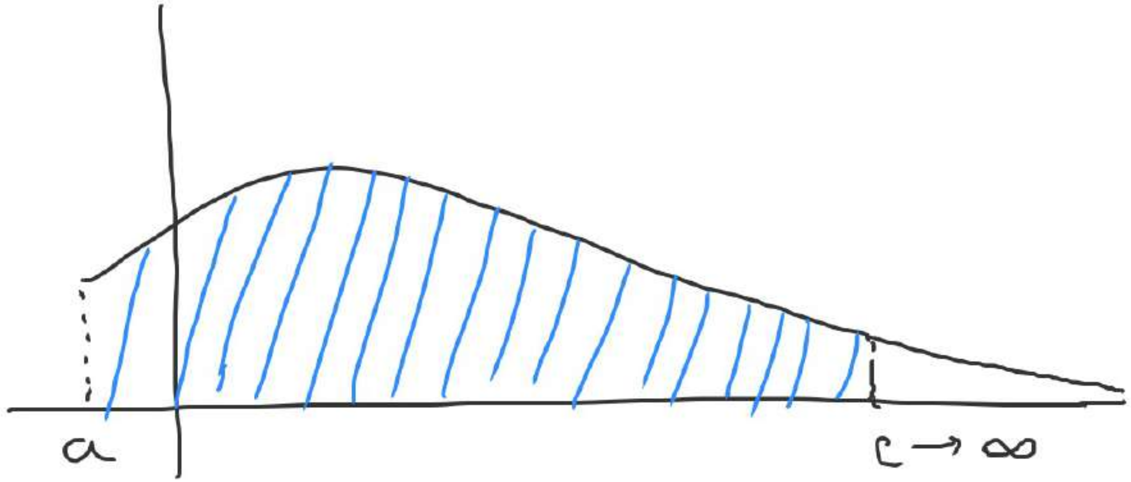
Integrale improprii pe intervale nemărginite

$f : [a, \infty) \to \mathbb{R}$ integrabila pe $[a, c]$, $\forall c > a$

Daca există

$$\lim_{c \to \infty} \int_a^c f(x)\,dx$$



aceasta s-n integrala improprie a fct. f pe $[a, \infty)$

si se not cu $\displaystyle\int_a^\infty f(x)\,dx$, adică

$$\int_a^\infty f(x)\,dx = \lim_{c \to \infty} \int_a^c f(x)\,dx.$$

Dacă limita este finită spunem că integrala este convergentă. Dacă limita nu există sau este infinită spunem că integrala este divergentă.

La fel dacă $f : (-\infty, b] \to \mathbb{R}$, f cont pe $[c, b]$, $\forall c < b$,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{c \to -\infty} \int_{c}^{b} f(x)\,dx$$

cu condiția ca limita din dreapta să existe.

Ex. Studiati convergenta integralei:

$$\int_0^{\infty} \frac{x^2}{x^3+1}\, dx$$

$$\lim_{c\to\infty} \int_0^{c} \frac{x^2}{x^3+1}\, dx = \lim_{c\to\infty} \frac{1}{3}\int_0^{c} \frac{(x^3+1)'}{x^3+1}\, dx$$

$$= \lim_{c\to\infty} \frac{1}{3} \ln(1+x^3)\Big|_0^{c} = \infty \Rightarrow \int_0^{\infty} \frac{x^2}{1+x^3}\, dx = \infty$$

deci int. este divergentă.

$$\int_0^{\infty} e^{-2x}\,dx$$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + C.$$

$$\int_0^{\infty} e^{-2x}\,dx = \lim_{c \to \infty} \int_0^{c} e^{-2x}\,dx = \lim_{c \to \infty} -\frac{e^{-2x}}{2}\Bigg|_0^{c} =$$

$$= \lim_{c \to \infty} \left(\frac{1}{2} - \frac{e^{-2c}}{2} \right) = \frac{1}{2}$$

int. este conv.

$$\int_{1}^{\infty} \frac{1}{x^2+3x+2}\,dx$$

$$x^2+3x+2=0 \begin{cases} x_1=-1 \\ x_2=-2 \end{cases} \Rightarrow x^2+3x+2=(x+1)(x+2)$$

$$\frac{1}{x^2+3x+2}=\frac{1}{(x+1)(x+2)}=\frac{A}{x+2}+\frac{B}{x+1}=\frac{A(x+1)+B(x+2)}{(x+1)(x+2)}$$

$$A(x+1)+B(x+2)=1, \quad \forall x$$

$$x(A+B)+A+2B=1 \begin{cases} A+B=0 \Rightarrow A=-B \\ A+2B=1 \Rightarrow A=-1, \; B=1 \end{cases}$$

$$\frac{1}{x^2+3x+1}=\frac{1}{x+1}-\frac{1}{x+2}$$

$$\int_1^c \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln(x+1)\Big|_1^c - \ln(x+2)\Big|_1^c =$$

$$= \ln(c+1) - \ln 2 - \ln(c+2) + \ln 3 = \ln \frac{c+1}{c+2} + \ln \frac{3}{2}$$

$$\int_1^\infty \frac{1}{x^2+3x+2} dx = \lim_{c \to \infty} \int_1^c \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln \frac{3}{2}$$

$$\int_a^\infty \frac{1}{x^\lambda}\,dx \quad , \quad a>0 . \quad \text{convergentă} \Longleftrightarrow \lambda>1 .$$

$$\lambda \neq 1. \quad \lim_{c\to\infty} \int_a^c \frac{1}{x^\lambda}\,dx = \lim_{c\to\infty} \left.\frac{x^{-\lambda+1}}{-\lambda+1}\right|_a^c =$$

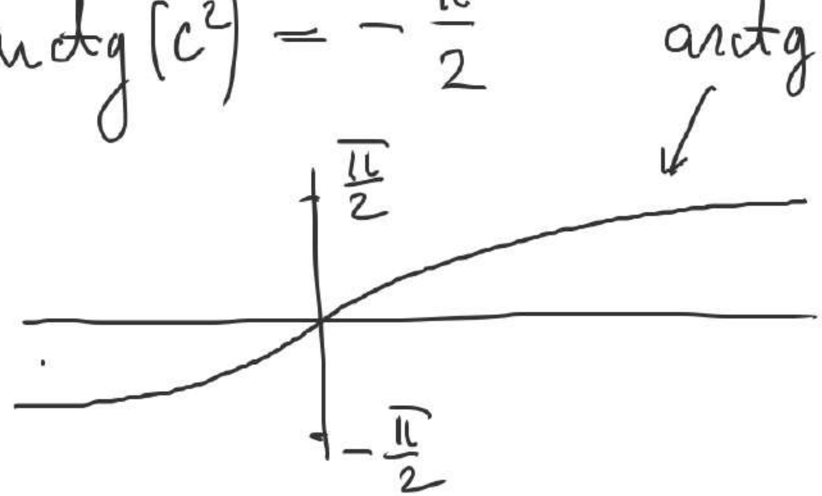
$$\lim_{c\to\infty} \left(\frac{c^{-\lambda+1}}{-\lambda+1} + \frac{a^{-\lambda+1}}{\lambda-1} \right) = \begin{cases} +\infty ; & \lambda<1 \\ \dfrac{a^{1-\lambda}}{\lambda-1} . & \lambda>1 \end{cases}$$

$$\lambda=1 , \quad \int_a^\infty \frac{1}{x}\,dx = \lim_{c\to\infty} \int_a^c \frac{1}{x}\,dx = \lim_{c\to\infty} \ln x \Big|_a^c =$$

$$= \lim_{c\to\infty} (\ln c - \ln a) = \infty$$

$$\int_{-\infty}^{0} \frac{2x}{1+x^4}\,dx = \lim_{c\to-\infty}\int_c^0 \frac{2x}{1+x^4}\,dx =$$

$$= \lim_{c\to-\infty} \operatorname{arctg}(x^2)\Big|_c^0 = 0 - \lim_{c\to-\infty}\operatorname{arctg}(c^2) = -\frac{\pi}{2}$$



