegrasyon alalım}}$

$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv}$ integrali için kısmi integrasyon alalım.

$* x = u \quad , \quad * e^x dx = dv.$

$dx = du \quad , \quad \int e^x dx = \int dv$

$e^x = v$

$\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x$

(*) $\Rightarrow \int x^2 e^x dx = x^2e^x - 2[xe^x - e^x] + c = x^2e^x - 2xe^x + 2e^x + c$

Ör: $\int_0^4 xe^{-x}dx = ?$

* $x = u$, $e^{-x}dx = dv$

$dx = du$

$\Downarrow$

$\int e^{-x}dx = \int dv$

$-e^{-x} = v$

$$\int_0^4 xe^{-x}dx = -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}dx$$

$$= (-4e^{-4} - 0) - e^{-x}\Big|_0^4$$

$$= -4e^{-4} - (e^{-4} - e^{0})$$

$$= -4e^{-4} - (e^{-4} - 1)$$

$$= -5e^{-4} + 1$$

$$= 1 - 5e^{-4}$$

Ör: $\int \cos^n x\,dx = ?$ (İndirgeme formülünü bulunuz)

$$\int \cos^n x\,dx = \int \underbrace{\cos^{n-1}x}_{u}\,\underbrace{\cos x\,dx}_{dv}$$

* $\cos^{n-1}x = u$, * $\cos x\,dx = dv$

$-(n-1)\cos^{n-2}x.\sin x\,dx = du$ $\quad \Downarrow$

$\int \cos x\,dx = \int dv$

$\sin x = v$

$$\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx = \cos^{n-1}x\sin x + (n-1)\int \sin^2 x.\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$n.\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$ (İndirgeme formülü)

Ör: $\int \cos^3 x\,dx = \dfrac{\cos^2 x\sin x}{3} + \dfrac{2}{3}\int \cos x\,dx$

$$= \frac{\cos^2 x\sin x}{3} + \frac{2}{3}\sin x + c.$$

Trigonometrik İntegraller

$\int \sin^m x \cos^n x \, dx$

* m tek, n çift ise.

$u = \cos x$
$du = -\sin x \, dx$ } dönüşümü ve $\sin^2 x + \cos^2 x = 1$ özdeşliği kullanılarak integral çözülür.

* n tek, m çift ise.

$u = \sin x$
$du = \cos x \, dx$ } dönüşümü ve $\sin^2 x + \cos^2 x = 1$ özdeşliği kullanılarak integral çözülür.

* m ve n ikiside çift ise.

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ } $\cos 2x$ 'in düşük katlarına indirgenir.

* m ve n tek ise herhangi birine u denir.

ör: $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x . \sin x \, dx$

$\cos x = u$
$-\sin x . dx = du$

$$= \int (1 - \cos^2 x) . \cos^2 x . \sin x . dx$$

$$= \int (1 - u^2) . u^2 . (-du)$$

$$= \int (u^4 - u^2) du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C.$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

ÖR: $\int \cos^5 x\, dx = \int \cos^4 x . \cos x . dx$

$\sin x = u$

$+\cos x . dx = du$

$= \int (\cos^2 x)^2 . \cos x\, dx$

$= \int (1-\sin^2 x)^2 . \cos x\, dx$

$= \int (1-u^2)^2 . du$

$= \int (u^4 - 2u^2 + 1)\, du$

$= \frac{u^5}{5} - \frac{2}{3} u^3 + u + C$

$= \frac{\sin^5 x}{5} - \frac{2}{3} \sin^3 x + \sin x + C$

ÖR: $\int \sin^2 x \cos^4 x . dx = \int \left(\frac{1-\cos 2x}{2}\right) . \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \int \left(\frac{1-\cos 2x}{2}\right) . \left(\frac{\cos^2 2x + 2\cos 2x + 1}{4}\right) dx$

$= \frac{1}{8} . \int (\cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x)\, dx$

$= \frac{1}{8} . \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$

$= \frac{1}{8} \left[x + \frac{\sin 2x}{2} - \int \cos^2 2x . dx + \int \cos^3 2x\, dx \right]$

