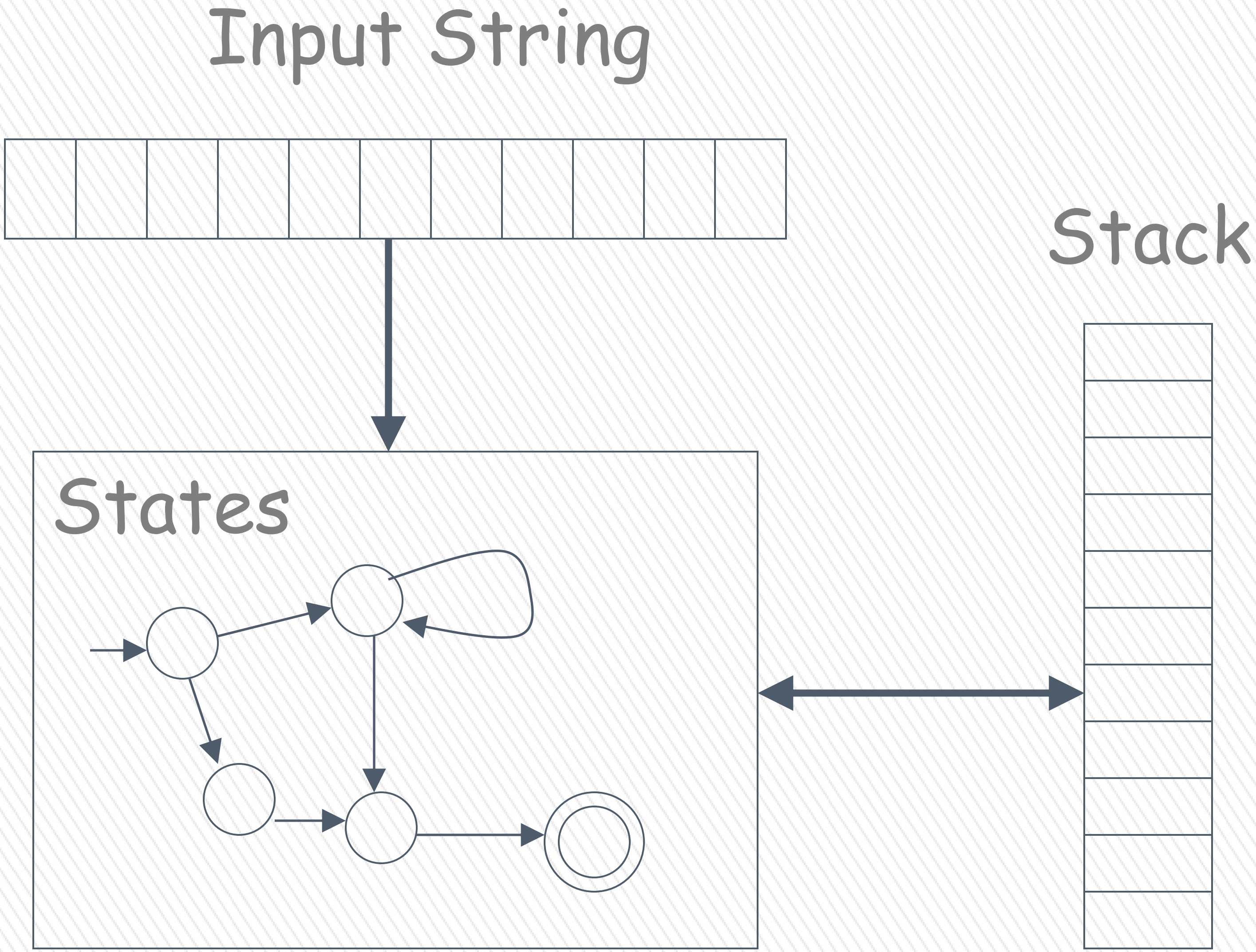
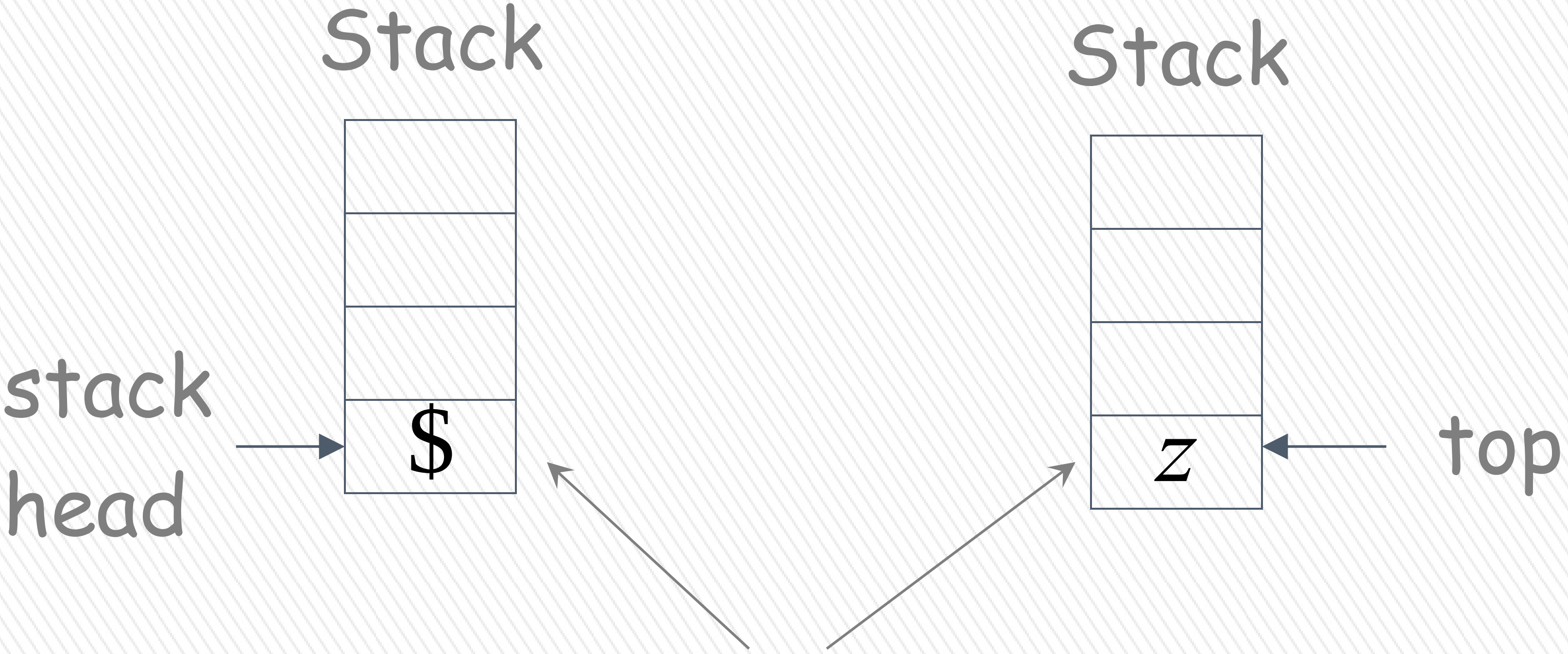
BLM2502 Theory of Computation

Pushdown Automata
PDA

Pushdown Automaton -- PDA



Initial Stack Symbol



Stack

Stack

stack head

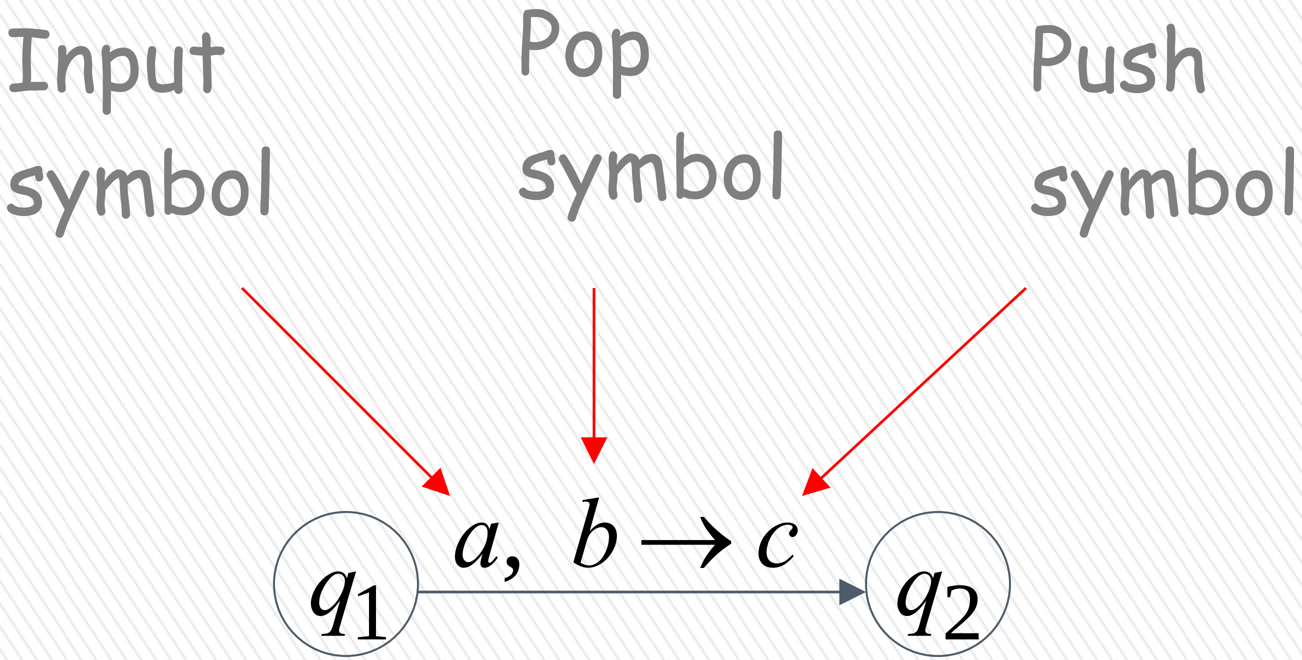
$\$$

z

top

bottom special symbol

Appears at time 0

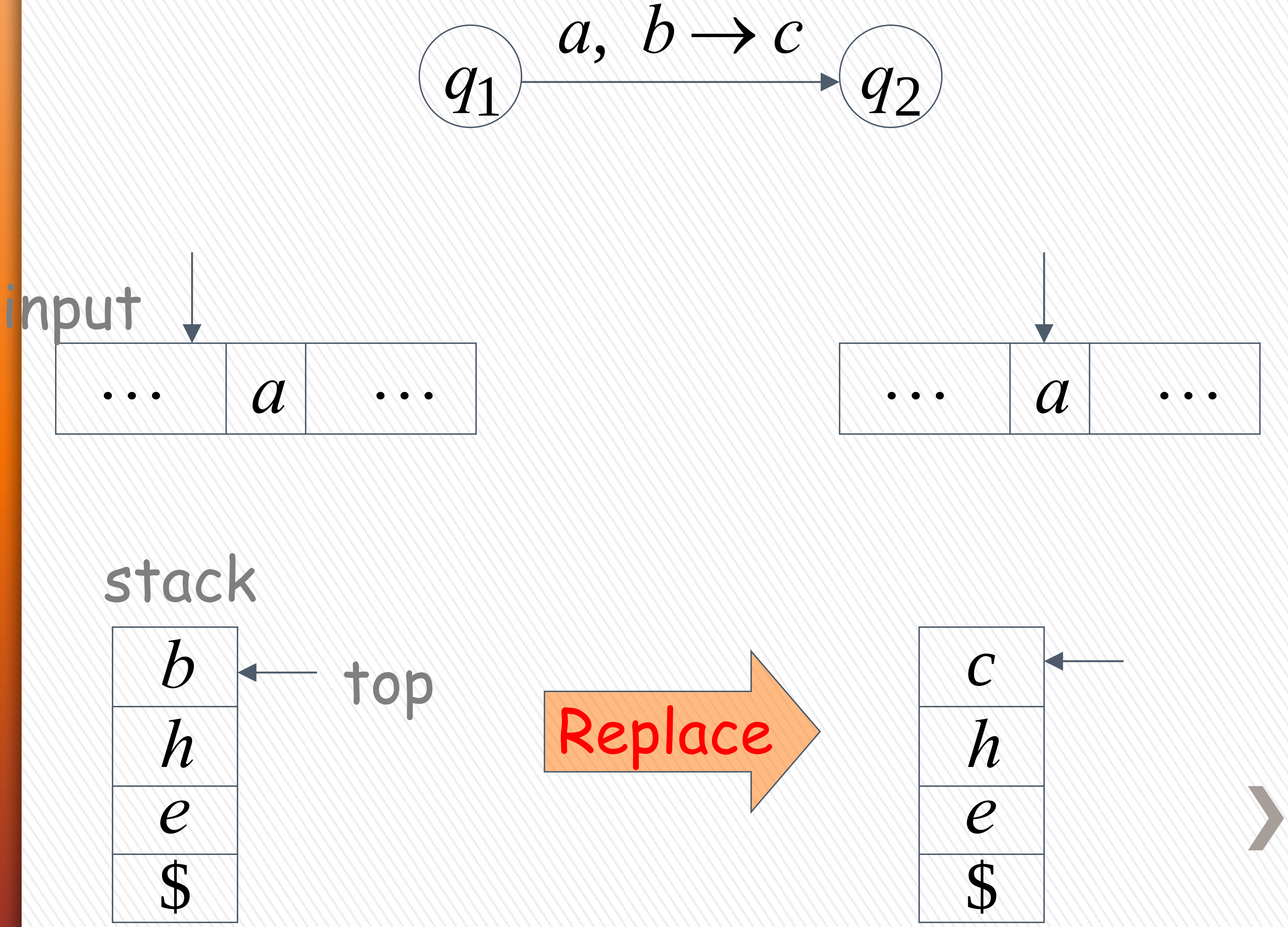


Input symbol

Pop symbol

Push symbol

The States



q_1 $\xrightarrow{a,\ b \to c}$ q_2

input

$\cdots$ a $\cdots$

$\cdots$ a $\cdots$

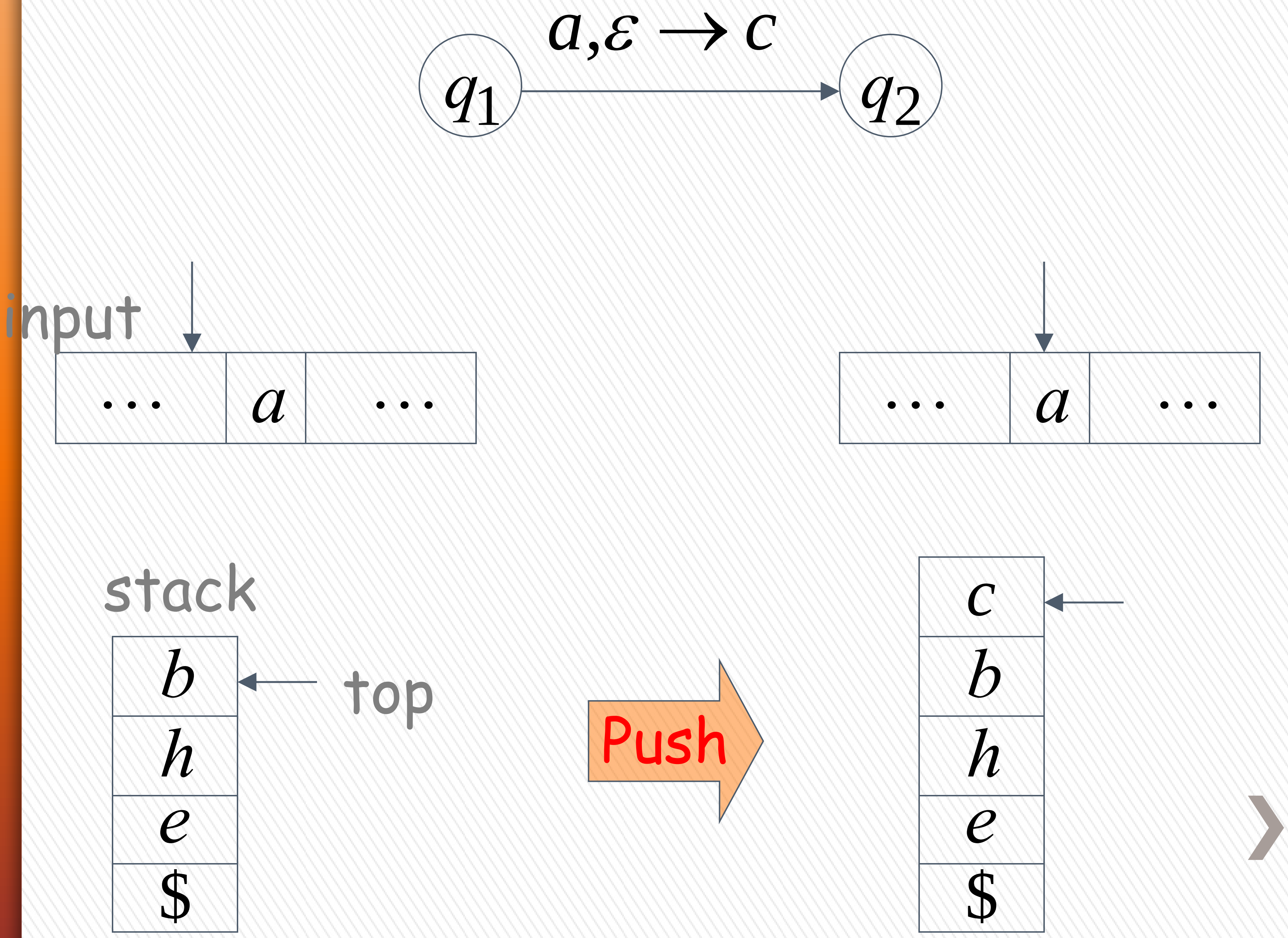
stack

b
h
e
$\$$

top

Replace

c
h
e
$\$$


$$q_1 \xrightarrow{a,\varepsilon \to c} q_2$$

input

$\cdots$ a $\cdots$

$\cdots$ a $\cdots$

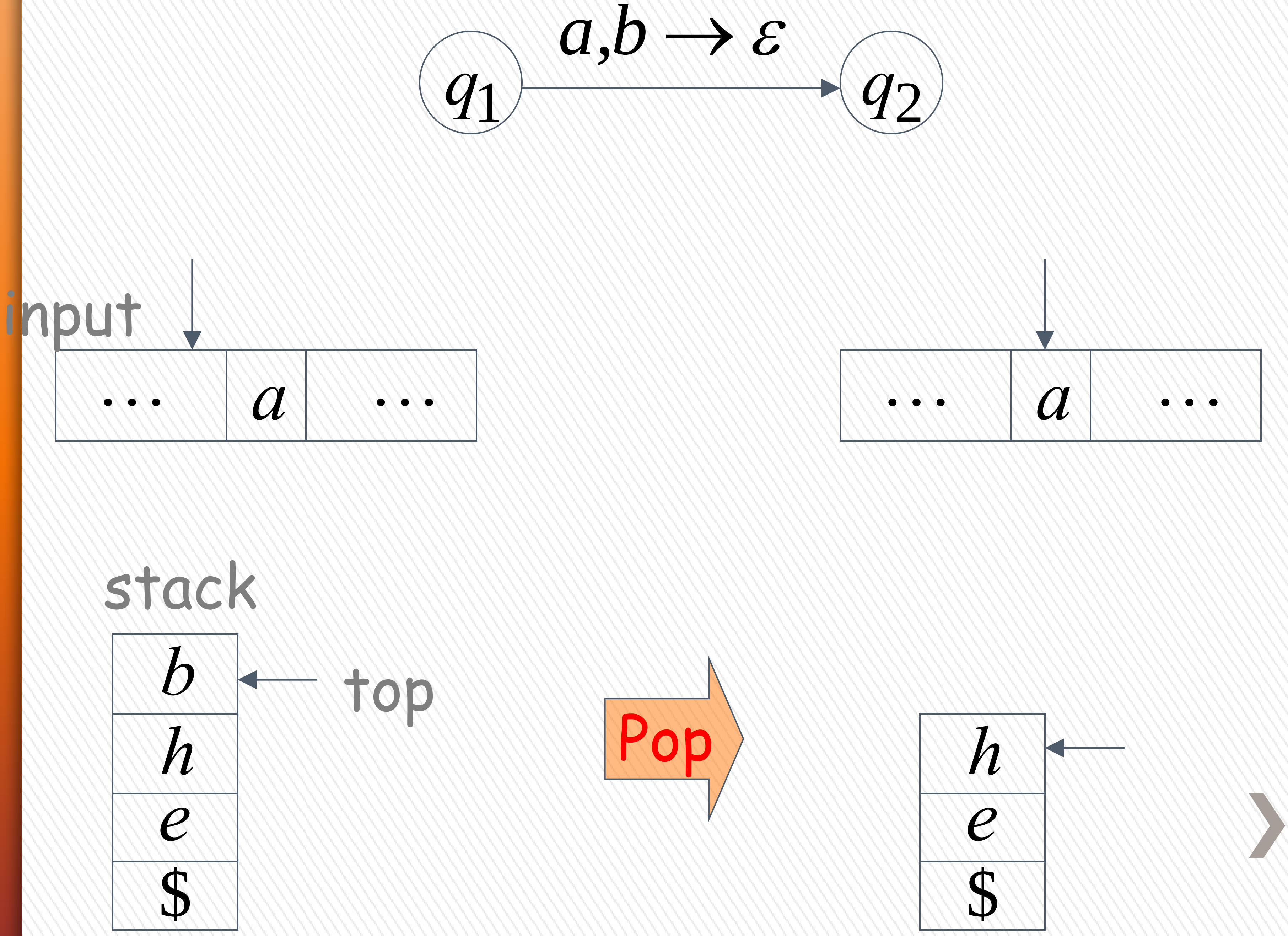
stack

b
h
e
$\$$

top

Push

c
b
h
e
$\$$



q_1

$a, b \to \varepsilon$

q_2

input

$\cdots$ a $\cdots$

$\cdots$ a $\cdots$

stack

top

b

h

e

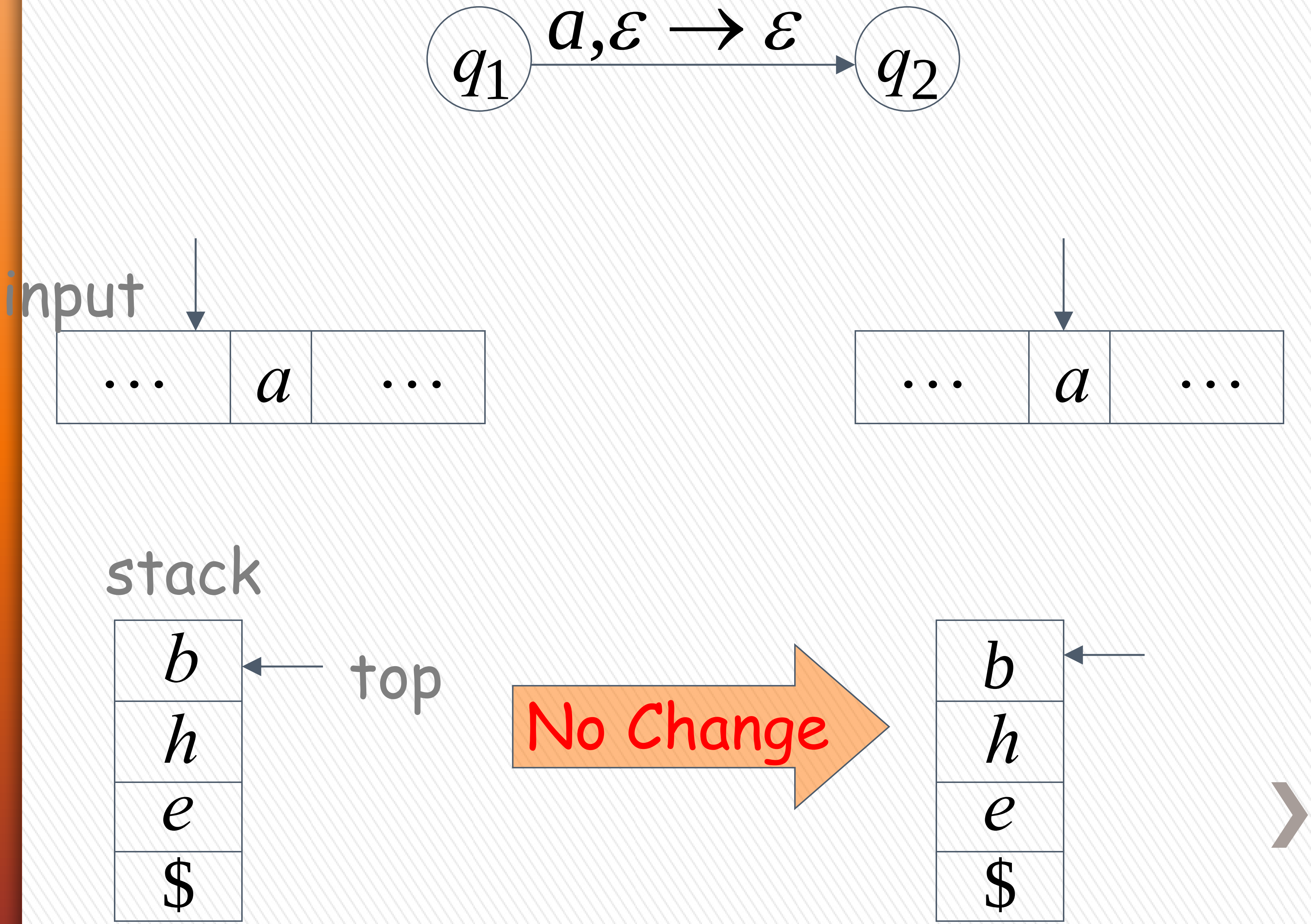
$\$$

Pop

h

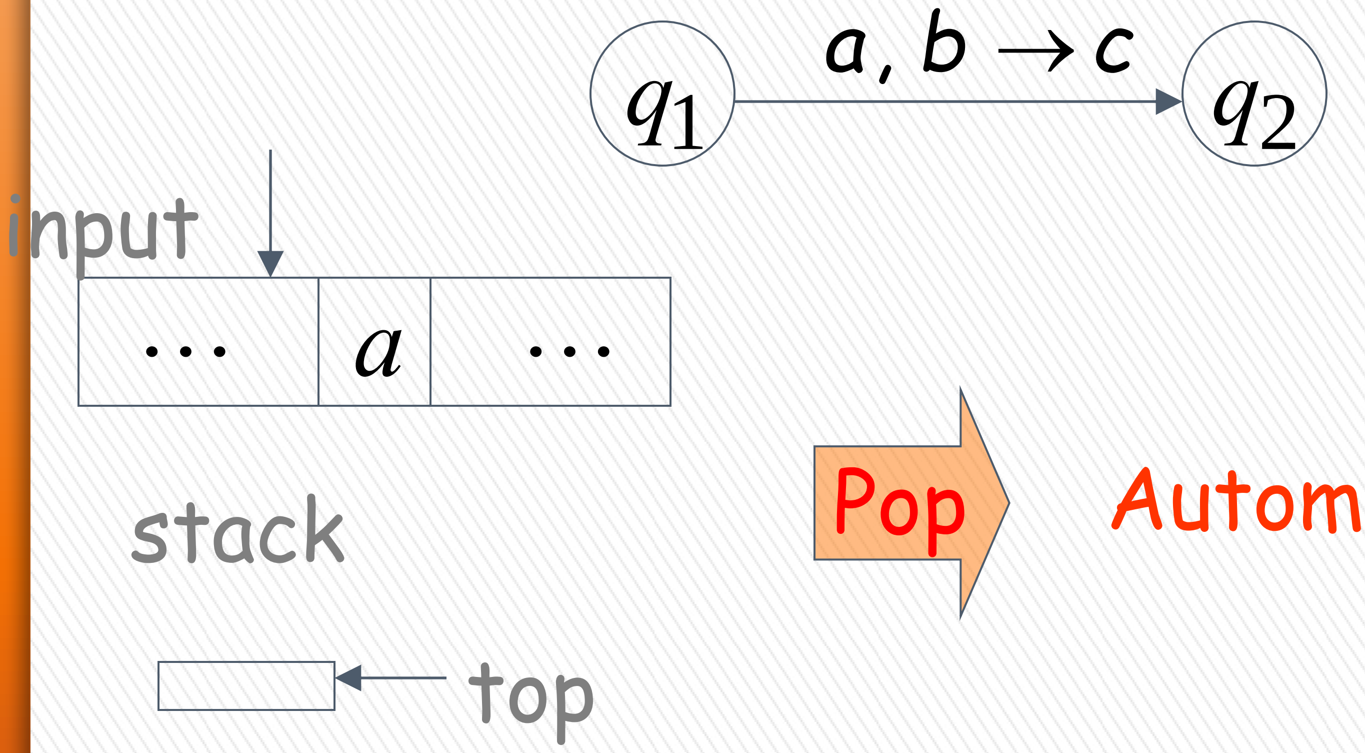
e

$\$$


q_1
$a, \varepsilon \rightarrow \varepsilon$
q_2
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$
stack
b
h
e
$\$$
top
b
h
e
$\$$

No Change

Pop from Empty Stack



q_1 $a, b \rightarrow c$ q_2

input

$\cdots$ a $\cdots$

stack

top

Pop

Stack underflow

Automaton halts!

