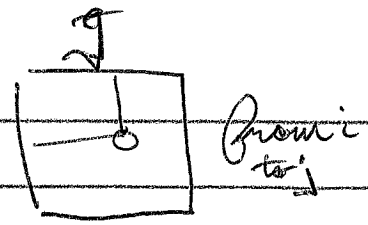
Calculating RSS



1. Normalize by dividing each row of the adjacency matrix by the row sum.

2. Calculate RSS iteratively.

Let $L(i,j,r)$ be the sum of all generalized relation strengths $R^*(i,j)$ of length r. Note that RSS is

$$S(i,j) = \sum_{s=1}^{r} L(i,j,s)$$

So for example, the RSS for $r = 4$ is

$$S(i,j) = \underbrace{L(i,j,1)}_{\text{path length } 1} + L(i,j,2) + L(i,j,3) + L(i,j,4)$$

$L(i,j,1) = a_{ij}$ where a_{ij} is the $(i,j)^{th}$ entry in the normalized adjacency matrix.

$$L(i,j,2) = \sum_{l=1}^{n} a_{il} \cdot a_{lj} - a_{ii}a_{ij} - a_{ij}a_{jj}$$

Put zeros on the diagonal and we get

$$= \sum_{l=1}^{n} a_{il} \cdot a_{lj}$$

$$= \text{sum}(A[i,] * A[,j])$$

$$L(i,j,3) = \sum_{l \notin \{i,j\}} \sum_{m \notin \{i,j,l\}} a_{im} \cdot a_{ml} \cdot a_{lj} + L(i,j,2) + L(i,j,2)$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} a_{im} \cdot a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \cdot \left[\sum_{m=1}^{n} a_{im} \cdot a_{ml} - a_{ij} \cdot a_{jl} \right]$$

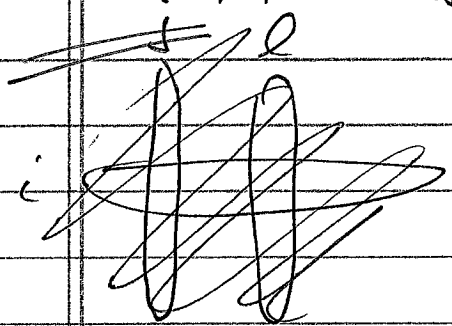
$$= \sum_{l \notin \{i,j\}} a_{lj} \left[L(i,l,2) - a_{ij} \cdot a_{jl} \right]$$

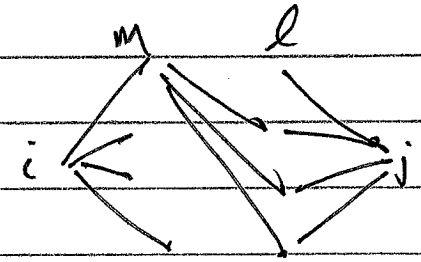
$$\text{sum}\left(A[,j] * \_\_\_\right)$$

$$= \text{sum}\left(A[,j] * \begin{bmatrix} L(i,1,2) - a_{ij} \cdot a_{j1} \\ L(i,2,2) - a_{ij} \cdot a_{j2} \\ \vdots \\ L(i,n,2) - a_{ij} \cdot a_{jn} \end{bmatrix} \right) - a_{ij} \left[\text{diag} - a_{ij} \cdot a_{ji} \right]$$

Exclude $a_{jj} = 0 \Rightarrow$ exclude j^{th} term from sum

$L(i,i,2) = a_{ij} a_{jl}$





$$i - p - m - l - j$$

$$L(i,j,4) = \sum_{l \notin \{i,j\}} \sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} a_{ip} a_{pm} a_{ml} a_{lj}$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} a_{ip} a_{pm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \left[\sum_{p \notin \{i,l,m\}} a_{ip} a_{pm} a_{ml} - a_{ij} a_{jm} a_{ml} \right] \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,l,m\}} a_{ip} a_{pm} a_{ml} - \sum_{m \notin \{i,j,l\}} a_{ij} a_{jm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,l\}} \sum_{p \notin \{i,l,m\}} a_{ip} a_{pm} a_{ml} - \sum_{p \notin \{i,j,l\}} a_{ip} a_{pj} a_{jl} - \sum_{m \notin \{i,j,l\}} a_{ij} a_{jm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[L(i,l,3) - a_{jl} \sum_{p \notin \{i,j,l\}} a_{ip} a_{pj} - a_{ij} \sum_{m \notin \{i,l,j\}} a_{jm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[L(i,l,3) - a_{jl} \underbrace{\text{sum}\left(A[i,] * A[,j]\right)}_{L(i,j,2)} + a_{jl} a_{il} a_{lj} \right.$$

$$\left. - a_{ij} * \underbrace{\text{sum}\left(A[j,] * A[,l]\right)}_{L(j,l,2)} + a_{ij} a_{ji} a_{il} \right]$$

