Linear Algebra

Matrix:

$$\left.\begin{array}{l} \to\ 2x+y=1 \\ \Rightarrow\ x-y=2 \end{array}\right\} \qquad \begin{array}{c} \text{column} \\ \downarrow \quad \downarrow \end{array}$$

$$\text{row} \quad \begin{array}{c} \to \\ \to \end{array} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{2\times 1}$$

Def A formal table

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n}$$

elements can be number, variable, ... functions.

$m\times n \Rightarrow$ order of matrix

ex

$$A = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}}_{3\times 2} \qquad B = \underbrace{\begin{bmatrix} \sqrt{3}+i & 0 \\ \sqrt{2}-i & 1 \end{bmatrix}}_{2\times 2}$$

$\Downarrow$ $a_{ij} \in \mathbb{R}$ $\qquad$ $\Downarrow$ complex matrix

Some Basic Matrix

1) If $m=1$, it's called "row matrix." $\to \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}}_{1\times 4}$

2) If $n=1$, it's called "column matrix" $\to B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

A $\to$ multiple choices
B .... ? ......
1 class questi.

3) If $\forall a_{ij} = 0$, it's called "zero matrix".

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \underset{1\times 1}{[0]}$$

4) If $m = n$, it's called "square matrix".

$a_{11}, a_{22}, \dots, a_{nn} \rightarrow$ prime diagonal elements.

5) If except the prime diagonal elements, all the other elements are zero, then it's called "diagonal matrix".

$$A = \underset{3\times 3}{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}} \qquad a_{11} = 1, \quad a_{22} = 0, \quad a_{33} = 2$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{diagonal matrix} \leftarrow B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

6) In the diagonal matrix, if $a_{11} = a_{22} = \dots = a_{nn} = 1$, then it's called "identity matrix".

$$I_2 = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_1 = [1]$$

Sum and Subtraction of matrix

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrix with the same order $m \times n$.

$$A \mp B = [a_{ij} \mp b_{ij}]$$

ex

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \\ 1 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 3 & 4 & -2 \\ 0 & 1 & 7 \\ 3 & 0 & -2 \end{bmatrix}$$

$B + A$

$$A - B = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & 2 \end{bmatrix}}$$

$0 - (-2)$

Properties of the matrix

Let A, B and C be $\overset{m\times n}{\text{matrix}}$ and λ_1, λ_2 be two scalars.

1) $A+B=B+A$

2) $A+(B+C)=(A+B)+C$

3) $A+O=A$

4) $\lambda_1(A+B)=\lambda_1 A+\lambda_1 B$

$$2\left\{\underbrace{[1 \quad 0 \quad -1]}_{A}+\underbrace{[1 \quad 1 \quad 1]}_{B}\right\}=\underbrace{[2 \quad 0 \quad -2]}_{2A}+\underbrace{[2 \quad 2 \quad 2]}_{2B}$$

5) $(\lambda_1\lambda_2)A=(\lambda_2\lambda_1)A=\lambda_2(\lambda_1 A)$

matrix multiplication

Let $A=[a_{ij}]_{m\times r}$ and $B=[b_{ij}]_{r\times n}$ be two matrix.

For $i:1,\dots,m$, $j:1,\dots,n$ $\quad AB=C=[c_{ij}]_{m\times n}$ $\quad c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{ir}b_{rj}=\sum_{k=1}^{n}a_{ik}b_{kj}$

ex

$$A=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}_{2\times 2} \quad B=\begin{bmatrix}0 & 2\\ 3 & 1\end{bmatrix}_{2\times 2} \quad AB=\begin{bmatrix}\boxed{0} & \boxed{4}\\ \boxed{3} & \boxed{3}\end{bmatrix}_{2\times 2}$$

$2.0+0.3=0$

$2.2+0.1=4$

$1.0+1.3=3$

$1.2+1.1=3$

$BA=$

$$B=\begin{bmatrix}0 & 2\\ 3 & 1\end{bmatrix}_{2\times 2} \quad A=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}_{2\times 2} \quad BA=\begin{bmatrix}\boxed{2} & \boxed{2}\\ \boxed{7} & \boxed{1}\end{bmatrix}_{2\times 2}$$