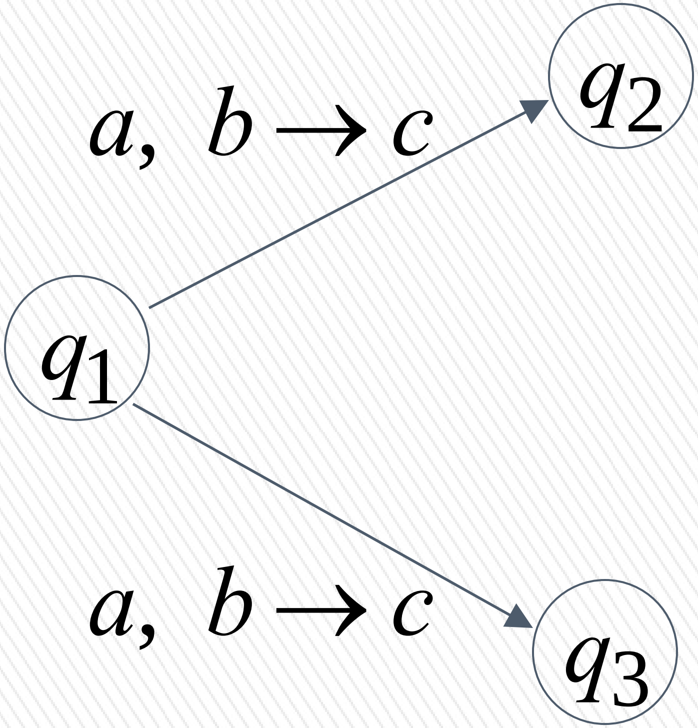
If the automaton attempts to pop from empty stack then it halts and rejects input

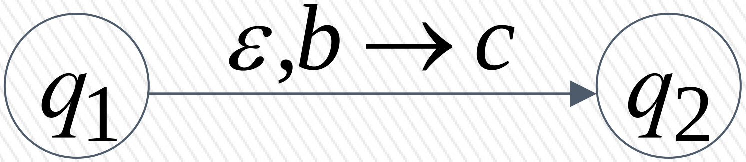
Non-Determinism

PDAs are non-deterministic

Allowed non-deterministic transitions

↳ NFA gibi





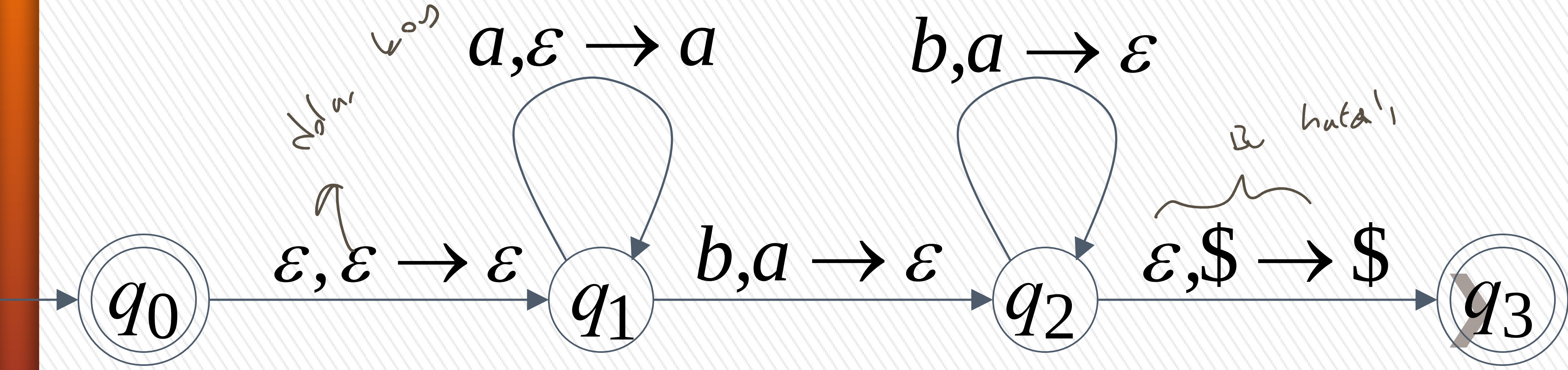
ε – transition

Example PDA

PDA M:

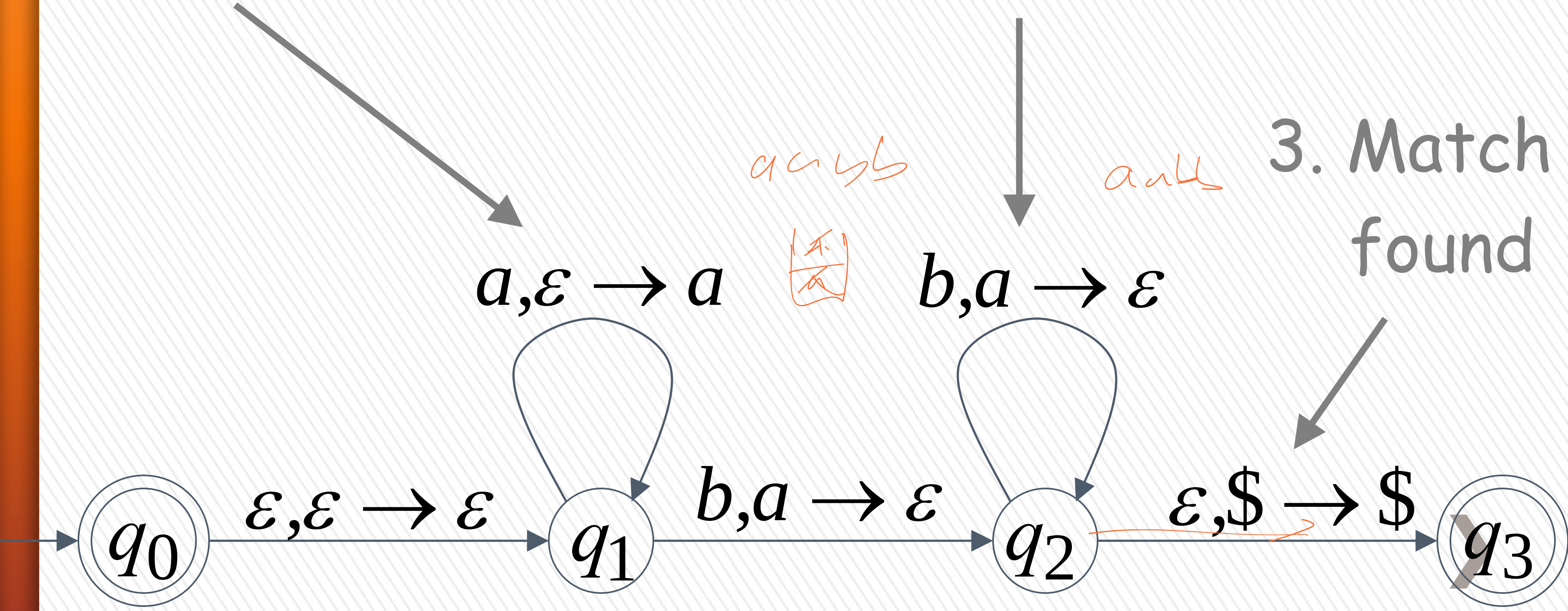
$$L(M) = \{a^n b^n : n \geq 0\}$$

a b


dolar koy
$a, \varepsilon \rightarrow a$
$b, a \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$b, a \rightarrow \varepsilon$
bu hatalı
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:



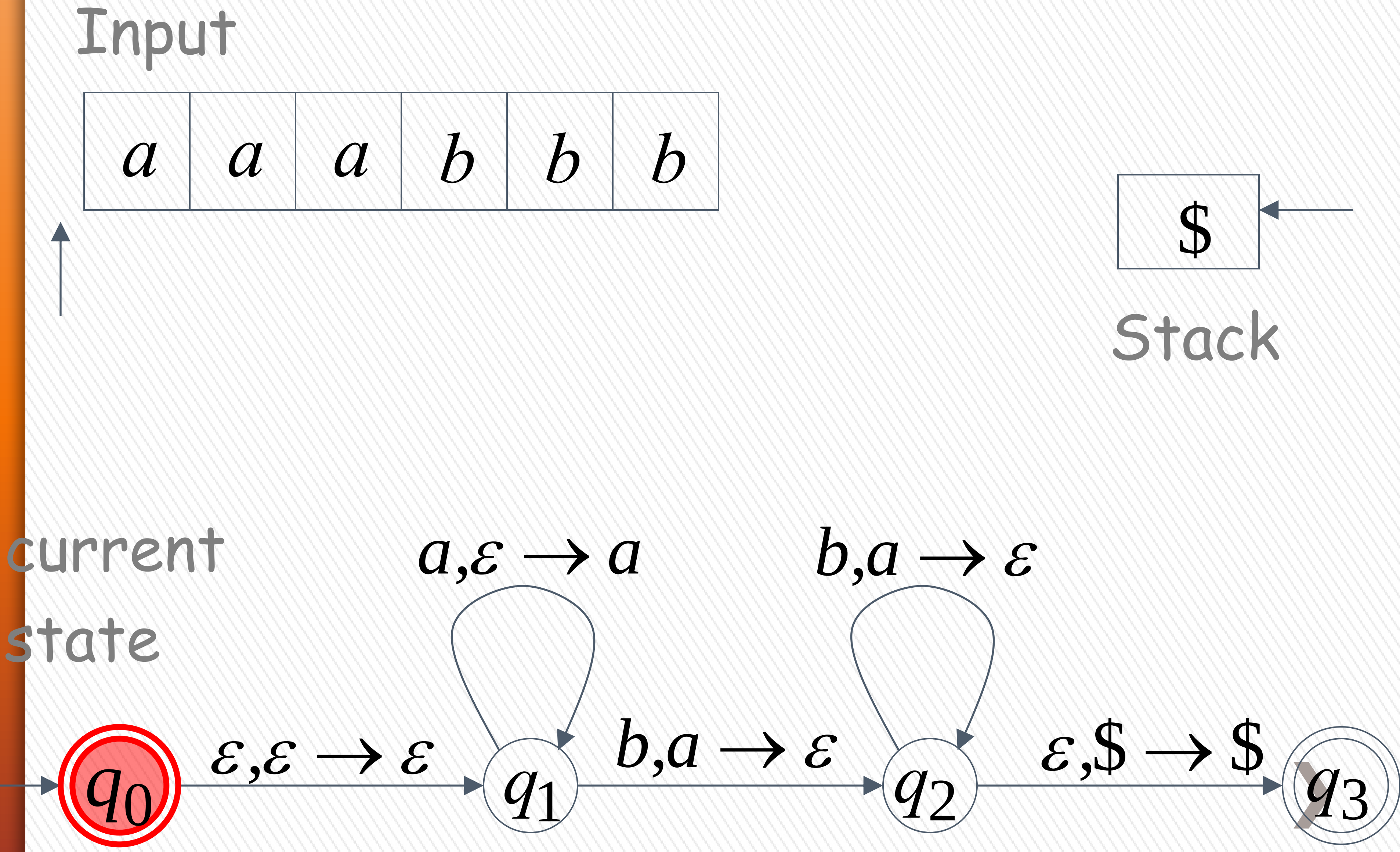
1. Push the a’s on the stack

2. Match the b’s on input with a’s on stack

3. Match found

$a, \varepsilon \rightarrow a$
$b, a \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$b, a \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2
q_3
aabb
aabb

Execution Example: Time 0



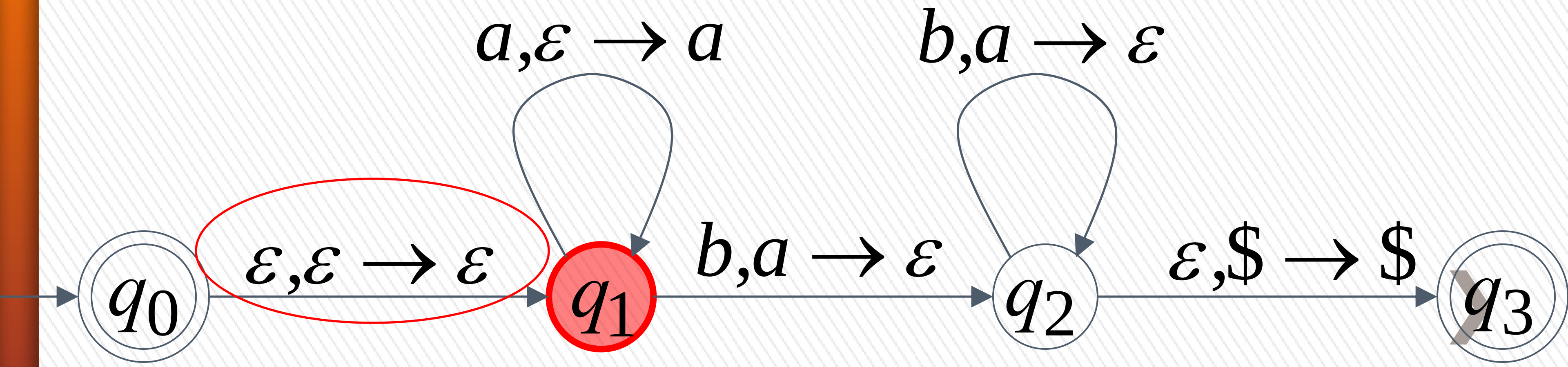
Time 1

Input

a	a	a	b	b	b

$\$$

Stack



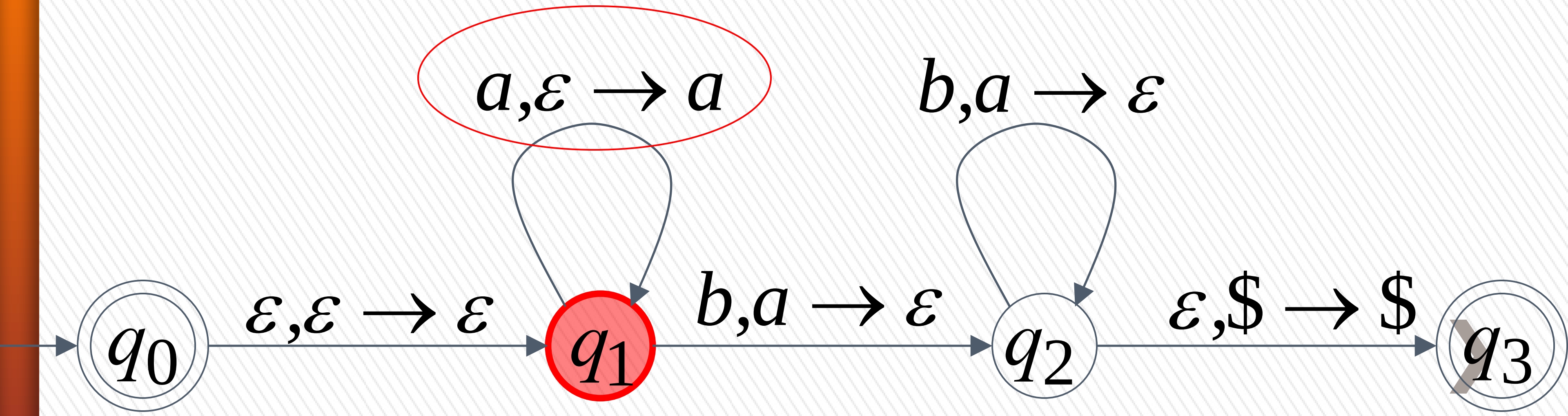
Time 2

Input

a	a	a	b	b	b

a
$\$$

Stack



Time 3

Input

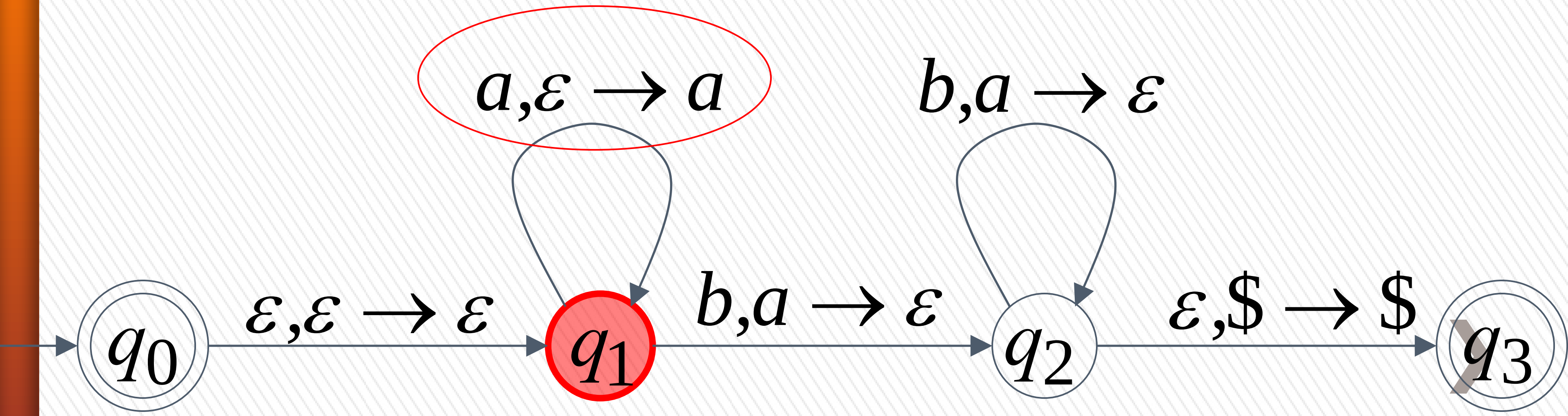
a	a	a	b	b	b

a

a

$\$$

Stack



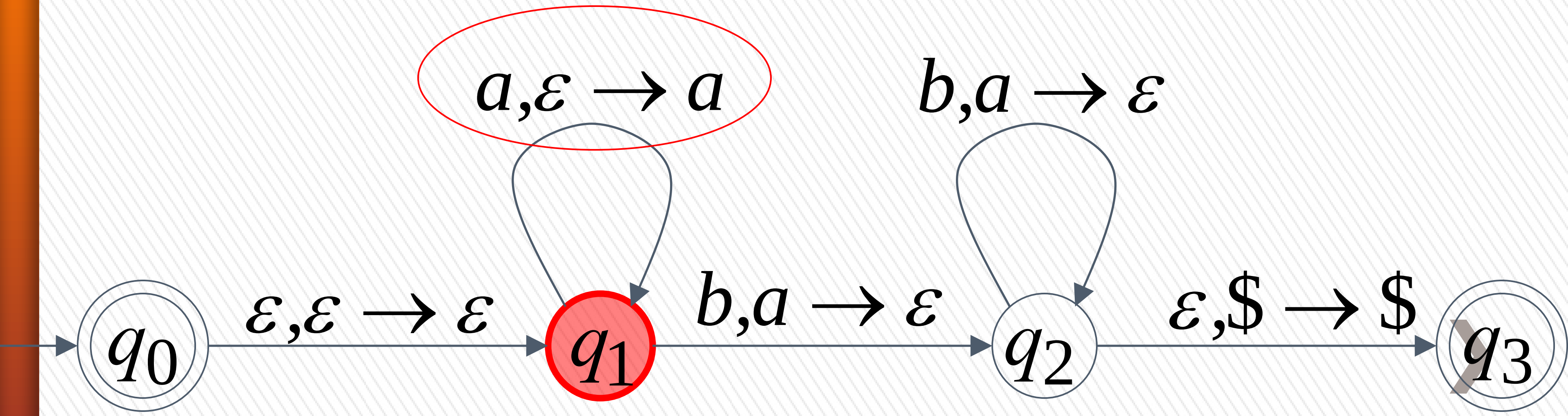
Time 4

Input

a	a	a	b	b	b

a
a
a
$\$$

Stack



Time 5

Input

a	a	a	b	b	b

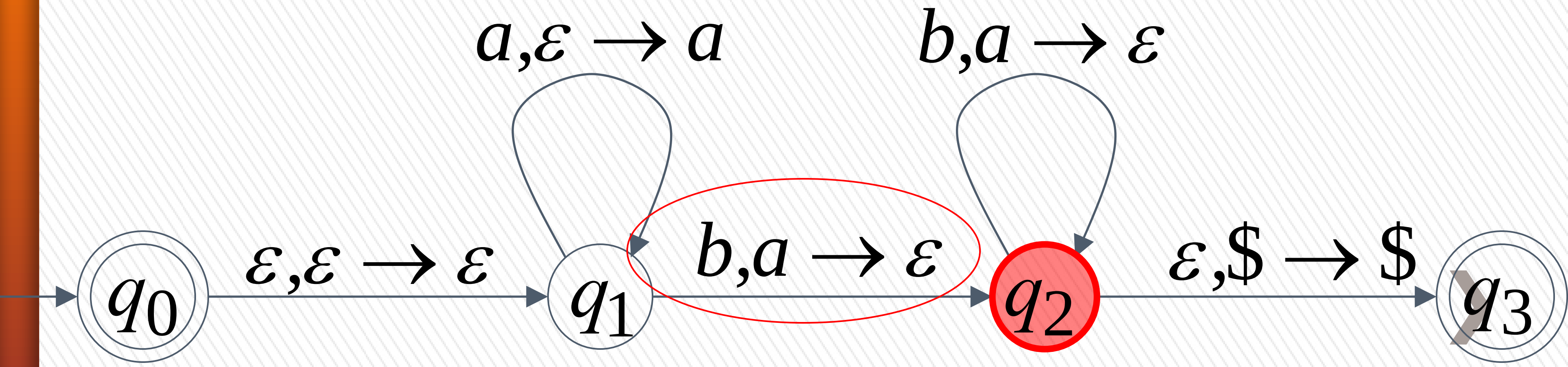
$\cancel{a}$

a

a

$\$$

Stack



q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_2
$b,a \rightarrow \varepsilon$
$\varepsilon,\$ \rightarrow \$$
q_3

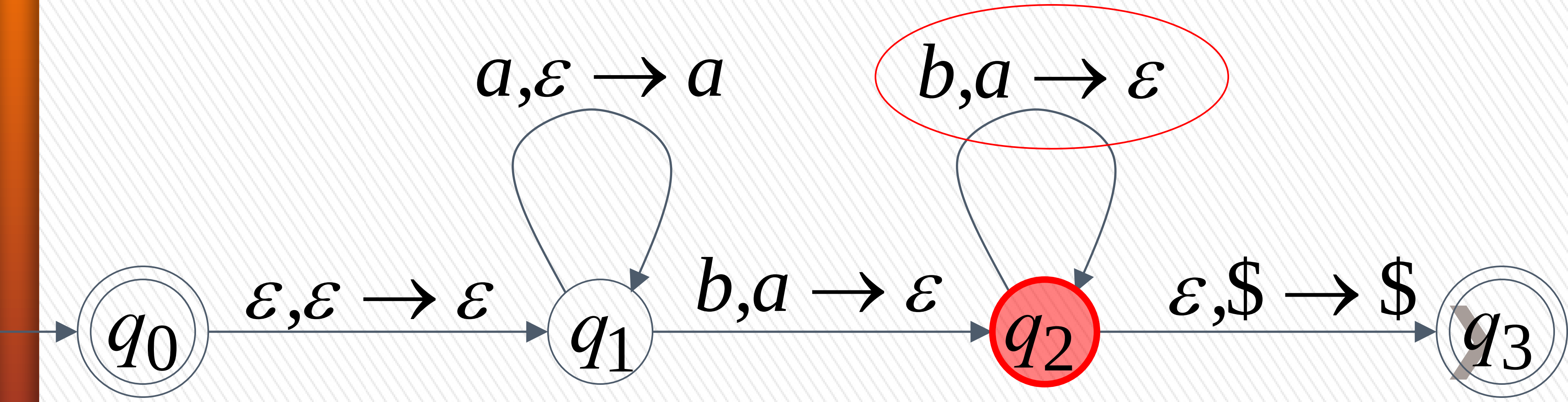
Time 6

Input

a	a	a	b	b	b

$\cancel{a}$
a
$\$$

Stack



$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$b,a \rightarrow \varepsilon$
q_2
$\varepsilon,\$ \rightarrow \$$
q_3