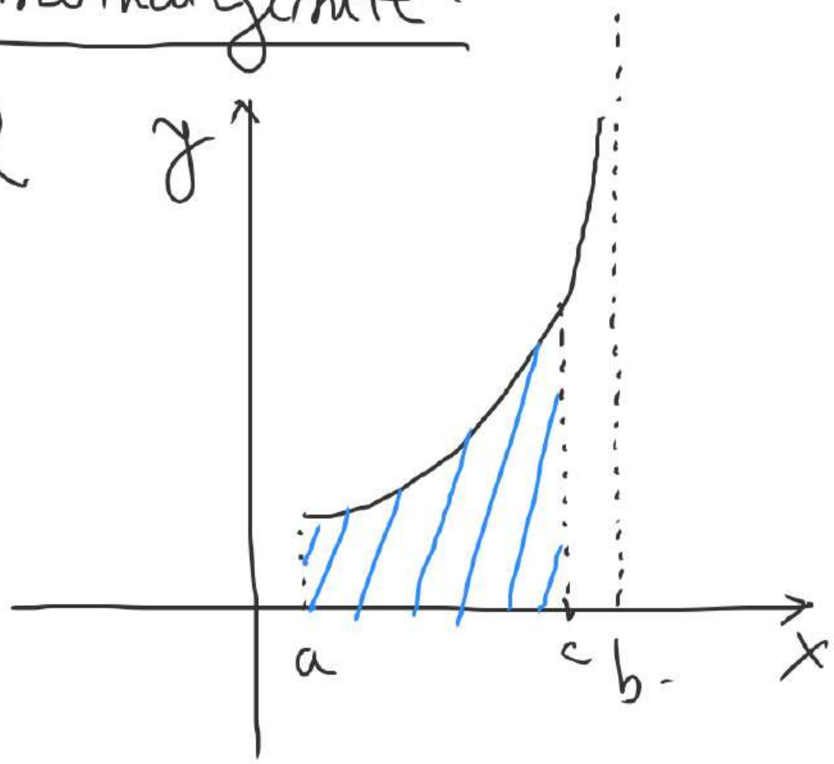
$$\int \frac{2x}{1+x^4}\,dx = \operatorname{arctg}(x^2)+C.$$

$x^2=t \qquad 2x\,dx=dt \qquad \int\frac{dt}{1+t^2} = \operatorname{arctg} t + C$

$$\operatorname{arctg} = \left(\operatorname{tg}\Big|_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}\right)^{-1} : \mathbb{R}\longrightarrow\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Integrale improprii pt. funcții nemărginite.

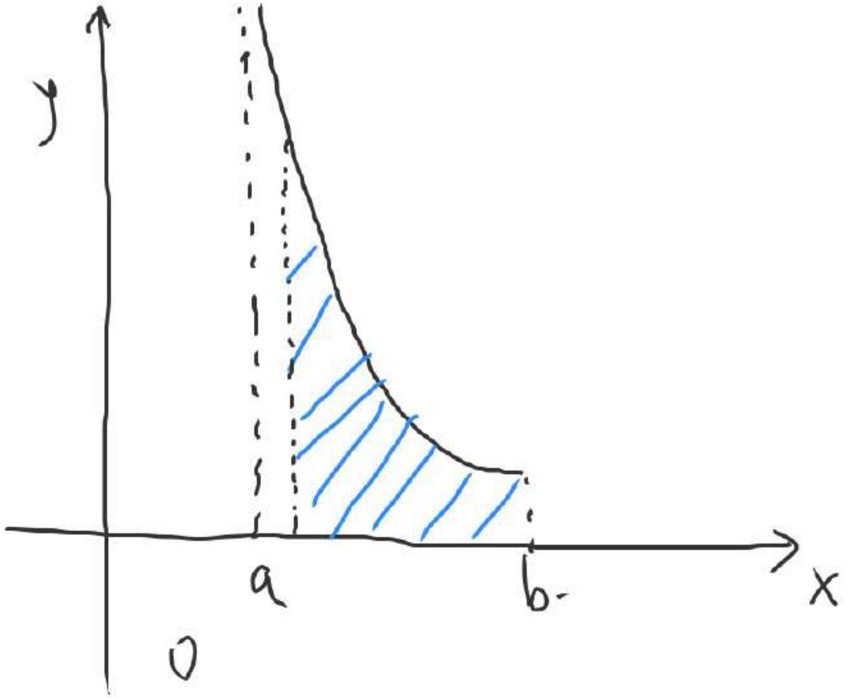
$$\lim_{x\to b} |f(x)| = \infty, \quad f: [a,b) \to \mathbb{R}$$



f int R. pe $[a,c]$, $a<c<b$

$$\int_a^b f(x)dx = \lim_{\substack{c\to b\\ c<b}} \int_a^c f(x)dx$$

cu condiția ca limita să existe. Dacă limita există și este finită spunem că $\int_a^b f(x)dx$ este conv. În caz contrar spunem că e divergentă



$$\int_a^b f(x)\,dx = \lim_{\substack{c\to a\\ c>a}} \int_c^b f(x)\,dx$$

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{c\to -1\\ c>-1}} \int_c^0 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{c\to -1\\ c>-1}} \arcsin x \Big|_c^0 =$$

$$\arcsin := \left(\sin\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)^{-1} \qquad = -\arcsin(-1) = \frac{\pi}{2}$$

$$\arcsin : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \arcsin y = x \Leftrightarrow \sin x = y$$

$$\int_0^1 \frac{1}{x^\lambda}\,dx \text{ convergenta} \iff \lambda < 1.$$

$$\lambda \neq 1; \quad \int_0^1 \frac{1}{x^\lambda}\,dx = \lim_{\substack{c \to 0 \\ c > 0}} \int_c^1 \frac{1}{x^\lambda}\,dx = \lim_{\substack{c \to 0 \\ c > 0}} \left.\frac{x^{-\lambda+1}}{-\lambda+1}\right|_c^1 =$$

$$= \lim_{\substack{c \to 0 \\ c > 0}} \left(\frac{1}{1-\lambda} - \frac{c^{-\lambda+1}}{-c+1} \right) = \begin{cases} \dfrac{1}{1-\lambda}; & \lambda < 1 \\ +\infty & \lambda > 1 \end{cases}$$

$$\lambda = 1. \quad \int_0^1 \frac{1}{x}\,dx = \lim_{\substack{c \to 0 \\ c > 0}} \left. \ln x \right|_c^1 = \infty.$$

