

# Decidable Languages

Recall that:

A language  $L$  is **Turing-Acceptable**  
if there is a Turing machine  $M$   
that accepts  $L$

Also known as: **Turing-Recognizable**  
or  
**Recursively-enumerable**  
languages

# Turing-Acceptable

For any input string  $w$  :

$w \in L \implies M$  halts in an accept state

$w \notin L \implies M$  halts in a non-accept state  
**or** loops forever

# Definition:

A language  $L$  is **decidable**  
if there is a Turing machine (**decider**)  $M$   
which accepts  $L$   
and halts on every input string

Also known as **recursive languages**

# Turing-Decidable

For any input string  $w$ :

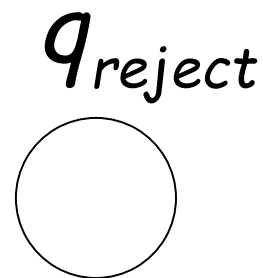
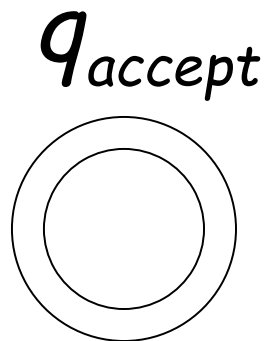
$w \in L \implies M$  halts in an accept state

$w \notin L \implies M$  halts in a non-accept state

Observation:

Every decidable language is Turing-Acceptable

Sometimes, it is convenient to have Turing machines with single accept and reject states

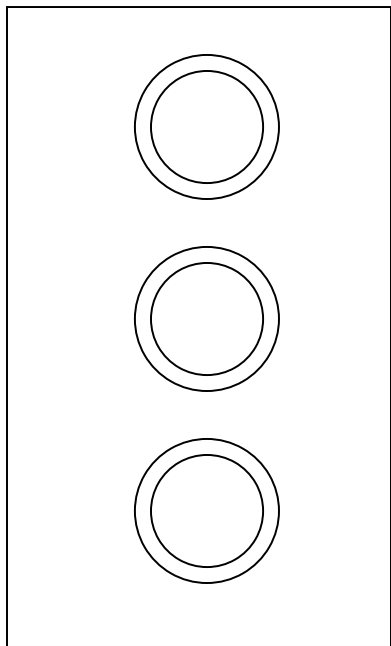


These are the only halting states

That result to possible  
halting configurations

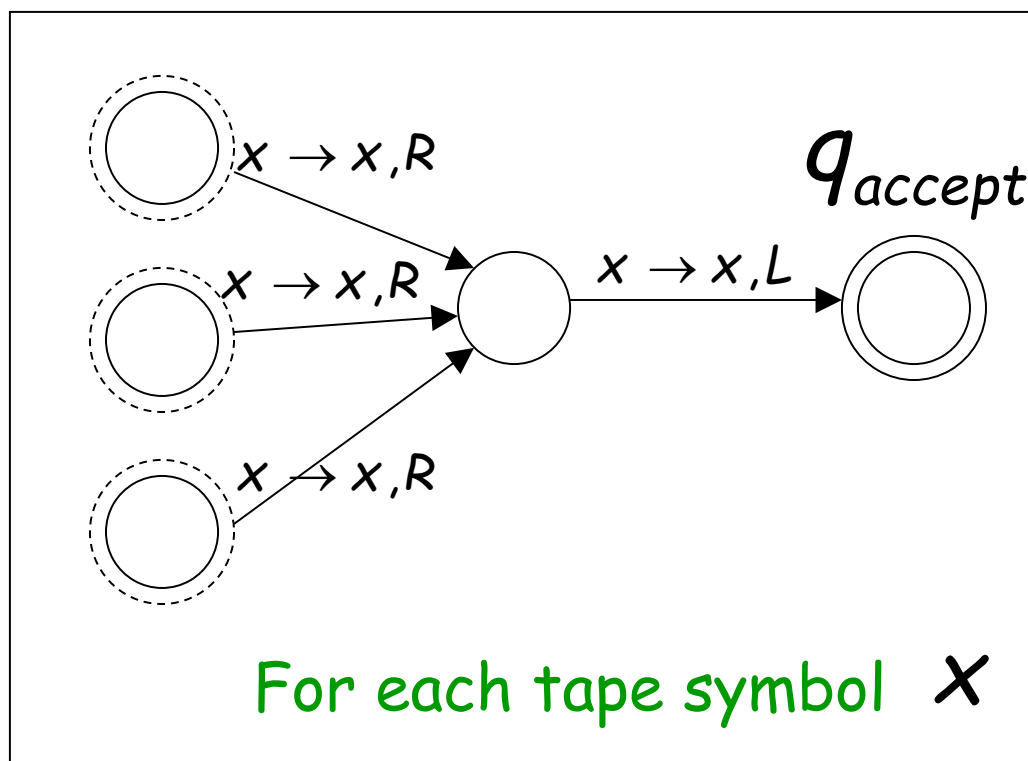
We can convert any Turing machine to have single accept and reject states