$l = j \Rightarrow a_{jj} = 0$ so no need to exclude
but 7 terms where $l = i$ is not zero

$$= \text{sum}\Big(A[,j] * \begin{cases} l=1: \\ \vdots \\ n: \end{cases} \begin{array}{l} L(i,l,3) - a_{jl} \cdot \text{sum}(A[i,] * A[,j]) \\ + a_{jl}\, a_{il}\, a_{lj} + a_{ij}\, a_{ji}\, a_{il} \\ - a_{ij}\, \text{sum}(A[j,] * A[,l]) \end{array} \Big)$$

$$+ a_{ij} \cdot a_{ji} \cdot \text{sum}(A[i,] * A[,j])$$

$$+ a_{ij} \cdot a_{ij}\, \text{sum}(A[j,] * A[,i])$$

$i - q - p - m - l - j$ (5)

$$L(i,j,5) = \sum_{l \notin \{i,j\}} \sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} \sum_{q \notin \{i,j,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} a_{lj}$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} \sum_{q \notin \{i,j,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} \left(\sum_{q \notin \{i,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} - a_{ij} a_{jp} a_{pm} a_{ml} \right) \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \left(\sum_{p \notin \{i,j,l,m\}} \sum_{q \notin \{i,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} - \sum_{p \notin \{i,j,l,m\}} a_{ij} a_{jp} a_{pm} a_{ml} \right) \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \left(\sum_{p \notin \{i,l,m\}} \sum_{q \notin \{i,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} - \sum_{q \notin \{i,l,m,j\}} a_{iq} a_{qj} a_{jm} a_{ml} - \sum_{p \notin \{i,j,l,m\}} a_{ij} a_{jp} a_{pm} a_{ml} \right) \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \left[\sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,l,m\}} \sum_{q \notin \{i,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml} - \sum_{m \notin \{i,j,l\}} \sum_{q \notin \{i,j,l,m\}} a_{iq} a_{qj} a_{jm} a_{ml} - \sum_{m \notin \{i,j,l\}} \sum_{p \notin \{i,j,l,m\}} a_{ij} a_{jp} a_{pm} a_{ml} \right]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \Bigg[\sum_{m \notin \{i,l\}} \sum_{p \notin \{i,l,m\}} \sum_{q \notin \{i,l,m,p\}} a_{iq} a_{qp} a_{pm} a_{ml}$$

$$- \sum_{p \notin \{i,j,l\}} \sum_{q \notin \{i,j,l,p\}} a_{iq} a_{qp} a_{pj} a_{jl}$$

$$- \sum_{m \notin \{i,j,l\}} a_{jm} a_{ml} \left(\sum_{q \notin \{i,j,l,m\}} a_{iq} a_{qj} \right)$$

$$\cancel{-} \sum_{m \notin \{i,j,l\}} a_{ij} a_{ml} \left(\sum_{p \notin \{i,j,l,m\}} a_{jp} a_{pm} \right) \Bigg]$$

$$= \sum_{l \notin \{i,j\}} a_{lj} \Bigg[L(i,l,4) - a_{jl} \sum_{p \notin \{i,j,l\}} \sum_{q \notin \{i,j,l,p\}} a_{iq} a_{qp} a_{pj}$$

$$- \sum_{m \notin \{i,j\}} a_{jm} a_{ml} \left(\sum_{q \notin \{l,m\}} a_{iq} a_{qj} \right)$$

$$- a_{ij} \sum_{m \notin \{i,j\}} a_{ml} \left(\sum_{p \notin \{i,l\}} a_{jp} a_{pm} \right) \Bigg]$$

$$\sum_{p \notin \{i,j,l\}} \sum_{q \notin \{i,j,l,p\}} a_{iq} a_{qp} a_{pj} = \sum_{p \notin \{i,j,l\}} \left(\sum_{q \notin \{i,j,p\}} a_{iq} a_{qp} a_{pj} - a_{il} a_{lp} a_{pj} \right)$$

$$= \sum_{p \notin \{i,j,l\}} \sum_{q \notin \{i,j,p\}} a_{iq} a_{qp} a_{pj} - \sum_{p \in \{i,j,l\}} a_{il} a_{lp} a_{pj}$$

$$= L(i,j,3) - \sum_{q \notin \{i,j,l\}} a_{iq} a_{ql} a_{lj} - \sum_{p \notin \{i,j,l\}} a_{il} a_{lp} a_{pj}$$

$$L(i,j,5) = \sum_{l \notin \{i,j\}} a_{lj} \Big[L(i,l,4)$$

$$- a_{jl} \left(L(i,j,3) - \sum'_{q \notin \{i,j,l\}} a_{iq} a_{ql} a_{lj} - a_{il} \sum'_{p \notin \{i,j,l\}} a_{lp} a_{pj} \right)$$

$$- \sum'_{m \notin \{i\}} a_{jm} a_{ml} \left(\sum'_{q \notin \{l,m\}} a_{iq} a_{qj} \right)$$

$$- a_{ij} \sum'_{m \notin \{i,j\}} a_{ml} \left(\sum'_{p \notin \{i,l\}} a_{jp} a_{pm} \right) \Big]$$

$$\sum'_{q \notin \{i,j,l\}} a_{iq} a_{ql} a_{lj} = a_{lj} \sum'_{q \notin \{i,j,l\}} a_{iq} a_{ql}$$

$$= a_{lj} \sum'_{q \notin \{j\}} a_{iq} a_{ql}$$

$$= a_{lj} \sum'_{q} a_{iq} a_{ql} - a_{lj} a_{ij} a_{jl}$$

$$= a_{lj} \left(\text{sum}(A[i,] * A[,l]) - a_{ij} a_{jl} \right)$$