$0.2+2.1=2$

$0.0+2.1=2$

$3.2+1.1=7$

$3.0+1.1=1$

ex

$$A=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\end{bmatrix}_{2\times 3} \quad B=\begin{bmatrix}0\\ -1\\ 0\end{bmatrix}_{3\times 1} \quad AB=C=\begin{bmatrix}\boxed{-1}\\ \boxed{1}\end{bmatrix}_{2\times 1}$$

$2.0+1(-1)+1.0=-1$

$1.0+(-1)(-1)+0.0=1$

Properties

Let A $m \times n$, B $n \times r$, C $n \times r$, D $r \times t$ be matrix and λ be a scalar.

1) $A(BD) = (AB)D \Rightarrow m \times t$

2) $A(B+C) = AB + AC$ $\qquad m \times n \quad [0]_{n \times ?}$

3) $\lambda(AB) = (\lambda A)B = A(\lambda B)$

4) $A \cdot 0 = 0$ $\qquad m \times n \quad n \times n$

5) $A \cdot I_n = A = I_n \cdot A$ $\qquad A \Rightarrow m = n$

ex

$A = \underbrace{[1 \quad 2 \quad 3]}_{1\times 3}$ $\quad B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}_{3\times 2}$ $\quad D = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2\times 1}$

$\underbrace{A}_{1\times 3}\underbrace{(BD)}_{3\times 1} = \begin{bmatrix} \quad \end{bmatrix}_{1\times 1}$

$BD = \begin{bmatrix} \boxed{7} \\ \boxed{0} \\ \boxed{4} \end{bmatrix}_{3\times 1}$

$1 \Rightarrow 1.1 + 2.3 = 7$

$2 \Rightarrow 0.1 + 0.3 = 0$

$3 \Rightarrow 1.1 + 1.3 = 4$

$A(BD) = [\boxed{19}]_{1\times 1}$

$1.7 + 2.0 + 3.4 = 19$

Transpose of matrix

Transpose of a matrix is shown by A^t, A^T or A'. and obtained by changing its rows into colums and its colums into rows.

$[A]_{m\times n} \qquad [A^t]_{n\times m}$

ex

$A = \begin{bmatrix} 1 & 4 \\ 4 & 2 \\ 1 & 0 \end{bmatrix}_{3\times 2}$ $\qquad A^t = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 0 \end{bmatrix}_{2\times 3}$

ex

$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}_{3\times 3}$ $\qquad A^t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix}_{3\times 3}$

Properties of transpose

Let A, B $m\times n$ matrix, λ scalar.

1) $(A^t)^t = A$

2) $(\lambda A)^t = \lambda A^t$

3) $(A+B)^t = A^t + B^t$

$$\{\underbrace{[1\ \ -1\ \ 1]}_{A} + \underbrace{[0\ \ 0\ \ 1]}_{B}\}^t = \{[1\ \ -1\ \ 2]\}^t = \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{A^t} + \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B^t} \longrightarrow \begin{bmatrix}1\\-1\\2\end{bmatrix}$$

4) If A and B be multiplicable matrix $(m=n)$

$(AB)^t = B^t A^t$

$A = \begin{bmatrix}2 & 1\\-1 & -2\end{bmatrix}$ $\quad B = \begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}$ $\quad AB = \begin{bmatrix}\boxed{2} & \boxed{2}\\\boxed{-4} & \boxed{-1}\end{bmatrix}_{2\times 2}$ $\quad (AB)^t = \begin{bmatrix}2 & -4\\2 & -1\end{bmatrix}$

$A^t = \begin{bmatrix}2 & -1\\1 & -2\end{bmatrix}$ $\quad B^t = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$ $\quad A^t . B^t = \begin{bmatrix}-1 & 4\\-2 & 2\end{bmatrix}$ $\quad B^t = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$ $\quad A^t = \begin{bmatrix}2 & -1\\1 & -2\end{bmatrix}$

$B^t A^t = \begin{bmatrix}2 & -4\\2 & -1\end{bmatrix}$

<u>**Def**</u>: Let A be a square matrix. If

1) $A^t = A$, it's called "symmetric matrix."

