EBRU week 6/lesson 5

Linear eq. Sys. 10/11/21 Joe

Linear Equation Systems

For $i=1,\dots,m$ and $j=1,\dots,n$; let a_{ij} and b_i be real numbers and $x_1, x_2, \dots, x_n$ be unknown variables

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \underbrace{\underset{m\times n}{A}\;\underset{n\times 1}{X}}_{m\times 1} = B$$

$$A = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}}_{m\times n} \text{ coefficient matrix} \qquad X = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} \text{ unknown matrix} \qquad B = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{n\times 1} \text{ right side matrix}$$

$AX = B \longrightarrow$ augumented m.

$ \longrightarrow$ cramer rule

$$[A : \underset{\downarrow\atop B}{I}]$$

Solutions of L.E.S using matrices

Let us consider the augumented matrix

$$[A : B] = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right]$$

we will try to find the reduced echelon form of it.

(+) If $r_A \neq r_{[A:B]}$, then system has $\boxed{\text{No solution.}}$

(++) If $r_A = r_{[A:B]} = r$ and $r = n$, then system has one solution (unique)

n: unknown vari number

(+++) If $r_A = r_{[A:B]} = r$ and $r < n$, then system has infinitely many solution depends on $(n-r)$ free variable

This method is called Gauss-Jordan method.

ex
$$\left.\begin{aligned} x+y&=-1 \\ 4x-3y&=3 \end{aligned}\right\} \quad [A:B] = \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 4 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & \frac{-7}{-7} & \frac{7}{-7} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

(-4)

$r_A = 2 = r_{[A:B]} = 2 = n = 2 \Rightarrow$ unique solution

ex
$$\left.\begin{aligned} x+y&=-1 \\ 4x-3y&=3 \end{aligned}\right\} \quad [A:B] = \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 4 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & \frac{-7}{-7} & \frac{7}{-7} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

(-4)

$r_A = 2 = r_{[A:B]} = 2 = n = 2 \Rightarrow$ unique solution

$$\begin{cases} x+y=-1 \\ y=-1 \end{cases} \quad x=0$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

ex

$$\left.\begin{aligned} x+3y+z+t&=3\\ 2x-2y+z+2t&=8\\ 3x+y+2z-t&=-1 \end{aligned}\right\}$$

z free

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 2 & -2 & 1 & 2 & 8\\ 3 & 1 & 2 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 0 & -8 & -1 & -4 & -10 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 0 & 0 & 0 & -4 & -12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3\\ 0 & 1 & 1/8 & 0 & -1/4\\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$r_A = 3 = r_{[A:D]} = 3 < n = 4$ we have infinitely many solutio depends on $(4-3)=1$ free var.

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$$\left\{\begin{aligned} x+3y+z+t&=3\\ y+\frac{z}{8}&=-\frac{1}{4}\\ t&=3 \end{aligned}\right\} \quad \begin{aligned} x&=\cancel{3}-\cancel{3}-z-3\left[-\frac{1}{4}-\frac{z}{8}\right]=\frac{3}{4}-\frac{5z}{8}\\ y&=-\frac{1}{4}-\frac{z}{8}\\ t&=3 \end{aligned}$$

$$X=\begin{bmatrix} x\\ y\\ z\\ t \end{bmatrix}=\begin{bmatrix} \frac{3}{4}-\frac{5k}{8}\\ -\frac{1}{4}-\frac{k}{8}\\ k\\ 3 \end{bmatrix};\ k\in\mathbb{R}$$

$$k=0 \Rightarrow X=\begin{bmatrix} \frac{3}{4}\\ -\frac{1}{4}\\ 0\\ 3 \end{bmatrix} \qquad k=8 \Rightarrow X=\begin{bmatrix} -\frac{17}{4}\\ -5/4\\ 8\\ 3 \end{bmatrix}$$

ex

$$\left.\begin{aligned} 3x+2y&=7\\ 17x+y&=0\\ 6x+4y&=3 \end{aligned}\right\} \left[\begin{array}{cc|c} 3 & 2 & 7\\ 17 & 1 & 0\\ 6 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & 7\\ 17 & 1 & 0\\ 0 & 0 & -11 \end{array}\right]$$