Old machine



Multiple  
accept states

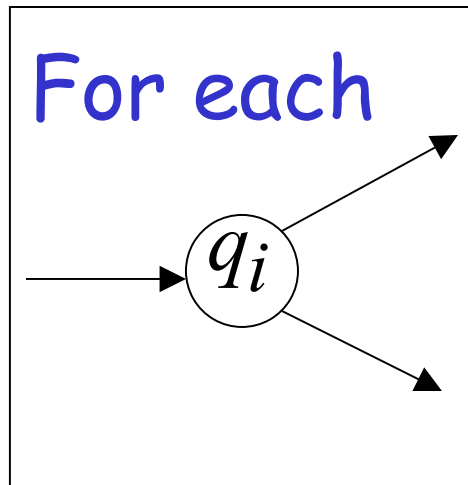
New machine



One accept state

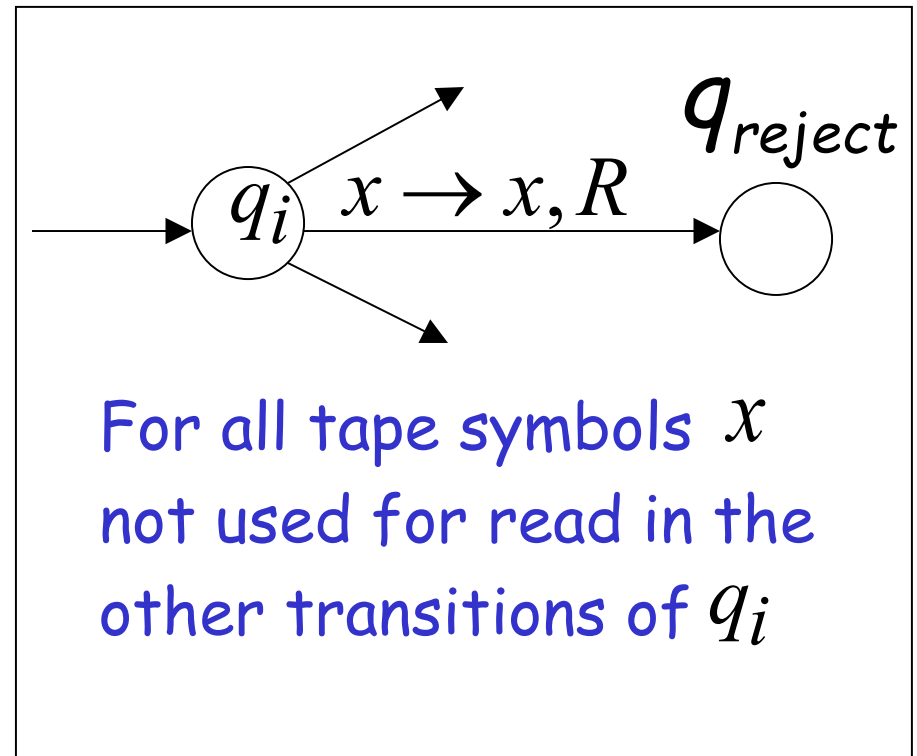
Do the following for each possible halting state:

Old machine



Multiple  
reject states

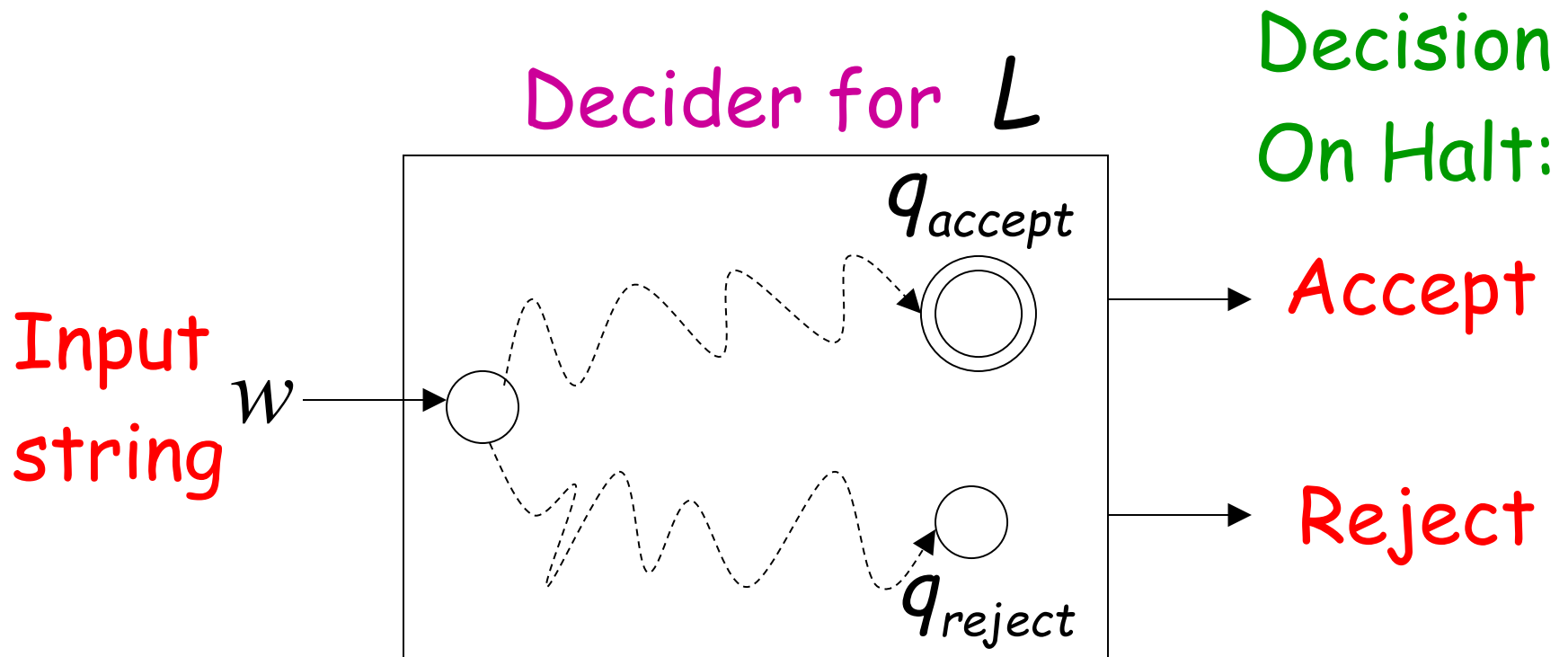
New machine



One reject state



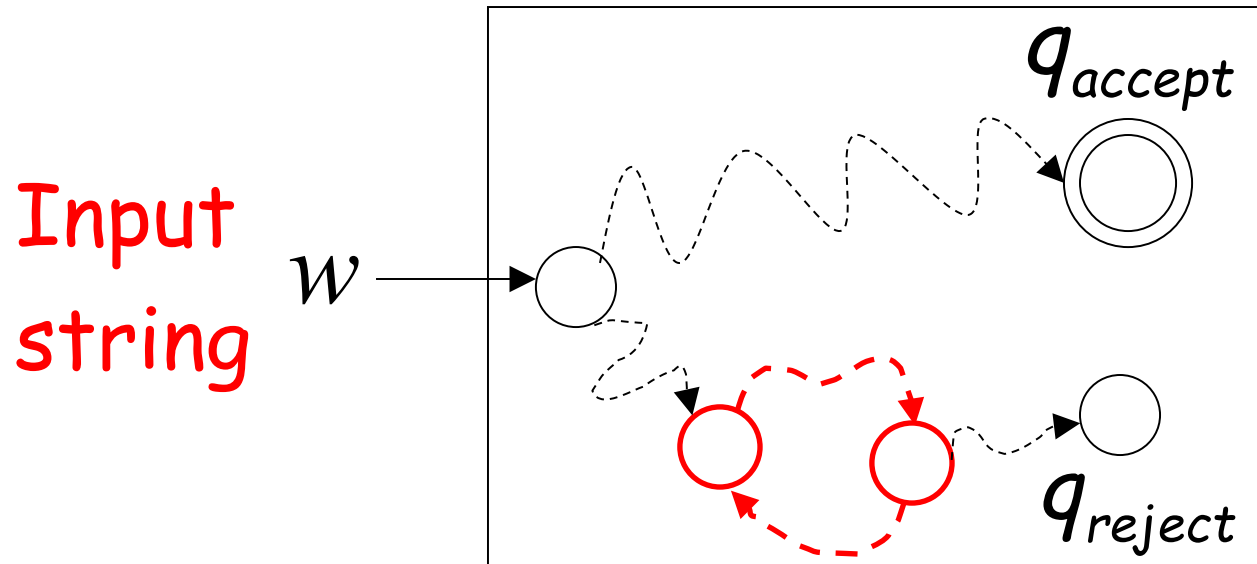
For a decidable language  $L$ :



For each input string, the computation halts in the accept or reject state

For a Turing-Acceptable language  $L$  :

