ESN Jacobian for Lyapunov Exponent Calculation

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The post training autonomous reservoir evolution equation is,

$$r(t + \Delta t) = \tanh\left[Ar(t) + W_{in}\left(W_{out,1}r(t) + W_{out,2}r^2(t)\right)\right] \tag{1}$$

In code, we have a single W_{out} matrix which acts on $r^*(t)$ where $r^*(t)$ is the vector $r(t)$ with elements at even indices squared. For calculating the Jacobian we separate out the W_{out} into $W_{out,1}$ and $W_{out,2}$ which act on odd and even indices respectively (with zeros at the even and odd indexed columns respectively). This procedure is descirbed in the Lyapunov exponent paper [1] page 5.

Now we calculate the tangent map, or the variational equation that gives the evolution

$$\delta r(t + \Delta t) = F\left[r(t), \delta r(t)\right] \tag{2}$$

This looks like follows:

$$\begin{aligned}
r(t + \Delta t) &= \tanh\left[Ar(t) + W_{in}\left(W_{out,1}r(t) + W_{out,2}r^2(t)\right)\right] \qquad (3)\\
\delta r(t + \Delta t) &= \delta\{\tanh\left[Ar(t) + W_{in}\left(W_{out,1}r(t) + W_{out,2}r^2(t)\right)\right]\} \qquad (4)\\
&= \left[1 - r^2(t + \Delta t)\right]\left[\left(A + W_{in}W_{out,1}\right)\delta r + \left(2W_{in}W_{out,2}r(t)\right)\delta r(t)\right] \qquad (5)
\end{aligned}$$

1 Refrences

1. Chaos 27, 121102 (2017); https://doi.org/10.1063/1.5010300

The reservoir network is as described in Sec. II with the parameters listed in Table III. In the training phase, Fig. 1(a), we evolve the reservoir according to Eq. (1) from $t = -T$ to $t = 0$. Next, we use Tikhonov regularized regression [see Eq. (3)] to compute the output parameters, $\mathbf{P}$ such that $\mathbf{W}_{out}(\mathbf{r}, \mathbf{P}) = \mathbf{P}\tilde{\mathbf{r}}(t) \simeq \mathbf{u}(t)$ for $-T \leq t < 0$. Here, $\tilde{\mathbf{r}}$ is a D_r-dimensional vector such that the i^{th} component of $\tilde{\mathbf{r}}$ is $\tilde{r}_i = r_i$ for half the reservoir nodes and $\tilde{r}_i = r_i^2$ for the remaining half. With the output parameters determined, we let the reservoir evolve autonomously for $t > 0$ as shown in Fig. 1(b) according to Eq. (4).

Figure 1: Page 5 of Ref [1]