Integrale duble

$f : D \to \mathbb{R}$

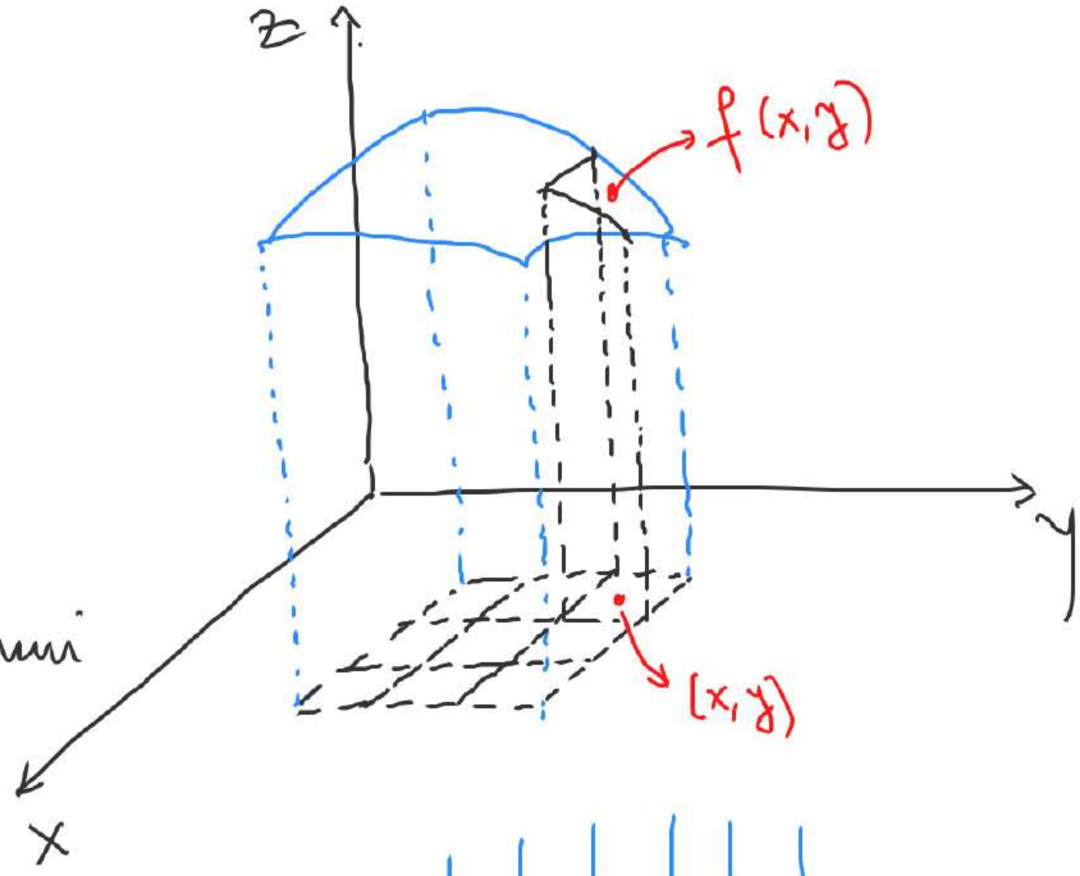
$\Delta = \{D_1, D_2, \dots, D_n\}$

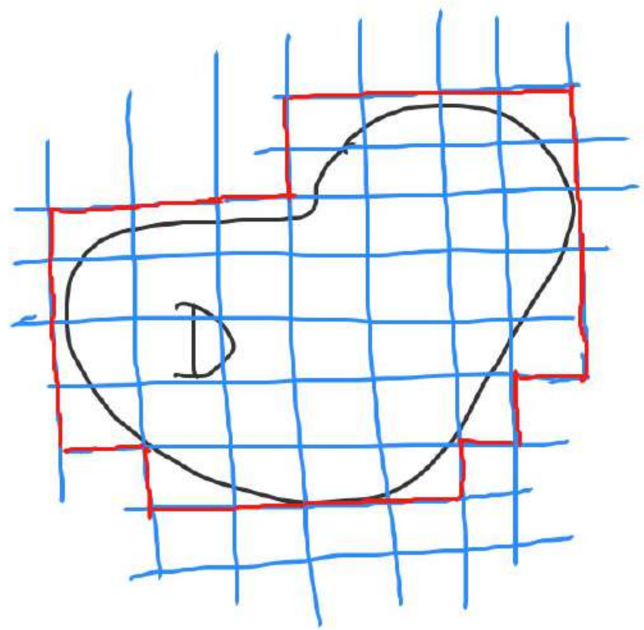
a.î. $D_1, D_2, \dots, D_n$ dreptunghiuri

și $D \subset D_1 \cup D_2 \cup \dots \cup D_n$

a.î.

$$\begin{cases} D \cap D_i \neq \emptyset, & \forall i \\ \operatorname{int}(D_i) \cap \operatorname{int}(D_j) = \emptyset \end{cases}$$





$$\operatorname{diam}(D_i) = \sup\left\{ \sqrt{(x-x')^2+(y-y')^2} \;\middle|\; (x,y),(x',y') \in D_i \right\}$$

$$\|\Delta\| = \max\left\{ \operatorname{diam}(D_i) \;\middle|\; 1 \le i \le n \right\}$$

$(x_i, y_i) \in D_i$ $\qquad \xi = \left\{ (x_i, y_i) \;\middle|\; (x_i, y_i) \in D_i \right\}$

$$\sum_{i=1}^{n} f(x_i, y_i) \cdot \operatorname{aria}(D_i) = \sigma_\Delta(f, \xi)$$

$$\iint_D f(x,y)\,dx\,dy = \lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi)$$ cu condiția ca

limita să existe și să fie finită.

<u>Interpretarea geom. a int. duble</u>

1) $f: D \to [0, \infty)$, $\iint_D f\,dxdy$ – volumul cuprins între graficul lui f și planul XOY

2) $\iint_D dxdy$ – aria lui D

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1). $f: D = \overset{x}{[a,b]} \times \overset{y}{[c,d]} \to \mathbb{R}$ continuă

$$\iint_D f(x,y)\,dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$\iint_D (x+2y)\,dx\,dy \qquad D=\underset{x}{[0,2]}\times\underset{y}{[0,1]}.$$

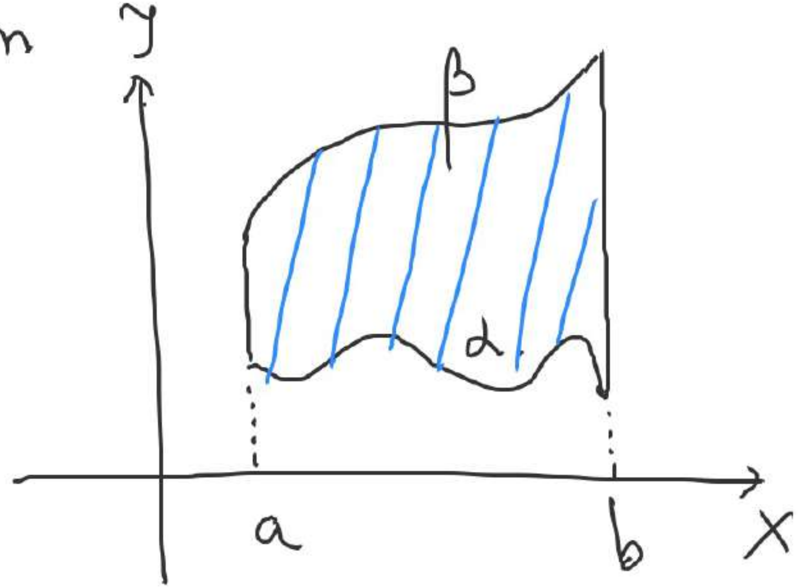
$$= \int_0^1 \Big(\underbrace{\int_0^2 (x+2y)\,dx}\Big)\,dy = \int_0^1 (2+4y)\,dy = 4$$

$$\int_0^2 (x+2y)\,dx = \left(\frac{x^2}{2}+2xy\right)\Bigg|_{x=0}^{x=2} = \frac{2^2}{2}-\frac{0}{2}+2y(2-0) = 2+4y$$

2) $f: D \to \mathbb{R}$ cont. $D = \{(x,y) \mid a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$

unde $\alpha, \beta: [a,b] \to \mathbb{R}$ int. Riemann

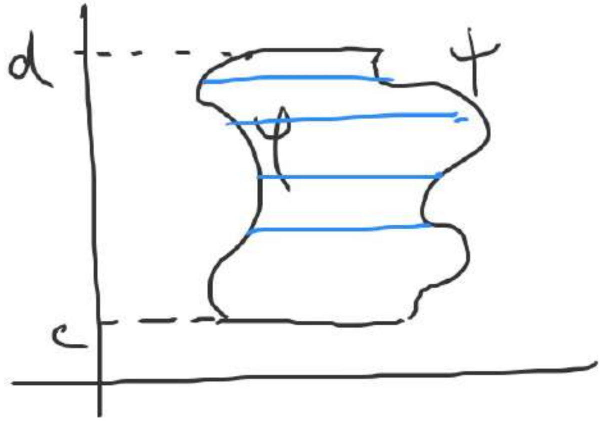
$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$



3) $f: D \to \mathbb{R}$, $D = \{(x,y) \mid c \le y \le d,\ \varphi(y) \le x \le \psi(y)\}$