$0\cdot x+0\cdot y=-11$ ✗

No solution

$$\left[\begin{array}{cc|c} 3 & 2 & 7\\ 0 & -\frac{31}{3} & -\frac{119}{3}\\ 0 & 0 & 1 \end{array}\right]$$

$r_A = 2 \neq r_{[A:D]} = 3$

$\underline{\underline{r_A}} = 2 \neq \underline{\underline{r_{[A:B]}}} = 3$

<u>ex</u>

$$\left.\begin{aligned} x+y+z-t&=2\\ x-y+t&=1 \end{aligned}\right\}$$

z, t free

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 1 & -1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & \frac{-2}{-2} & \frac{-1}{-2} & \frac{2}{-2} & \frac{-1}{-2} \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2\\ 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \end{array}$$

$r_A = 2 = r_{[A:B]} = 2 < n = 4 \implies 4-2=2$ free varia.

$$\left.\begin{aligned} x+y+z-t&=2\\ y+\frac{z}{2}-t&=\frac{1}{2} \end{aligned}\right\} \begin{aligned} x &= 2+\cancel{t}-z-\frac{1}{2}+\frac{z}{2}-\cancel{t} = \frac{3}{2}-\frac{z}{2}\\ y &= \frac{1}{2}-\frac{z}{2}+t \end{aligned}$$

$$X = \begin{bmatrix} \frac{3}{2}-\frac{k}{2}\\ \frac{1}{2}-\frac{k}{2}+m\\ k\\ m \end{bmatrix} \qquad k, m \in \mathbb{R}$$

<u>ex</u>

$$\left.\begin{aligned} x+y+(k^2-5)z &= k\\ x+2y+z &= 3\\ x+y-z &= 2 \end{aligned}\right\}$$

for which value(s) of k the system has

a) No solution

b) One solution

c) Infinitely many sol.

$$\left[\begin{array}{ccc|c} 1 & 1 & k^2-5 & k\\ 1 & 2 & 1 & 3\\ 1 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 1 & k^2-5 & k\\ 0 & 1 & 6-k^2 & 3-k\\ 0 & 0 & \underbrace{4-k^2}_{(2-k)(2+k)} & 2-k \end{array}\right]$$

c)

+ $k=2$ $\quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad r_A = 2 = r_{[A:B]} = 2 < n = 3$

$3-2=1$ free inf. may solut.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 0 & \underbrace{4-k^2}_{(2-k)(2+k)} & 2-k \end{array}\right]$$

c

+ $k=2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_A = 2 = r_{[A:B]} = 2 < n = 3$

$3-2=1$ free inf. many solut.

a

$k=-2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$r_A = 2 \neq r_{[A:B]} = 3$

No solution

b

$k \neq -2, 2$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & -5 & 0 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 4 & 2 \end{array}\right]}_{\text{any example}}$$

$r_A = 3 = r_{[A:B]} = 3 = n = 3$

One solution

Linear Homogeneous E.S

$AX = 0$

$r_A = r_{[A:B]} = r$. so $\underbrace{X=0}_{\text{trivial solution}}$ is one solution for $AX=0$

+ If $r=n$, then system has only trivial solution.
+ If $r<n$, " " " infinitely many solution depends on $(n-r)$ free variable

\

ex

$$\left.\begin{aligned} x+y+z+t&=0 \\ 2x+y-z+3t&=0 \\ x-2y+z+t&=0 \end{aligned}\right\}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & -3 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & \frac{9}{9} & \frac{-3}{9} & 0 \end{array}\right]$$

