Solving a System of Coupled Harmonic Oscillators

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For the interested reader, I present the explicit calculation behind the single line of JavaScript. The problem is nothing special, but after years it was fun to play around with a Lagrangian and some Linear Algebra. Enjoy!

Keywords: coupled harmonic oscillator, Lagrangian mechanics, principal axis theorem.

Here we restrict to the case of two masses with fixed boundaries. Although it might seem complicated, an analytical solution is known. This holds true for the case of a chain of N masses. Each additional mass adds merely a new dimension, thus the procedure presented below can be applied as well for many masses. Even for an infinite number of masses[?] the system stays solvable als long the interacting force stays linear.

I. THE SYSTEM

We consider the case of two identical springs at the boundaries with a spring constant k_1 and a connecting spring with a spring constant k_2. Further, for the masses we set $m_1 = m_2 = m$. The system is shown in figure TODO.

The displacement from the equilibrium point is described by a real number for each mass, thus the configuration space is 2-dim.. The Lagrangian is given by the kinetic energy T minus the potential energy U, i.e. we have

$$L = T - U, \tag{1}$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \tag{2}$$

$$U = -\frac{k_1}{2}(x^2 + y^2) - \frac{k_2}{2}(x - y)^2. \tag{3}$$

A. The equations of motion

First, we will rewrite the potential energy to the form

$$U = -\vec{x}A\vec{x}, \tag{4}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} k_1 + k2 & -k_2 \\ -k_2 & k_1 + k_2 \end{pmatrix}. \tag{5}$$

Having in mind, that the norm appearing in the kinetic energy T is invariant under rotations, we can decouple the system by a principal axis transformation of matrix A. In the eigenbasis matrix A is given by

$$A_{eb} = \begin{pmatrix} k_1/2 + k2 & \\ 0 & k_1/2 \end{pmatrix}. \tag{6}$$

Plugging the transformed Lagrangian into the Euler-Lagrange-equation

$$\partial_{x_i} L - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \tag{7}$$

We end up with two decoupled harmonic oscillators

$$\ddot{u} = \frac{1}{m}(k_1/2 + k2) \tag{8}$$

$$\ddot{v} = \frac{1}{m}k_1 vv = . \tag{9}$$

The solution of a single uncoupled harmonic oscillator is well known. Now, to get back the old coordinates we have to transform the solutions by the transformation matrix T_r.

$$\vec{x} = T_r^T \begin{pmatrix} u \\ v \end{pmatrix}. \tag{10}$$

And thats it, we are done here.

a) Also at GitHub, https://github.com/BLyndon