Derivate parțiale. Diferențiale

$I \subset \mathbb{R}$ interval, $a \in I$, $f : I \to \mathbb{R}$. Spunem că f este derivabilă în a dacă există și este finită

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Fie $D \subset \mathbb{R}^3$ deschisă $f : D \to \mathbb{R}$ și $(x_0, y_0, z_0) \in D$. Spunem că f este derivabilă parțial în (x_0, y_0, z_0) în raport cu x (resp. y, resp z) dacă limita

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{x \to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0}$$ există și este finită

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ - derivata parțială a lui f în raport cu x în punctul (x_0, y_0, z_0).

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \lim_{y \to y_0} \frac{f(x_0, y, z_0) - f(x_0, y_0, z_0)}{y - y_0}$$

$$\frac{\partial f}{\partial z}(x_0, y_0, z_0) = \lim_{z \to z_0} \frac{f(x_0, y_0, z) - f(x_0, y_0, z_0)}{z - z_0}$$

$f : D \subset \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in D$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}; \quad \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

$$f(x,y) = x^2 y + e^{x+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = 2xy + e^{x+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = x^2 + 2y\, e^{x+y^2}$$

$$\left(\left(e^{x^2+1} \right)' = e^{x^2+1} \cdot (x^2+1)' = 2x\, e^{x^2+1} \right) \qquad \left(\frac{1}{x} \right)' = -\frac{1}{x^2}$$

$$\ln(u(x))' = \frac{u'(x)}{u(x)}.$$

$$f(x,y,z) = x^2 y z + \frac{x^2}{y} + \ln(x+z^2)$$

$$\frac{\partial f}{\partial x}(x,y,z) = 2xyz + \frac{2x}{y} + \frac{1}{x+z^2}$$

$$\frac{\partial f}{\partial y}(x,y,z) = x^2 z - \frac{x^2}{y^2}, \quad \frac{\partial f}{\partial z}(x,y,z) = x^2 y + \frac{2z}{x+z^2}$$

$T: \mathbb{R}^n \to \mathbb{R}$ este liniara dacă $\forall u_1, u_2 \in \mathbb{R}^n$ și $\forall \alpha \in \mathbb{R}$.

$T(u_1+u_2) = T(u_1) + T(u_2)$, $T(\alpha u_1) = \alpha T(u_1)$

$T(x,y,z) = 2x+3y-z$

$T((x_1,y_1,z_1)+(x_2,y_2,z_2)) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= 2(x_1+x_2)+3(y_1+y_2)-(z_1+z_2) = 2x_1+3y_1-z_1+2x_2+3y_2-z_2$

$= T(x_1,y_1,z_1)+T(x_2,y_2,z_2)$

$T(\alpha(x,y,z)) = T(\alpha x, \alpha y, \alpha z) = 2\alpha x+3\alpha y-\alpha z = \alpha(2x+3y-z) = \alpha T(x,y,z)$

$T : \mathbb{R}^n \to \mathbb{R}$ liniara $\Leftrightarrow \exists\, a_1, a_2, \ldots, a_n \in \mathbb{R}$ ai.

$$T(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

<u>Definiție</u>. Spunem că $f : D \subset \mathbb{R}^n \to \mathbb{R}$ este diferențiabilă în $a \in D$ dacă există $T : \mathbb{R}^n \to \mathbb{R}$ liniară a.î.

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\|x-a\|} = 0.$$

$f : D \subset \mathbb{R}^3 \to \mathbb{R}$, (x_0, y_0, z_0), f diferențiabilă în (x_0, y_0, z_0) dacă $\exists\, T : \mathbb{R}^3 \to \mathbb{R}$ liniară a.î.

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{f(x,y,z)-f(x_0,y_0,z_0)-T(x-x_0,\, y-y_0,\, z-z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0,$$

Daca T ca mai sus exista este unică, se not cu $df(x_0,y_0,z_0)$ si se numeste diferentiala functiei f în (x_0,y_0,z_0)

<u>Propozitie</u>- Dacă f este diferentiabilă într-un punct at. f este cont. în acel punct.