$= \frac{1}{8} . \left[x + \cancel{\frac{\sin 2x}{2}} - \frac{x}{2} - \frac{\sin 4x}{8} - \cancel{\frac{\sin 2x}{2}} + \frac{\sin^3 2x}{6} \right] + C = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$

$\int \cos^2 2x\, dx = \int \left(\frac{1+\cos 4x}{2}\right) dx = \frac{x}{2} + \frac{\sin 4x}{8}$

$\int \cos^3 2x\, dx = \int \cos^2 2x . \cos 2x\, dx = \int (1-\sin^2 2x) . \cos 2x\, dx = \int (1-u^2) . \frac{du}{2}$

$(\sin 2x = u$

$2 . \cos 2x\, dx = du)$

$= \frac{1}{2} . (u - \frac{u^3}{3}) + C = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}$

Kare Köklerden Kurtulmak

Hatırlatma

$* \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$* 2\cos^2\theta = 1+\cos 2\theta$

Ör: $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = ?$

$4x = 2\theta$ desek. $\Rightarrow 2x = \theta \Rightarrow 2dx = d\theta$.

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/2} \sqrt{1+\cos 2\theta}\, \frac{d\theta}{2}$$

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{1+\cos 2\theta}\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{2\cos^2\theta}\, d\theta$$

$$= \frac{1}{2} \sqrt{2} \cdot \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{\sqrt{2}}{2} \left(\sin\theta \Big|_0^{\pi/2} \right)$$

$$= \frac{\sqrt{2}}{2} \cdot \left(\sin\frac{\pi}{2} - \sin 0 \right)$$

$$= \frac{\sqrt{2}}{2}.$$

Tanx ve Secx Kuvvetlerinin İntegralleri

$* \ 1+\tan^2 x = \sec^2 x$

ör: $\int \tan^2 x . dx = ?$

I. yol $\int \tan^2 x \, dx = \int (\tan^2 x + 1 - 1) dx = \int (\tan^2 x + 1) dx - \int dx$

$= \tan x - x + C$

II. yol $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \int \sec^2 x \, dx - \int dx$

$= \tan x - x + C.$

ön: $\int \tan^4 x \, dx = \int \tan^2 x . \tan^2 x \, dx = \int \tan^2 x (\sec^2 x - 1) dx$

$= \underbrace{\int \tan^2 x \sec^2 x \, dx}_{I_1} - \underbrace{\int \tan^2 x . dx}_{I_2}$

$= \dfrac{\tan^3 x}{3} - \tan x + x + C$

$I_1 = \int \tan^2 x \sec^2 x . dx = \int u^2 du = \dfrac{u^3}{3} + C = \dfrac{\tan^3 x}{3} + C$

$\tan x = u$

$\sec^2 x \, dx = du$

$I_2 = \int \tan^2 x \, dx = \tan x - x + C$

ör: $\int \sec^3 x\,dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{dv}$

* $\sec x = u$, $\sec^2 x\,dx = dv$

$\sec x \cdot \tan x\,dx = du$ $\quad \tan x = v$

$$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + c.$$

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + c.$$

* $\int \sin mx \sin nx\,dx$, $\int \sin mx \cos nx\,dx$, $\int \cos mx \cos nx\,dx$

$$\sin mx . \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\cdot\left[\cos(m-n)x + \cos(m+n)x\right]$$

$$\sin mx . \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

ör: $\int \sin 3x \cos 5x\,dx = ?$

$$\int \sin 3x \cos 5x\,dx = \int \frac{1}{2}\cdot\left[\sin(-2x) + \sin 8x\right]dx$$

$$= \frac{1}{2}\int(-\sin 2x + \sin 8x)\,dx$$

$$= \frac{1}{2}\left(\frac{\cos 2x}{2} - \frac{\cos 8x}{8}\right) + C.$$

$$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C.$$

Trigonometrik Değişken Dönüşümleri

$x = a\tan\theta$ dönüşümü,

$$\Downarrow$$

$$\sqrt{a^2+x^2}$$

$$\Downarrow$$

$$\sqrt{a^2+a^2\tan^2\theta}$$

$$\Downarrow$$

$$\sqrt{a^2\cdot(1+\tan^2\theta)}$$

$$a\cdot|\sec\theta|$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

aralığında

$\tan\theta$ nın tersi vardır

$x = a\tan\theta$,

$$\theta = \tan^{-1}\left(\frac{x}{a}\right)$$

ya da

$$\theta = \arctan\left(\frac{x}{a}\right)$$

dır.