2) $A^t = -A$, " " "inverse symmetric matrix."

$A = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$ $\quad -A = \begin{bmatrix}-1 & 0\\1 & -1\end{bmatrix}$

3) $\underbrace{A^2}_{A.A} = A$, it's called "idempotent matrix".

4) $A^2 = I$, it's called "involutory matrix."

ex

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad A^t = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 0 \end{bmatrix} = A$$

sym. matrix

ex

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad A^t = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$$

inverse sym. matrix

ex

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$$

$1+3-5=-1$

$-3-9+15=3$

$-5-15+25=5$

idempotent matrix.

ex If $A = \begin{bmatrix} 4 & 3 & k \\ -1 & 0 & -1 \\ m & -4 & -3 \end{bmatrix}$ is an involutory matrix then what is $k+m$?

a) 0 b) 1 c) -1 d) 2 e) None of them

$$A^2 = \begin{bmatrix} 4 & 3 & k \\ -1 & 0 & -1 \\ m & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 3 & k \\ -1 & 0 & -1 \\ m & -4 & -3 \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$12+0-4k=0$

$4k=12$

$\boxed{k=3}$

$3m+12=0$

$\boxed{m=-4}$

$k+m=-4+3$
$=-1$

Conjugate of a matrix

In a matrix whose elements are complex numbers, the matrix obtained by replacing each element with it's conjugate is called Conjugate of the matrix. It's shown by $\bar{A}$.

ex

$$A = \begin{bmatrix} 0 & -i & 2+3i \\ \sqrt{2} & -\sqrt{2}i & 1 \\ 0 & 3 & 1-\sqrt{3}i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 0 & i & 2-3i \\ \sqrt{2} & \sqrt{2}i & 1 \\ 0 & 3 & 1+\sqrt{3}i \end{bmatrix}$$

$\alpha + i\beta$ — α: real part, β: imaginer part

$\left.\begin{matrix}\alpha+i\beta \\ \alpha-i\beta\end{matrix}\right\}$ conjugate $\quad (\alpha+i\beta)(\alpha-i\beta)=\alpha^2+\beta^2$

ex

$A = \begin{bmatrix} -1-i & 2 & 2i \end{bmatrix}$ $\quad \overline{A} = \begin{bmatrix} -1+i & 2 & -2i \end{bmatrix}$

Properties of $\overline{A}$

Let A, B $m \times n$ matrix k be a scalar

1) $(\overline{\overline{A}}) = A$

2) $(\overline{kA}) = \overline{k}\,\overline{A}$

3) $(\overline{A+B}) = \overline{A} + \overline{B}$

4) $(\overline{AB}) = \cancel{\overline{A}\,\overline{B}}$ $\quad (m = n)$

$\overline{B}\,\overline{A}$

$$A = \begin{bmatrix} 1-2i & 3i & 1 \\ 0 & -i & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -2i \end{bmatrix} \quad AB = \begin{bmatrix} 3i & 0 & 1-4i \\ -i & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad (\overline{AB}) = \begin{bmatrix} -3i & 0 & 1+4i \\ i & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\overline{A} = \begin{bmatrix} 1+2i & -3i & 1 \\ 0 & +i & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \quad \overline{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & +2i \end{bmatrix}$$

Def Let A be a square matrix. If

1) $(\bar{A})^t = A$, it's called Hermitian matrix.

2) $(\bar{A})^t = -A$, " " inverse " "

ex $A = \begin{bmatrix} 5 & 2-i & 3 \\ 2+i & 0 & -i \\ 3 & i & 7 \end{bmatrix}$ $B = \begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$

ex

$\begin{bmatrix} i & 1-i & m \\ -1-i & 3i & i \\ -2 & n & 0 \end{bmatrix}$

If A is an invers hermi mat what is $m \cdot n$?

a) 2 b) -2 c) 2i d) -2i e) 0