Propoziție. Dacă f este diferențiabilă în (x_0, y_0, z_0) at.

există $\frac{\partial f}{\partial x}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial y}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ și

$$\underbrace{df(x_0, y_0, z_0)}_{T}(\underbrace{u, v, w}_{\mathbb{R}^3}) = \frac{\partial f}{\partial x}(x_0, y_0, z_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0, z_0) v + \frac{\partial f}{\partial z}(x_0, y_0, z_0) w$$

$$\lim_{x \to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0) - df(x_0, y_0, z_0)(x - x_0, 0, 0)}{|x - x_0|} = 0.$$

$$\lim_{x \to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0) - (x - x_0)\, df(x_0, y_0, z_0)(1, 0, 0)}{x - x_0} = 0.$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x,y_0,z_0) - f(x_0,y_0,z_0)}{x - x_0} = df(x_0,y_0,z_0)(1,0,0).$$

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = df(x_0,y_0,z_0)(1,0,0) = df(x_0,y_0,z_0)(e_1).$$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

$(u,v,w) = u e_1 + v e_2 + w e_3$

$$\frac{\partial f}{\partial y}(x_0,y_0,z_0) = df(x_0,y_0,z_0)(e_2)$$

$$\frac{\partial f}{\partial z}(x_0,y_0,z_0) = df(x_0,y_0,z_0)(e_3)$$

Deci, dacă f este dif. în (x_0, y_0, z_0)

$$df(x_0,y_0,z_0)(u,v,w) = df(x_0,y_0,z_0)(e_1)\cdot \underbrace{u}_{dx(u,v,w)} + df(x_0,y_0,z_0)(e_2)\cdot \underbrace{v}_{dy(u,v,w)} +$$

$$+ df(x_0,y_0,z_0)(e_3)\cdot \underbrace{w}_{dz(u,v,w)}$$

$$= \frac{\partial f}{\partial x}(x_0,y_0,z_0)\cdot u + \frac{\partial f}{\partial y}(x_0,y_0,z_0)\cdot v + \frac{\partial f}{\partial z}(x_0,y_0,z_0)\cdot w$$

$dx;\ dy,\ dz : \mathbb{R}^3 \to \mathbb{R}$, $\quad dx(u,v,w) = u$

$dy(u,v,w) = v \qquad dz(u,v,w) = w$

$$df(x_0,y_0,z_0) = \frac{\partial f}{\partial x}(x_0,y_0,z_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0,z_0)\,dy + \frac{\partial f}{\partial z}(x_0,y_0,z_0)\,dz$$

$f : D \subset \mathbb{R}^2 \to \mathbb{R}$ $(x_0, y_0, z_0) \in D$, f dif în (x_0, y_0)

$$df(x_0,y_0)(u,v) = \frac{\partial f}{\partial x}(x_0,y_0)\cdot u + \frac{\partial f}{\partial y}(x_0,y_0)\cdot v \qquad \begin{array}{l} dx(u,v) = u \\ dy(u,v) = v \end{array}$$

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy$$

<u>Teoremă</u> Fie $D \subset \mathbb{R}^2$ deschisă, $(x_0, y_0) \in D$ și $f : D \to \mathbb{R}$

Dacă există o vecinătate V a lui (x_0, y_0) cu prop. că $\frac{\partial f}{\partial x}$ și $\frac{\partial f}{\partial y}$ există în orice pct din V și sunt continue în (x_0, y_0) atunci f este diferențiabilă în (x_0, y_0) și

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy$$

$$f(x,y,z) = x^3 y + (x+3z)^{100} + z^2 .$$

$$\left(u(x)^n\right)' = n\,u(x)^{n-1} \cdot u'(x)$$

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 y + 100(x+3z)^{99}$$

$$\frac{\partial f}{\partial y}(x,y,z) = x^3 , \quad \frac{\partial f}{\partial z}(x,y,z) = 100(x+3z)^{99} \cdot 3 + 2z$$

$$= 300(x+3z)^{99} + 2z$$

$$df(1,2,0)(u,v,w) = \frac{\partial f}{\partial x}(1,2,0)u + \frac{\partial f}{\partial y}(1,2,0)v + \frac{\partial f}{\partial z}(1,2,0)w$$

$$\frac{\partial f}{\partial x}(1,2,0) = 6 + 100 = 106 , \quad \frac{\partial f}{\partial y}(1,2,0) = 1, \quad \frac{\partial f}{\partial z}(1,2,0) = 300 .$$

$$df(1,2,0)(u,v,w) = 106u + v + 300w .$$

$$df(1,2,0) = 106\,dx + dy + 300\,dz$$

Derivate parțiale pt funcții compuse.