## Turing Machine for $L$



It is possible that for some input string the machine enters an infinite loop

A computational problem is decidable  
if the corresponding language is decidable

We also say that the problem is solvable

**Problem:** Is number  $x$  prime?

Corresponding language:

$$PRIMES = \{1, 2, 3, 5, 7, \dots\}$$

We will show it is decidable

Decider for *PRIMES* :

On input number  $x$  :

Divide  $x$  with all possible numbers  
between 2 and  $\sqrt{x}$

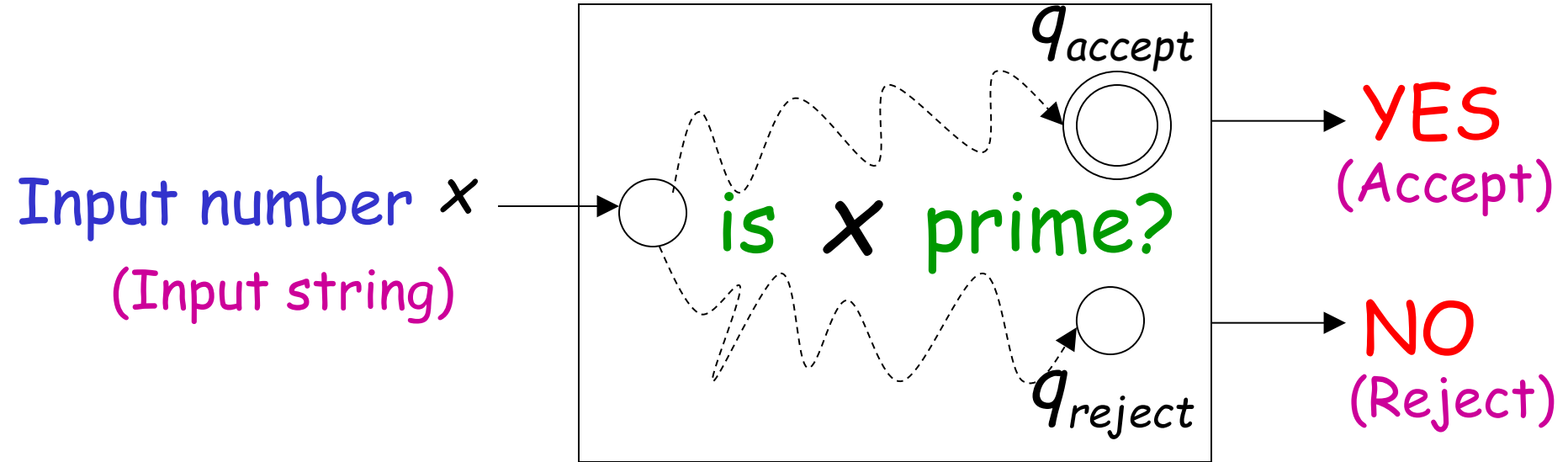
If any of them divides  $x$

Then reject

Else accept

the decider Turing machine can be designed based on the algorithm

### Decider for *PRIMES*



**Problem:** Does DFA  $M$  accept  
the empty language  $L(M) = \emptyset$ ?

Corresponding Language: (Decidable)

$EMPTY_{DFA} =$

$\{\langle M \rangle : M \text{ is a DFA that accepts empty language } \emptyset\}$

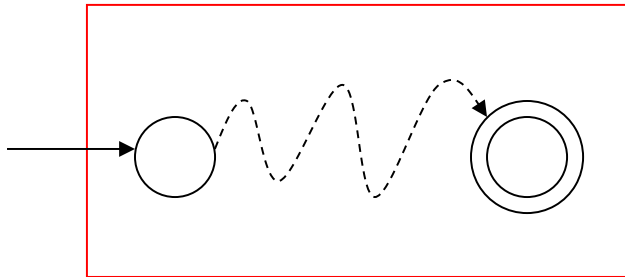
↑  
Description of DFA  $M$  as a string  
(For example, we can represent  $M$  as a  
binary string, as we did for Turing machines)

# Decider for $EMPTY_{DFA}$ :

On input  $\langle M \rangle$ :

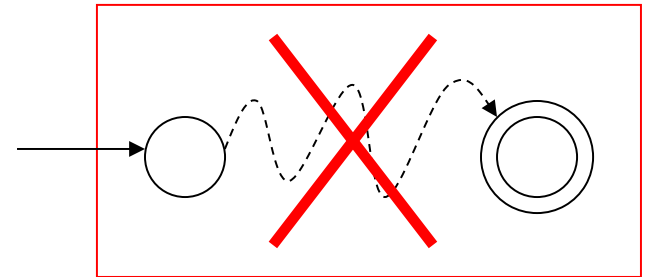
Determine whether there is a path from the initial state to any accepting state