Time 7

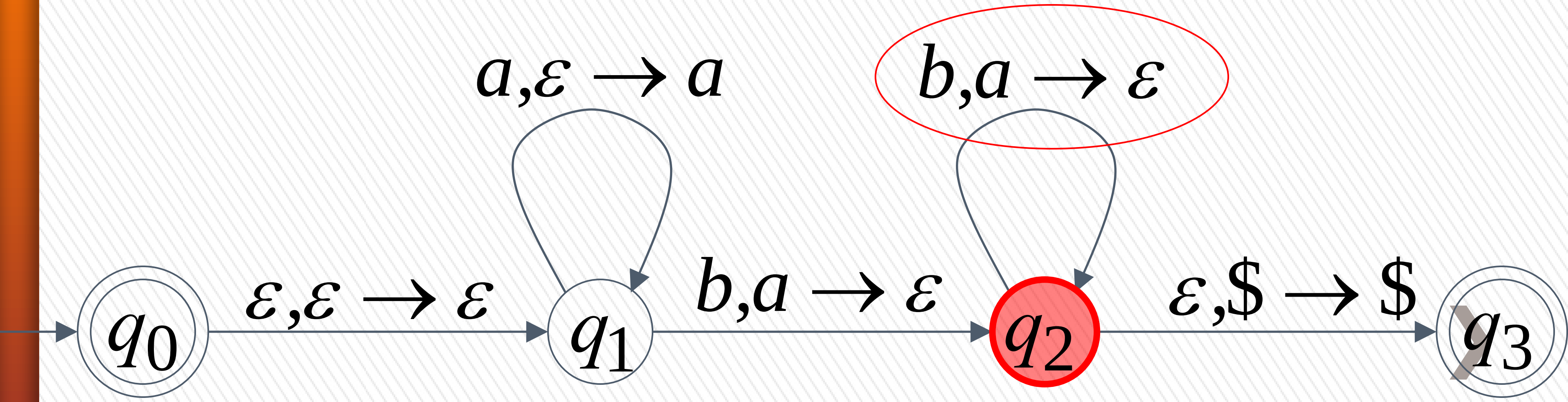
Input

a	a	a	b	b	b

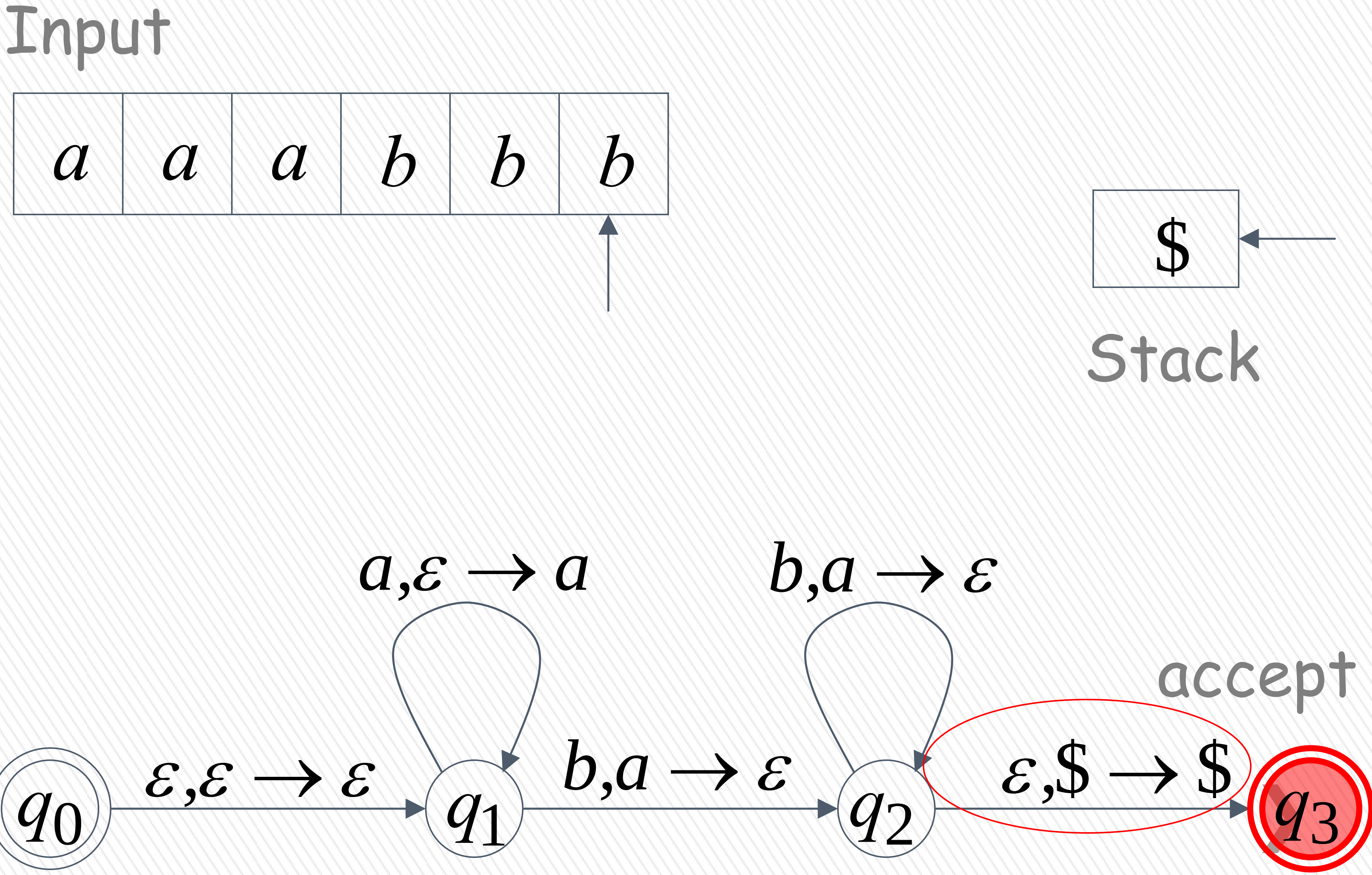
$\cancel{a}$

$\$$

Stack



Time 8



Input

a	a	a	b	b	b

$\$$

Stack

$a,\varepsilon \rightarrow a$

$b,a \rightarrow \varepsilon$

q_0 $\varepsilon,\varepsilon \rightarrow \varepsilon$ q_1 $b,a \rightarrow \varepsilon$ q_2 $\varepsilon,\$ \rightarrow \$$ q_3

accept

A string is accepted if there is
a computation such that:

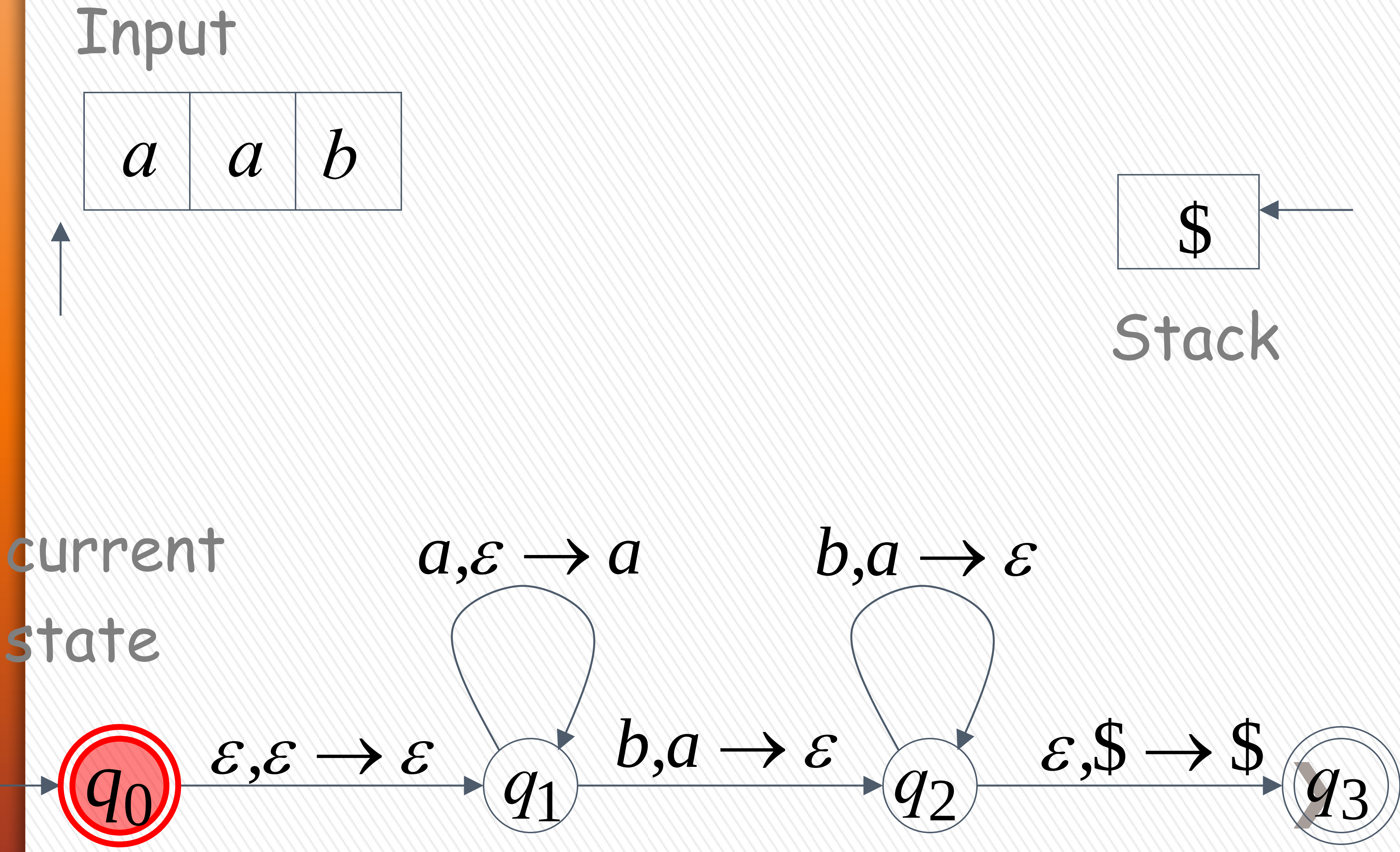
All the input is consumed

AND

The last state is an accepting state

we do not care about the stack contents
at the end of the accepting computation

Rejection Example: Time 0



Input

a	a	b

$\$$

Stack

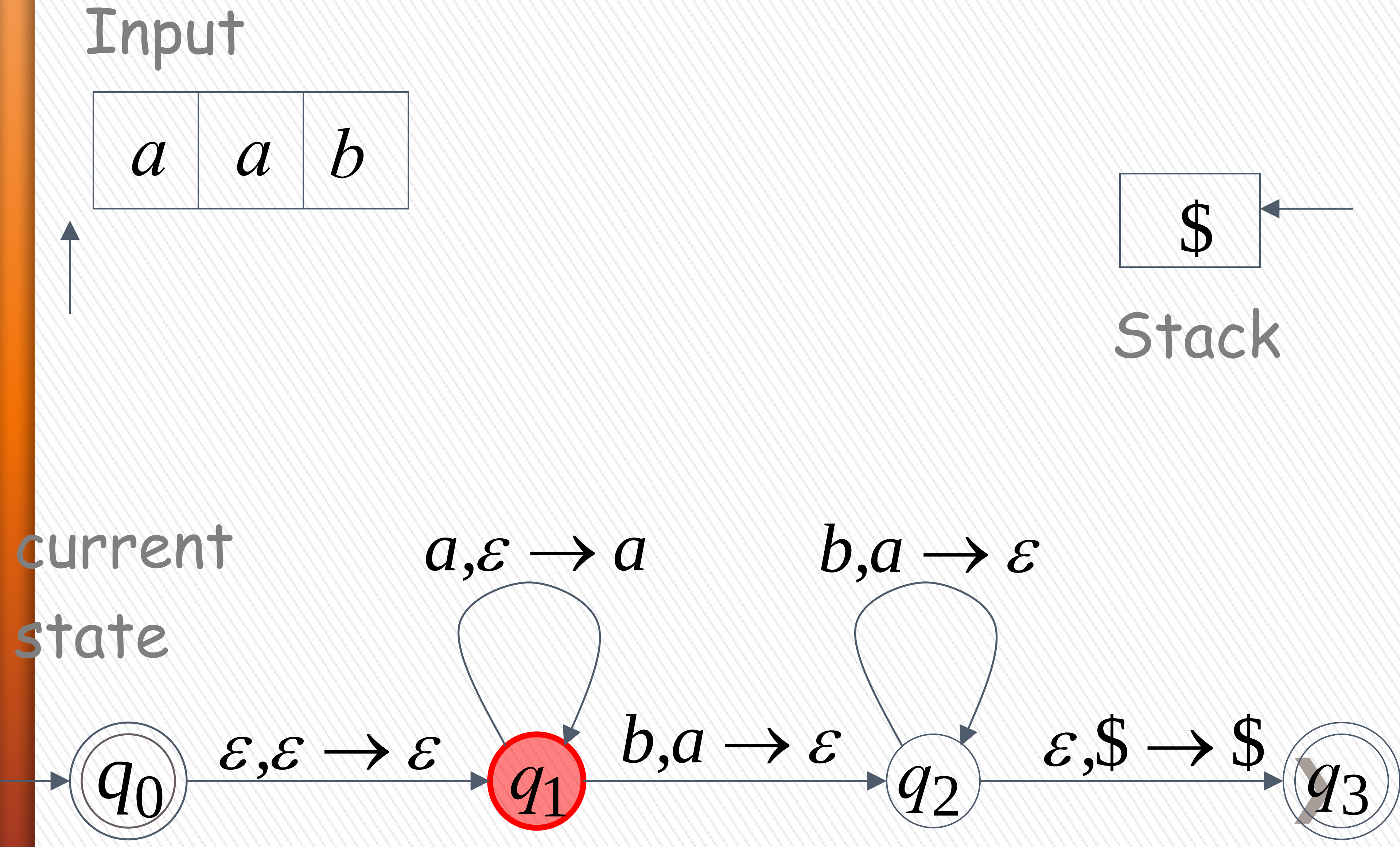
current state

$a,\varepsilon \rightarrow a$

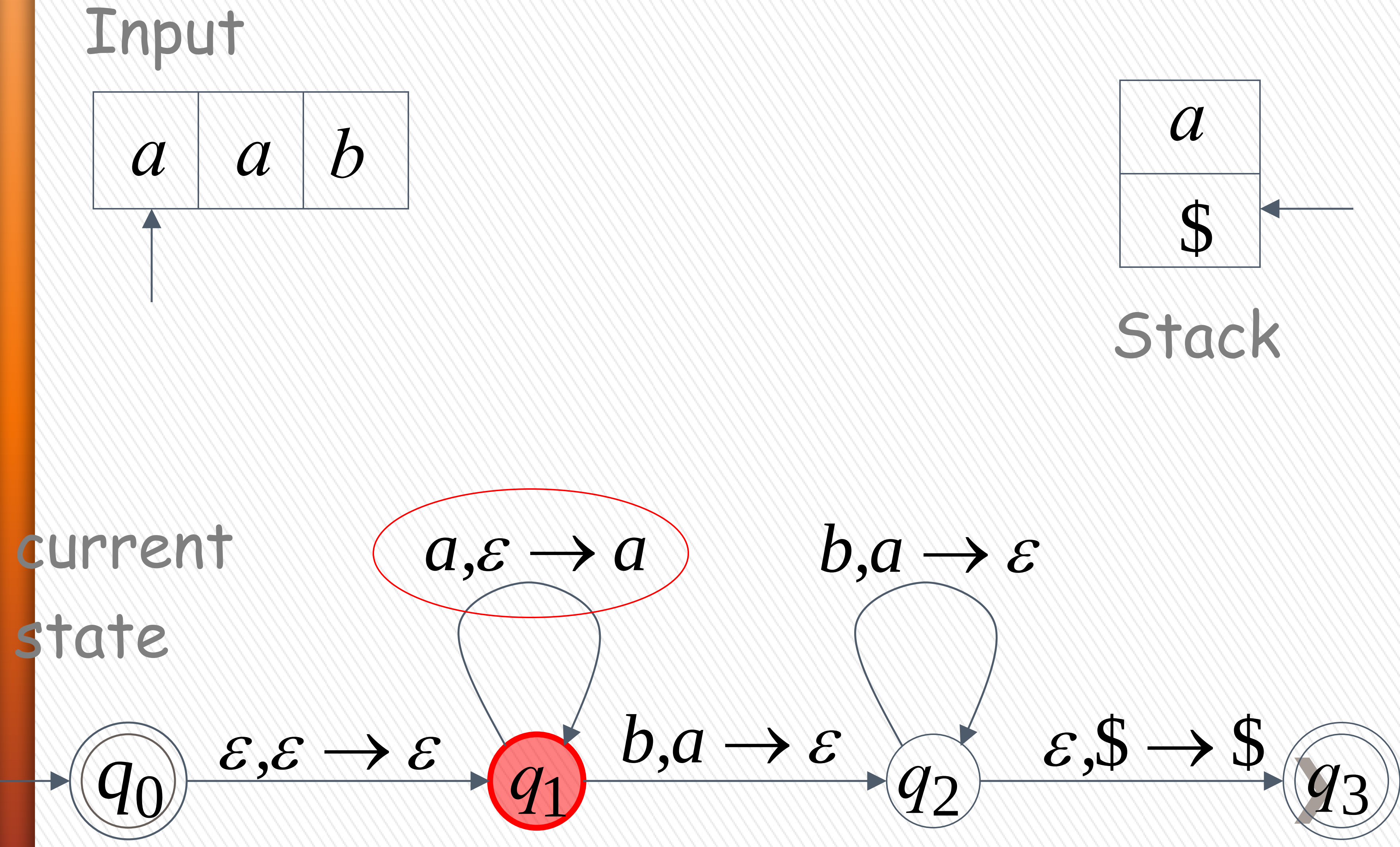
$b,a \rightarrow \varepsilon$

q_0 $\varepsilon,\varepsilon \rightarrow \varepsilon$ q_1 $b,a \rightarrow \varepsilon$ q_2 $\varepsilon,\$ \rightarrow \$$ q_3

Rejection Example: Time 1


Input
a
a
b
$\$$
Stack
current
state
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$b,a \rightarrow \varepsilon$
q_2
$\varepsilon,\$ \rightarrow \$$
q_3

Rejection Example: Time 2


Input
a
a
b
a
$\$$
Stack
current state
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$b,a \rightarrow \varepsilon$
q_2
$\varepsilon,\$ \rightarrow \$$
q_3

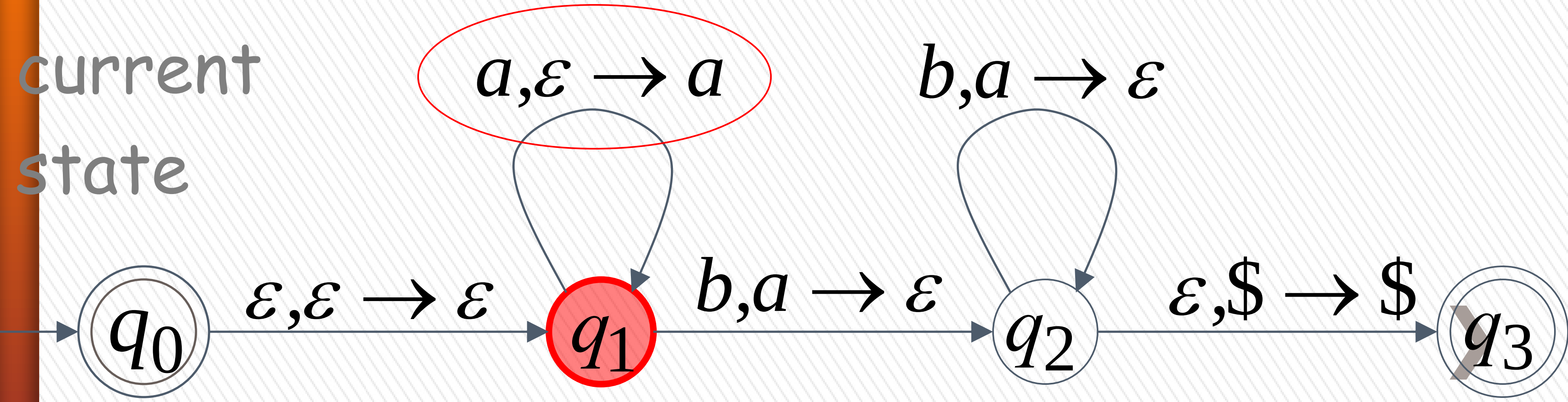
Rejection Example: Time 3

Input

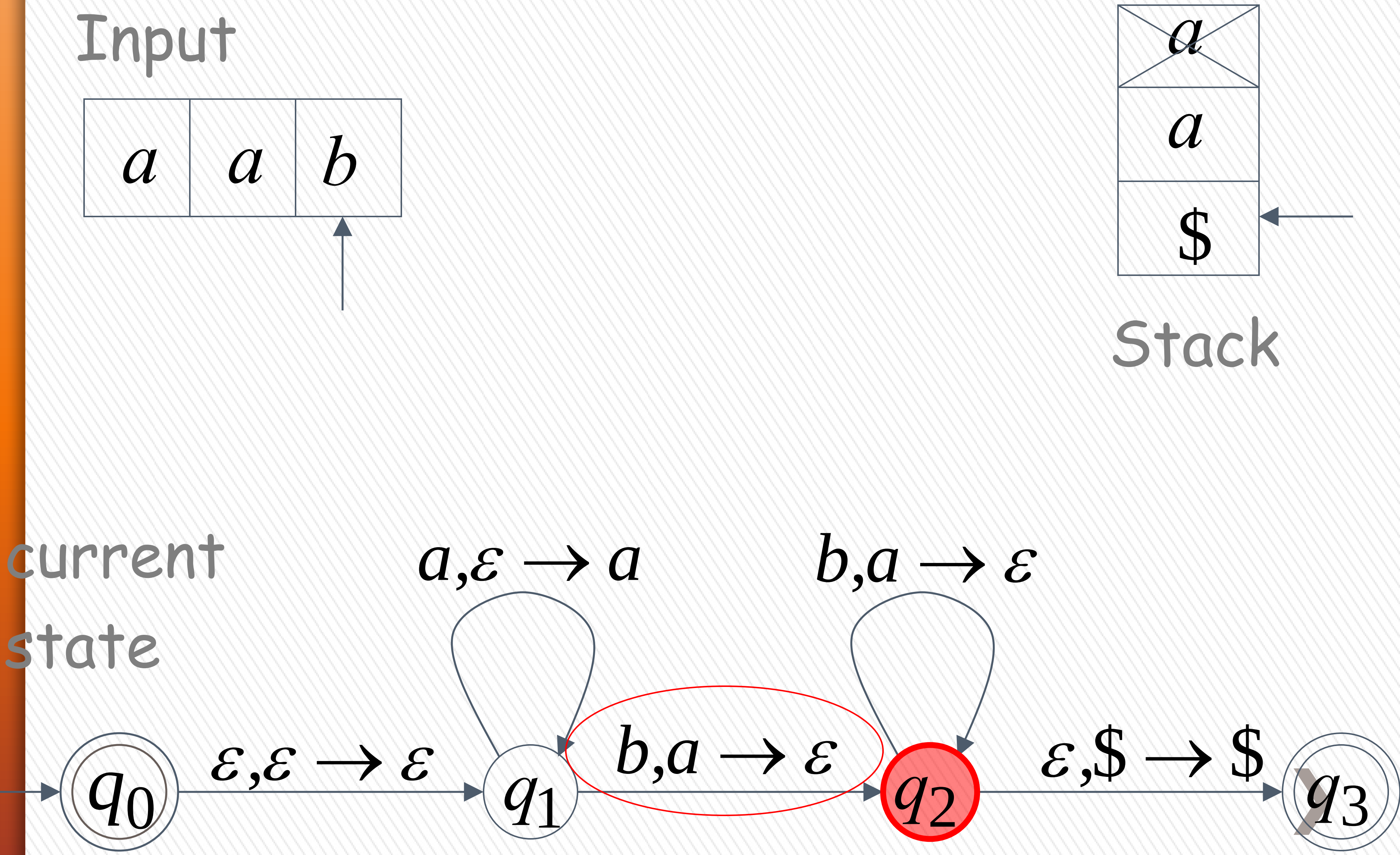
a	a	b

a
a
$\$$

Stack

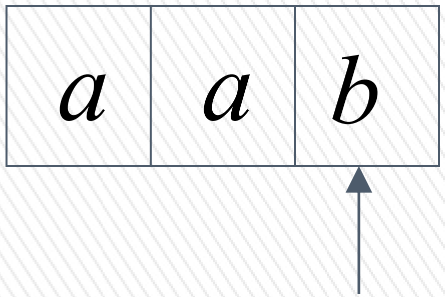

current
state
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$b,a \rightarrow \varepsilon$
q_2
$\varepsilon,\$ \rightarrow \$$
q_3

Rejection Example: Time 4

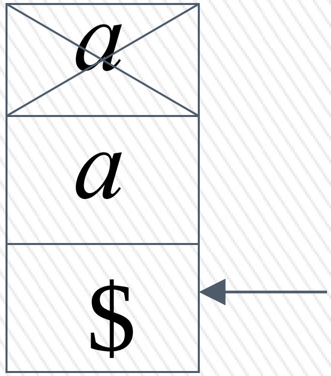

Input
a a b
~~a~~
a
$\$$
Stack
current
state
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_2
$b,a \rightarrow \varepsilon$
$\varepsilon,\$ \rightarrow \$$
q_3

Rejection Example: Time 4

Input



a	a	b



$\cancel{a}$

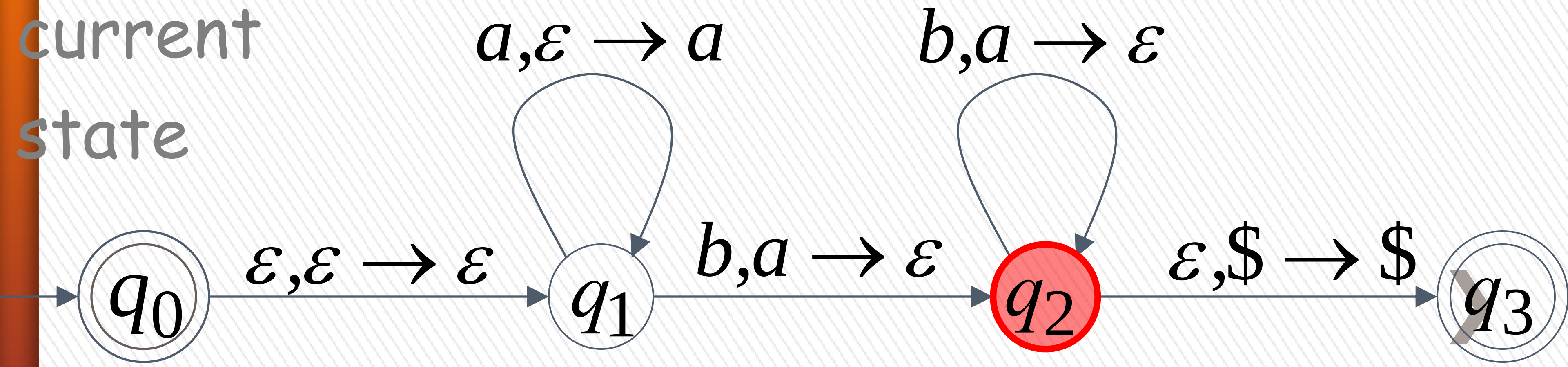
a

$\$$

Stack

reject

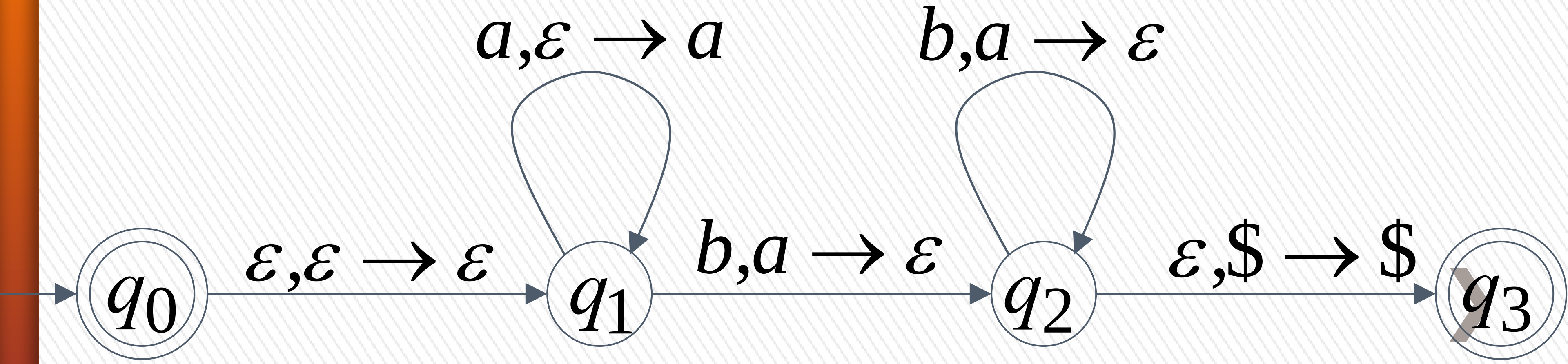
current state



q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_2
$b,a \rightarrow \varepsilon$
$\varepsilon,\$ \rightarrow \$$
q_3