(−2), (−1), (−3)

t free

$1 \quad -\frac{1}{3}$

$r_A = r_{[A:B]} = 3 < n = 4 \qquad (4-3) = 1$ free vari

$$\left.\begin{aligned} x+y+z+t&=0 \\ y+3z-t&=0 \\ z-\tfrac{1}{3}t&=0 \end{aligned}\right\} \quad \begin{aligned} x &= t - \tfrac{t}{3} - 0 = \tfrac{2t}{3} \\ y &= t - 3\cdot\tfrac{t}{3} = 0 \\ z &= \tfrac{t}{3} \end{aligned}$$

$$X = \begin{bmatrix} -4k/3 \\ 0 \\ k/3 \\ k \end{bmatrix} \quad k \in \mathbb{R}$$

Previous cikmis sorular \

12. $\left.\begin{aligned} x + 2y + z &= -1 \\ 2x + 5y + 3z &= -4 \\ -x - 7y + az &= a^2 + 2 \end{aligned}\right\}$ Lineer Denklem Sisteminin

a. Tek çözümünün

b. Sonsuz çözümünün

c. Çözümsüz olması için a ne olmalıdır?

13. $\left.\begin{aligned} 3x_1 - 2x_2 - x_4 &= 7 \\ 2x_2 + 2x_3 + x_4 &= 5 \\ x_1 - 2x_2 - 3x_3 - 2x_4 &= -1 \end{aligned}\right\}$ Lineer Denklem Sistemini çözünüz.

14. $\left.\begin{aligned} x + 2y + (1-m)z &= 0 \\ x - my + 2z &= 0 \\ 2x + (1-m)y + 3z &= -1 \end{aligned}\right\}$ Lineer Denklem Sisteminin

a. Tek çözümünün

b. Sonsuz çözümünün

c. Çözümsüz olması için m ne olmalıdır?

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14. $\left.\begin{aligned} x + 2y + (1-m)z &= 0 \\ x - my + 2z &= 0 \\ 2x + (1-m)y + 3z &= -1 \end{aligned}\right\}$ ~~Lineer Denklem Sisteminin~~ which value(s) of m

a. ~~Tek çözümünün~~ One solution

b. ~~Sonsuz çözümünün~~ infinitely many sol.

c. ~~Çözümsüz olması için m ne olmalıdır?~~ No sol.

$$\left[\begin{array}{ccc|c} 1 & 2 & (1-m) & 0 \\ 1 & -m & 2 & 0 \\ 2 & (1-m) & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1-m & 0 \\ 0 & -m-2 & 1+m & 0 \\ 0 & -3-m & 1+2m & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1-m & 0 \\ 0 & 1 & -m & 1 \\ 0 & -3-m & 1+2m & -1 \end{array}\right]$$

(a)

$m = -3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -1 \end{array}\right] \qquad r_A = 3 = r_{[A:B]} = 3 = n = 3$$

One sol.

$m = -\frac{1}{2}$

$$\left[\begin{array}{cccc} 1 & 2 & 3/2 & 0 \\ 0 & 1 & 1/2 & 1 \\ 0 & -\frac{5}{2} & 0 & -1 \end{array}\right] \qquad r_A = 3 = r_{[A:B]} = 3 = n = 3$$

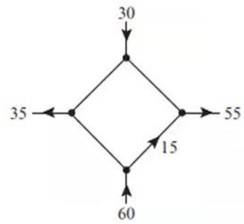
$\frac{5}{2}$

$$\left[\begin{array}{cccc} 1 & 2 & 3/2 & 0 \\ 0 & 1 & 1/2 & 1 \\ 0 & & 0 & -1 \end{array}\right]$$

$0 \quad -\frac{5}{2} \quad \frac{3}{2}$

Modelling

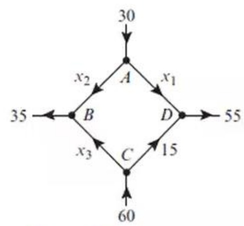
Application of E. Sys.