DFA  $M$



$$L(M) \neq \emptyset$$

DFA  $M$



$$L(M) = \emptyset$$

Decision: **Reject**  $\langle M \rangle$

**Accept**  $\langle M \rangle$



**Problem:** Does DFA  $M$  accept a finite language?

Corresponding Language: (Decidable)

$FINITE_{DFA} =$

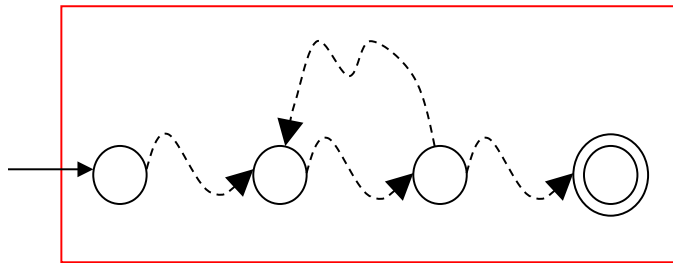
$\{\langle M \rangle : M \text{ is a DFA that accepts a finite language}\}$

# Decider for $FINITE_{DFA}$ :

On input  $\langle M \rangle$ :

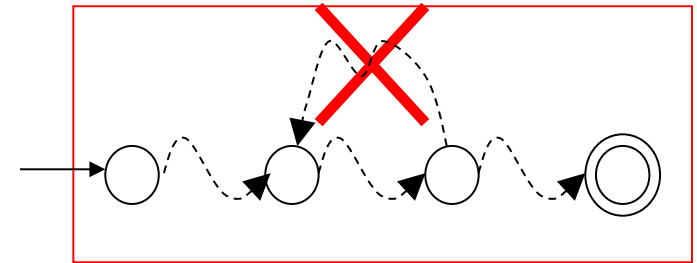
Check if there is a walk with a cycle  
from the initial state to an accepting state

DFA  $M$



infinite

DFA  $M$



finite

Decision: **Reject**  $\langle M \rangle$   
(NO)

**Accept**  $\langle M \rangle$   
(YES)

**Problem:** Does DFA  $M$  accept string  $w$  ?

Corresponding Language: (Decidable)

$A_{DFA} =$

$\{\langle M, w \rangle : M \text{ is a DFA that accepts string } w\}$

Decider for  $A_{DFA}$  :

On input string  $\langle M, w \rangle$ :

Run DFA  $M$  on input string  $w$

If  $M$  accepts  $w$

Then accept  $\langle M, w \rangle$  (and halt)

Else reject  $\langle M, w \rangle$  (and halt)

**Problem:** Do DFAs  $M_1$  and  $M_2$   
accept the same language?

Corresponding Language: (Decidable)

$EQUAL_{DFA} =$

$\{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are DFAs that accept the same languages}\}$

## Decider for $EQUAL_{DFA}$ :

On input  $\langle M_1, M_2 \rangle$  :

Let  $L_1$  be the language of DFA  $M_1$

Let  $L_2$  be the language of DFA  $M_2$

Construct DFA  $M$  such that:

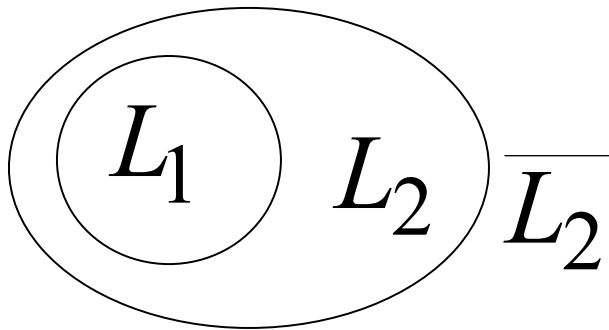
$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$$

(combination of DFAs)

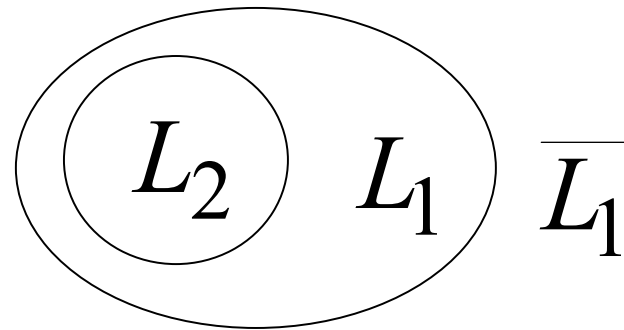
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$$



$$L_1 \cap \overline{L_2} = \emptyset \quad \text{and} \quad \overline{L_1} \cap L_2 = \emptyset$$



$$L_1 \subseteq L_2$$



$$L_2 \subseteq L_1$$



$$L_1 = L_2$$

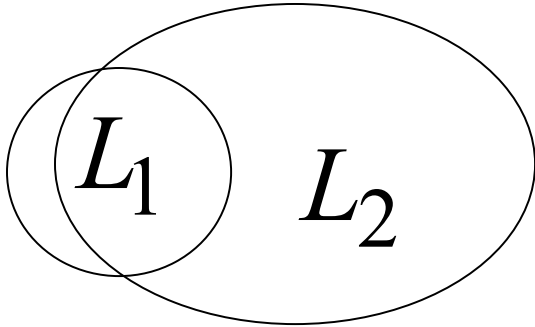
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \emptyset$$