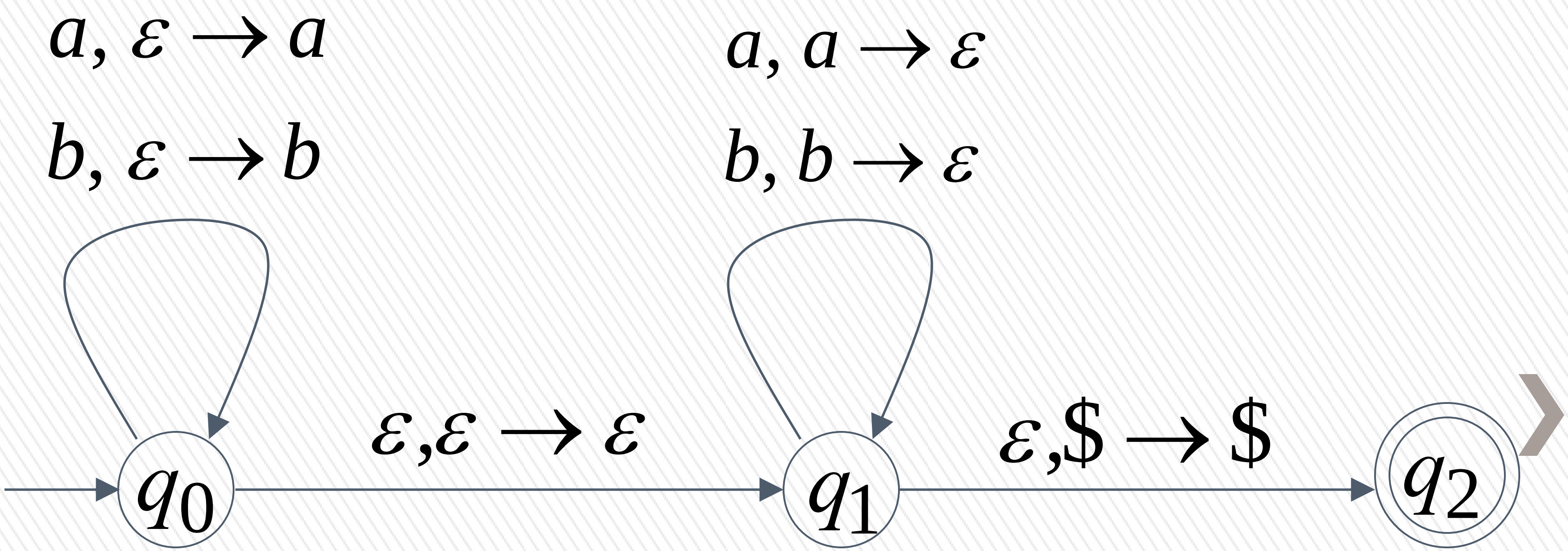
There is no accepting computation for aab

The string aab is rejected by the PDA


$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
q_0
$\varepsilon,\varepsilon \rightarrow \varepsilon$
q_1
$b,a \rightarrow \varepsilon$
q_2
$\varepsilon,\$ \rightarrow \$$
q_3

Another PDA example

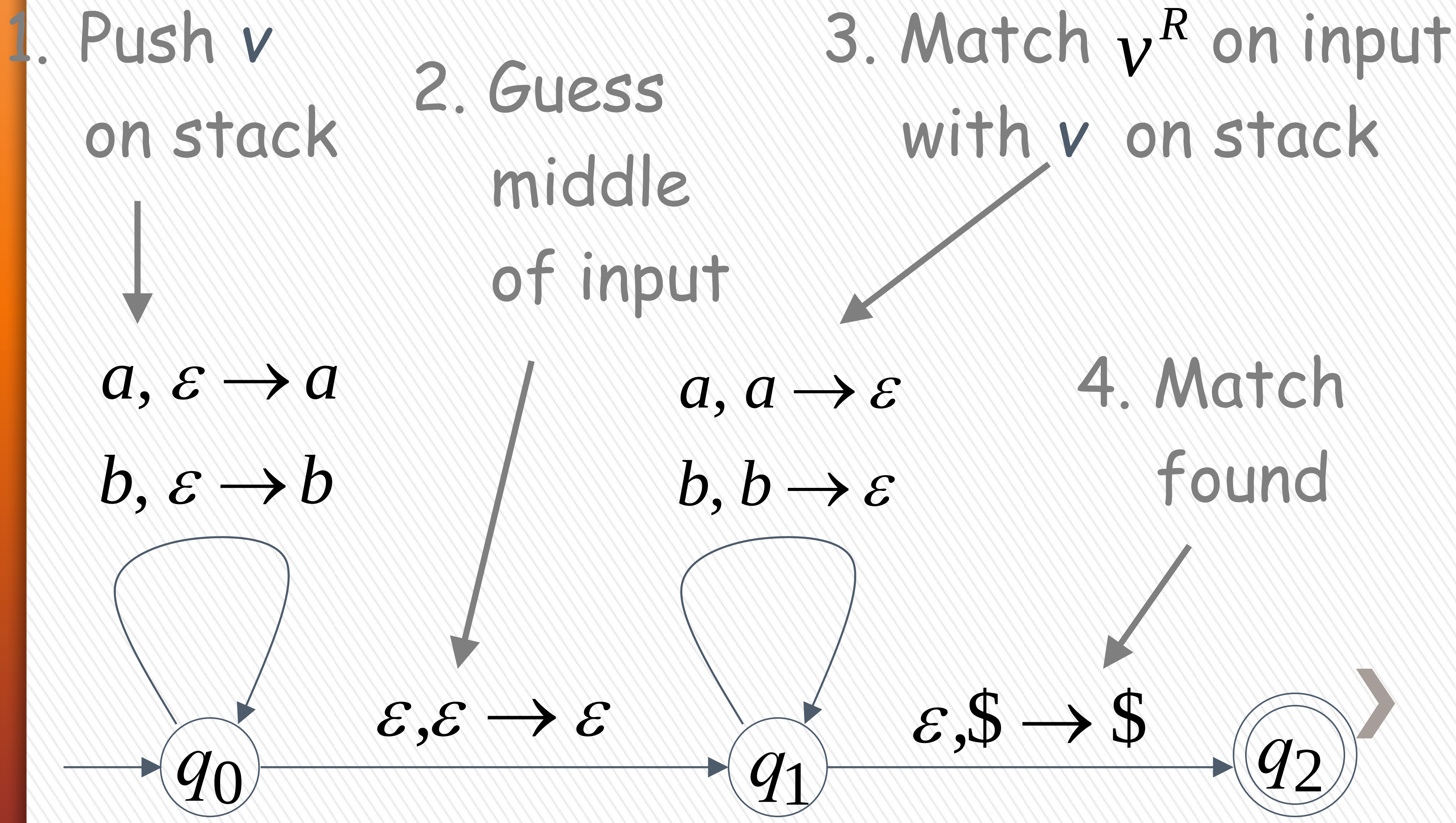
PDA M: $L(M)=\{vv^R : v\in\{a,b\}^*\}$



$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

Basic Idea:

$$L(M) = \{vv^R : v \in \{a,b\}^*\}$$



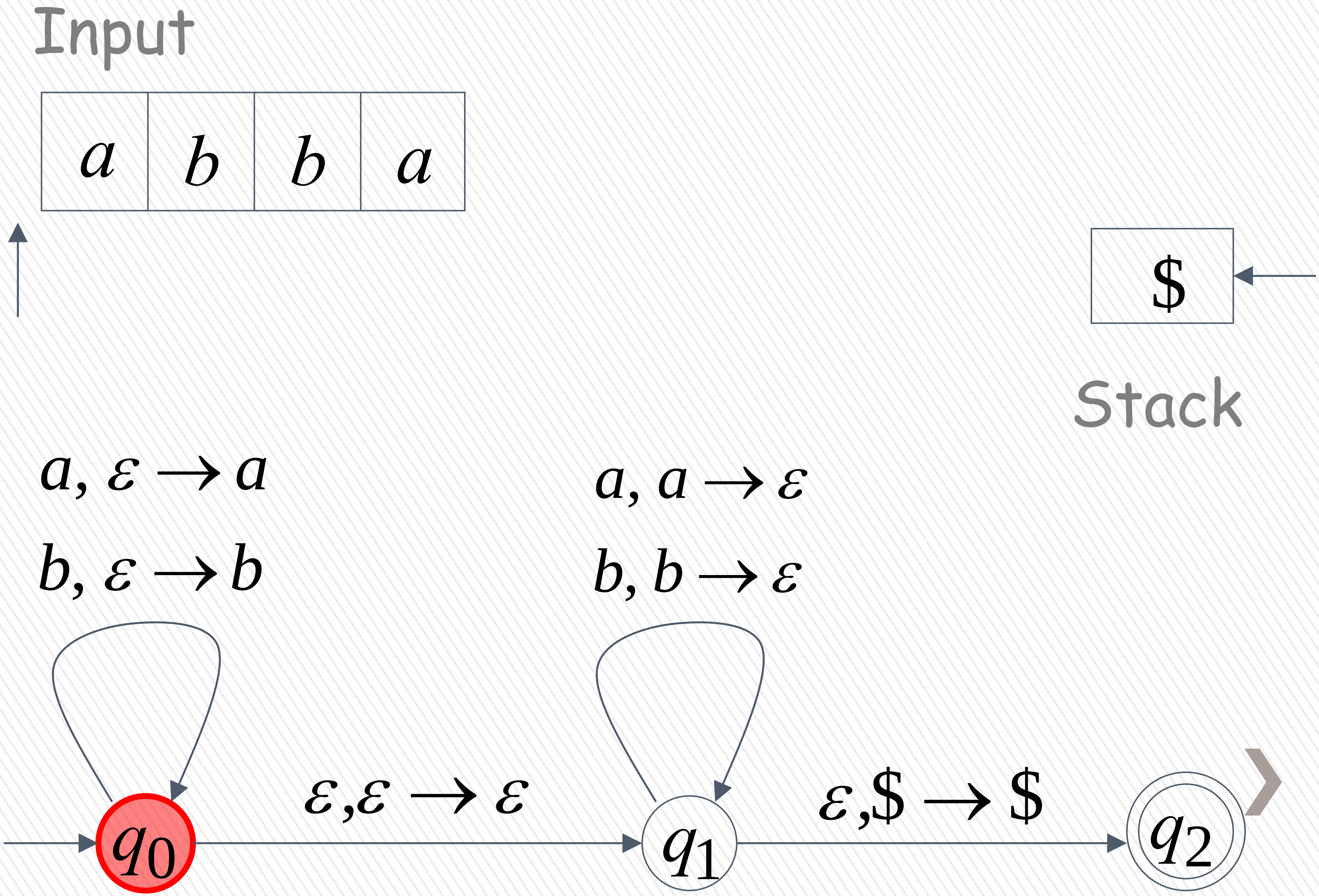
1. Push v on stack

2. Guess middle of input

3. Match v^R on input with v on stack

4. Match found

Execution Example: Time 0



Input

a	b	b	a

$\$$

Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

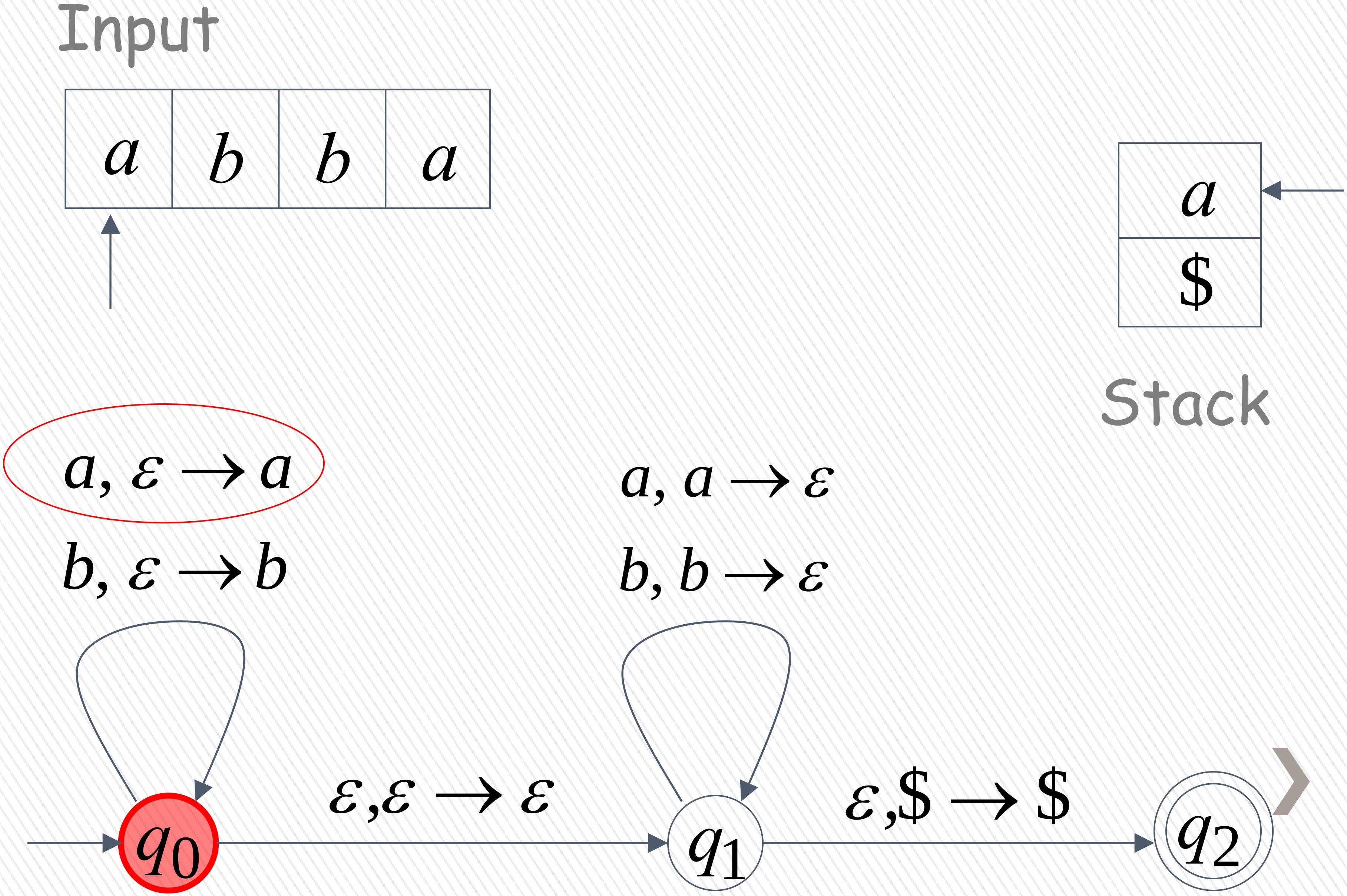
$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \$ \to \$$

Time 1



Input

a	b	b	a

Stack

a
$\$$

$a, \varepsilon \to a$

$b, \varepsilon \to b$

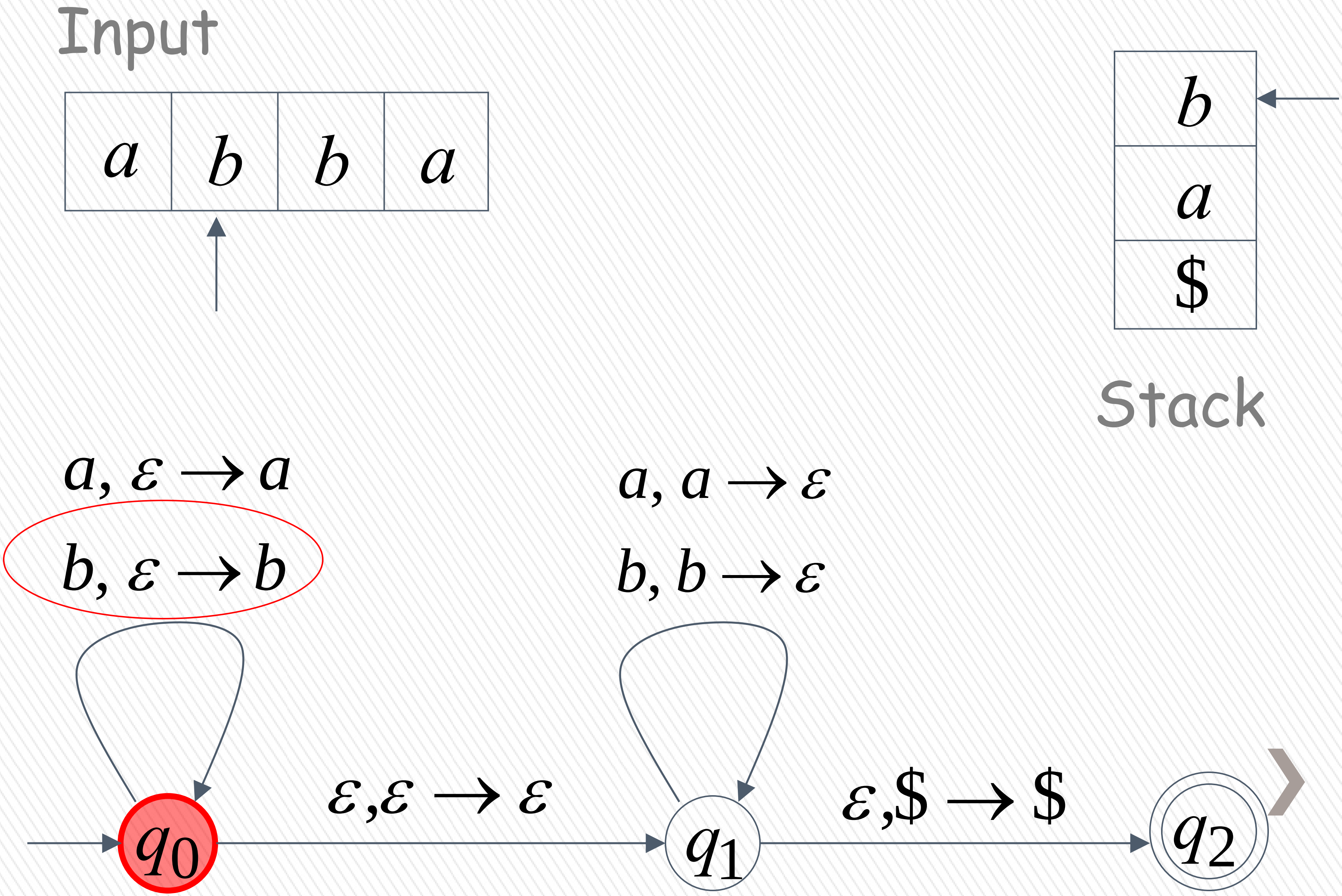
$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \$ \to \$$

Time 2



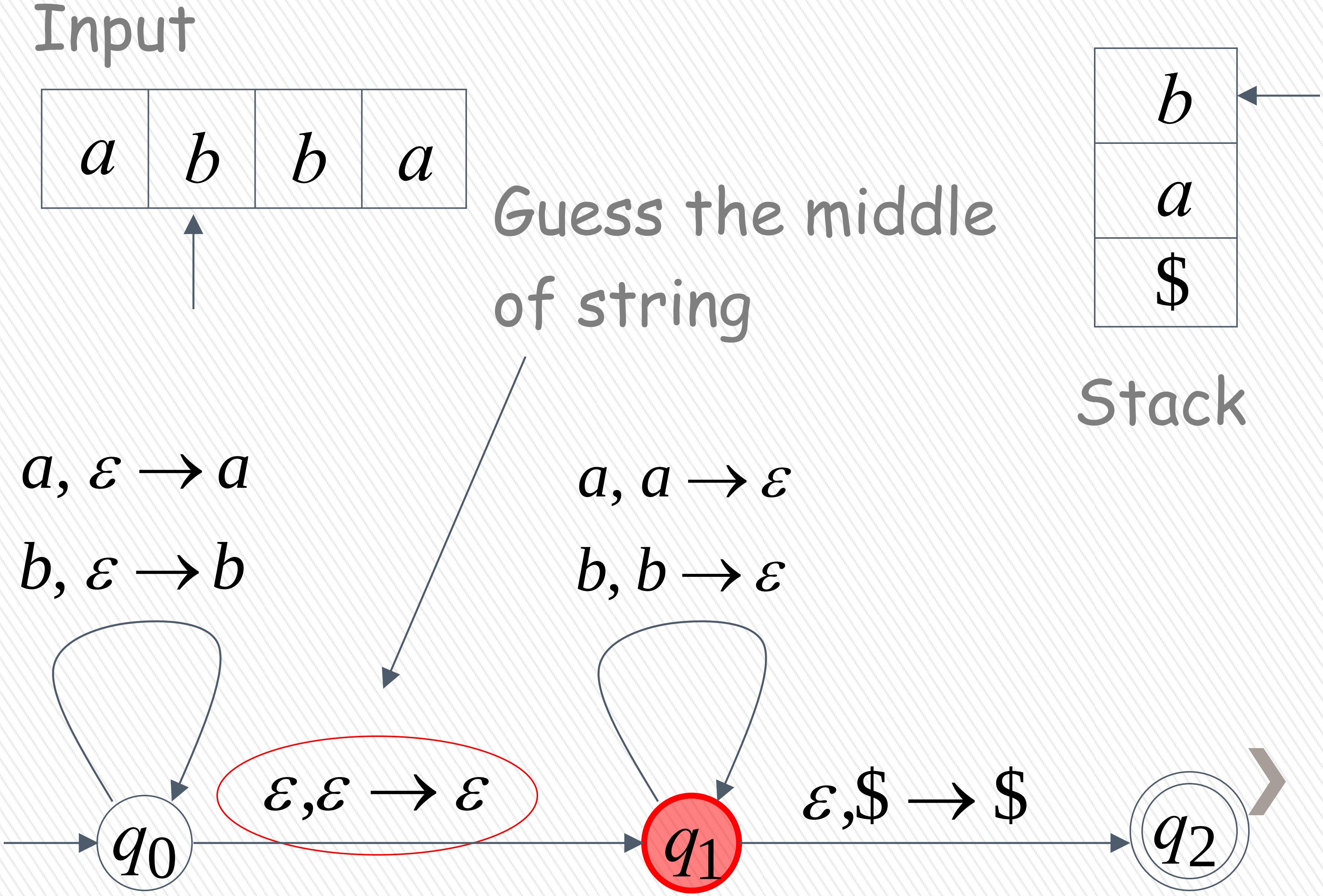
Input

a	b	b	a

Stack

b
a
$\$$

Time 3



Input

a	b	b	a

Guess the middle of string

Stack

b
a
$\$$

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

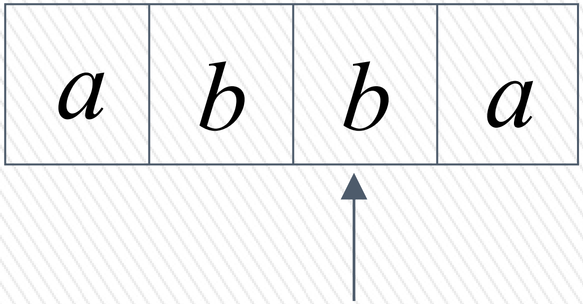
$\varepsilon, \varepsilon \rightarrow \varepsilon$