$$f(x,y,z) = g\big(u(x,y,z), v(x,y,z)\big) \qquad g(u,v)$$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial g}{\partial u}\big(u(x,y,z), v(x,y,z)\big) \cdot \frac{\partial u}{\partial x}(x,y,z) +$$

$$+ \frac{\partial g}{\partial v}\big(u(x,y,z), v(x,y,z)\big) \frac{\partial v}{\partial x}(x,y,z)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$f(x,y) = g(u(x,y,z), v(x,y,z), w(x,y,z)) \qquad g(u,v,w)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial y}.$$

$$f(x,y,z) = g(\overbrace{x^2+y^2}^{u}, \overbrace{xyz^2}^{v}, \overbrace{xy+yz}^{w}). \qquad g(u,v,w)$$

$$u(x,y,z) = x^2+y^2$$

$$v(x,y,z) = xyz^2$$

$$w(x,y,z) = xy+yz$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial g}{\partial u}\cdot 2x + \frac{\partial g}{\partial v}\cdot yz^2 + \frac{\partial g}{\partial w}\cdot y$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial g}{\partial u}\cdot 2y + \frac{\partial g}{\partial v}\cdot xz^2 + \frac{\partial g}{\partial w}\cdot(x+z).$$

$$\frac{\partial f}{\partial z} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial z} = \frac{\partial g}{\partial v}\cdot 2xyz + \frac{\partial g}{\partial w}\cdot y.$$

Exercitiu: Arătați că f

$$f(x,y) = g(x^2-y^2, e^{x^2-y^2})$$

$g(u,v)$.

$$\frac{\partial v}{\partial y} = e^{x^2-y^2}\cdot(-2y)$$

$$\frac{\partial v}{\partial x} = e^{x^2-y^2}\cdot 2x$$

verifică $y\cdot\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial g}{\partial u}\cdot 2x + \frac{\partial g}{\partial v}\cdot 2x\ e^{x^2-y^2} \quad \Big|\cdot y$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial g}{\partial u}\cdot(-2y) + \frac{\partial g}{\partial v}\cdot(-2y)\, e^{x^2-y^2} \quad \Big|\cdot x$$

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

Derivate parțiale de ordin superior.

$f : D \subset \mathbb{R}^2 \to \mathbb{R}$, $\frac{\partial f}{\partial x}$ există pe D. Dacă există.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) \overset{\text{not}}{=} \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

$$\frac{\partial^3 f}{\partial x \partial y^2}(x_0, y_0, z_0) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)(x_0, y_0).$$

$$f(x,y) = x^2 y + \cos(x+2y).$$

$$\frac{\partial f}{\partial x}(x,y) = 2xy - \sin(x+2y), \quad \frac{\partial f}{\partial y}(x,y) = x^2 - 2\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 2y - \cos(x+2y), \quad \frac{\partial^2 f}{\partial y^2} = -4\cos(x+2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x - 2\cos(x+2y)$$

$$\|$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x - 2\cos(x+2y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Teoremă (Schwarz). $f: D \subset \mathbb{R}^2 \to \mathbb{R}$. Dacă derivatele $\frac{\partial^2 f}{\partial x \partial y}$ și $\frac{\partial^2 f}{\partial y \partial x}$ există într-o vecinătate a lui (x_0, y_0) și sunt continue în (x_0, y_0) atunci

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Dacă $\frac{\partial^2 f}{\partial x \partial y}$ și $\frac{\partial^2 f}{\partial y \partial x}$ există și sunt continue pe D.

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y), \quad \forall (x, y) \in D.$$

Teoremă (Fermat) Fie $D \subset \mathbb{R}^3$ deschisă $f : D \to \mathbb{R}$ diferențiabilă în $(x_0, y_0, z_0) \in D$. Dacă (x_0, y_0, z_0) este pct de extrem local atunci $df(x_0, y_0, z_0) = 0$ $\left(\text{deci } \frac{\partial f}{\partial x}(x_0, y_0, z_0) = 0,\right.$ $\left.\frac{\partial f}{\partial y}(x_0, y_0, z_0) = 0,\ \frac{\partial f}{\partial z}(x_0, y_0, z_0) = 0\right)$.