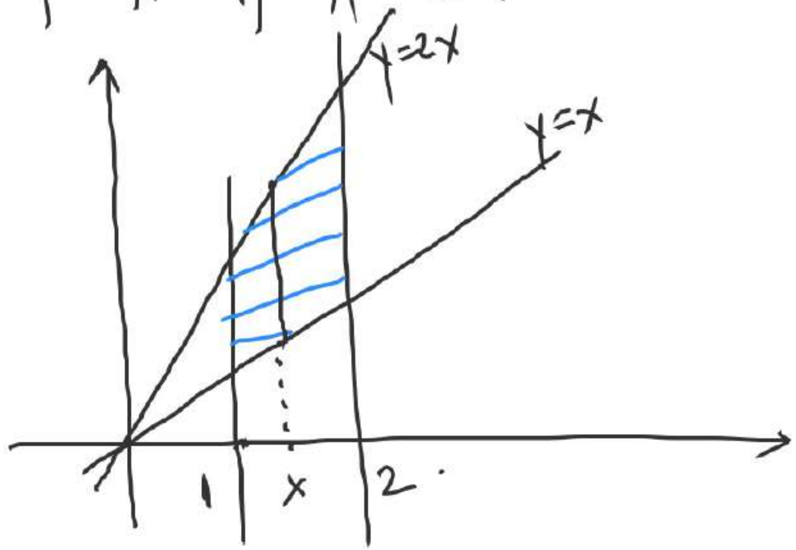
$\varphi, \psi: [c,d] \to \mathbb{R}$ int. R.

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx \right) d$$



$$I = \iint_D xy \, dx\, dy$$, D este mărginită de dreptele

$y = 2x$, $y = x$, $x = 1$, $x = 2$.

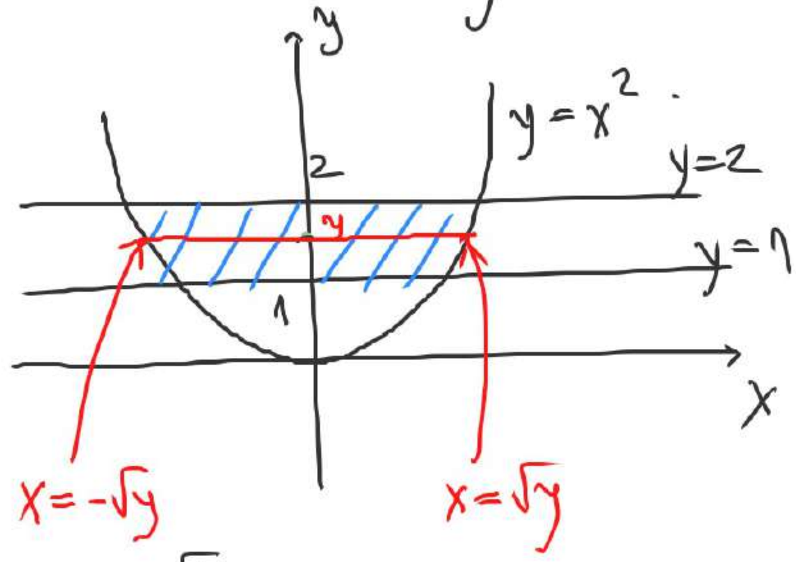


$$D = \{ (x,y) \mid 1 \le x \le 2, \; x \le y \le 2x \}$$

$$I = \int_1^2 \left(\int_x^{2x} xy \, dy \right) dx =$$

$$= \int_1^2 \left(\frac{xy^2}{2} \Big|_{y=x}^{y=2x} \right) dx = \int_1^2 \frac{x \cdot (4x^2 - x^2)}{2} dx = \int_1^2 \frac{3x^3}{2} dx = \frac{3}{2} \cdot \frac{x^4}{4} \Big|_1^2 = \ldots$$

$$J=\iint_D (y-x)\,dx\,dy$$ D mărginit de curbele $y=x^2$ $y=1,\ y=2$



$$D:\ \begin{cases} 1\le y\le 2 \\ -\sqrt{y}\le x\le \sqrt{y}\end{cases}$$

$$J=\int_1^2\left(\int_{-\sqrt{y}}^{\sqrt{y}}(y-x)\,dx\right)dy=\int_1^2\left(xy-\frac{x^2}{2}\right)\Bigg|_{x=-\sqrt{y}}^{x=\sqrt{y}}dy=$$

$$=\int_1^2 2y\sqrt{y}\,dy=\int_1^2 2\cdot y^{\frac{3}{2}}\,dy=2\cdot\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Bigg|_1^2=\ldots\ldots$$

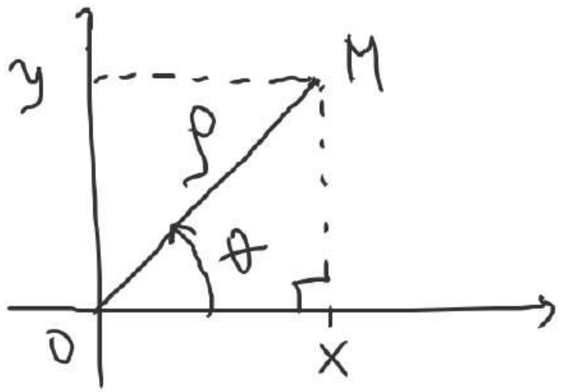
$T : \Omega \longrightarrow D$ bijectiva, au derivate partiale cont.

$T(\rho,\theta) = (x,y)$ $\qquad$ $T : \begin{cases} x = x(\rho,\theta) \\ y = y(\rho,\theta) \end{cases}$

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} \neq 0 \quad \text{pe } \Omega.$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{\Omega = T^{-1}(D)} f(x(\rho,\theta), y(\rho,\theta)) \underbrace{\left|\frac{D(x,y)}{D(\rho,\theta)}\right| d\rho\, d\theta}_{dx\,dy}.$$

Trecerea la coordonate polare.



$$\rho = d(O, M) = \sqrt{x^2+y^2}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \begin{array}{l} \rho \in [0, \infty) \\ \theta \in [0, 2\pi]. \end{array}$$

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

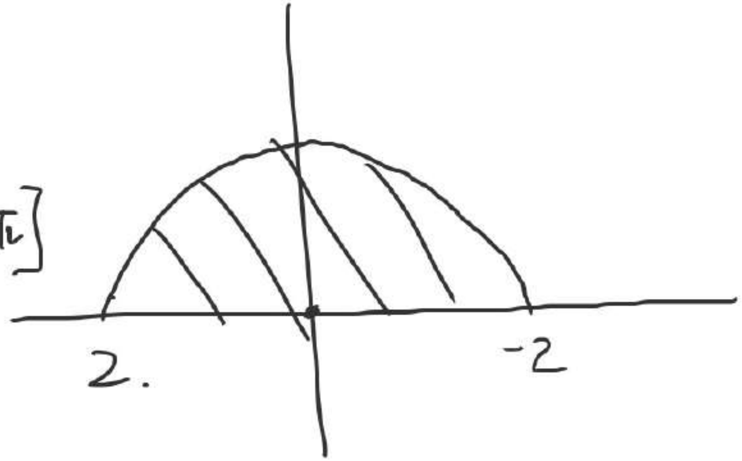
$$\boxed{dx\,dy = \rho\,d\rho\,d\theta}$$

$$D \subset \mathbb{R}^2; \quad \Omega = \{(\rho,\theta) \mid (\underset{x}{\rho\cos\theta}, \underset{y.}{\rho\sin\theta}) \in D\}$$

$$\iint_D f(x,y)\,\underline{dx\,dy} = \iint_\Omega f(\rho\cos\theta, \rho\sin\theta)\cdot \underbrace{\rho\,d\rho\,d\theta}_{dx\,dy\,.}$$

$$\iint_D x \, dx \, dy \qquad D = \{(x,y) \mid x^2+y^2 \leq 4, \quad y \geq 0\}$$

$$\begin{cases} x = \rho\cos\theta & \rho \in [0,2] \\ y = \rho\sin\theta & \theta \in [0,\pi] \end{cases} \qquad D \longleftrightarrow [0,2]\times[0,\pi]$$



$$\left(\begin{array}{l} x^2+y^2 \leq 4 \iff \rho^2 \leq 4 \iff \rho \in [0,2]. \\ y \geq 0 \iff \rho\sin\theta \geq 0 \iff \theta \in [0,\pi] \end{array} \right) \qquad dx\,dy = \rho \, d\rho \, d\theta$$