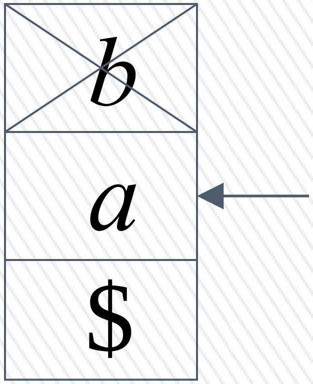
$\varepsilon, \$ \rightarrow \$$

q_0 q_1 q_2

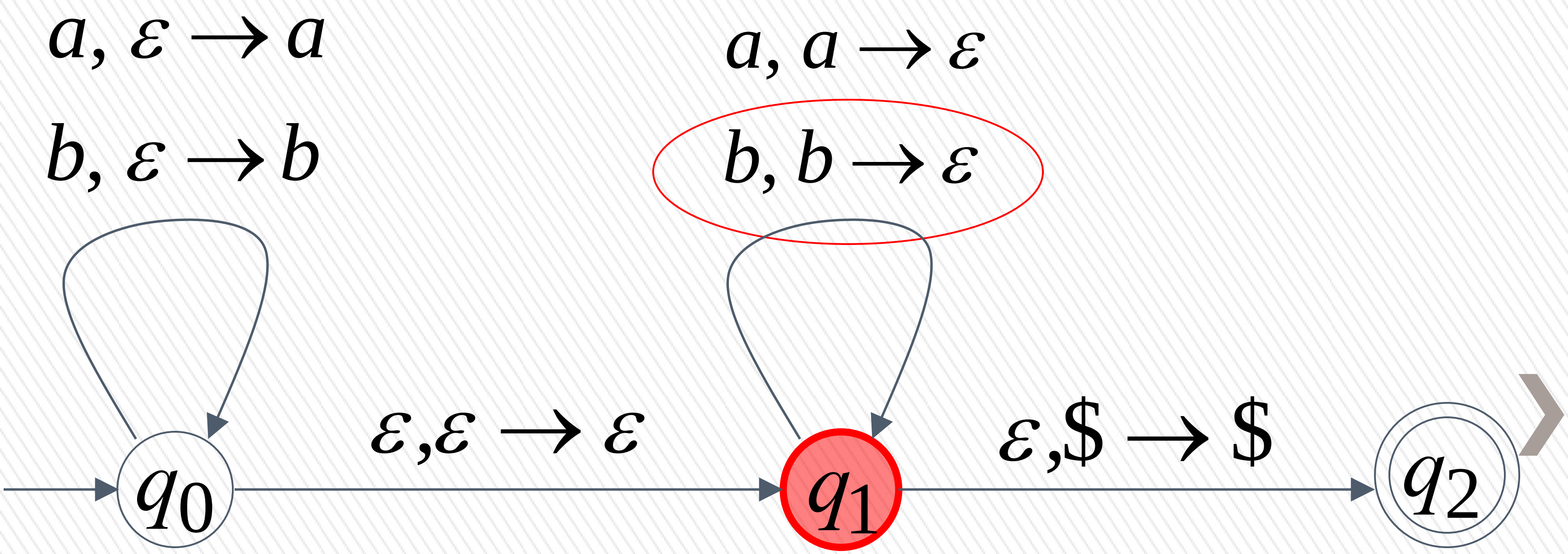
Time 4

Input



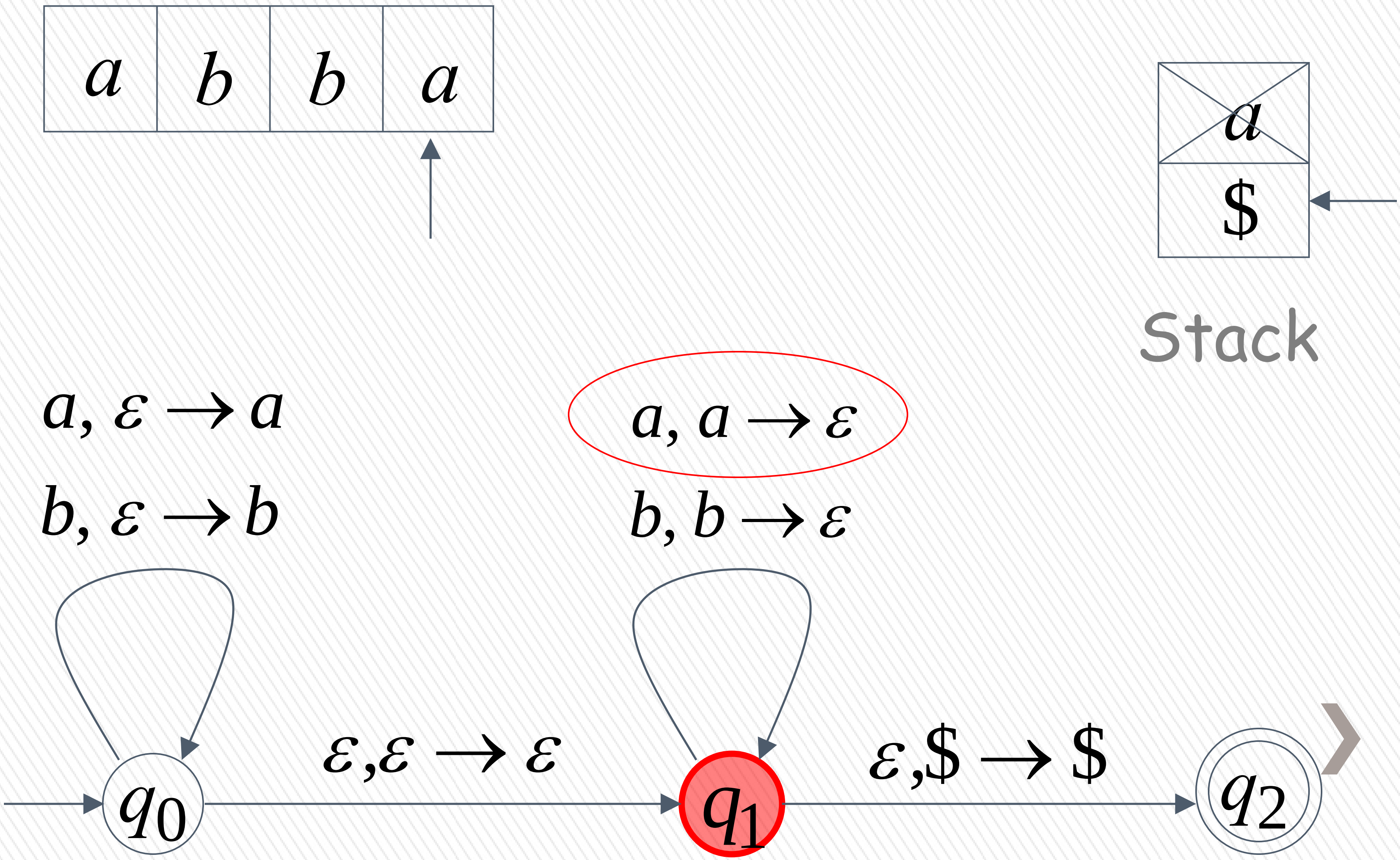


Stack


$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

Input



a b b a

$\xcancel{a}$

$\$$

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

q_0 q_1 q_2

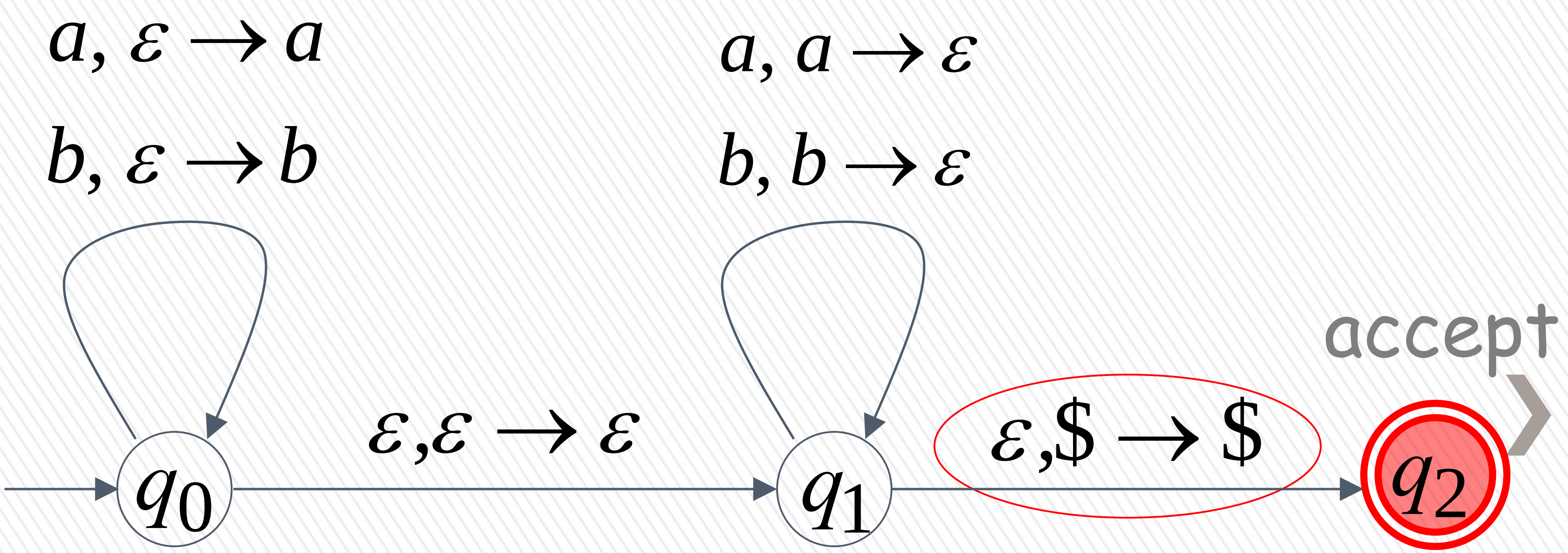
Time 6

Input

a	b	b	a

$\$$

Stack



$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
accept
q_0
q_1
q_2

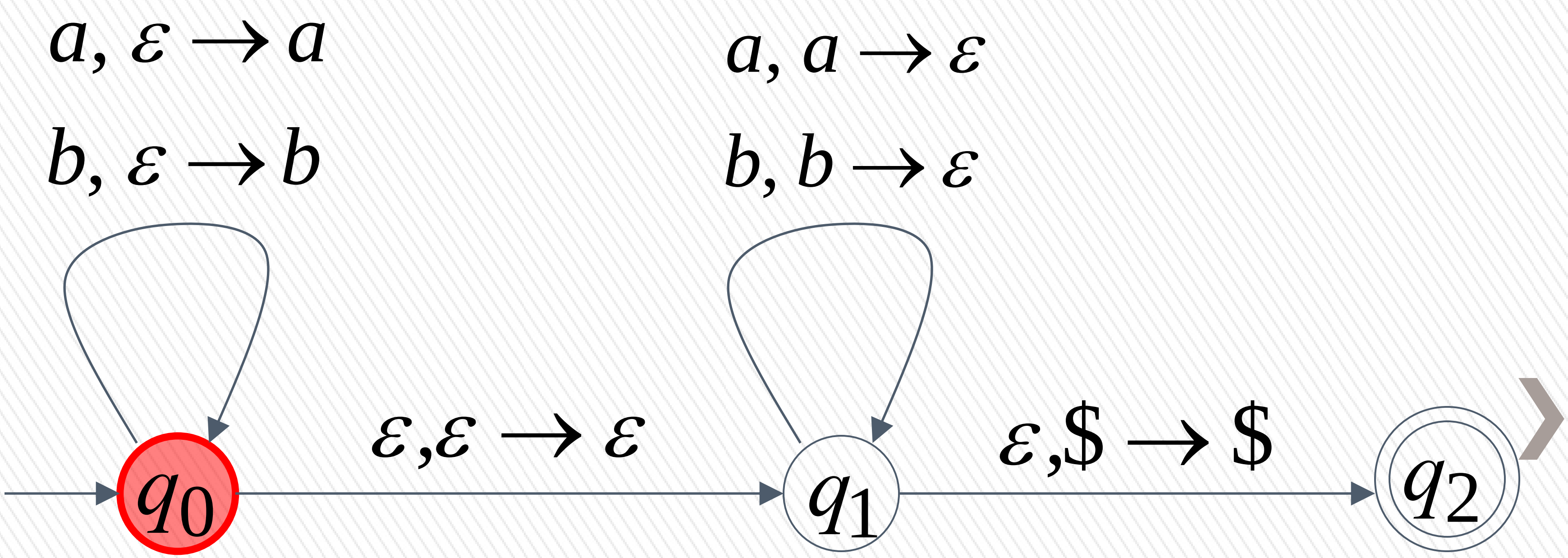
Rejection Example: Time 0

Input

a	b	b	b

$\$$

Stack



$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

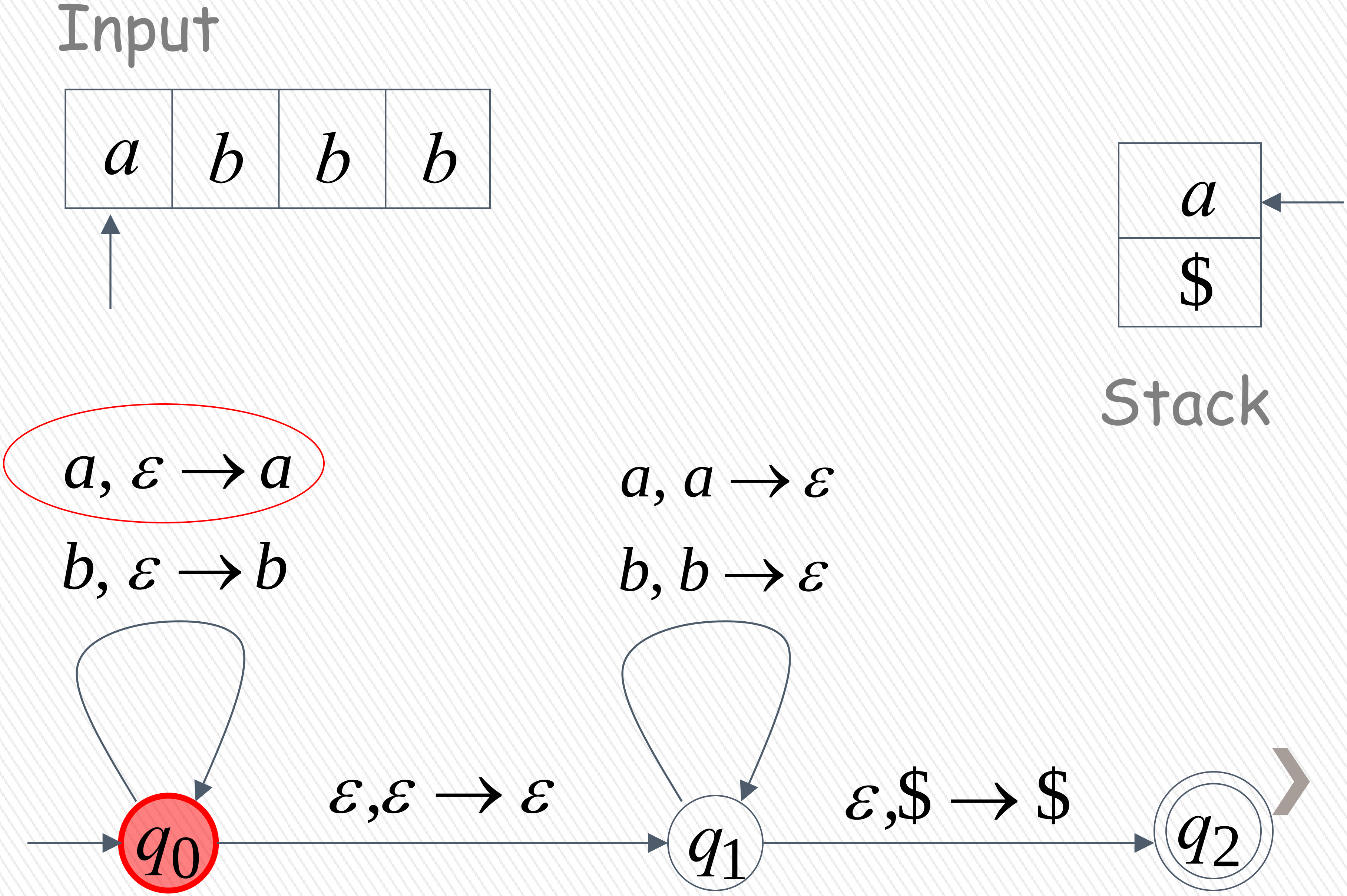
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

Time 1



Input

a b b b

Stack

a

$\$$

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

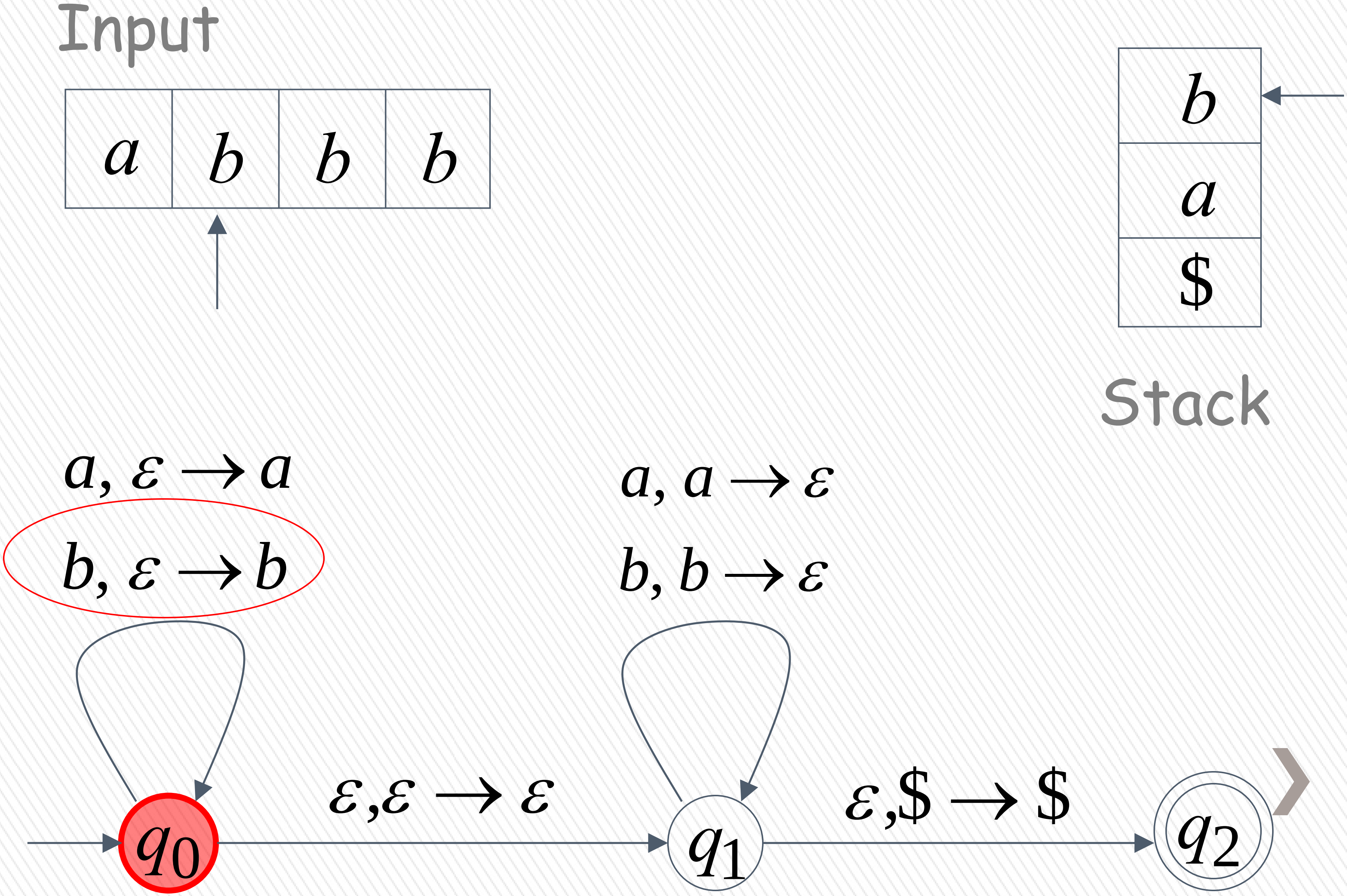
$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

Time 2



Input

a	b	b	b

b
a
$\$$

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

q_0

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

q_1

$\varepsilon, \$ \rightarrow \$$

q_2

Time 3

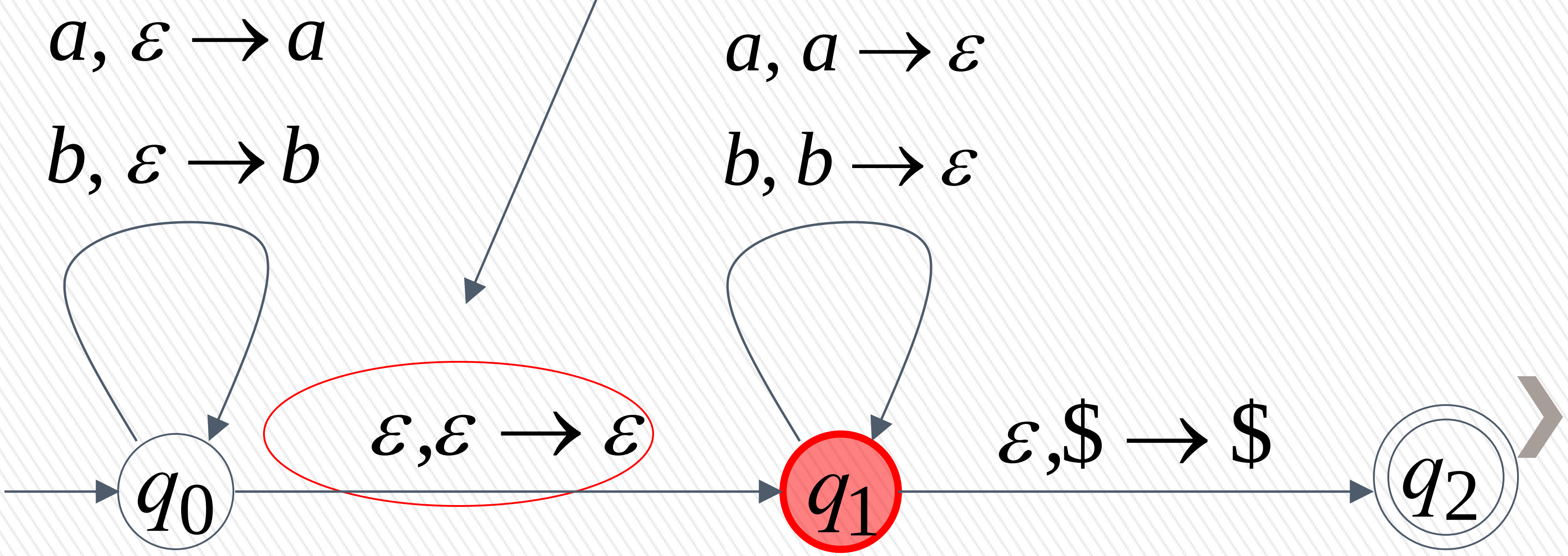
Input

a	b	b	b

Guess the middle of string

b
a
$\$$

Stack



$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
q_0
$\varepsilon, \varepsilon \rightarrow \varepsilon$
q_1
$\varepsilon, \$ \rightarrow \$$
q_2

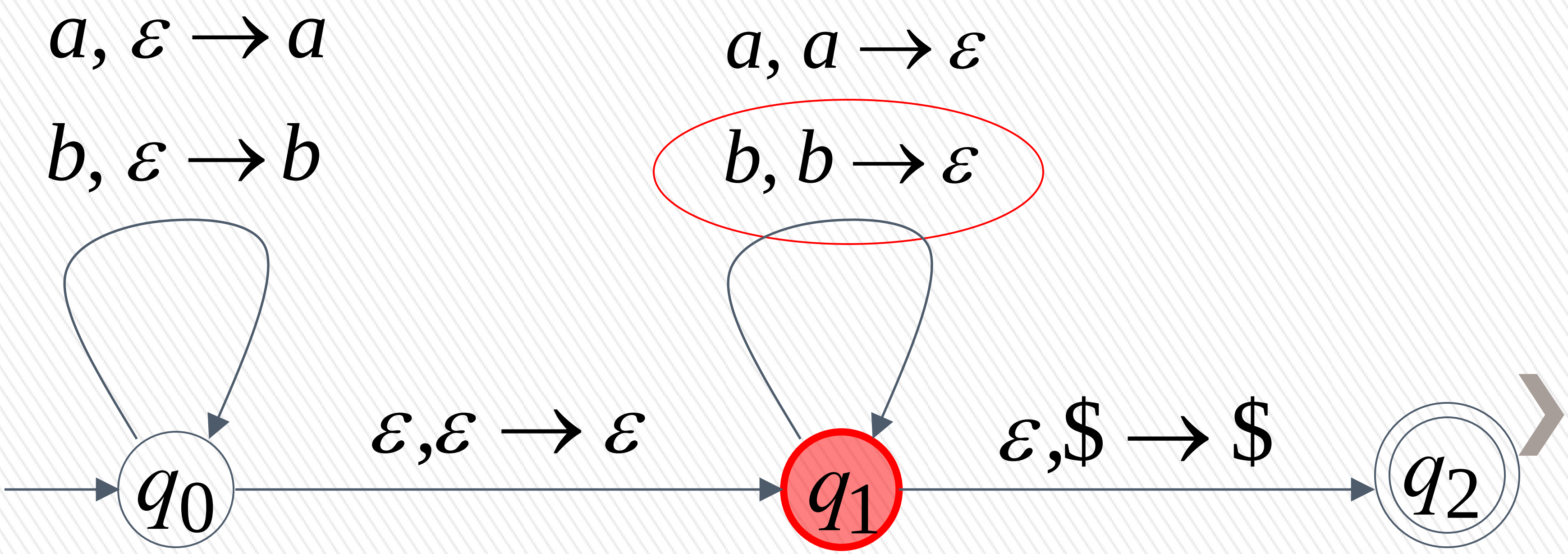
Time 4

Input

a	b	b	b

~~b~~
a
$\$$

Stack

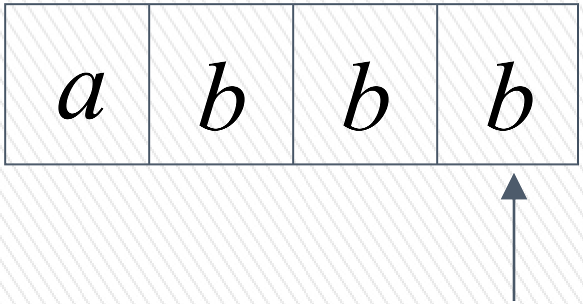


$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

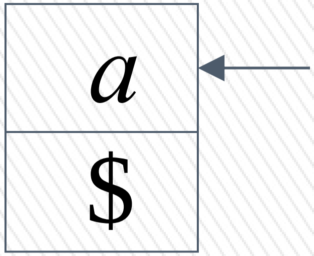
Time 5

There is no possible transition.

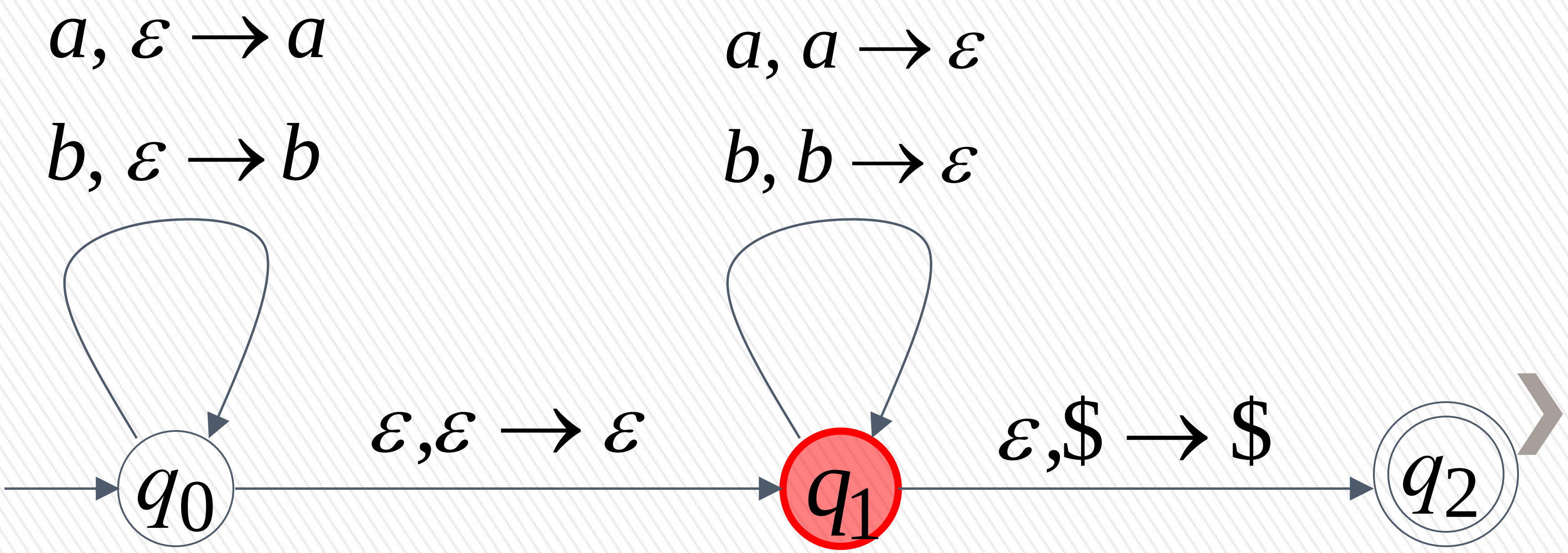
Input



Input is not consumed



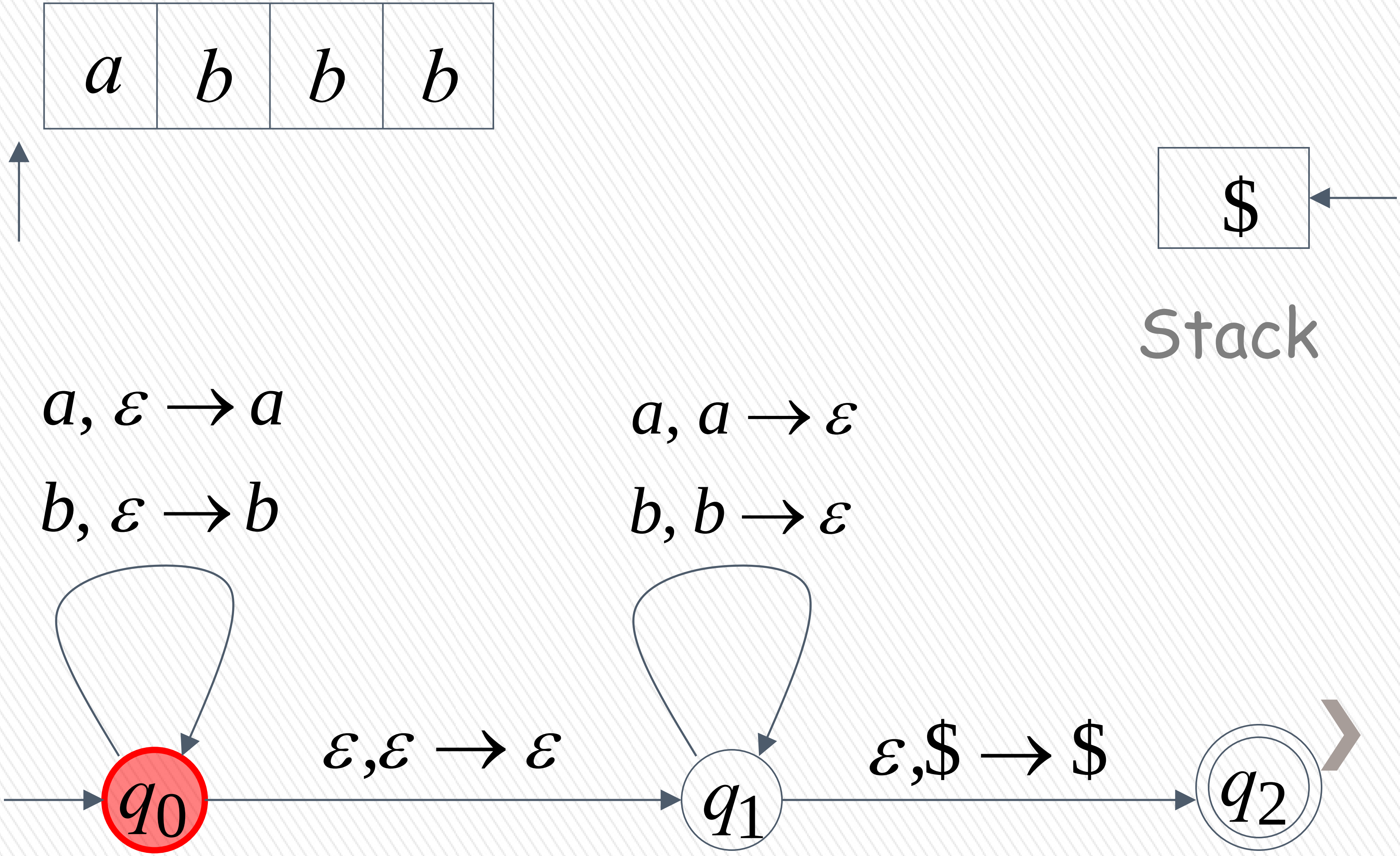
Stack


$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

Another computation on same string:

Input

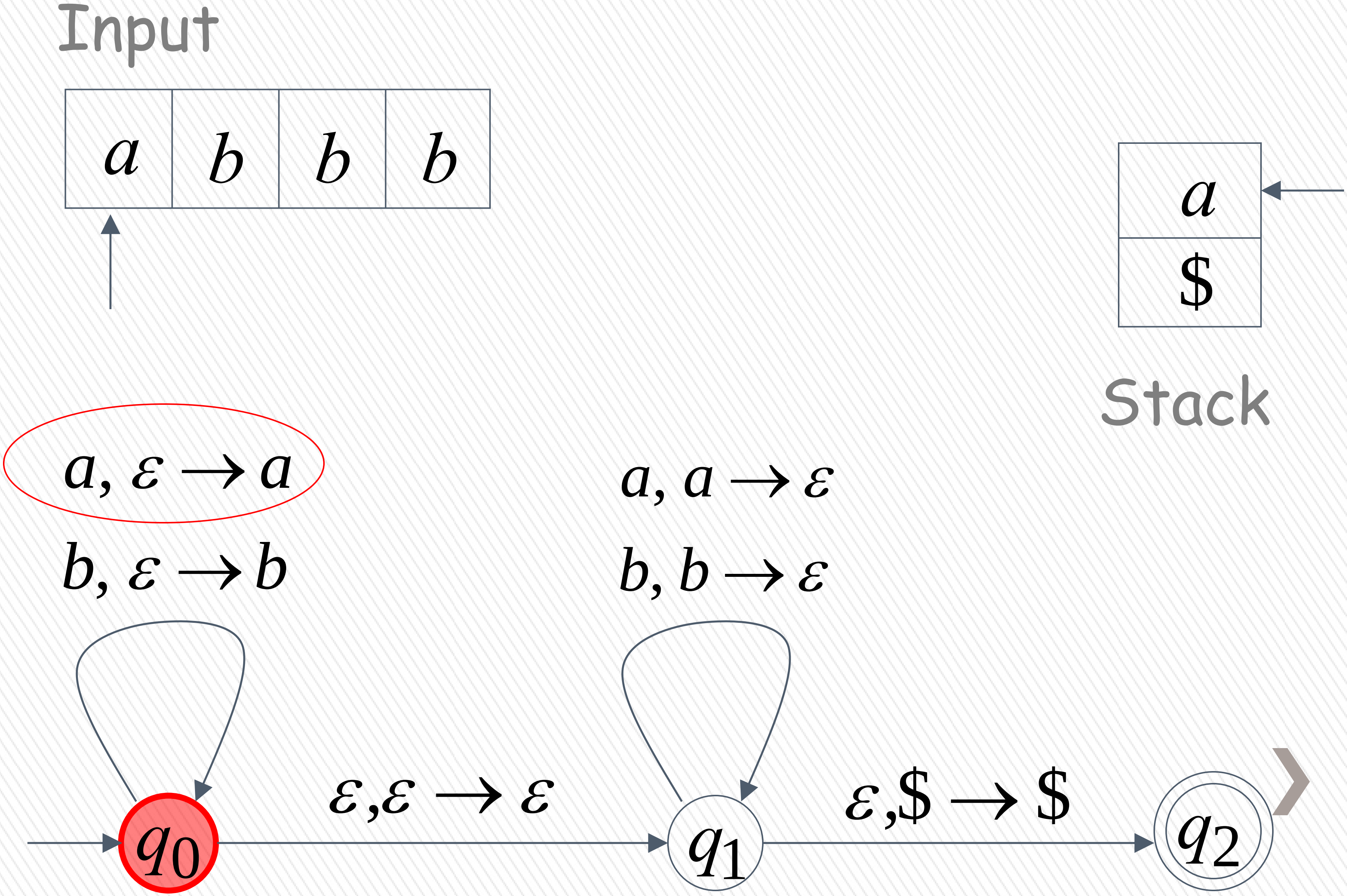
Time 0



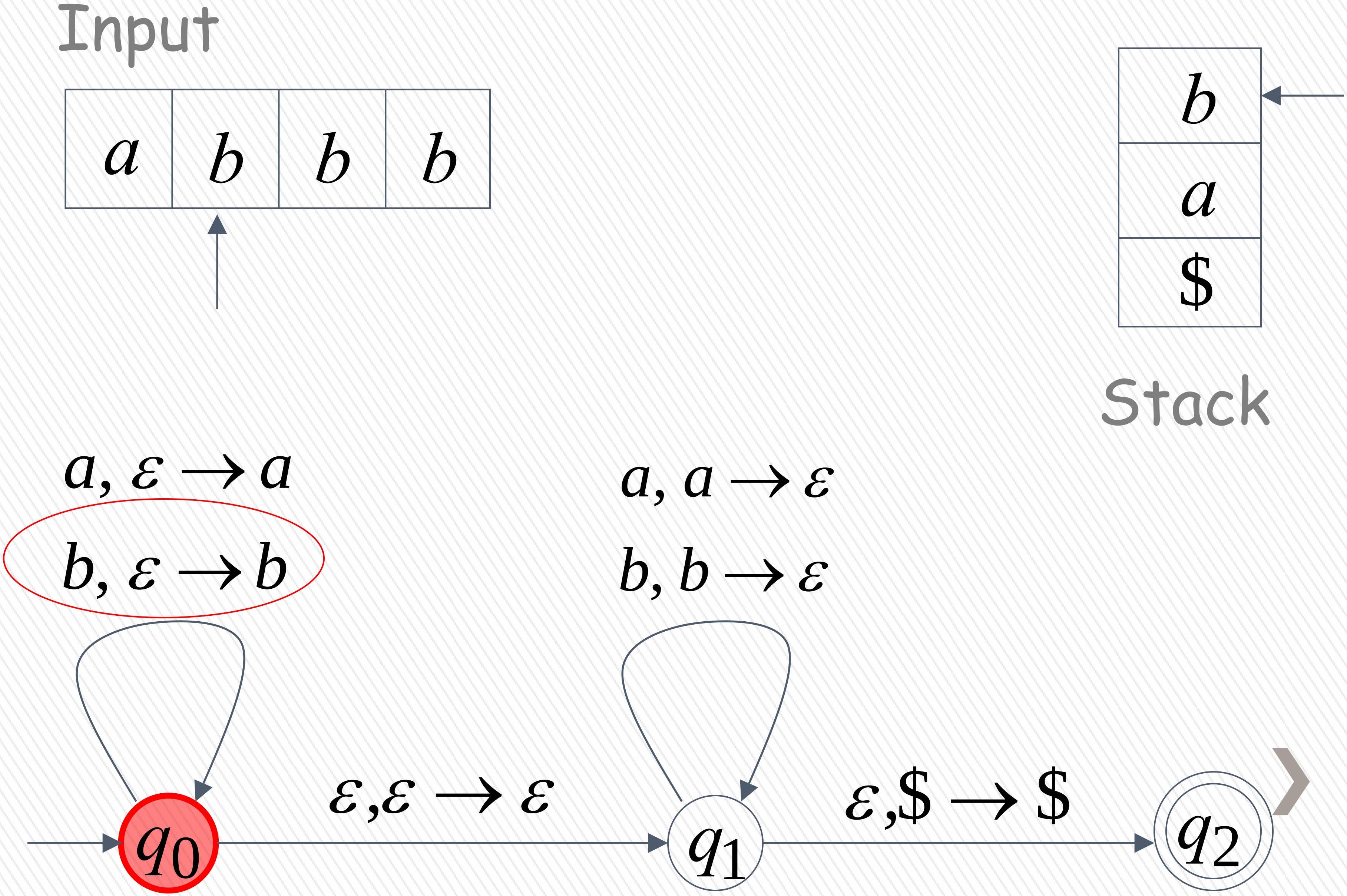
Stack

a
b
b
b
$
$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

Time 1


Input
a b b b
a
$\$$
Stack
$a, \varepsilon \to a$
$b, \varepsilon \to b$
$a, a \to \varepsilon$
$b, b \to \varepsilon$
$\varepsilon, \varepsilon \to \varepsilon$
$\varepsilon, \$ \to \$$
q_0
q_1
q_2

Time 2



Input

a	b	b	b

b
a
$\$$

Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

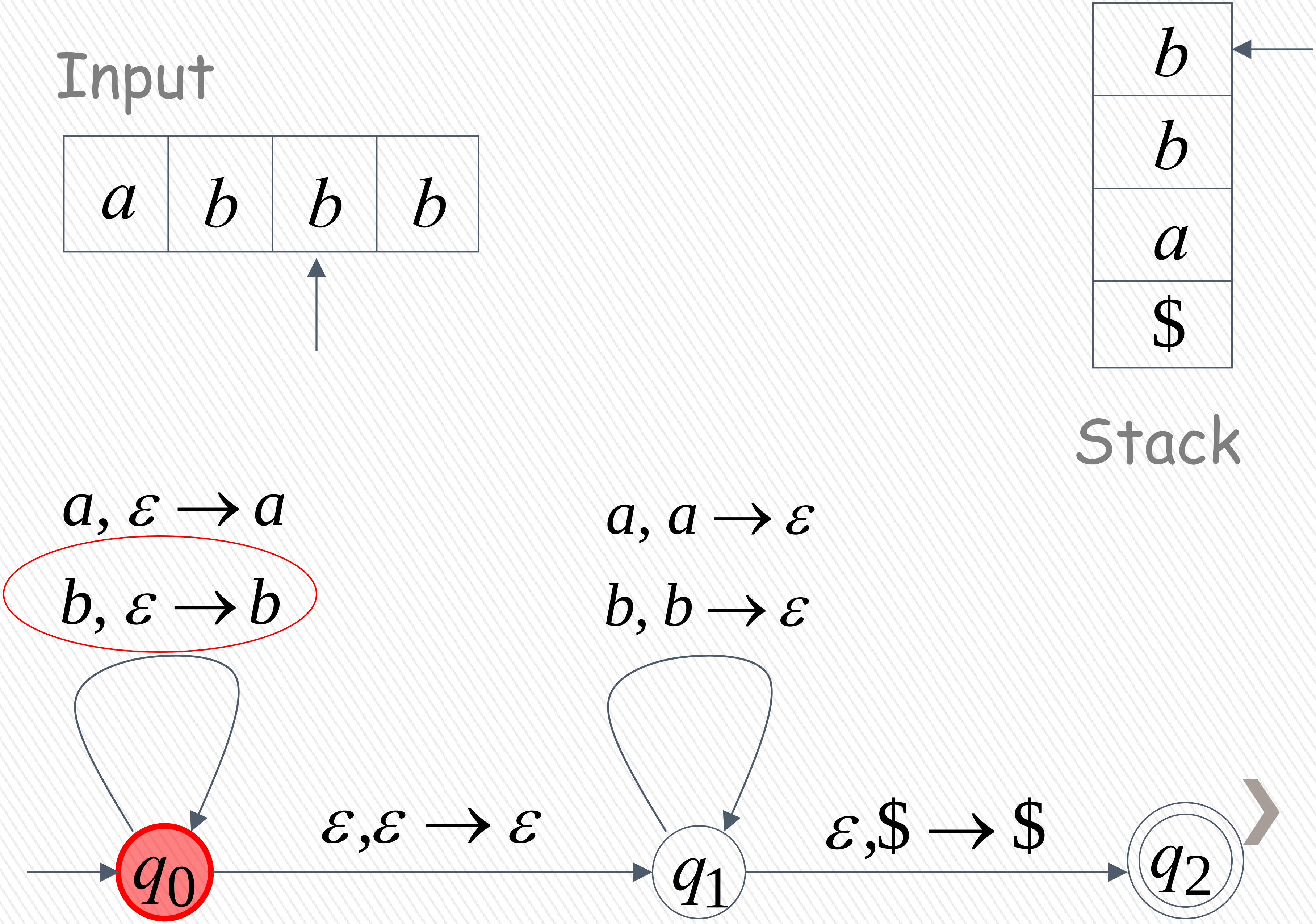
$\varepsilon, \$ \to \$$

q_0

q_1

q_2

Time 3



Input

a	b	b	b

Stack

b
b
a
$\$$

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

q_0

q_1

q_2

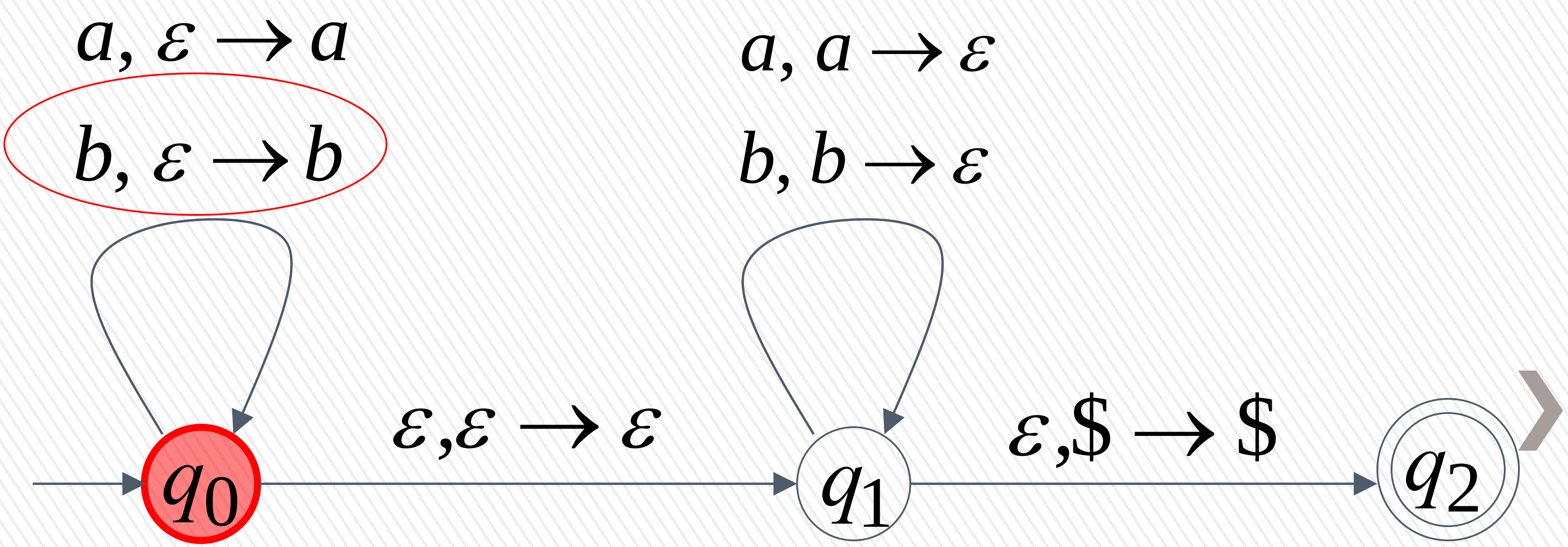
Time 4

Input

a	b	b	b

b
b
b
a
$\$$

Stack


$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

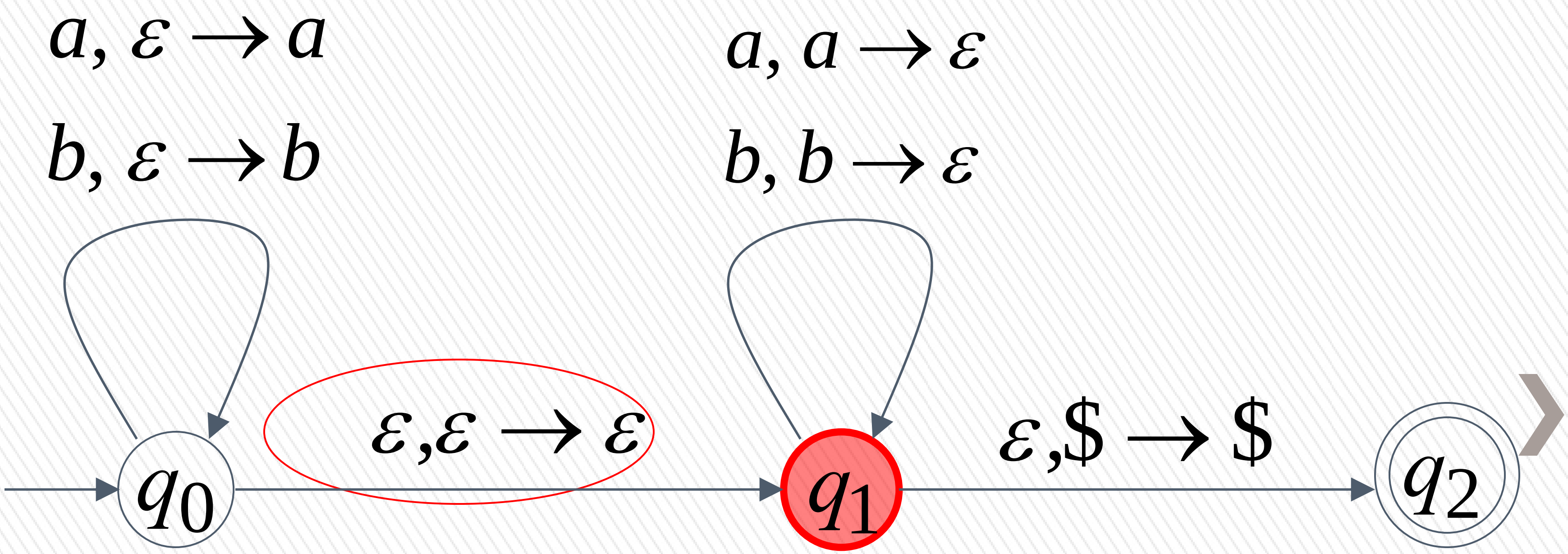
Input

a	b	b	b

No accept state is reached

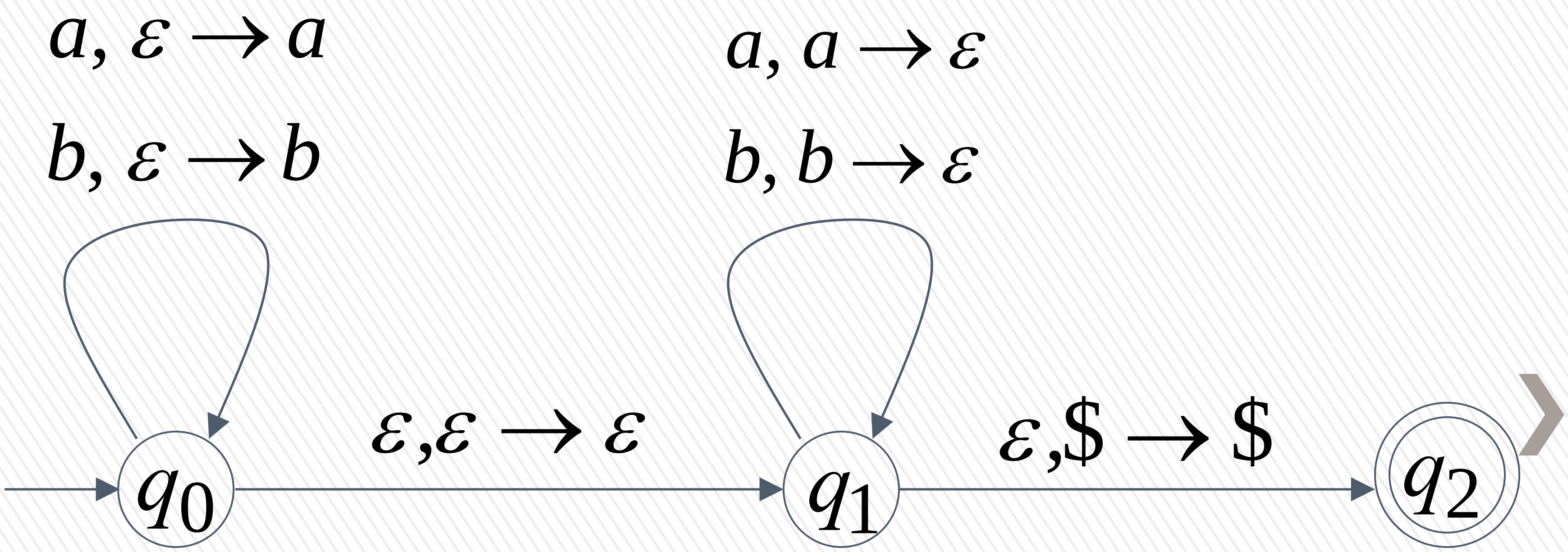
b
b
b
a
$\$$

Stack

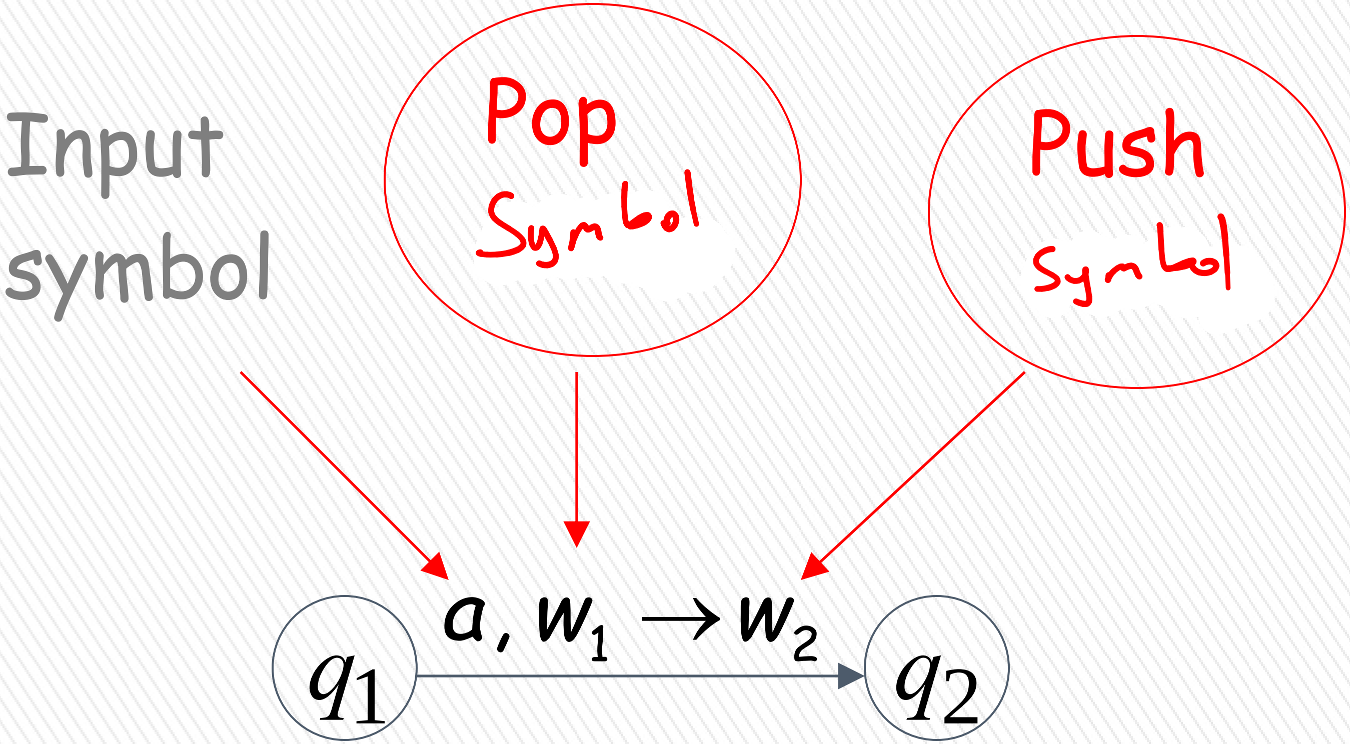

$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$
$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$\varepsilon, \$ \rightarrow \$$
q_0
q_1
q_2

There is no computation that accepts string $abbb$

$$abbb \notin L(M)$$



Pushing & Popping Strings



Input symbol

Pop Symbol

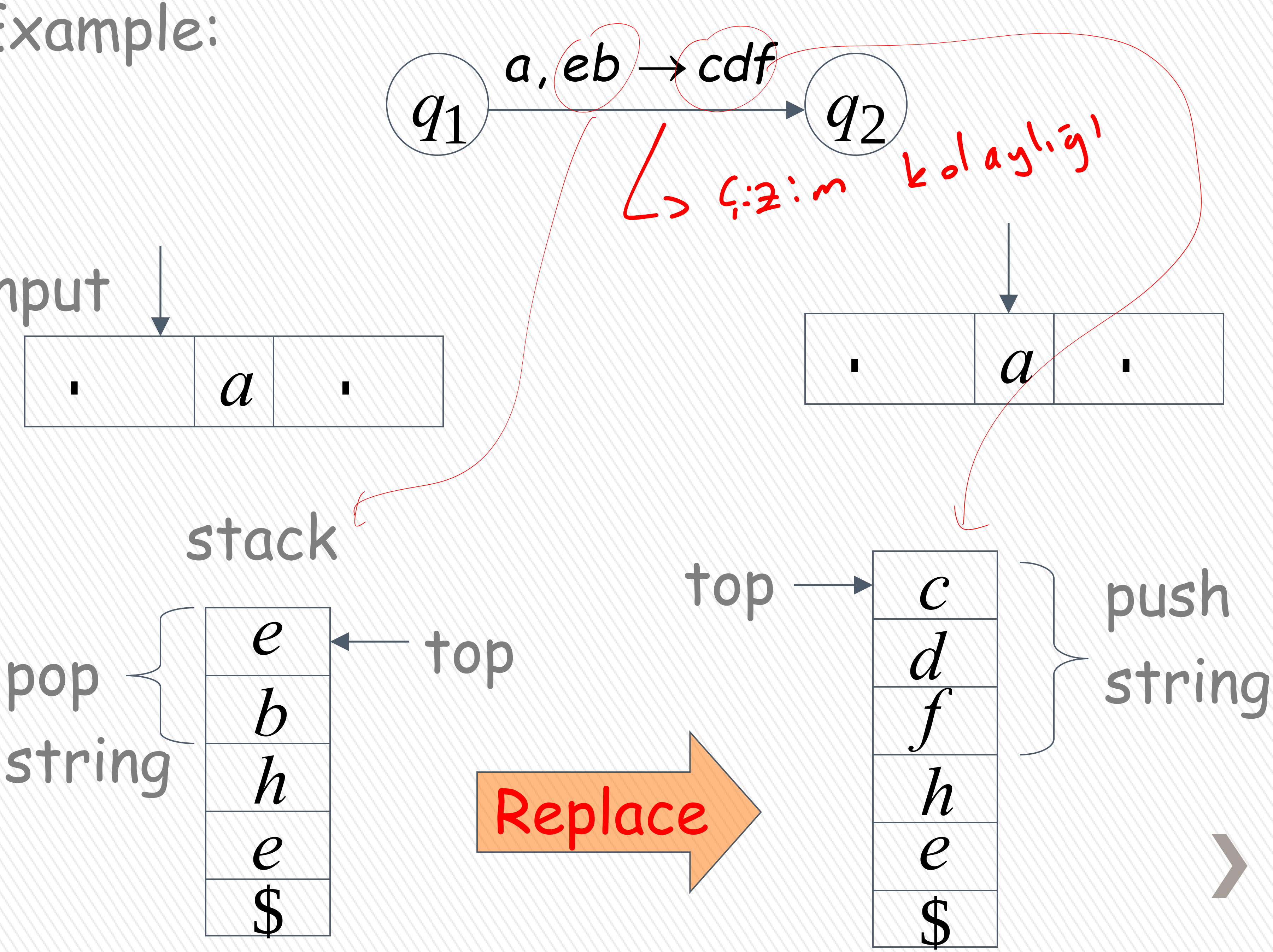
Push Symbol

$a, w_1 \to w_2$

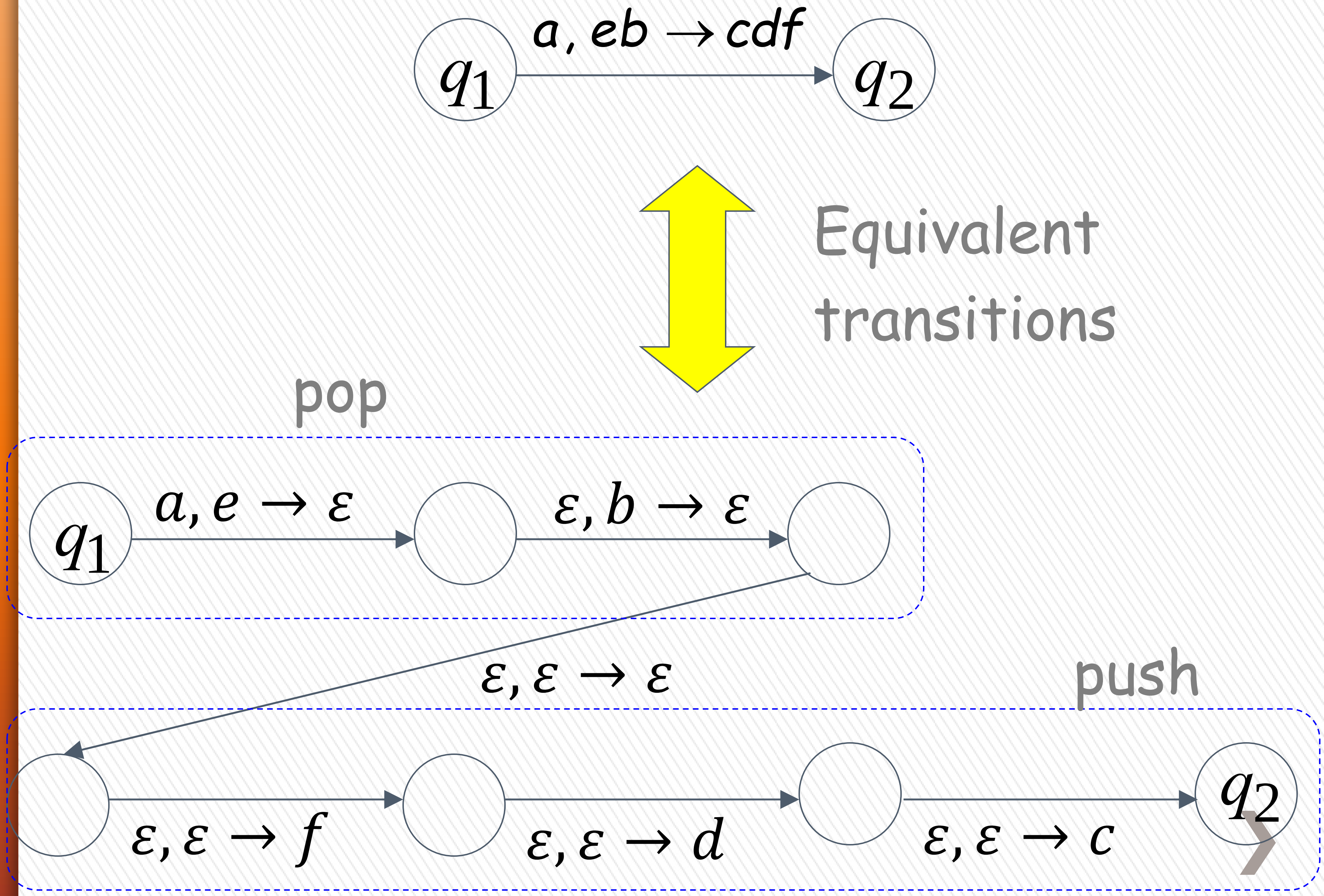
q_1

q_2

Example:



q_1
$a, eb \to cdf$
q_2
çizim kolaylığı
input
a
a
stack
pop
string
top
e
b
h
e
$
Replace
top
c
d
f
h
e
$
push
string



q_1 $\xrightarrow{a,\, eb \to cdf}$ q_2

Equivalent transitions

pop

q_1 $\xrightarrow{a,\, e \to \varepsilon}$ $\xrightarrow{\varepsilon,\, b \to \varepsilon}$

