MA 279 Final Exam

The equations of motion for the flipper locomotor (See Section 4.1.6 of the posted book, Chapter 4) is

$$\begin{aligned} &J\ddot{\theta} + D\dot{\theta} + v\Lambda\theta = Bu, \\ &m\dot{v} + d(\theta)v + \dot{\theta}^{\mathsf{T}}\Lambda\theta = 0. \end{aligned} \tag{1}$$

where $\theta(t) \in \mathbb{R}^n$ are the tail joint angles, $v(t) \in \mathbb{R}$ is the velocity of the center of mass, and

$$\begin{aligned} &J := LML/3 + F^{\mathsf{T}}MF, \quad D := LC_nL/3 + F^{\mathsf{T}}C_nF, \\ &\Lambda := F^{\mathsf{T}}C_o + \mathrm{diag}((F^{\mathsf{T}} - eh^{\mathsf{T}}/m)C_te), \quad F := (B^{\mathsf{T}})^{-1}AL, \\ &d(\theta) := c_{t_o} + e^{\mathsf{T}}C_te + \theta^{\mathsf{T}}C_o\theta, \quad C_o := C_n - C_t, \quad h := F^{\mathsf{T}}Me, \\ &C_t = \mathrm{diag}(c_{t_1}, \ldots, c_{t_n}), \quad C_n = \mathrm{diag}(c_{n_1}, \ldots, c_{n_n}), \\ &L = \mathrm{diag}(\ell_1, \ldots, \ell_n), \quad M = \mathrm{diag}(m_1, \ldots, m_n), \end{aligned}$$

$$B = \begin{bmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

The flipper body is formed by a head link and n tail links, with the parameter values given by

$$\begin{aligned} &n = 5, \quad m = 0.001 \text{ kg}, \quad \ell = 0.1 \text{ m}, \\ &c_{n_i} = 0.009 \text{ N/(m/s)}, \quad c_{t_i} = 0.0006 \text{ N/(m/s)}, \quad c_{t_o} = 0.0006 \text{ N/(m/s)}, \\ &\ell_i = \ell/(2n+2), \quad m_i = m/(n+1), \quad \ell_o = \ell/(2n+2), \quad m_o = m/(n+1), \end{aligned}$$

for $i = 1, \ldots, n$.

For the power optimal gait with the average velocity $v_o = -0.15$ m/s, your task is to design a distributed feedback controller that achieves the optimal gait for the closed-loop system. For each of the following questions, explain the design process and write down the result (especially the part underlined), with the information enough to reproduce your result.

(a) Define the linear time-invariant plant by the first equation in (1) with constant $v(t) \equiv v_o$. Let this plant be described by

$$\dot{\mathfrak{x}} = \mathfrak{A}\mathfrak{x} + \mathfrak{B}u, \quad y = \mathfrak{C}\mathfrak{x},$$

where the output and state vectors are $y := \phi$ and $\mathfrak{x} := \mathrm{col}(\phi, \dot{\phi})$, and $\phi := B^{\mathsf{T}}\theta$ is the vector of relative joint angles.[1] Find <u>analytical expressions</u> for the state space matrices $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ in terms of the given parameters. <u>By a permutation of the state variables, convert the system into another state space realization</u>

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \quad y = \mathcal{C}x,$$

where the state vector is $x := \mathrm{col}(\zeta_1, \ldots, \zeta_n)$, with $\zeta_i := \mathrm{col}(\phi_i, \dot{\phi}_i)$, and numerically verify that your $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ matrices match with (Ap,Bp,Cp) in the data file finaldata.mat posted on CCLE. <u>State</u> whether your calculation verified this or not.

[1] The notation $\mathrm{col}(a, b, c)$ means that it is the matrix or vector obtained by stacking a, b, and c in a column.

(b) Suppose the target optimal gait is given by $\phi(t) = \Im[\hat{\phi}e^{j\omega t}]$. The state trajectory corresponding to this gait can be expressed as $x(t) = \mathfrak{X}e^{\Omega t}\eta$, and the control input that generates this gait is given by $u(t) = \mathfrak{U}e^{\Omega t}\eta$. Find analytical expressions for $\mathfrak{X}$, $\mathfrak{U}$, Ω, and η in terms of $\hat{\phi}$, ω, v_o, and the plant parameters in (1). The data (phih,w) in finaldata.mat gives the optimal gait $(\hat{\phi}, \omega)$. For this data, numerically verify that the regulator equation

$$\mathcal{A}\mathfrak{X} + \mathcal{B}\mathfrak{U} = \mathfrak{X}\Omega$$

is satisfied. State whether your calculation verified this or not.