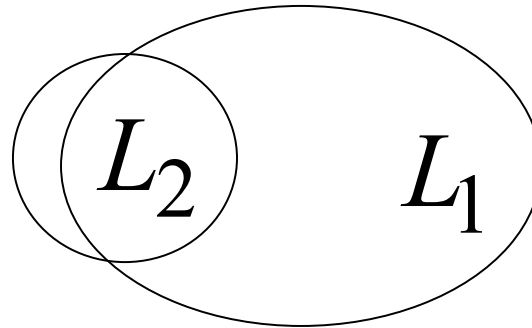
$$L_1 \cap \overline{L_2} \neq \emptyset$$

or

$$\overline{L_1} \cap L_2 \neq \emptyset$$



$$L_1 \not\subseteq L_2$$



$$L_2 \not\subseteq L_1$$



$$L_1 \neq L_2$$



Therefore, we only need  
to determine whether

$$L(M) = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$$

which is a solvable problem for DFAs:

*EMPTY*<sub>DFA</sub>

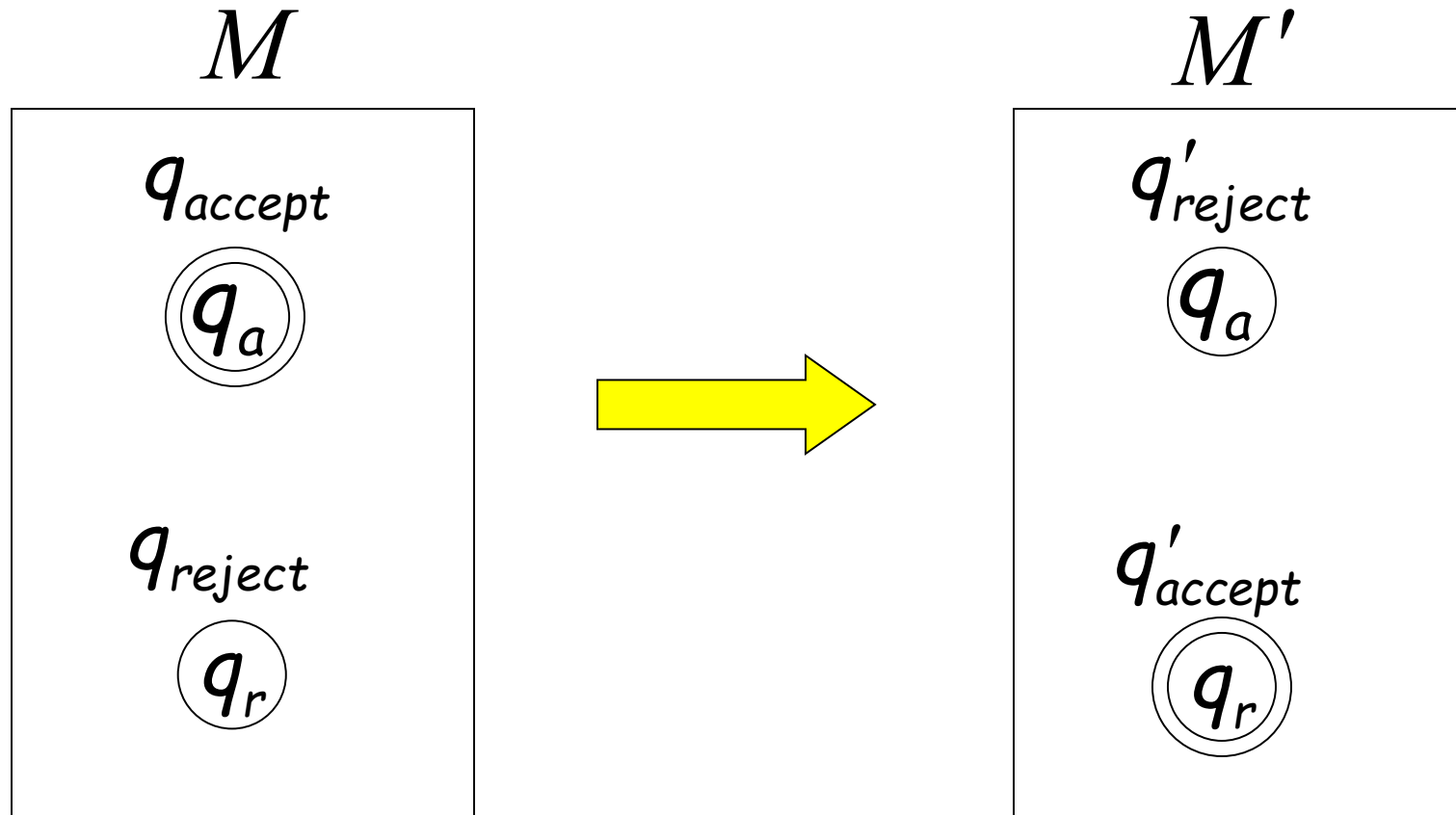
## Theorem:

If a language  $L$  is decidable,  
then its complement  $\bar{L}$  is decidable too

## Proof:

Build a Turing machine  $M'$  that  
accepts  $\bar{L}$  and halts on every input string  
( $M'$  is decider for  $\bar{L}$ )

# Transform accept state to reject and vice-versa



# Turing Machine $M'$

On each input string  $w$  do:

1. Let  $M$  be the decider for  $L$
2. Run  $M$  with input string  $w$   
If  $M$  accepts then reject  
If  $M$  rejects then accept

Accepts  $\bar{L}$  and halts on every input string

# Undecidable Languages

# Undecidable Languages

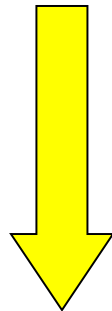
An undecidable language has no decider:  
Any Turing machine that accepts  $L$   
does not halt on some input string

We will show that:

There is a language which is  
Turing-Acceptable and undecidable

We will prove that there is a language  $L$ :

- $L$  is Turing-acceptable
- $\overline{L}$  is **not** Turing-acceptable  
(not accepted by any Turing Machine)



the complement of a  
decidable language is decidable

Therefore,  $L$  is undecidable

Non Turing-Acceptable  $\overline{L}$

Turing-Acceptable  $L$

Decidable



Consider alphabet  $\{a\}$

Strings of  $\{a\}^+$  :

$a, aa, aaa, aaaa, \dots$

$a^1 \quad a^2 \quad a^3 \quad a^4 \quad \dots$

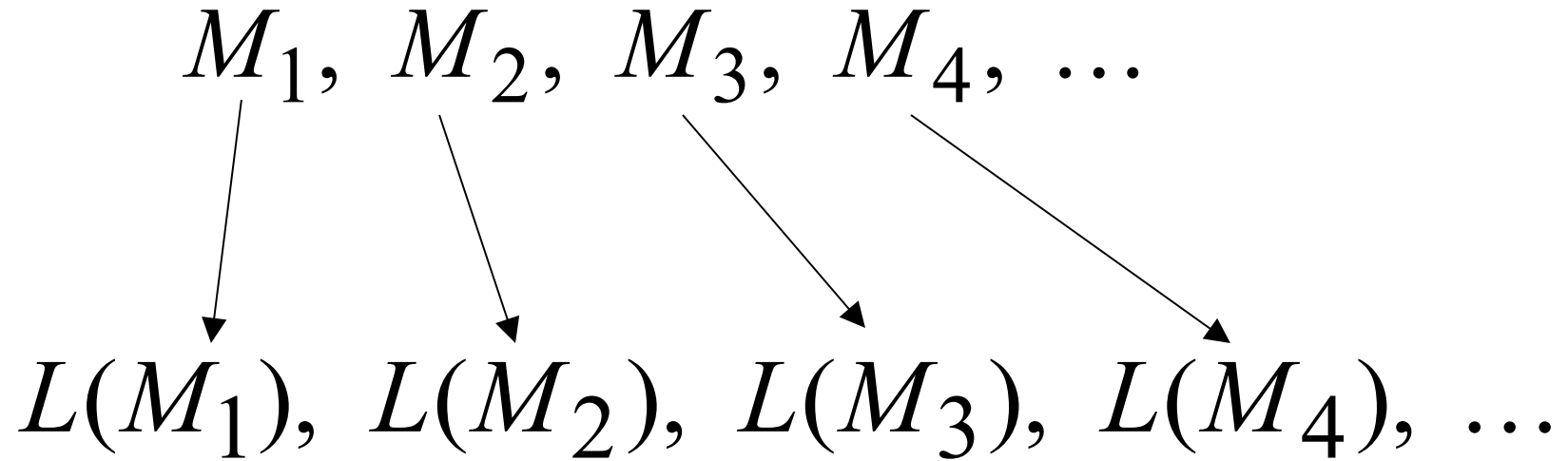
Consider Turing Machines  
that accept languages over alphabet  $\{a\}$