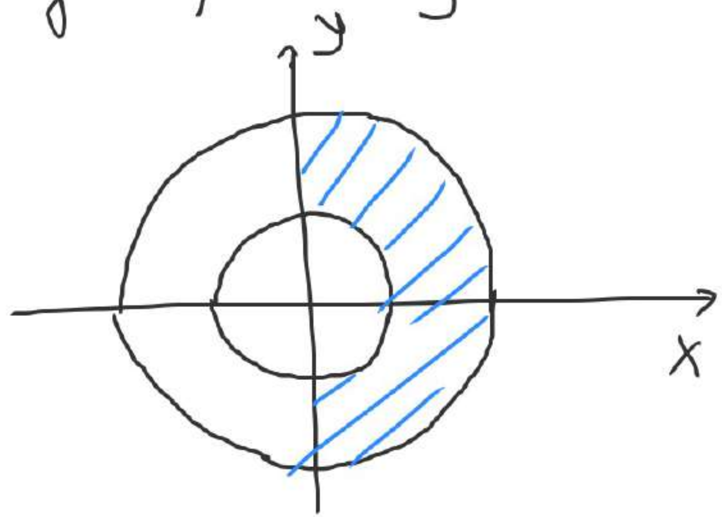
$$\iint_D x \, dx \, dy = \iint_{[0,2]\times[0,\pi]} \rho\cos\theta \cdot \rho \, d\rho \, d\theta = \int_0^2 \left(\int_0^\pi \rho^2 \cos\theta \, d\theta \right) d\rho$$

$$= \int_0^2 \rho^2 \sin\theta \Big|_{\theta=0}^{\theta=\pi} d\rho = \int_0^2 0 \, d\rho = 0.$$

Obs: $\displaystyle\iint_{[a,b]\times[c,d]} g(x)h(y)\,dx\,dy = \int_a^b \left(\int_c^d g(x) \underbrace{h(y)\,dy} \right) dx =$

$$= \int_a^b \left[g(x) \cdot \int_c^d h(y)\,dy \right] dx = \left(\int_a^b g(x)\,dx \right) \cdot \left(\int_c^d h(y)\,dy \right)$$

$$J = \iint_D y^2 \, dx\, dy \qquad D = \{(x,y) \mid 1 \le x^2 + y^2 \le 4,\ x \ge 0\}$$



$$D : \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad \begin{array}{l} \rho \in [1, 2] \\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{array}$$

$$dx\,dy = \rho\, d\rho\, d\theta$$

$$J = \int_{[1,2]\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \rho^2 \sin^2\theta \cdot \underbrace{\rho\, d\rho\, d\theta}_{dx\,dy} = \int_1^2 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^3 \sin^2\theta\, d\theta \right) d\rho$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^2 \sin^2\theta \, d\theta = \rho^3 \int_{-\pi/2}^{\pi/2} \sin^2\theta \, d\theta = \rho^3 \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1-\cos 2\theta}{2} \, d\theta =$$

$$\left(\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2} \right)$$

$$= \frac{\rho^3}{2} \left(\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin 2\theta}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) = \frac{\pi \rho^3}{2}$$

$$J = \int_1^2 \frac{\pi \rho^3}{2} \, d\rho = \frac{\pi \rho^4}{8} \Big|_1^2 = \frac{15\pi}{8}$$

Coordonate polare generalizate. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\le r^2\right)$

$$\begin{cases} x = a\rho\cos\theta & \rho\in[0,\infty) \\ y = b\rho\sin\theta & \theta\in[0,2\pi]. \end{cases}$$

$\rho \geq 0$. !

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{vmatrix} = ab\rho.$$

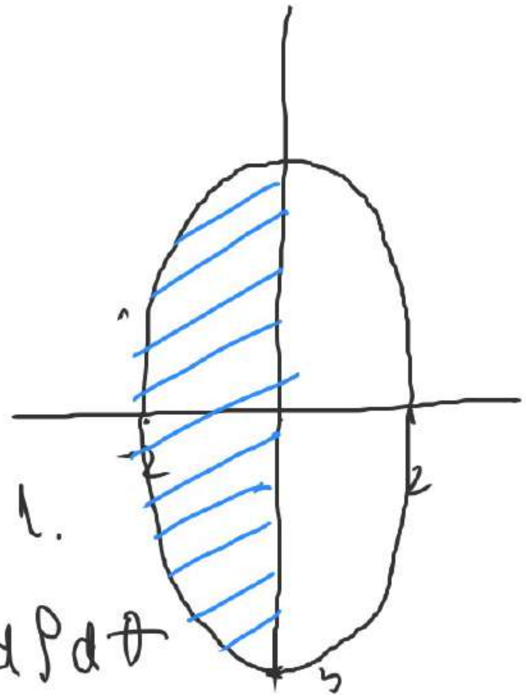
$$dx\,dy = ab\rho\, d\rho\, d\theta;$$

$$I = \iint_D y\,dx\,dy \qquad D = \left\{(x,y)\,\middle|\, \frac{x^2}{4}+\frac{y^2}{9}\le 1,\ x\le 0\right\}$$

$$\begin{cases} x = 2\rho\cos\theta\ ; & \rho\in[0,1]. \\ y = 3\rho\sin\theta & \theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right] \end{cases}$$

$$\frac{x^2}{4}+\frac{y^2}{9} = \rho^2\le 1.$$

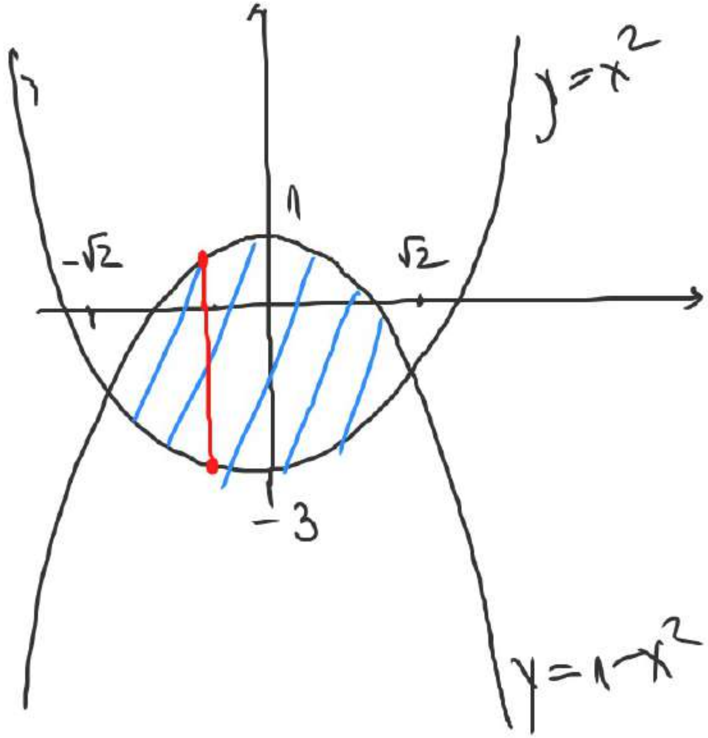
$$dx\,dy = 2\cdot 3\,\rho\, d\rho\, d\theta$$



$$I = \iint_{[0,1]\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]} 3\rho\sin\theta \cdot 6\rho \, d\rho \, d\theta = \int_0^1 \left(\int_{\pi/2}^{3\pi/2} 18\rho^2 \sin\theta \, d\theta \right) d\rho =$$