(c) Design a network of coupled n Andronov-Hopf oscillators with states $\xi_i(t) \in \mathbb{R}^2$ $(i = 1, 2, \ldots, n)$ so that $\xi_i = h_i$ with

$$h_i(t) = \begin{bmatrix} \sin(\omega t + \beta_i) \\ \cos(\omega t + \beta_i) \end{bmatrix}, \qquad i = 1, 2, \ldots, n$$

is an orbitally stable limit cycle, where β_i are the phase angles of the optimal gait, i.e., $\beta_i = \angle\hat{\phi}_i$. The network should be a chain of oscillators with nearest neighbor coupling, i.e., the i^{th} oscillator is allowed to directly communicate only with the $(i+1)^{\text{th}}$ and $(i-1)^{\text{th}}$ oscillators. Explain the design process and write down the equations and parameter values to describe the network oscillator you designed, with the information enough to reproduce your result.

Simulate the network with initial state $\xi_1(0) = [1; 1]$ and $\xi_i(0) = 0$ for $i = 2, \ldots, n$. Give a plot of $\xi_{i1}(t)$ (the first entry of $\xi_i(t) \in \mathbb{R}^2$) for $i = 1, \ldots, n$ and another plot of $\xi_{n1}(t)$ and $\xi_{n2}(t)$. The response should converge in roughly 2 s or less and the plots should be shown for $0 \le t \le 2$. An example is given in Fig. 1.

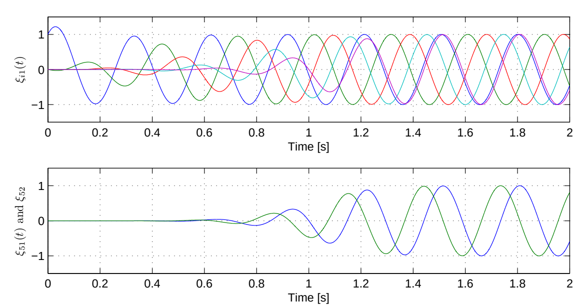

Figure 1: Coupled Andronov-Hopf oscillators

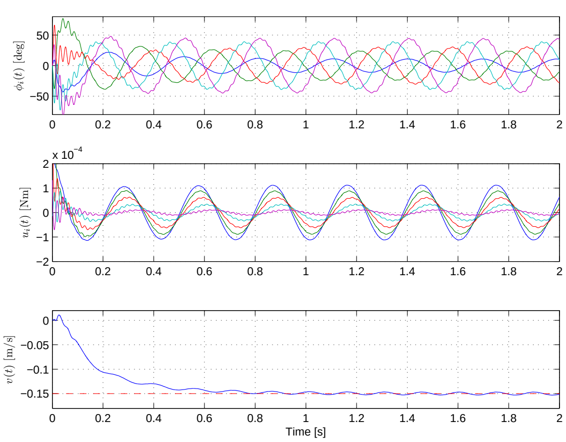

Figure 2: Output Regulator

(d) Design a decentralized output regulator of the form

$$u = U\xi + Q_1(y - \mathcal{C}X\xi), \qquad \dot{\xi} = \varphi(\xi),$$

to achieve convergence to the optimal gait $x(t) \to \mathfrak{X}e^{\Omega t}\eta$, where $\dot{\xi} = \varphi(\xi)$ is the network oscillator you designed in part (c). The controller structure should be decentralized in the sense that u_i depends only on y_i and ξ_i. With the stable limit cycle $h(t) \in \mathbb{R}^{2n}$ of the network, (X, U) should be chosen so that $\mathfrak{X} = XH$ and $\mathfrak{U} = UH$ are satisfied, where H is the matrix such that

$h = He^{\Omega t}\eta$ for some $\eta \in \mathbb{R}^2$. The gain Q_1 can be chosen to be a constant matrix representing the joint stiffness of the flipper. Describe the process for designing (X, U) and Q_1, and give the equations for these parameters with numerical data enough to specify their values.