They are countable:

$$M_1, M_2, M_3, M_4, \dots$$

(There is an enumerator that generates them)

Each machine accepts some language over  $\{a\}$



Note that it is possible to have

$$L(M_i) = L(M_j) \quad \text{for } i \neq j$$

Since, a language could be accepted by more than one Turing machine

Example language accepted by  $M_i$

$$L(M_i) = \{aa, aaaa, aaaaaa\}$$

$$L(M_i) = \{a^2, a^4, a^6\}$$

Binary representation

	$a^1$	$a^2$	$a^3$	$a^4$	$a^5$	$a^6$	$a^7$	$\dots$
$L(M_i)$	0	1	0	1	0	1	0	$\dots$

# Example of binary representations

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

Consider the language

$$L = \{a^i : a^i \in L(M_i)\}$$

$L$  consists of the 1's in the diagonal

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L = \{a^3, a^4, \dots\}$$

Consider the language  $\overline{L}$

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

$$L = \{a^i : a^i \in L(M_i)\}$$

$\overline{L}$  consists of the 0's in the diagonal



	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$\overline{L} = \{a^1, a^2, \dots\}$$

## Theorem:

Language  $\overline{L}$  is not Turing-Acceptable

## Proof:

Assume for contradiction that

$\overline{L}$  is Turing-Acceptable

Let  $M_k$  be the Turing machine  
that accepts  $\overline{L}$  :  $L(M_k) = \overline{L}$

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

Question:  $M_k = M_1$ ?

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

$$a^1 \in L(M_k)$$

$$a^1 \notin L(M_1)$$

Answer:

$$M_k \neq M_1$$

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

Question:  $M_k = M_2$  ?

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

$$a^2 \in L(M_k)$$

$$a^2 \notin L(M_2)$$

**Answer:**  $M_k \neq M_2$

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

Question:  $M_k = M_3$  ?

	$a^1$	$a^2$	$a^3$	$a^4$	$\dots$
$L(M_1)$	0	1	0	1	$\dots$
$L(M_2)$	1	0	0	1	$\dots$
$L(M_3)$	0	1	1	1	$\dots$
$L(M_4)$	0	0	0	1	$\dots$

$$L(M_k) = \bar{L} = 1100\dots$$

$$a^3 \notin L(M_k)$$

$$a^3 \in L(M_3)$$

$$M_k \neq M_3$$

Answer:



Similarly:  $M_k \neq M_i$  for any  $i$

Because either:

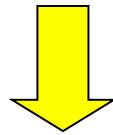
$$a^i \in L(M_k)$$

or

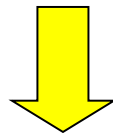
$$a^i \notin L(M_k)$$

$$a^i \notin L(M_i)$$

$$a^i \in L(M_i)$$



the machine  $M_k$  cannot exist



$\overline{L}$  is not Turing-Acceptable

End of Proof

Non Turing-Acceptable

$\overline{L}$

Turing-Acceptable

Decidable

We will prove that the language

$$L = \{a^i : a^i \in L(M_i)\}$$

Is Turing-  
Acceptable



There is a  
Turing machine  
that accepts  $L$

Undecidable



Each machine  
that accepts  $L$   
doesn't halt  
on some input string

**Theorem:** The language

$$L = \{a^i : a^i \in L(M_i)\}$$

Is Turing-Acceptable

**Proof:** We will give a Turing Machine that accepts  $L$

# Turing Machine that accepts $L$

For any input string  $w$

- Suppose  $w = a^i$
- Find Turing machine  $M_i$   
(using the enumerator for Turing Machines)
- Simulate  $M_i$  on input string  $a^i$
- If  $M_i$  accepts, then accept  $w$

End of Proof

Therefore:

Turing-Acceptable

$$L = \{a^i : a^i \in L(M_i)\}$$

Not Turing-acceptable

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

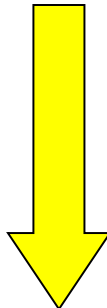
Non Turing-Acceptable  $\overline{L}$

Turing-Acceptable  $L$

Decidable

$L?$

**Theorem:**  $L = \{a^i : a^i \in L(M_i)\}$   
is undecidable

**Proof:** If  $L$  is decidable  
 the complement of a  
decidable language is decidable

Then  $\overline{L}$  is decidable

However,  $\overline{L}$  is not Turing-Acceptable!

**Contradiction!!!!**



Not Turing-Acceptable  $\overline{L}$

Turing-Acceptable  $L$

Decidable