Solving a System of Coupled Harmonic Oscillators

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For the interested reader, I present the explicit calculation behind the single line of JavaScript. The problem is nothing special, but after years it was fun to play around with a Lagrangian and some Linear Algebra. Enjoy!

Keywords: coupled harmonic oscillator, Lagrangian mechanics, principal axis theorem.

Here we restrict to the case of two masses with fixed boundaries. Although it might seem complicated, an analytical solution is known. This holds true for the case of a chain of N masses. Each additional mass adds merely a new dimension, thus the procedure presented below can be applied as well for $N > 2$. Even for an infinite number of masses? the system remains solvable als long the interacting force stays linear.

I. THE SYSTEM

We consider the case of two identical springs at the boundaries with a spring constant k_1 and a connecting spring with a spring constant k_2. Further, for the masses we set $m_1 = m_2 = m$.

The displacement from the equilibrium point is described by a real number for each mass, thus the configuration space is 2-dim. The Lagrangian is given by the kinetic energy T minus the potential energy U, i.e. we have

$$L = T - U, \tag{1}$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \tag{2}$$

$$U = -\frac{k_1}{2}(x^2 + y^2) - \frac{k_2}{2}(x - y)^2. \tag{3}$$

II. SOLUTION

First, we will rewrite the potential energy to the form

$$U = -\vec{x}A\vec{x}, \tag{4}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} k_1 + k2 & -k_2 \\ -k_2 & k_1 + k_2 \end{pmatrix}. \tag{5}$$

This expression is equivalent to the old one, convince yourself by performing the matrix multiplication. Having in mind, that the norm $\|\dot{\vec{x}}\|$ appearing in the kinetic energy T is invariant under rotations, we can decouple the system by a principal axis transformation of matrix A

$$U = -\vec{x}A\vec{x} = -\vec{x}T_r T_r^T A T_r T_r^T \vec{x} = \vec{u}^T A_{eb}\vec{u}, \tag{6}$$

where we know T_r from the principle axis theorem. Thus we have

$$A_{eb} = \begin{pmatrix} k_1/2 + k2 & \\ 0 & k_1/2 \end{pmatrix}, \tag{7}$$

$$T_r = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \vec{u} = T_r\vec{x}. \tag{8}$$

Plugging the transformed Lagrangian into the Euler-Lagrange-equation

$$\partial_{x_i} L - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \tag{9}$$

Finally we have two decoupled harmonic oscillators

$$\ddot{u} = -\frac{1}{m}(k_1/2 + k2) \tag{10}$$

$$\ddot{v} = -\frac{1}{m}k_1 v. \tag{11}$$

The solution of a single uncoupled harmonic oscillator is well known

$$\vec{u} = \begin{pmatrix} A_0 \sin(\omega_0 t) + B_0 \cos(\omega_0 t) \\ A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t) \end{pmatrix} \tag{12}$$

Now, we are almost done, since we are interested in the displacement of the masses, so we need the result in the more suitable basis we started from. So, in the very last step we transform back to the old basis

$$\vec{x} = T_r^T \begin{pmatrix} u \\ v \end{pmatrix}. \tag{13}$$

For given initial conditions you have to determine the values for the constants A_0, A_1, B_0 and B_1.

[a)] Also at GitHub, https://github.com/BLyndon