Determinant

Let A be a square matrix. The determinant of A is given by $\det(A)$ or $|A|$.

(i) $\underset{1\times 1}{A=[a]} \Longrightarrow |A| = a$

(ii) $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow |A| = ad - bc$

ex

$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Longrightarrow |A| = 1.4 - 3.2 = 4-6 = -2$

(iii) $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

I. way

Sarrus rule

$$\begin{array}{lccc|l} & a_{11} & a_{12} & a_{13} & \\ & a_{21} & a_{22} & a_{23} & \\ & a_{31} & a_{32} & a_{33} & \\ - & a_{11} & a_{12} & a_{13} & + \\ - & a_{21} & a_{22} & a_{23} & + \\ - & & & & + \end{array}$$

$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$
$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$

ex

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1.1.1 + 2(-2)(-1) + 3.0(-1) - (-1).1.3 - (-1)(-2).1 - 1.0.2$$

$$= 1 + 4 + 3 - 2 = 6$$

2.way

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

ex

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{vmatrix}$$

$1 + 3 = 4$

1. $$1. \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = (1-2) - (-4-3) = -1 + 7 = 6$$

2.

2. $$A = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & -2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = (1+3) + 2(-1+2) = 4 + 2 = 6$$

a_{23} $\qquad 2 + 3 = 5$

Laplace expansion

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Laplace expansion

Def: Let A be a $n \times n$ square matrix and $n \geq 2$.

For $i = 1, \dots, n$ $\quad j = 1, \dots, n$

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$$

Here A_{ij} = cofactor

Def Let $A = [a_{ij}]$ be a $n \times n$ square matrix and M_{ij} denotes the $(n-1) \times (n-1)$ order matrix obtained from A by deleting the row and column containing a_{ij}. M_{ij} is called "minor of a_{ij} element".

$A_{ij} = (-1)^{i+j} |M_{ij}|$ is called cofactor of a_{ij}

$\underline{\underline{ex}}$ $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} \cancel{2} & \cancel{1} & \cancel{0} \\ 4 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

$a_{11} \Rightarrow M_{11} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_{11} = (-1)^2 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$

$a_{12} \Rightarrow M_{12} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$ $A_{12} = (-1)^3 \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$

$a_{12} \Rightarrow M_{12} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$ $\underline{\underline{A_{12}}} = (-1)^3 \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$

$a_{13} = M_{13} = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$ $\underline{\underline{A_{13}}} = (-1)^4 \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = (4-0) = 4$

$a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$

$|A| = 2A_{11} + 1 \cdot A_{12} + 0 \cdot A_{13} = 2(-2) + 2 = \underline{-2}$

Properties of Determinant

1) If a square matrix has no inverse, $|A| = 0$

   " " is invertible, $|A| \neq 0$.

2) The product of pivots is the determinant

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} (-3) \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \Rightarrow |A| = 1.(-2) = -2$

ex $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow |A| = \underline{1}.(\underline{-1}).\underline{3} = -3$

3) The determinant changes sign when two rows (columns) are exchanged.

ex $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ $|A| = 1.(-1)^2 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$

a_{11}

a_{12}

$$B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad |B| = 2(-1)^3 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2.2 = \underline{\underline{-4}}$$

a_{21}

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad |C| = 2(-1)^3 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2.2 = \underline{\underline{-4}}$$

4) $|I_n| = 1$

5) If two rows (columns) are equal. $|A| = 0$

$$\begin{aligned} 2x + y + z &= 1 \\ x - y + z &= 3 \\ x - y + z &= 4 \end{aligned}$$

3 unknow.

$\underline{\underline{2}}$

6) If all elemets are 0 in a row (column), $|A| = 0$

(A) 7) substracting a multiple of one row (colum) from another row (column) leaves |A| unchanged.

ex

a_{22}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad |A| = 1(-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = \underline{\underline{(-1)}}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \overset{(-2)}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -6 \\ 1 & 0 & 2 \end{bmatrix} \qquad |B| = 2(-1)^3 \underbrace{\begin{vmatrix} -2 & -6 \\ 1 & 2 \end{vmatrix}}_{(-4+6)} - 3(-1)^4 \underbrace{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}_{(2-3)}$$

$$|B| = -2.2 - 3(-1) = -4 + 3 = \underline{\underline{-1}}$$

8) If we multiply one row (column) with a skalar k, then $|A|$ is multiplied with k.

ex

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad |A| = 1$$

$$B = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \quad |B| = 2 \longrightarrow |B| = 2|A|$$

9) Let A, B be square matrix.

$$|AB| = |A| . |B|$$

10) $|\underbrace{A . A^{-1}}_{I}| = |A||A^{-1}| = 1 \implies |A^{-1}| = \dfrac{1}{|A|}$

11) $|A| = |A^t|$

11) $|A| = |A^t|$ $\rightarrow$ A n×n order.

$$\frac{1}{\frac{1}{2}} = 2$$

ex If $|A| = \frac{1}{2}$, $\underbrace{|2A|}_{2^n \cdot \frac{1}{2}}$, $\underbrace{|-A|}_{(-1)^n \cdot \frac{1}{2}}$, $\underbrace{|A^2|}_{|A||A|}$, $|A^{-1}|$ = ?

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$A = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}$$ $$2A = \underbrace{\begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix}}$$ $$|2A| = 2^{2}|A|$$

$|A| = 2 + 1$

$= 3$

$8 + 4 = 12$

Inverse of a matrix

The transpose of the matrix obtained by taking the cofactors of that element instead of the element of a square matrix is called Adjoint matrix of the first matrix and

$$\begin{bmatrix} \cancel{8} & 1 \\ 1 & 1 \end{bmatrix}^t$$

Adjoint matrix of the first matrix and given by adj(A) or A^*.

$$A^{-1} = \frac{Adj(A)}{|A|}$$

ex

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad A^{-1} = \frac{Adj(A)}{|A|}$$

$$|A| = 3 - (-2) = 5 \qquad Adj(A) = \begin{bmatrix} 3 & (+1) \\ -2 & 1 \end{bmatrix}^t = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

ex

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \qquad |A| = 6$$

$$Adj(A) = \begin{bmatrix} 2 & 0 \\ +1 & 3 \end{bmatrix}^t = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} \end{bmatrix}$$

ex

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A^{-1} = ? \qquad |A| = 2 \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \cdot (-1)^6 = -16$$

$$Adj(A) = \begin{bmatrix} -4 & -8 & 0 \\ -2 & 4 & 0 \\ 1 & -2 & -8 \end{bmatrix}^t = \begin{bmatrix} -4 & -2 & 1 \\ -8 & 4 & -2 \\ 0 & 0 & -8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

ex

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of a matrix

The determinant of one of matrices obtained by deleting some rows and columns in a nonzero matrix A is nonzero, but if the determinants of all square matrices longer that $r \times r$ are zero then rank (A) = r

ex

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 6 & 2 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}_{3\times 4}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 6 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 1(6-6) - 2(2-2) - (6-6) = 0$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 6 & 2 \\ 1 & 3 & 2 \end{vmatrix} = 1(12-6) - 2(4-2) - (6-6)$$

$$= 6 - 4 = 2 \neq 0$$

$\text{rank}(A) = 3$

ex

$$A = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 2 & -5 & 1 \end{bmatrix}$$

$$|A| = 2(-2-15) + 4(1+6) - 6(-5+4) = 0$$

$\text{rank}(A) = 2$

$-4+4$

$12-12$

$-5+4 = -1 \neq 0$