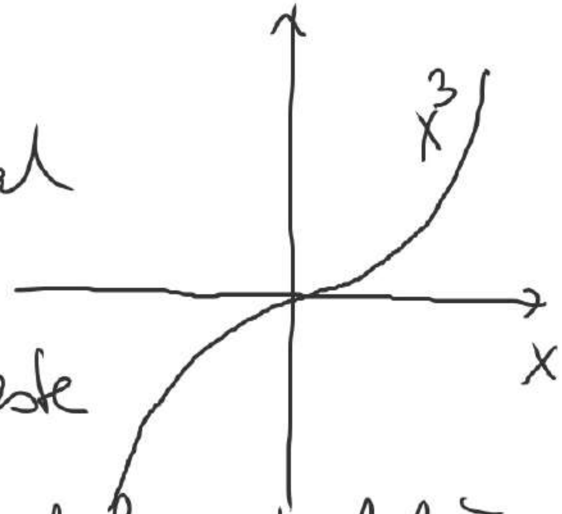
Obs Nu orice pct critic este pct de extrem local.

$f(x,y) = x^2 - y^2$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -2y$ $(0,0)$ – pct critic

$(0,0)$ nu este pct. de extrem local. $f(0,0) = 0$.

$f\left(\frac{1}{n}, 0\right) = \frac{1}{n^2} > 0$, $f\left(0, \frac{1}{n}\right) = -\frac{1}{n^2} < 0$.

$f(x)=x^3 \qquad f'(x)=3x^2$



0 - pct critic dar nu este pct de extrem local

Definiție. $f: A\subset\mathbb{R}^n$. Spunem că $a\in A$ este punct critic (staționar) pt f dacă f este diferențiabilă în a și $df(a)=0$. (adică $\frac{\partial f}{\partial x_i}(a)=0, \forall i$).

Obs: $f: I\subset\mathbb{R}\to\mathbb{R}$ f diferențiabila în $x_0\in I \iff$ f derivabilă în x_0 și $df(x_0)(u)=f'(x_0)\cdot u$.

$T:\mathbb{R}\to\mathbb{R}$ liniara $\iff \exists\, a\in\mathbb{R}$ $T(x)=ax$

Obs. $f, g : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^4$ $g(x,y) = x^2 - y^4$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 4y^3$, $\frac{\partial g}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -4y^3$

$(0,0)$ - pct critic.

$(0,0)$ - nu este pct de extrem local pt g.

$(0,0)$ - minim global pt f $\left(f(x,y) \geq 0 = f(0,0) \right)$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 g}{\partial x \partial y} \equiv 0$, $\frac{\partial^2 f}{\partial y^2} = 12y^2$, $\frac{\partial^2 g}{\partial y^2} = -12y^2$.

$H_f(0,0) = H_g(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$.

Propoziție. Fie $D \subset \mathbb{R}^2$ deschisă, $f : D \to \mathbb{R}$ de clasă C^2 (adică f are toate derivatele parțiale de ordinul 2 continue pe D), $(x_0, y_0) \in D$ pct critic. (adică $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$). Fie

$$H_f(x_0, y_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{pmatrix} \quad \Delta_1 = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$$

$$\Delta_2 = \det H_f(x_0, y_0)$$

(1) $\Delta_1 > 0$, $\Delta_2 > 0 \Rightarrow (x_0, y_0)$ pct de minim local

(2) $\Delta_1 < 0$, $\Delta_2 > 0 \Rightarrow (x_0, y_0)$ pct de maxim local

(3) $\Delta_2 < 0 \Rightarrow (x_0, y_0)$ nu este pct de extrem local

(4) dacă $\Delta_2 = 0$ nu putem trage nicio concluzie.

Exemplu Determinați pct de extrem local ale funcției

$f : \mathbb{R}^2 \to \mathbb{R}, \; f(x,y) = 2x^2 + y^4 - 4xy$

I. $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0. \end{cases} \iff \begin{cases} 4x - 4y = 0 \Rightarrow x = y \\ 4y^3 - 4x = 0 \Rightarrow x = y^3 \Rightarrow x = x^3 \end{cases}$

$x(x^2-1)=0 \Rightarrow x_1 = 1, y_1 = 1; \; x_2 = -1, y_2 = -1; \; x_3 = 0, y_3 = 0$

$\dfrac{\partial f}{\partial x} = 4x - 4y, \; \dfrac{\partial f}{\partial y} = 4y^3 - 4x$

Pct critice : $(1,1), (-1,-1), (0,0)$.

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = 4, \; \dfrac{\partial^2 f}{\partial y^2} = 12y^2, \; \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = -4.$

II. $H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & 12y^2 \end{pmatrix}$

$H_f(1,1) = \begin{pmatrix} 2 & -4 \\ -4 & 12 \end{pmatrix}$ $\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = \begin{vmatrix} 2 & -4 \\ -4 & 12 \end{vmatrix} = 8 > 0. \end{array}\right| \Rightarrow$ $(1,1)$ pct de minim local.