$\varepsilon, \varepsilon \to \varepsilon$

push

$\xrightarrow{\varepsilon,\, \varepsilon \to f}$ $\xrightarrow{\varepsilon,\, \varepsilon \to d}$ $\xrightarrow{\varepsilon,\, \varepsilon \to c}$ q_2

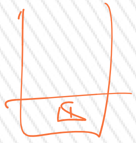
Another PDA example

a sayısı = b sayısı

$$L(M)=\{w\in\{a,b\}^{*}:\quad n_a(w)=n_b(w)\}$$

PDA M

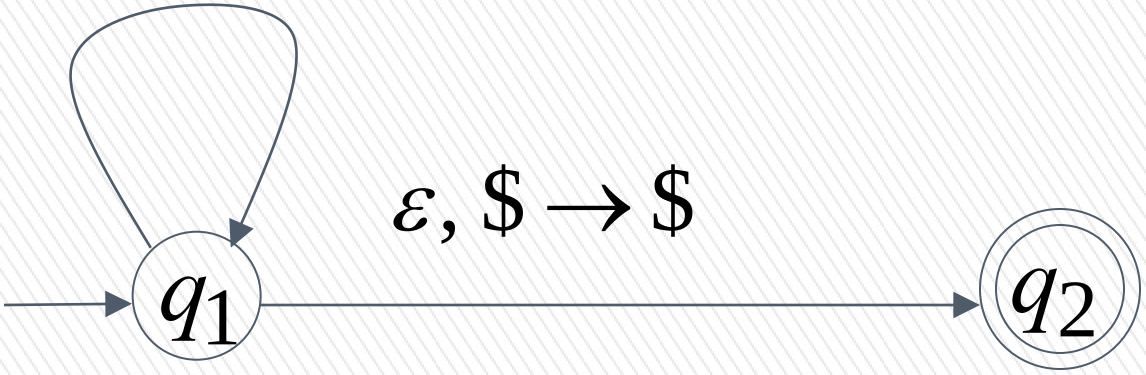
a a a a



$a,\$\rightarrow 0\$$ $\qquad$ $b,\$\rightarrow 1\$$

$a,0\rightarrow 00$ $\qquad$ $b,1\rightarrow 11$

$a,1\rightarrow \varepsilon$ $\qquad$ $b,0\rightarrow \varepsilon$


$\varepsilon,\$\rightarrow \$$

Execution Example: Time 0

Input

a	b	b	b	a	a

$\$$

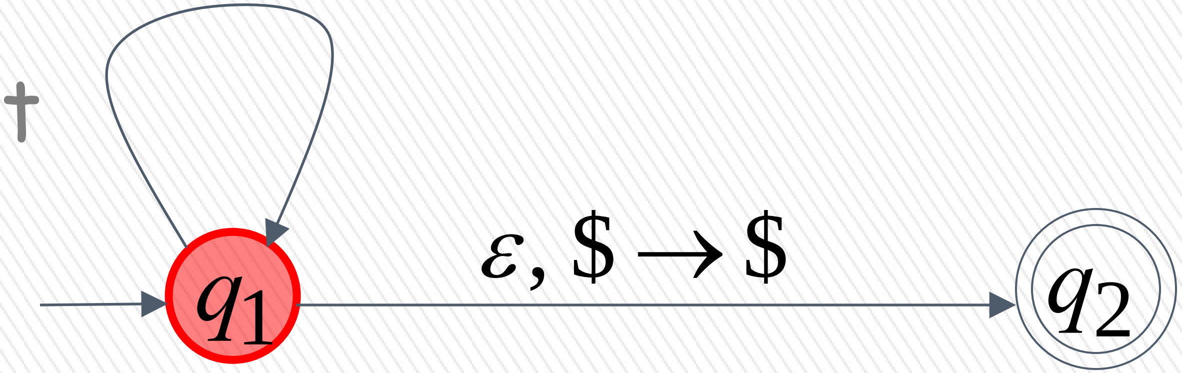
Stack

$a, \$ \rightarrow \cancel{0}a\$$ $\quad b, \$ \rightarrow \cancel{1}b\$$

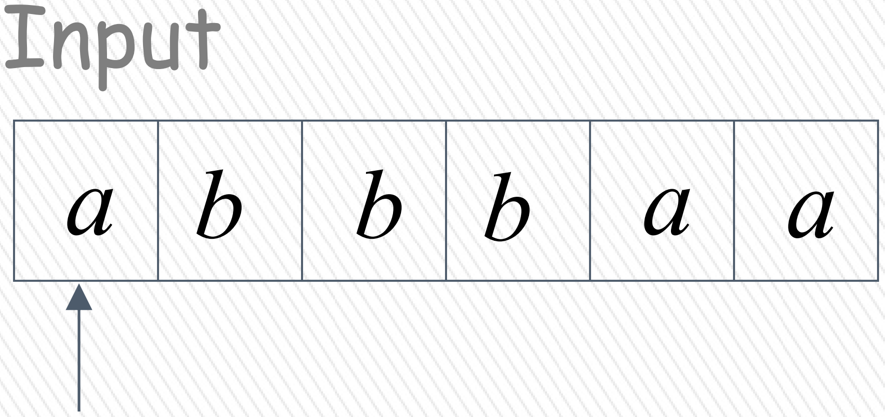
$a, \cancel{0}a \rightarrow \cancel{00}aa$ $\quad b, \cancel{1}b \rightarrow \cancel{11}bb$

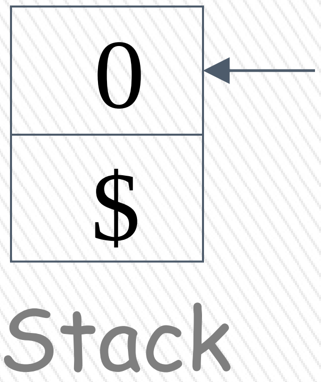
$a, \cancel{1}b \rightarrow \varepsilon$ $\quad b, \cancel{0}a \rightarrow \varepsilon$

current state



Time 1

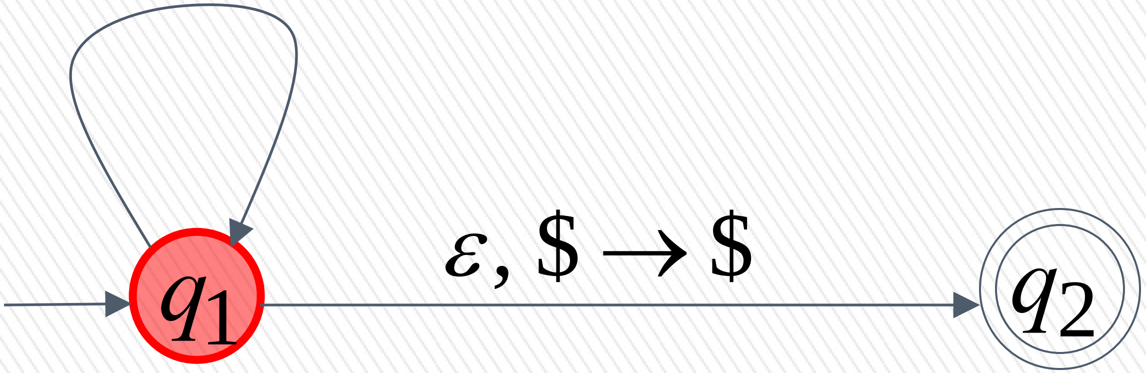




$$\boxed{a, \$ \to 0\$} \qquad b, \$ \to 1\$$$

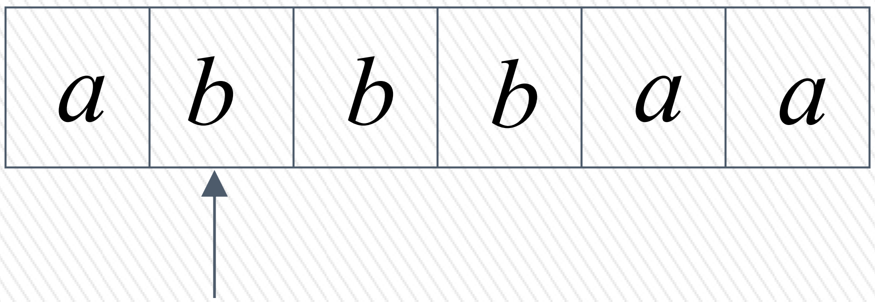
$$a, 0 \to 00 \qquad b, 1 \to 11$$

$$a, 1 \to \varepsilon \qquad b, 0 \to \varepsilon$$

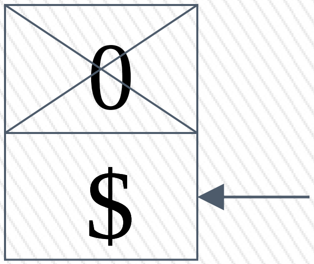


Time 3

Input



a	b	b	b	a	a



$\cancel{0}$

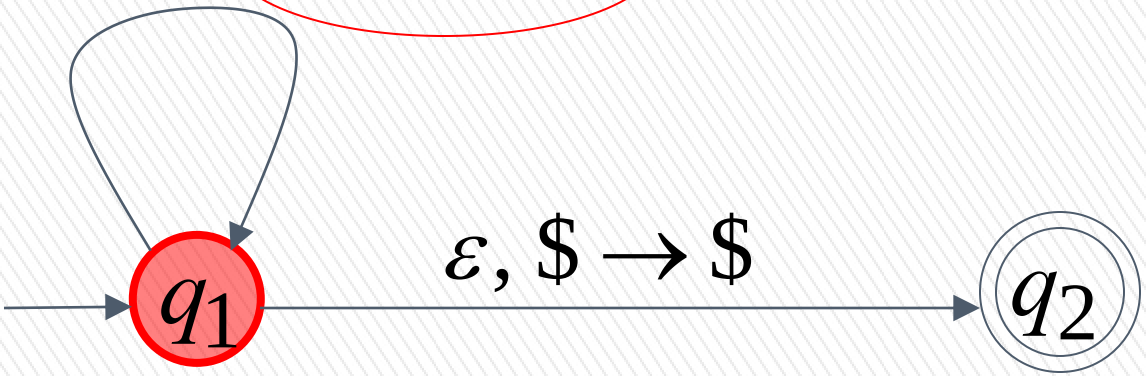
$\$$

Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a,1 \rightarrow \varepsilon$ $\quad$ $b,0 \rightarrow \varepsilon$



Time 4

Input

a	b	b	b	a	a

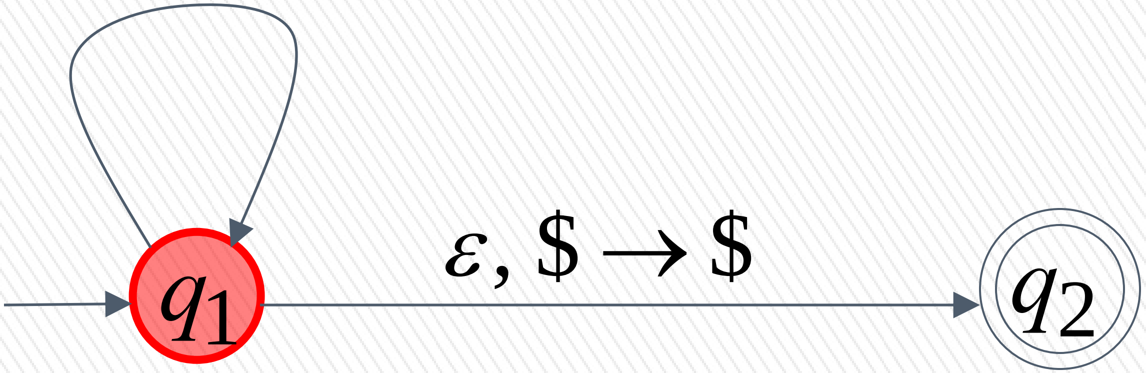
1
\$

Stack

$a, \$ \to 0\$$ $\quad$ $b, \$ \to 1\$$

$a, 0 \to 00$ $\quad$ $b, 1 \to 11$

$a, 1 \to \varepsilon$ $\quad$ $b, 0 \to \varepsilon$



$\varepsilon, \$ \to \$$

Time 5

Input

a	b	b	b	a	a

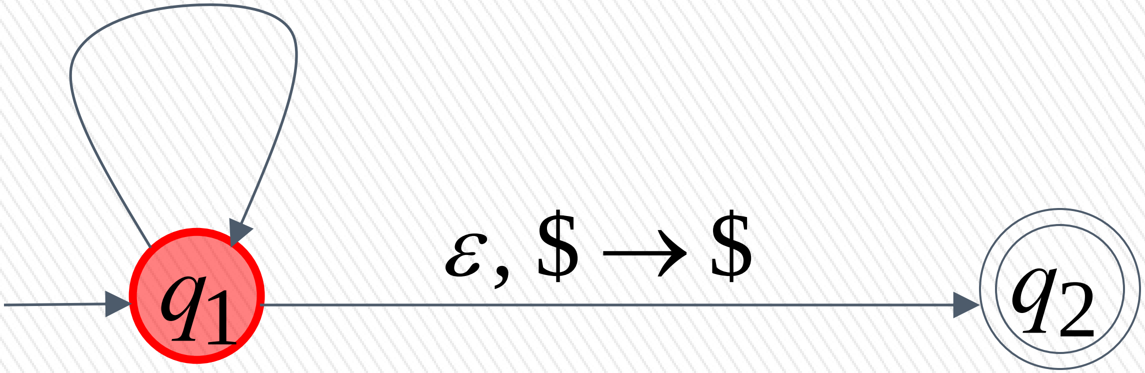
1
1
\$

Stack

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

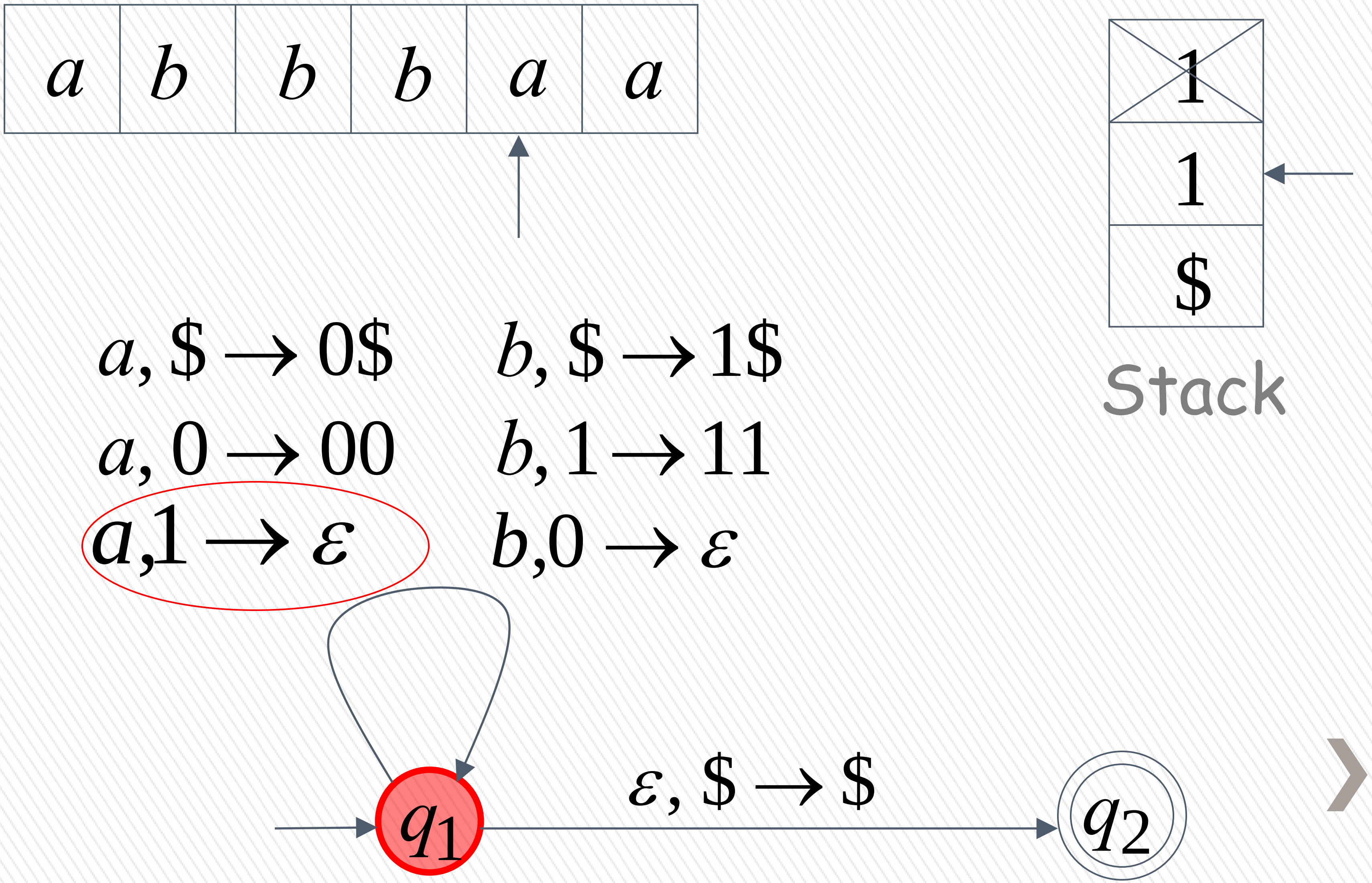
$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $b, 0 \rightarrow \varepsilon$

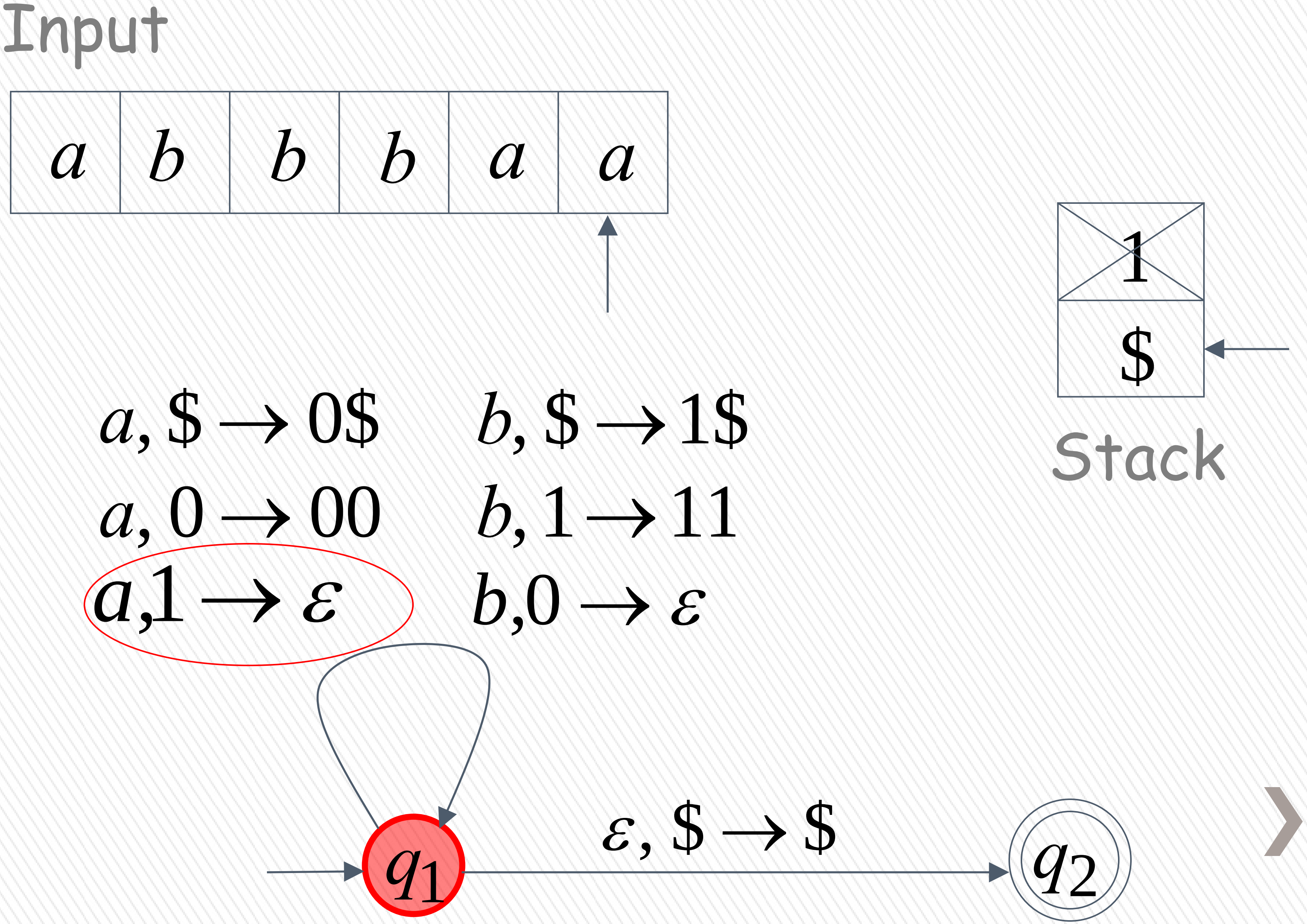


Time 6

Input


a
b
b
b
a
a
$\cancel{1}$
1
$\$$
Stack
$a, \$ \rightarrow 0\$$
$b, \$ \rightarrow 1\$$
$a, 0 \rightarrow 00$
$b, 1 \rightarrow 11$
$a, 1 \rightarrow \varepsilon$
$b, 0 \rightarrow \varepsilon$
q_1
$\varepsilon, \$ \rightarrow \$$
q_2

Time 7



Input

a	b	b	b	a	a

$\cancel{1}$
$\$$

Stack

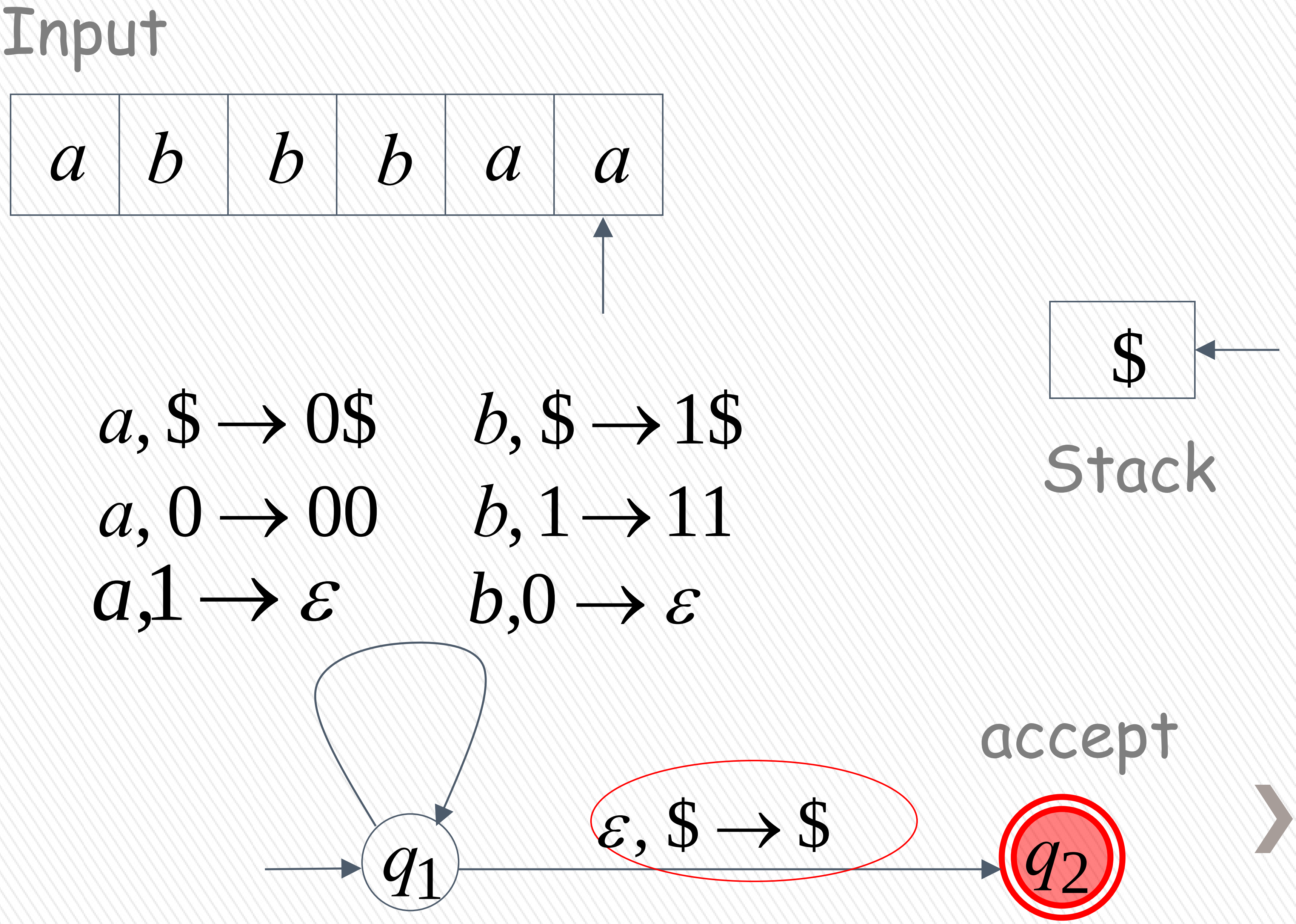
$a,\$ \rightarrow 0\$$ $\quad$ $b,\$ \rightarrow 1\$$