$x = a\sin\theta$ dönüşümü,

$$\Downarrow$$

$$\sqrt{a^2-x^2}$$

$$\Downarrow$$

$$\sqrt{a^2-a^2\sin^2\theta}$$

$$\Downarrow$$

$$\sqrt{a^2\cdot(1-\sin^2\theta)}$$

$$a\cdot|\cos\theta|$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

aralığında

$\sin\theta$ nın tersi vardır.

$x = a\sin\theta$

$$\theta = \sin^{-1}\left(\frac{x}{a}\right)$$

ya da

$$\theta = \arcsin\left(\frac{x}{a}\right)$$

dır

$x = a\sec\theta$ dönüşümü.

$$\Downarrow$$

$$\sqrt{x^2-a^2}$$

$$\Downarrow$$

$$\sqrt{a^2\sec^2\theta-a^2}$$

$$\Downarrow$$

$$\sqrt{a^2\cdot(\sec^2\theta-1)}$$

$$a\cdot|\tan\theta|$$

$$0 \le \theta < \frac{\pi}{2}, \quad \frac{x}{a} \ge 1$$

$$\frac{\pi}{2} < \theta \le \pi, \quad \frac{x}{a} \le -1$$

aralıklarında

$\sec\theta$ nın tersi vardır.

$x = a\sec\theta$

$$\theta = \sec^{-1}\left(\frac{x}{a}\right)$$

ya da

$$\theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$$

dır.

ör: $\int \frac{dx}{\sqrt{4+x^2}} = ?$

$x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

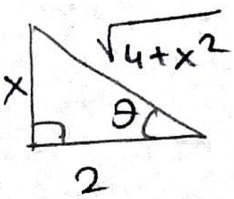
$dx = 2\sec^2\theta \, d\theta$.

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4.\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}}}$$

$$= \int \frac{\cancel{2}\sec^2\theta}{\cancel{2}\,|\sec\theta|} \, d\theta$$

$$= \int \frac{\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}} \, d\theta$$



$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C.$$

$$= \ln\left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c.$$

Ör: $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}} = ?$

$x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 3\cos\theta\, d\theta$

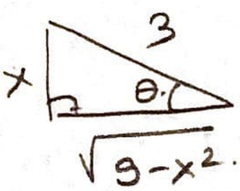
$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \frac{27\sin^2\theta\cos\theta}{\sqrt{9.(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}}\, d\theta$$

$$= \int \frac{\cancel{27}\sin^2\theta\cos\theta}{\cancel{3}\,|\cos\theta|}\, d\theta$$

$$= \int \frac{9\sin^2\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta$$

$$= \int 9\sin^2\theta\, d\theta$$



$$= \int 9.\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\underbrace{\sin 2\theta}_{2.\sin\theta.\cos\theta} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{\cancel{9}^{3}}{\cancel{4}_{2}}.\cancel{2}.\frac{x}{\cancel{3}}.\frac{\sqrt{9-x^2}}{\cancel{3}} + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{\sqrt{9-x^2}}{2} + C.$$

ôn: $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}} = ?$

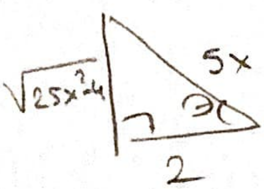
$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta.\tan\theta\, d\theta$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\frac{2}{5}\sec\theta\,\tan\theta.d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= \int \frac{\frac{2}{5}\sec\theta.\tan\theta.d\theta}{\sqrt{4.(\underbrace{\sec^2\theta-1}_{\tan^2\theta})}}$$

$$= \int \frac{\cancel{\frac{2}{5}}\sec\theta.\cancel{\tan\theta}.d\theta}{\cancel{2}\cancel{\tan\theta}}$$



$$= \frac{1}{5}\int \sec\theta.d\theta$$

$$= \frac{1}{5}\ln|\sec\theta+\tan\theta|+C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right|+C$$