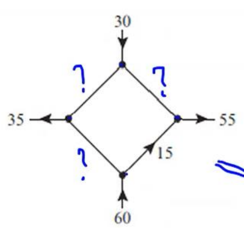
▲ Figure 1.9.1

▶ EXAMPLE 1 Network Analysis Using Linear Systems

Figure 1.9.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.



▲ Figure 1.9.2



▲ Figure 1.9.1

▶ EXAMPLE 1 Network Analysis Using Linear Systems

Figure 1.9.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

A: $30 = x_1 + x_2$

B: $x_2 + x_3 = 35$

C: $60 = x_3 + 15$

D: $x_1 + 15 = 55$

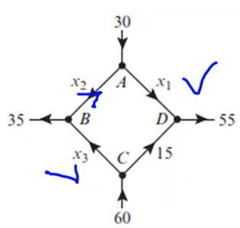
$x_1 + x_2 = 30$

$x_2 + x_3 = 35$

$x_3 = 45$

$x_1 = 40$

$x_2 = -10$



▲ Figure 1.9.2

$x_1 = 40$

$x_1 = -10$

$x_3 = 45$

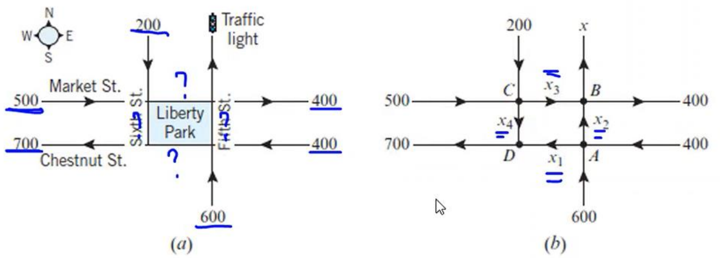
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 30 \\ 0 & 1 & 1 & 35 \\ 0 & 0 & 1 & 45 \\ 0 & 1 & 0 & -10 \end{array}\right]$$

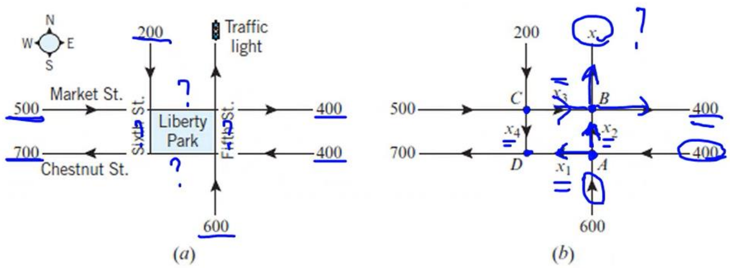
▶ **EXAMPLE 2 Design of Traffic Patterns**

The network in Figure 1.9.3 shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?





A ✓

$600 + 400 = x_1 + x_2$

B

$x_2 + x_3 = x + 400$

C

$200 + 500 = x_3 + x_4$

D

$x_1 + x_4 = 700$

$$\left.\begin{aligned} x_1 + x_2 \qquad\quad &= 1000 \\ x_3 + x_4 &= 700 \\ x_1 \qquad\quad + x_4 &= 700 \\ x_2 + x_3 \qquad &= x + 400 \end{aligned}\right\}$$

$$(-1)\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & x+400 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \\ 0 & 1 & 1 & 0 & x+400 \end{array}\right]$$

After solving all of this hoca was like ahaha get rekt

$$(-1)\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & x+400 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \\ 0 & 1 & 1 & 0 & x+400 \end{array}\right]$$

Because there was an easier solution

$$\mathrm{I} + \underline{\mathrm{II}} = \underline{\mathrm{III}} + \underline{\mathrm{IV}} \implies 1700 = x + 1100$$

$$x = 600$$

(B)

$$-1\left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1000 \end{array}\right] \sim +\left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \\ 0 & 1 & 1 & 0 & 1000 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & -1 & 0 & 1 & -300 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 700 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 700 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 + x_4 = 700 \\ x_2 - x_4 = 300 \\ x_3 + x_4 = 700 \end{array}\right\}$$