$$= \int_0^1 -18\rho^2 \cos\theta \Big|_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} d\rho = 0$$

$$\iint_D x(y-1)\,dx\,dy$$ D mărg. de curbele $y=1-x^2$ și $y=x^2-3$



$$1-x^2=x^2-3 \iff 2x^2=4 \iff x=\pm\sqrt{2}$$

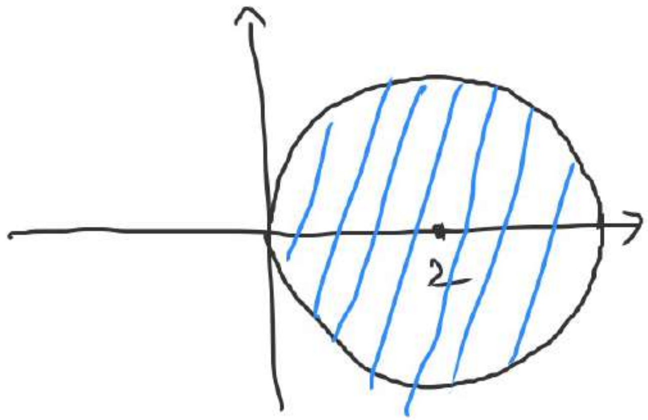
$$D=\left\{(x,y)\,\middle|\,-\sqrt{2}\le x\le\sqrt{2};\quad x^2-3\le y\le 1-x^2\right\}$$

$$\iint_D x(y-1)\,dx\,dy=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\int_{x^2-3}^{1-x^2} x(y-1)\,dy\right)dx=$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}} x\cdot\left(\frac{y^2}{2}-y\right)\Bigg|_{y=x^2-3}^{y=1-x^2}dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{x}{2} \cdot (1-x^2)^2 - \frac{x}{2}(x^2-3)^2 - x(1-x^2) + x(x^2-3) \right) dx = \dots$$

$$I = \iint_D xy \, dx\,dy \qquad D: \ x^2+y^2 \le 4x$$



$$\Updownarrow$$

$$x^2 - 4x + 4 + y^2 \le 4$$

$$(x-2)^2 + y^2 \le 4$$

$$\begin{cases} x-2 = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \iff \begin{cases} x = \rho\cos\theta + 2 \\ y = \rho\sin\theta \end{cases} \qquad \begin{matrix} \rho \in [0,2] \\ \theta \in [0,2\pi] . \end{matrix}$$

$$dx\,dy = \rho\, d\rho\, d\theta, \quad I = \int_0^2 \left(\int_0^{2\pi} \left(\rho\cos\theta + 2 \right) \rho\sin\theta \cdot \rho\, d\theta \right) d\rho = \dots$$

Integrale triple

$$\iiint_V dx\,dy\,dz = vol(V).$$

$$\overset{x}{}\quad \overset{y}{}\quad \overset{z}{}$$

1) $f : V \to \mathbb{R}$, $V = [a,b] \times [c,d] \times [k,p]$ f continuă

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_c^d \left(\int_k^p f(x,y,z)\,dz \right) dy \right) dx$$

$$\iiint_V (x+yz^2)\,dx\,dy\,dz \qquad V=[0,2]\times[0,1]\times[0,3].$$

$$\| $$

$$\int_0^2\left(\int_0^1\left(\int_0^3 (x+yz^2)\,dz\right)dy\right)dx = \int_0^2\left(\int_0^1 \left(xz+y\frac{z^3}{3}\right)\Bigg|_{z=0}^{z=3} dy\right)dx$$

$$= \int_0^2\left(\int_0^1 (3x+9y)\,dy\right)dx = \int_0^2 \left(3xy+\frac{9y^2}{2}\right)\Bigg|_{y=0}^{y=1} dx = \int_0^2 \left(3x+\frac{9}{2}\right)dx =$$

$$= \frac{3x^2}{2}\Bigg|_0^2 + \frac{9x}{2}\Bigg|_0^2 = 15$$

$$\iiint_V (xz + x\ln y)\,dx\,dy\,dz \qquad V = [0,2]\times[1,2]\times[2,4]$$

$$\|$$

$$\int_1^2 \left(\int_0^2 \left(\int_2^4 (xz + x\ln y)\,dz \right) dx \right) dy = \int_1^2 \left(\int_0^2 \left(\frac{xz^2}{2} + x\ln y \cdot z \right) \Bigg|_{z=2}^{z=4} dx \right) dy$$

$$= \int_1^2 \left(\int_0^2 (6x + 2x\ln y)\,dx \right) dy = \int_1^2 (3x^2 + x^2\ln y) \Big|_{x=0}^{x=2} dy =$$

$$= \int_1^2 (12 + 4\ln y)\,dy = \int_1^2 12\,dy + 4\int_1^2 \ln y\,dy = 12 + 4\int_1^2 \ln y\,dy$$

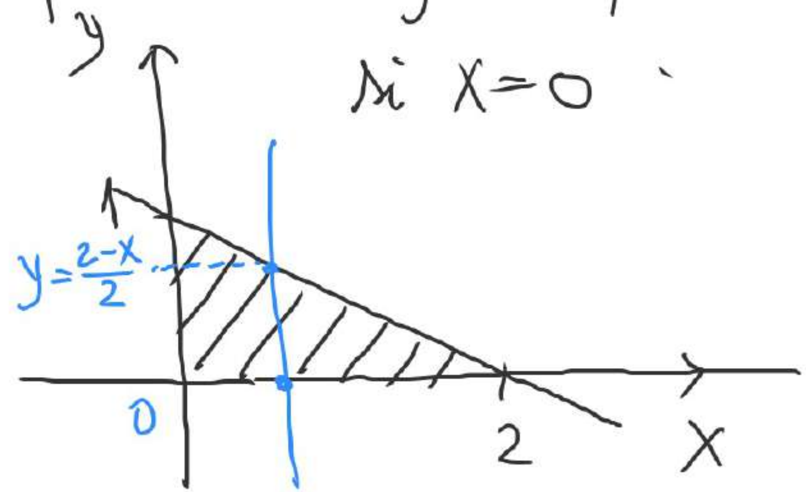
$$\int \ln y \, dy = \int y' \cdot \ln y \, dy = y \ln y - \int y \cdot (\ln y)' dy =$$

$$= y \ln y - \int dy = y \ln y - y + C$$

$$\int f' g \, dx = f \cdot g - \int f g'$$

$$I = 12 + 4 \int_1^2 \ln y \, dy = 12 + 4 \left(y \ln y - y \right) \Big|_1^2 = 12 + 4 \left(2 \ln 2 - 2 + 1 \right)$$

$$\iint_D (x+1)\,dx\,dy \quad D \text{ este mărg. de dreptele } x+2y=2,\ y=0 \text{ și } x=0.$$



$$D: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \dfrac{2-x}{2}. \end{cases}$$

(dreapta $x=4$: $\{(4,y) \mid y \in \mathbb{R}\}$)