$H_f(-1,-1) = H_f(1,1) \Rightarrow (-1,-1)$ minim local.

$H_f(0,0) = \begin{pmatrix} 2 & -4 \\ -4 & 0 \end{pmatrix}$ $\Delta_1 = 2$

$\Delta_2 = -16 < 0 \Rightarrow (0,0)$ nu este pct de extrem local.

Propoziție. Fie $D \subset \mathbb{R}^3$ deschisă $f : D \to \mathbb{R}$ de clasă C^2 și $a = (x_0, y_0, z_0) \in D$ pct critic pt f. $\left(\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = \frac{\partial f}{\partial z}(a) = 0 \right)$

$$H_f(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a) & \frac{\partial^2 f}{\partial x \partial y}(a) & \frac{\partial^2 f}{\partial x \partial z}(a) \\ \frac{\partial^2 f}{\partial x \partial y}(a) & \frac{\partial^2 f}{\partial y^2}(a) & \frac{\partial^2 f}{\partial y \partial z}(a) \\ \frac{\partial^2 f}{\partial x \partial z}(a) & \frac{\partial^2 f}{\partial y \partial z}(a) & \frac{\partial^2 f}{\partial z^2}(a) \end{pmatrix}$$

$$\Delta_1 = \frac{\partial^2 f}{\partial x^2}(a)$$

$$\Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a) & \frac{\partial^2 f}{\partial x \partial y}(a) \\ \frac{\partial^2 f}{\partial x \partial y}(a) & \frac{\partial^2 f}{\partial y^2}(a) \end{vmatrix}$$

$$\Delta_3 = \det H_f(a).$$

1) $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0 \Rightarrow a$ pct de minim local

2) $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0 \Rightarrow a$ pct de maxim local.

3) dacă $(\Delta_1 \geq 0, \Delta_2 \geq 0, \Delta_3 \geq 0)$ sau $(\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0)$ dar

există $i \in \{1,2,3\}$ a.î. $\Delta_i = 0$ atunci nu putem decide.

(4) în orice altă situație a nu este pct de extrem local.

Obs: În particular, dacă $\Delta_1, \Delta_2, \Delta_3 \neq 0$ dar nu suntem nici în cazul 1) nici în cazul 2) și nici 3) pct a nu este pct de extrem local

Exercitiu $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^6 + 6xy + 3y^2 + z^2 + 2z$

Determinati punctele de extrem local ale lui f

$$\frac{\partial f}{\partial x} = 6x^5 + 6y, \quad \frac{\partial f}{\partial y} = 6x + 6y, \quad \frac{\partial f}{\partial z} = 2z + 2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 30x^4, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6, \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6, \quad \frac{\partial^2 f}{\partial x \partial z} = 0, \quad \frac{\partial^2 f}{\partial y \partial z} = 0.$$

I
$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0. \end{cases} \Leftrightarrow \begin{cases} 6x^5 + 6y = 0 \Rightarrow y = -x^5 \\ 6x + 6y = 0 \Rightarrow y = -x. \\ 2z + 2 = 0 \Rightarrow z = -1. \end{cases}$$

$$-x^5 = -x \Rightarrow x^5 = x \Leftrightarrow x(x^4 - 1) = 0$$

$x_1 = 1$, $x_2 = -1$, $x_3 = 0$

$y_1 = -1$, $y_2 = 1$, $y_3 = 0$

Pct critice $(1,-1,-1)$, $(-1,1,-1)$, $(0,0,-1)$

$$H_f(x,y,z) = \begin{pmatrix} 30x^4 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H_f(1,-1,-1) = H_f(-1,1,-1) = \begin{pmatrix} 30 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 30 > 0$

$\Delta_2 = \begin{vmatrix} 30 & 6 \\ 6 & 6 \end{vmatrix} > 0.$

$\Delta_3 = \begin{vmatrix} 30 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 2 \end{vmatrix} > 0$

$\Rightarrow$ $(1,-1,-1)$ și $(-1,1,-1)$ pct de min. local

$$H_f(0,0,-1) = \begin{pmatrix} 0 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