Set the initial condition $\phi(0) = 0$, $\dot{\phi}(0) = 0$, and $v(0) = 0$, and simulate the closed-loop system containing the nonlinear plant dynamics with velocity $v(t)$ as one of the variables. In the simulation, add the sensor noise $\varrho(t) := 0.05\sin(300t)$ by modifying the output as $y = (1+\varrho)\phi$. Adjust the initial state $\xi(0)$ for the controller to achieve fast convergence to the steady oscillation near the optimal gait with average velocity $v(t) \cong v_o$. The sensor noise should be sufficiently attenuated and barely visible in the steady state. Explain how you choose $\xi(0)$. Show the plots of $\phi_i(t)$, $u_i(t)$, and $v(t)$. An example of a successful design is shown in Fig. 2.

(e) To the controller you designed in part (d), add a sensory feedback for the network oscillator with gain Q_2 so that the controller becomes

$$\begin{bmatrix} u \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} U\xi \\ \psi(\xi) \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} (y - \mathcal{C}X\xi).$$

Maintain the distributed architecture for the controller by choosing a constant gain Q_2 so that $\dot{\xi}_i$ is directly determined by y_i and ξ_i. The controller should improve the one in part (d) in terms of the recovery performance after a disturbance in the following sense.

Suppose the disturbance torque applied to the tail is modeled as

$$u(t) = d\Big(U\xi + Q_1(y - \mathcal{C}X\xi)\Big), \qquad d = \begin{cases} 20, & (2 < t < 2.1) \\ 1, & \text{(otherwise)} \end{cases}$$

where the constant $d \in \mathbb{R}$ represents the disturbance when its value is away from 1. Without sensory feedback ($Q_2 = 0$), the controller in part (d) may poorly react to the disturbance (Fig. 3), but with sensory feedback ($Q_2 \neq 0$), the transient can be improved (Fig. 4).

Explain how you design Q_2 and show its numerical value. Simulate the closed-loop system with the above disturbance (without the sensor noise $\varrho(t) \equiv 0$) for each of your two controllers with/without Q_2, starting from the same initial condition as the one in part (d). Show the plots of $\phi_i(t)$, $v(t)$, and undisturbed $u_i(t)$ (i.e., $u(t)$ with $d = 1$), following the format in Figs. 3 and 4.

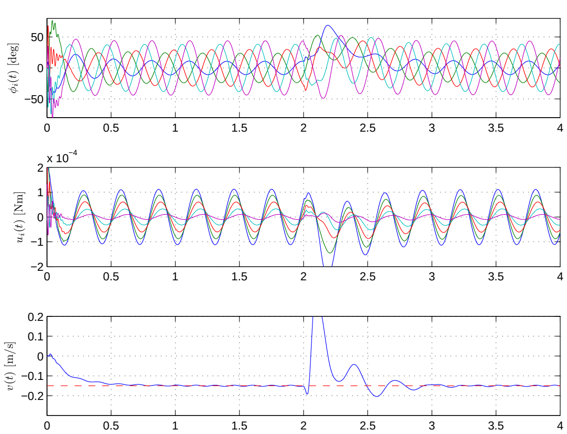

Figure 3: Output Regulator under disturbance

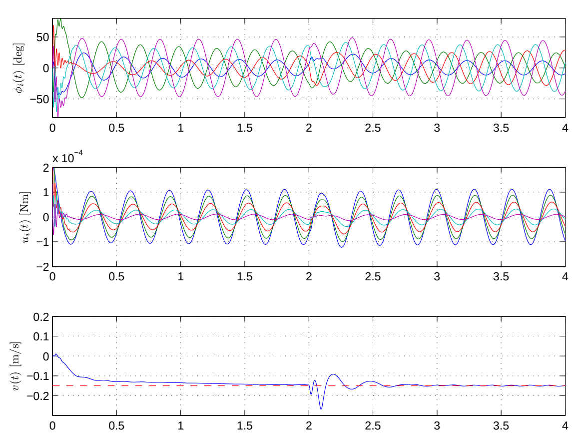

Figure 4: Output Regulator with Feedback