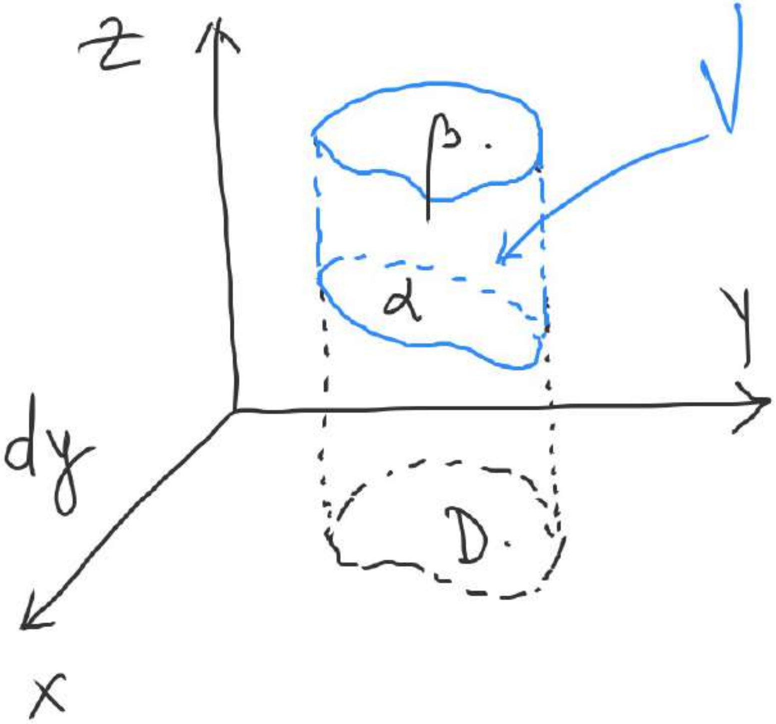
$$\iint_D (x+1)\,dx\,dy = \int_0^2 \left(\int_0^{\frac{2-x}{2}} (x+1)\,dy \right) dx$$

$$= \int_0^2 (x+1)\cdot y \Big|_{y=0}^{y=\frac{2-x}{2}} dx = \int_0^2 (x+1)\cdot \frac{2-x}{2}\,dx = \cdots$$

2) $f : V \to \mathbb{R}$ continuă

$$V = \{ (x,y,z) \in \mathbb{R}^3 \mid \alpha(x,y) \le z \le \beta(x,y) \ ; \ (x,y) \in D \}$$

$\alpha, \beta : D \to \mathbb{R}$ continue



$$\iiint_V f(x,y,z)\, dx\, dy\, dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz \right) dx\, dy$$

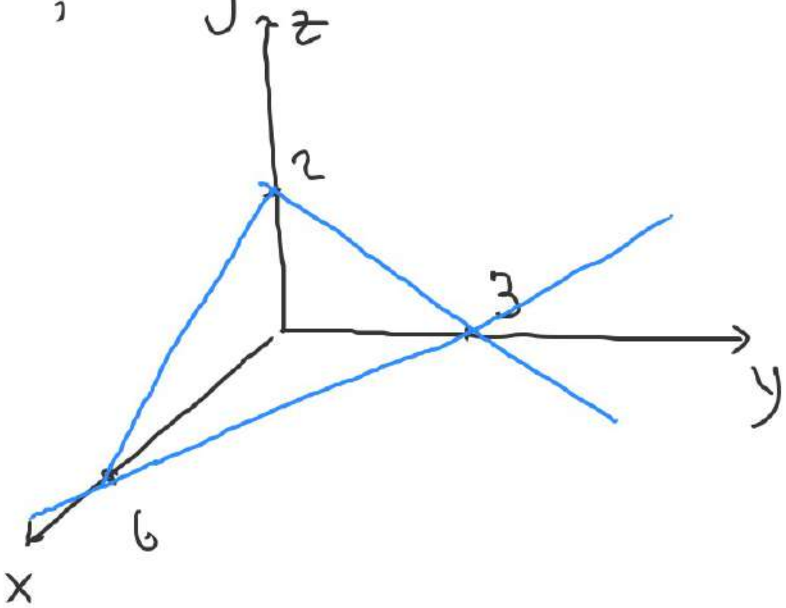
$ax+by+cz+d=0$ - ecuația generală a unui plan în $\mathbb{R}^3$.

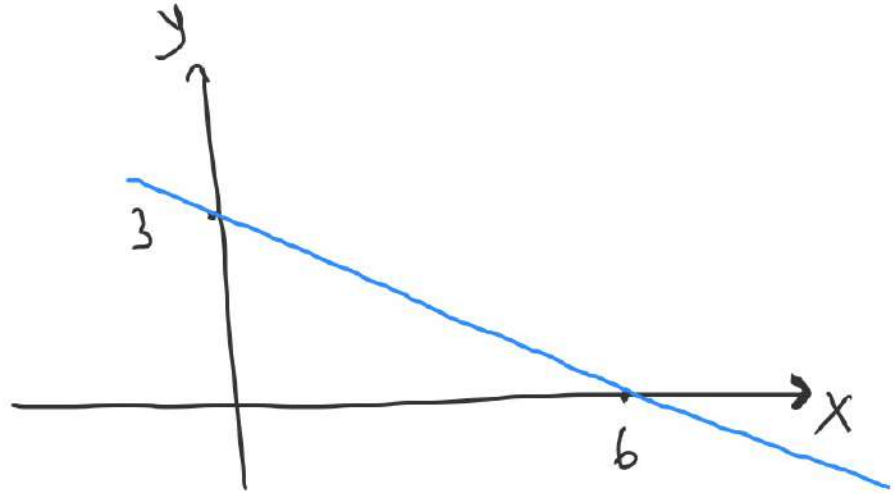
$x+2y+3z-6=0$.

$x=0, y=0 \Rightarrow z=2 \quad (0,0,2)$

$x=0, z=0 \Rightarrow y=3 \quad (0,3,0)$

$y=0, z=0 \Rightarrow x=6. \quad (6,0,0)$



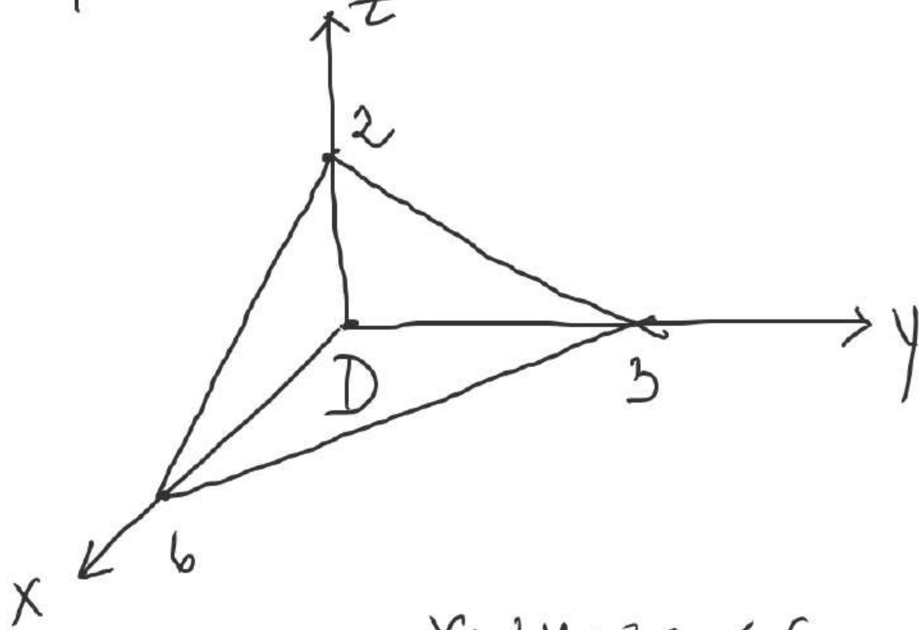


$x+2y-6=0$. in $\mathbb{R}^2$.