$a,0 \rightarrow 00$ $\quad$ $b,1 \rightarrow 11$

$a,1 \rightarrow \varepsilon$ $\quad$ $b,0 \rightarrow \varepsilon$

$\varepsilon,\$ \rightarrow \$$

Time 8



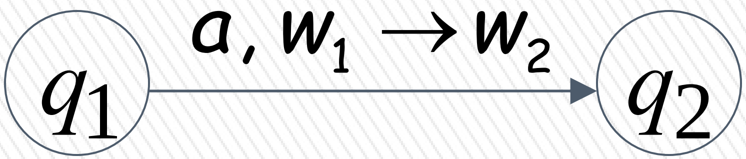
$a, \$ \to 0\$$ $\quad$ $b, \$ \to 1\$$

$a, 0 \to 00$ $\quad$ $b, 1 \to 11$

$a, 1 \to \varepsilon$ $\quad$ $b, 0 \to \varepsilon$

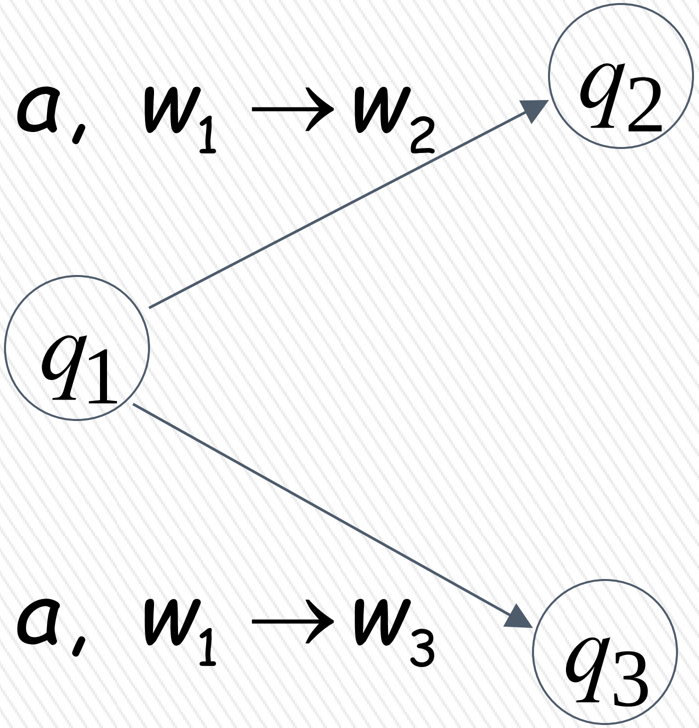
$\varepsilon, \$ \to \$$

Formalities for PDAs



Transition function:

$$\delta(\overset{\text{state \&}}{q_1}, \underset{\text{input}}{a}, \overset{\text{push-pop}}{w_1}) = \{(\overset{\text{state}}{q_2}, \overset{\text{push-pop}}{w_2})\}$$

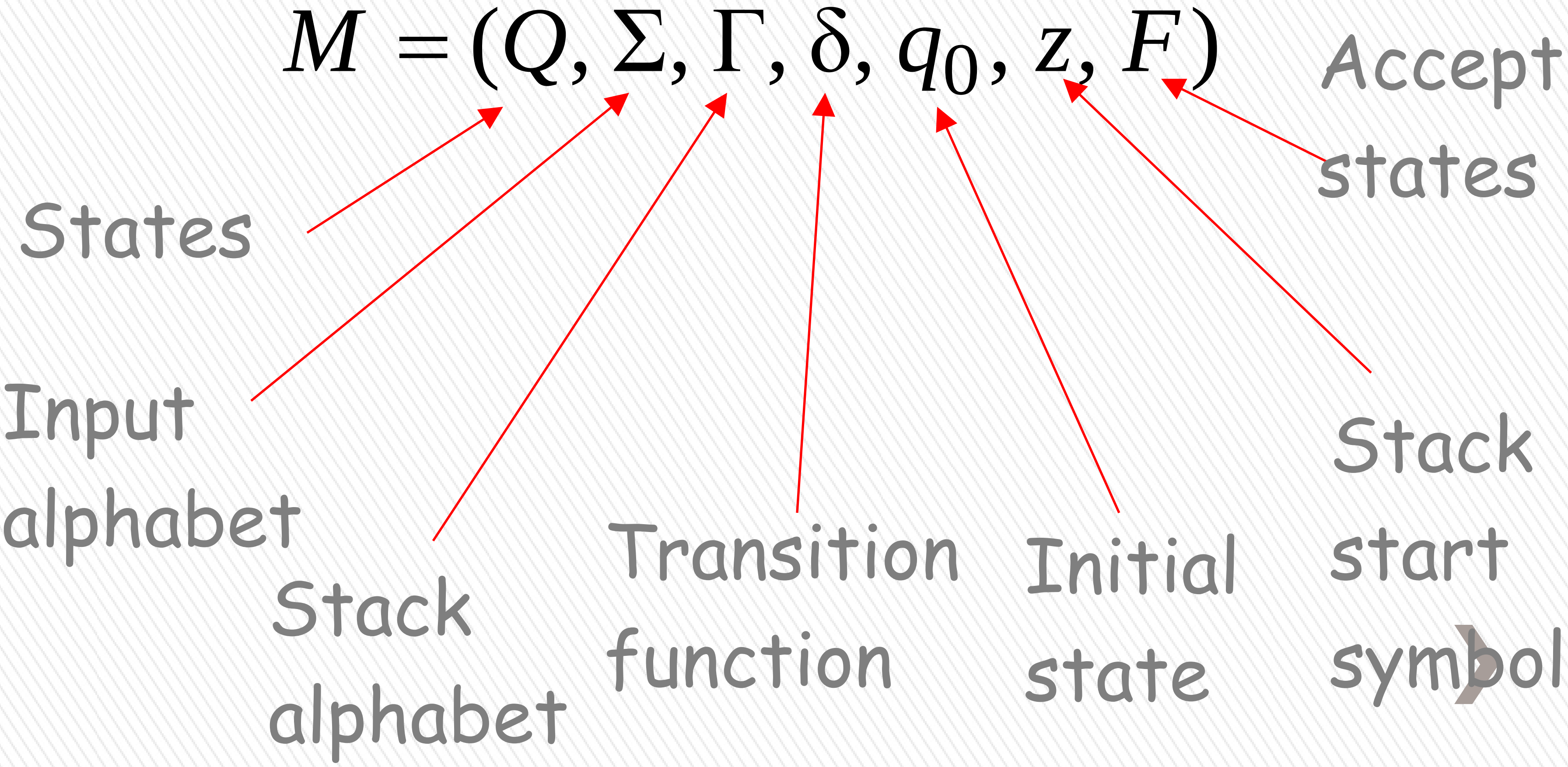


Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2), \ (q_3, w_3)\}$$

Formal Definition

Pushdown Automaton (PDA)



$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

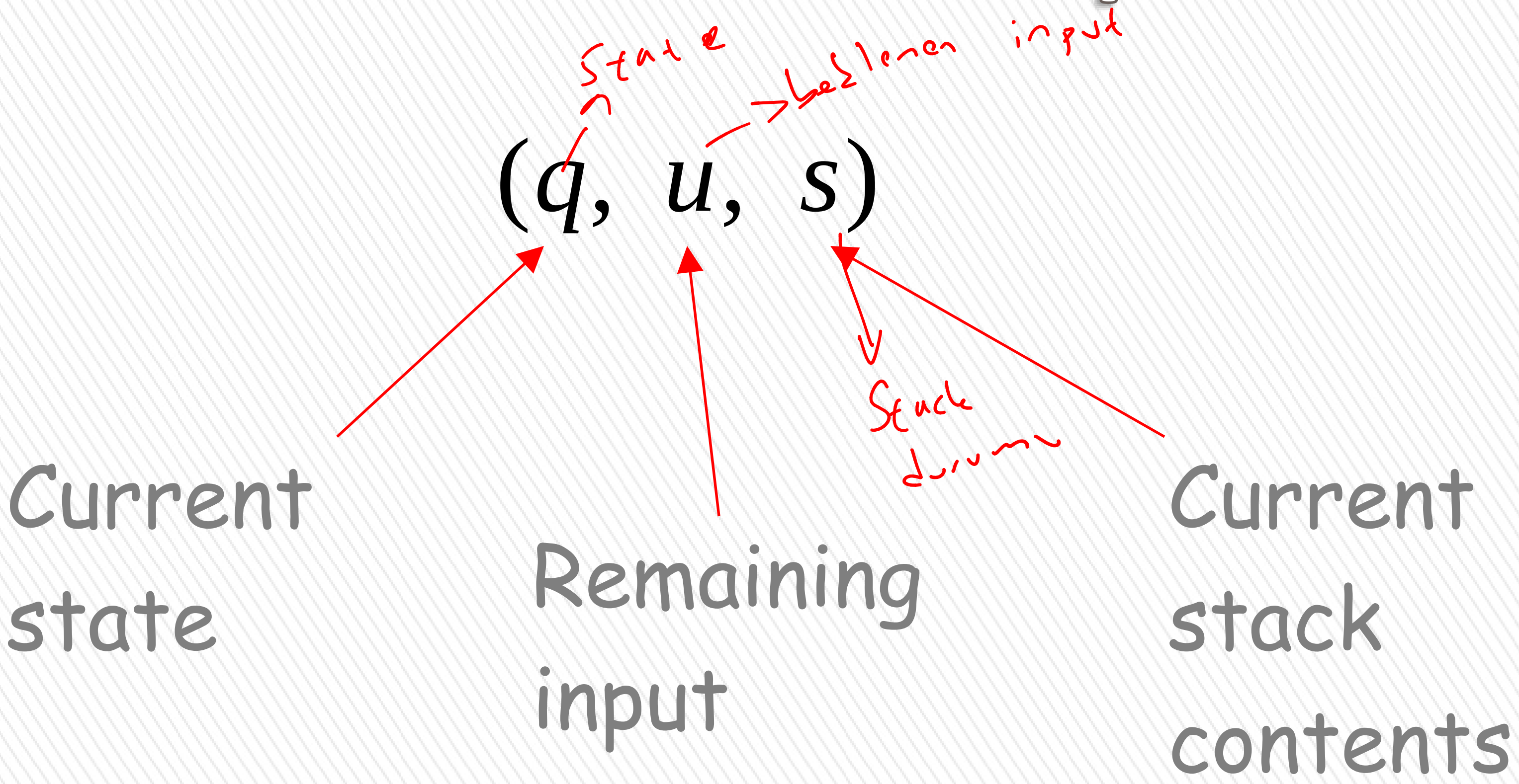
Transition function

Initial state

Stack start symbol

Accept states

Instantaneous Description



State

beklenen input

$$(q,\ u,\ s)$$

Stack durumu

Current state

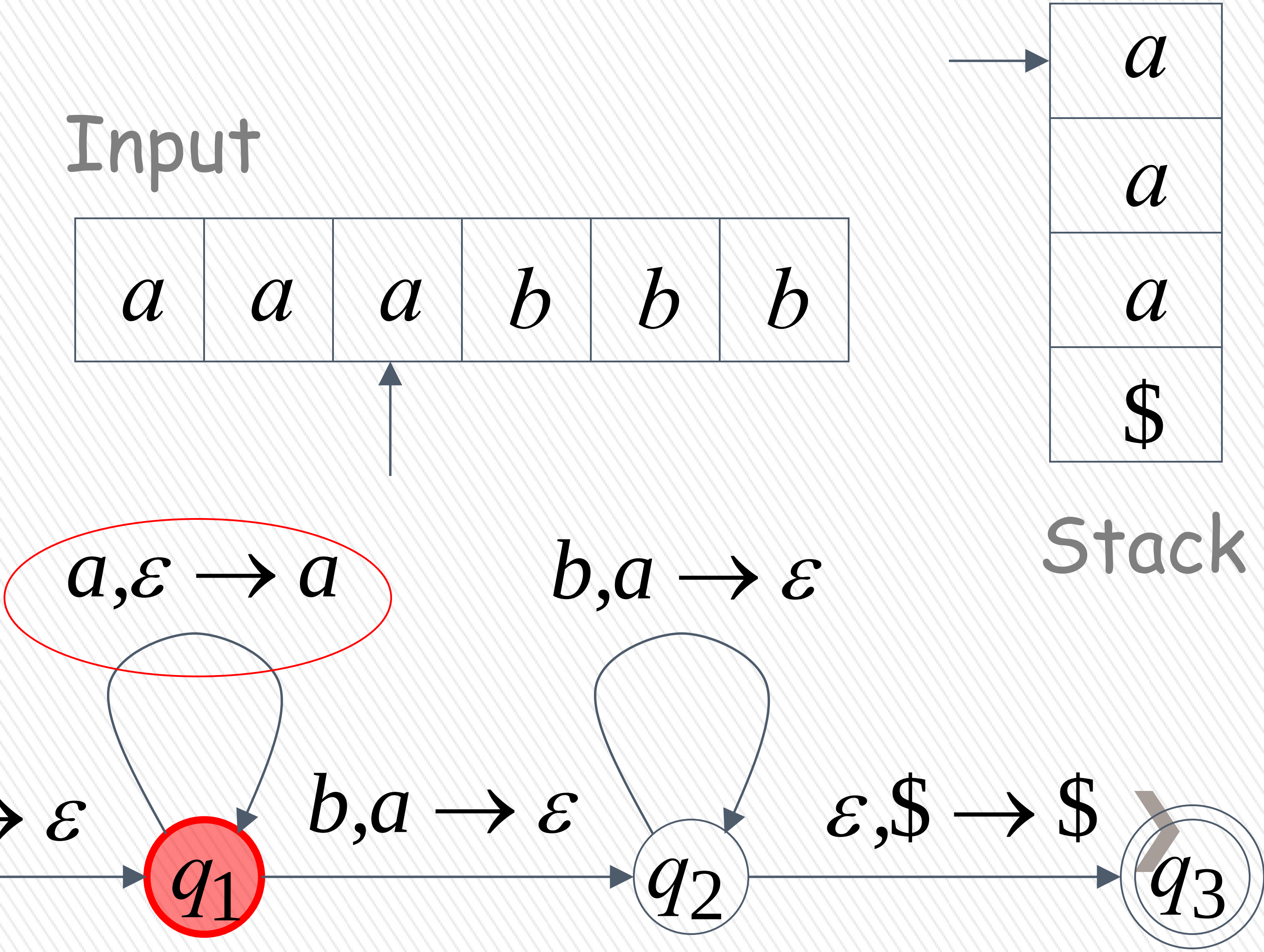
Remaining input

Current stack contents

Example: **Instantaneous Description**

$$(q_1, bbb, aaa\$)$$

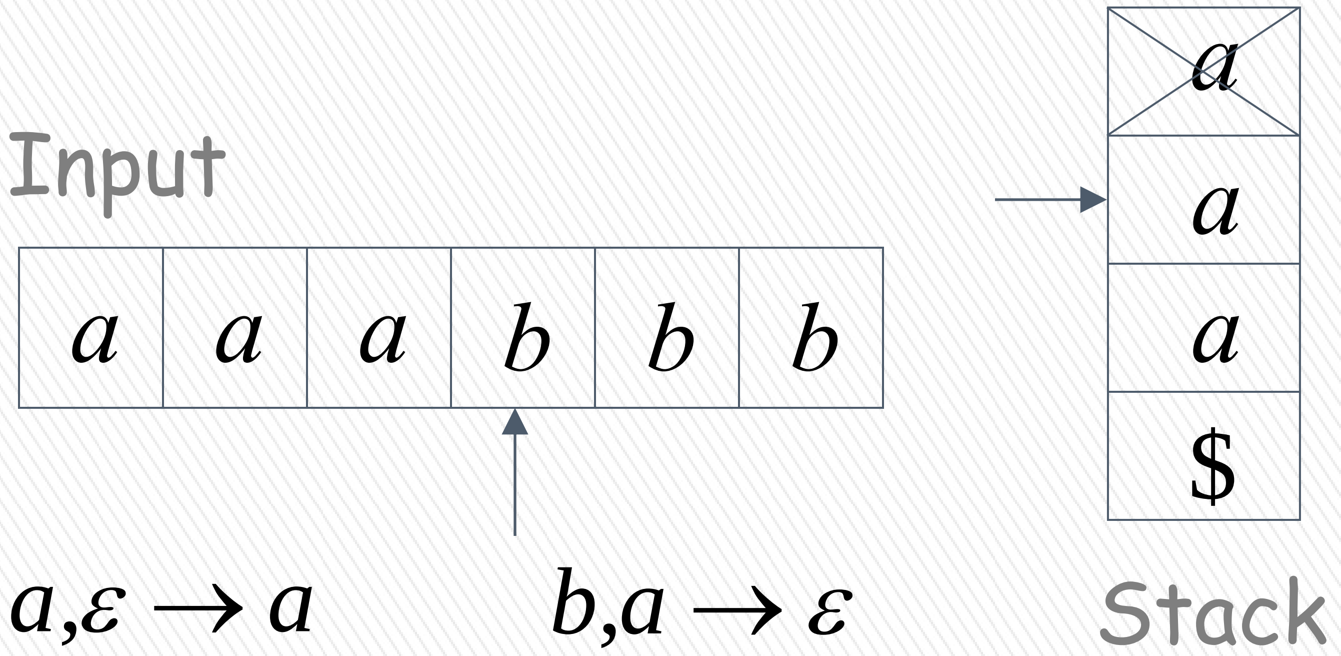
Time 4:

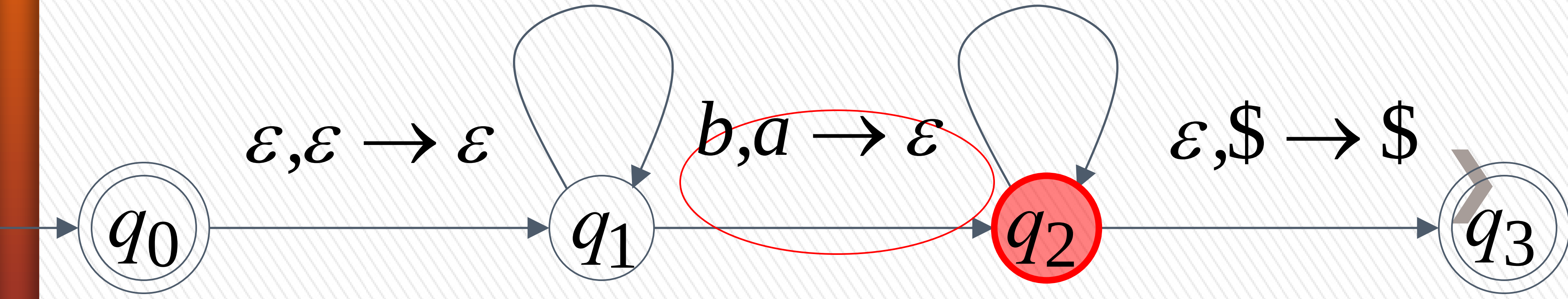

Input
a a a b b b
a
a
a
$
Stack
$a,\varepsilon \rightarrow a$
$b,a \rightarrow \varepsilon$
$\varepsilon,\varepsilon \rightarrow \varepsilon$
$b,a \rightarrow \varepsilon$
$\varepsilon,\$ \rightarrow \$$
q_0
q_1
q_2
q_3

Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

Time 5:




$\varepsilon,\varepsilon \rightarrow \varepsilon$
$b,a \rightarrow \varepsilon$
$\varepsilon,\$ \rightarrow \$$
q_0
q_1
q_2
q_3

We write:

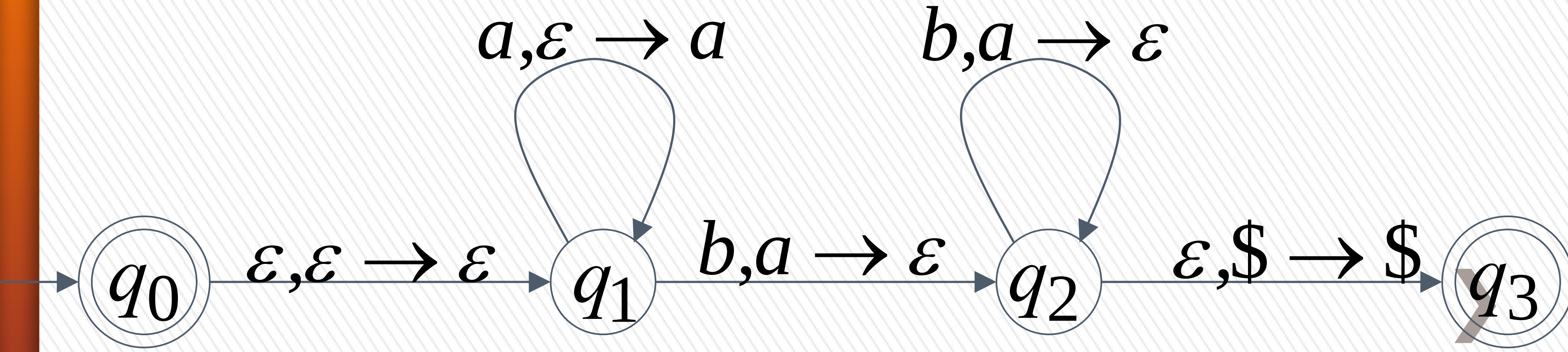
$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4

Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

Language of PDA → CFL'ler inki çizilir

Language $L(M)$ accepted by PDA M :

$$L(M)=\{w: \quad (q_0, w, z) \overset{*}{\succ} (q_f, \varepsilon, s)\}$$

Initial state

Accept state

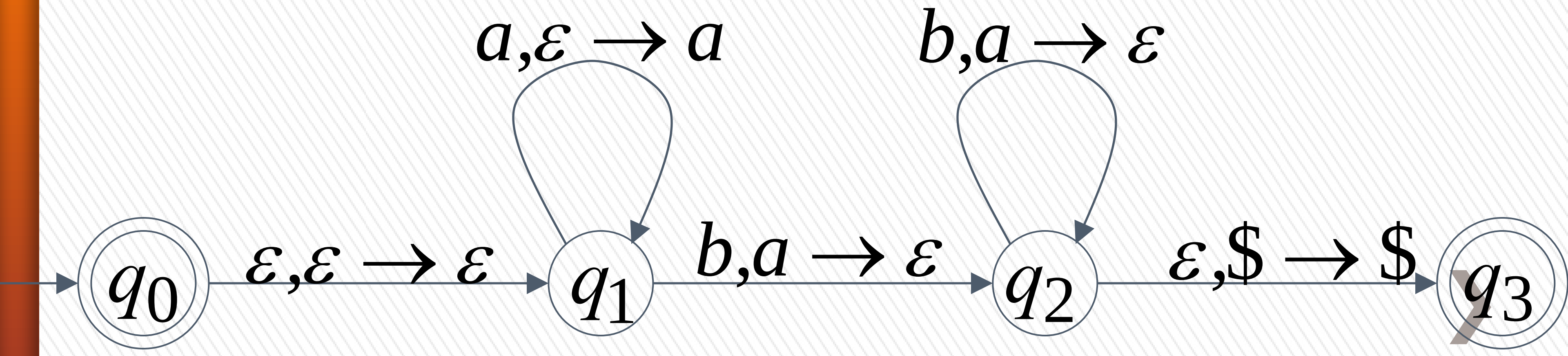
❯

Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

$$aaabbb \in L(M)$$

PDA **M** :

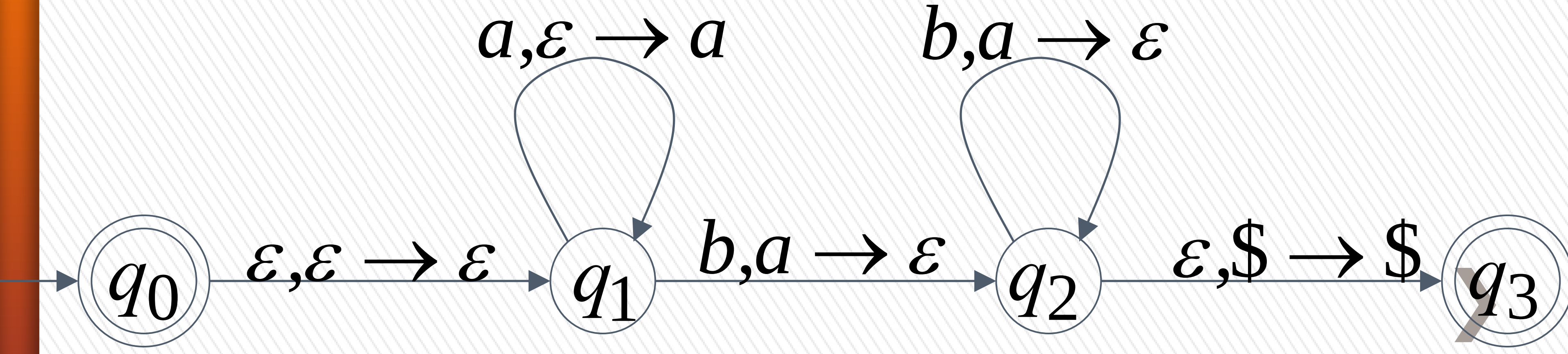


$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

$$\Downarrow$$

$$a^n b^n \in L(M)$$

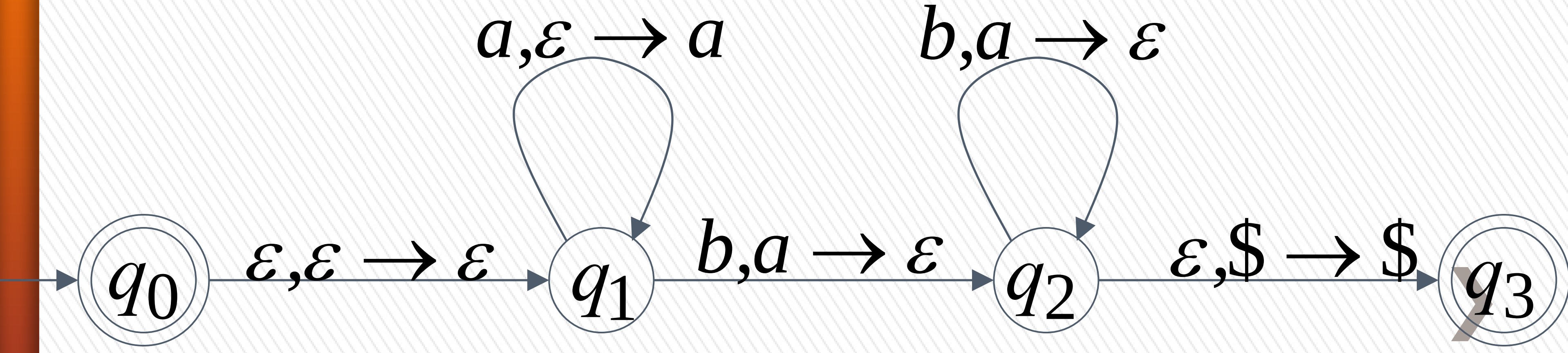
PDA M :



Therefore:

$$L(M) = \{a^n b^n : n \geq 0\}$$

PDA M :


$a, \varepsilon \rightarrow a$
$b, a \rightarrow \varepsilon$
q_0
$\varepsilon, \varepsilon \rightarrow \varepsilon$
q_1
$b, a \rightarrow \varepsilon$
q_2
$\varepsilon, \$ \rightarrow \$$
q_3