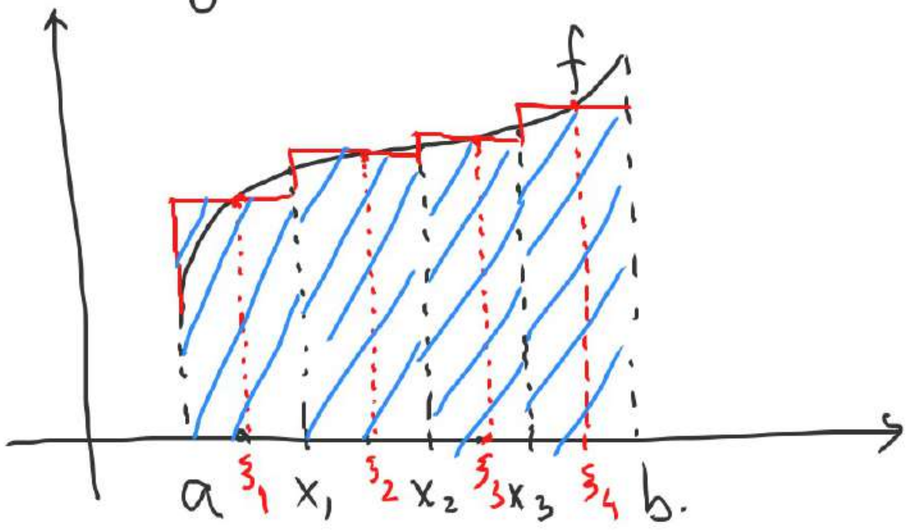
$\Delta_1 = 0.$

$\Delta_2 = -36.$

$\Delta_3 = -72$

$\Rightarrow$ $(0,0,-1)$ nu este pct de extrem local.

Integrala Riemann



$$\|\Delta\| = \max\{x_i - x_{i-1} \mid 1 \le i \le n\}$$

$$\xi = \{\xi_i \mid 1 \le i \le n;\ \xi_i \in [x_{i-1}, x_i]\}$$

$\Delta = \{a = x_0 < x_1 < \dots < x_n = b\}$ diviziune a lui $[a,b]$.

$$\sigma_\Delta(f, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$ - suma Riemann asoc. fct. f, div. Δ și sist. de pct. intermediare ξ.

Spunem că $f : [a,b]$ este integrabilă Riemann dacă $\exists\, I \in \mathbb{R}$ a.î. $\forall\, \varepsilon > 0$, $\exists\, \eta_\varepsilon > 0$ cu prop. că $\forall\, \Delta$ diviziune a lui $[a,b]$ cu $\|\Delta\| < \eta_\varepsilon$ și $\forall\, \xi$ sist. de puncte interm. asociat lui Δ, avem:

$$\left| \sigma_\Delta(f, \xi) - I \right| < \varepsilon$$

Numărul I (dacă există) este unic, se numește integrala Riemann a lui f pe $[a,b]$ și se not. $\int_a^b f(x)\,dx$.

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi) = \int_a^b f(x)\,dx.$$

f cont. și F primitivă a lui f (adică F deriv. și $F' = f$)

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{(Leibniz-Newton)}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C, \quad \int \sin x\,dx = -\cos x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad \int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + C.$$

$$\int u^n(x)\cdot u'(x)\,dx = \frac{u^{n+1}(x)}{n+1} + C, \quad \int e^{u(x)} u'(x)\,dx = e^{u(x)} + C$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

$$\int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x) \Big|_a^b - \int_a^b f(x)\, g'(x)\, dx$$

$$\int x e^x\, dx = \int x\, (e^x)'\, dx = x e^x - \int \underset{\overset{\shortparallel}{1}}{x'} \cdot e^x\, dx = x e^x - e^x + C$$

$$\int e^{2x}\, dx = \frac{e^{2x}}{2} + C$$

$2x = t,$

$2\,dx = dt$

$dx = \frac{1}{2}\, dt$

$$\int \frac{1}{2} e^t\, dt = \frac{1}{2} e^t + C.$$

$$\int e^{\alpha x}dx = \frac{e^{\alpha x}}{\alpha} + C$$

$$\int \cos\alpha x\, dx = \frac{\sin\alpha x}{\alpha} + C \ ; \ \int \sin\alpha x\, dx = -\frac{\cos\alpha x}{\alpha} + C$$

$$\int x e^{x^2} dx = \frac{1}{2}\int \underbrace{(x^2)'}_{2x} e^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

$x^2 = t$

$2x\,dx = dt$ $\quad \frac{1}{2}\int e^t dt = \frac{e^t}{2} + C.$ $\quad t = x^2$

$$\int x \sin 2x\, dx = ? \qquad \int x \ln x\, dx = ?$$