$x=0 \Rightarrow y=3 \quad (0,3)$

$y=0 \Rightarrow x=6. \quad (6,0)$

$$\iiint_V y\,dx\,dy\,dz \quad ; \quad V = \{(x,y,z) \mid x+2y+3z \le 6,\ x,y,z \ge 0\}$$

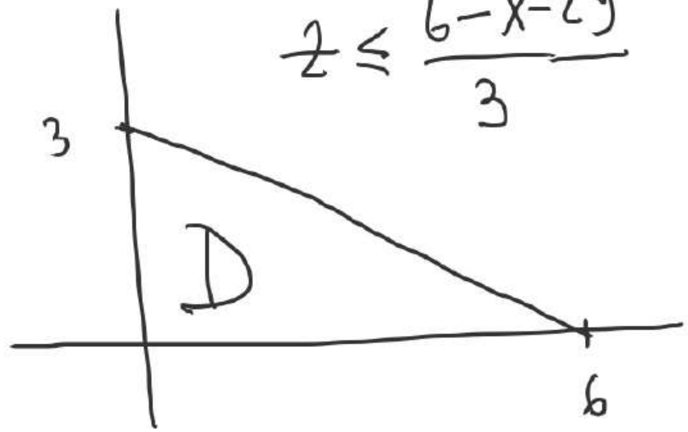


$$V: \begin{cases} (x,y) \in D. \\ 0 \le z \le \dfrac{6-x-2y}{3} \end{cases}$$

$\alpha(x,y)$ $\qquad\qquad$ $\beta(x,y)$

$x+2y+3z \le 6$

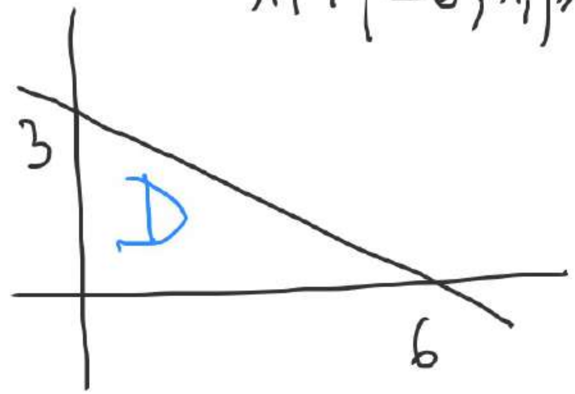
$z \le \dfrac{6-x-2y}{3}$



$$Pr_{xoy} V = D : \begin{cases} x+2y \le 6. \\ x,y \ge 0. \end{cases}$$

$$\iiint_V y\,dx\,dy\,dz = \iint_D \left(\int_0^{\frac{6-x-2y}{3}} y\,dz \right) dx\,dy$$

$x+2y \leq 6,\ x,y \geq 0$



$$D: \begin{cases} 0 \leq x \leq 6 \\ 0 \leq y \leq \frac{6-x}{2} \end{cases}$$

$$= \iint_D y \cdot \frac{6-x-2y}{3}\,dx\,dy$$

$$= \int_0^6 \left(\int_0^{\frac{6-x}{2}} y \cdot \frac{6-x-2y}{3}\,dy \right) dx = \cdots$$

$$V = \{ (x,y,z) \mid x+2y+3z \leq 6, \ x,y,z \geq 0 \}$$

$$\mathrm{vol}(V) = ?$$

$$\iiint_V dx\,dy\,dz = \iint_D \left(\int_0^{\frac{6-x-2y}{3}} dz \right) dx\,dy = \iint_D \frac{6-x-2y}{3}\,dx\,dy$$

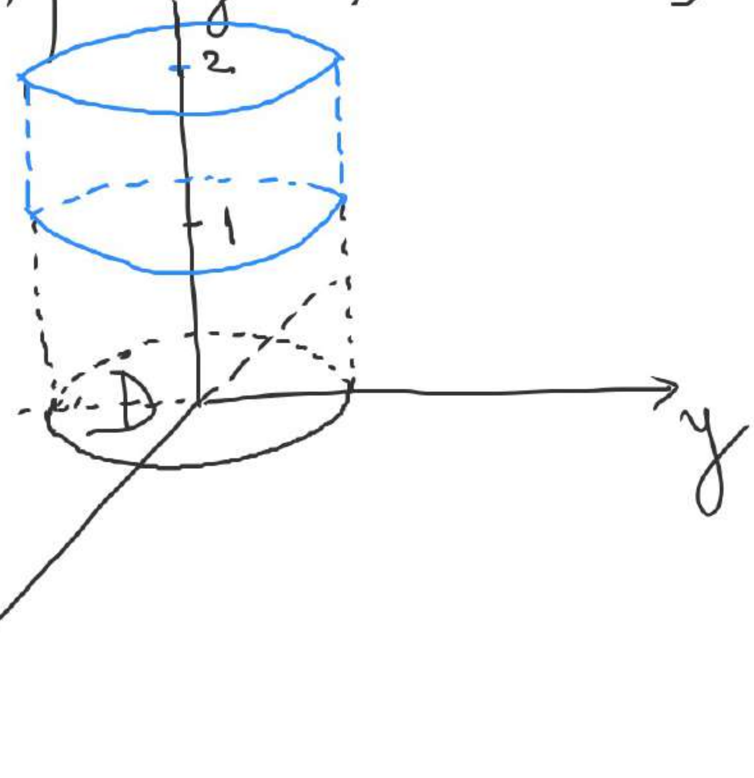
$$= \int_0^6 \left(\int_0^{\frac{6-x}{2}} \frac{6-x-2y}{3}\,dy \right) dx = \int_0^6 \left(\int_0^{\frac{6-x}{2}} \left(\frac{6-x}{3} - \frac{2y}{3} \right) dy \right) dx$$

$$= \int_0^6 \left(\frac{6-x}{3} \cdot y - \frac{2y^2}{6} \right) \Bigg|_{y=0}^{\frac{6-x}{2}} = \int_0^6 \left[\frac{(6-x)^2}{6} - \frac{1}{3} \cdot \frac{(6-x)^2}{4} \right] dx = \frac{1}{12} \int_0^6 (6-x)^2\,dx$$

$$= \frac{1}{12} \cdot \frac{(x-6)^3}{3} \Big|_0^6 = \frac{1}{12} \cdot \frac{6^3}{3} = \frac{36 \cdot 6}{36} = 6$$

$$\mathrm{Vol}(V) = \frac{1}{3} \cdot \text{aria}(D) \cdot h = \frac{1}{3} \cdot \frac{3 \cdot 6}{2} \cdot 2 = 6.$$

$$\iiint_V z(x^2+y^2)\,dx\,dy\,dz \; ; \quad V=\{(x,y,z) \mid x^2+y^2 \le 1,\ 1 \le z \le 2\}$$



$$V=\{(x,y,z) \mid (x,y)\in D,\ 1\le z\le 2\}$$

$$\alpha(x,y)=1, \quad \beta(x,y)=2$$

$$D=\{(x,y) \mid x^2+y^2 \le 1\}$$

$$\iiint_V z(x^2+y^2)\,dx\,dy\,dz = \iint_D \left(\int_1^2 z(x^2+y^2)\,dz \right) dx\,dy = \iint_D (x^2+y^2)\cdot \frac{z^2}{2}\Big|_{z=1}^{z=2} dx\,dy$$

$$= \iint_D \frac{3}{2}(x^2+y^2)\,dx\,dy \quad , \; D: \; x^2+y^2 \le 1$$

$$\begin{cases} x = \rho\cos\theta & , \; \rho \in [0,1] \\ y = \rho\sin\theta & \theta \in [0, 2\pi] \end{cases} \qquad dx\,dy = \rho\,d\rho\,d\theta$$

$$= \int_0^1 \left(\int_0^{2\pi} \frac{3}{2}\rho^2 \cdot \rho\,d\theta \right) d\rho = \int_0^1 3\pi \cdot \rho^3\,d\rho = 3\pi \frac{\rho^4}{4}\Bigg|_0^1 = \frac{3\pi}{4}.$$

$$\int_0^{\infty} x e^{-2x}\,dx = \lim_{c\to\infty} \int_0^{c} x e^{-2x}\,dx = (*)$$

$$\int x e^{-2x}\,dx = \int x \cdot \left(\frac{e^{-2x}}{-2}\right)' dx = \frac{x e^{-2x}}{-2} - \int x' \cdot \frac{e^{-2x}}{-2}\,dx$$

$$= -\frac{x e^{-2x}}{2} + \int \frac{e^{-2x}}{-2}\,dx = -\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$(*) = \lim_{c\to\infty} \left(-\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4}\right)\Bigg|_0^{c} = \lim_{c\to\infty} \left(-\frac{c e^{-2c}}{2} - \frac{e^{-2c}}{4} + \frac{1}{4}\right) = \frac{1}{4}.$$