CS340 - Analysis of Algorithms

Review: Discrete Math, Complexity, and Graphs

Logistics:

Office hours:

- Alternating Tuesday and Thursdays 7--9pm
- Cecilia will post each week on Piazza

Join the piazza and gradescope if you have not already

Make sure you have a goldengate account

- Email David Diaz ddiaz1@brynmawr.edu if you encounter issues

Upcoming deadlines:

Lab0 due tomorrow

HW1 due Monday (9/15)

Lab1 tomorrow

Quiz next class

Agenda

1. **Stable matching algorithm review**
2. Some math review
3. Complexity
4. Data structures
5. Graphs

Proof of Correctness:

For this algorithm to be correct, it must produce a perfect and stable matching.

A matching is perfect if every element of X and Y occurs in some pair.

Proof by contradiction: suppose there is some employer e who was not matched.

Function findStableMatching(E, A)
  All $e \in E$ and $a \in A$ are unpaired
  while *there is an e unpaired that hasn't made an offer to every a* **do**
    choose such an e
    let a be the highest-ranked applicant in e's preference list who e has not made an offer to yet
    if *a is unpaired* **then**
      pair a and e
    end
    else
      a is currently paired with e'
      if *a prefers e' to e* **then**
        e remains unpaired
      end
      else
        a is paired with e and e' becomes unpaired
      end
    end
  end
  return the set of pairs

Proof of Correctness:

Proof of stability by contradiction: Suppose there exists some e and some a’ such that e prefers a’ to its match a and a’ prefers e to its match e.

Either: (1) e never offered a job to a’ or (2) a’ rejected the offer

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Simplify the following equations:

1. $(3x^4y^2)(2x^3y^5)$

2. $(6x^7y^5) / (2x^3y^2)$

3. $((x^2y^3) / (z^4))^3$

Exponent Rules For $a \neq 0, b \neq 0$	
Product Rule	$a^x \times a^y = a^{x+y}$
Quotient Rule	$a^x \div a^y = a^{x-y}$
Power Rule	$\left(a^x\right)^y = a^{xy}$
Power of a Product Rule	$(ab)^x = a^x b^x$
Power of a Fraction Rule	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
Zero Exponent	$a^0 = 1$
Negative Exponent	$a^{-x} = \frac{1}{a^x}$
Fractional Exponent	$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

Logarithms

1. log(5x) + log(2y)

2. log(100) - log(4)

3. 3·log(x) – ½·log(y)

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p \log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

Polynomials

$f(n) = n^c$

- C is a constant
- Adding polynomials: combine like terms and simplify coefficients
- Example 1: $(3x^2 + 5x + 2) + (4x^2 - 3x + 7)$
  - $= (3x^2 + 4x^2) + (5x - 3x) + (2 + 7)$
  - $= 7x^2 + 2x + 9$
- Example 2: $(2x^2y + 3xy^2 + y) + (5x^2y - 2xy^2 + 4y)$
  - $7x^2y + xy^2 + 5y$

Exponentials

$f(n) = r^n$

- Constant base r
- Grows *much faster* than polynomials (constant factor, variable base)

Summations

- Constant series

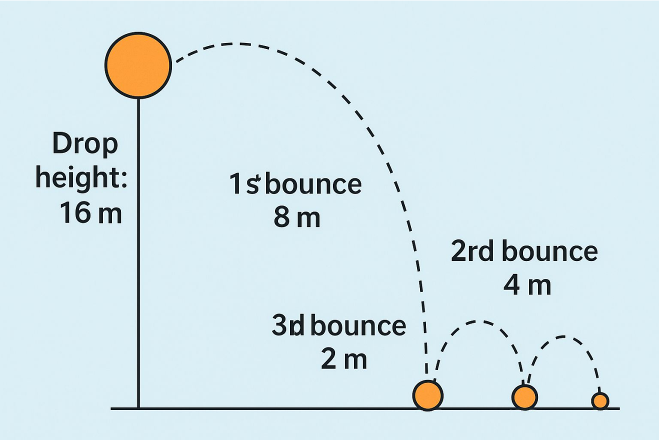
$$\sum_{i=a}^{b} 1 = \max(b - a + 1, 0)$$

- Arithmetic series

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad O(n^2)$$

- Geometric series

$$\sum_{i=0}^{n} c^i = 1 + c + c^2 \ldots + c^n = \frac{c^{n+1} - 1}{c - 1} \in \begin{cases} O(1), & c < 1 \\ O(c^n), & c > 1 \end{cases}$$



Summations

- Quadratic series

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

O(n^3)

- Linear-geometric series

$$\sum_{i=1}^{n} ic^i = c + 2c^2 + \cdots + nc^n$$

$$= \frac{(n-1)c^{n+1} - nc^n + c}{(c-1)^2}$$

O(nc^n)

Summations

- Harmonic series

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + O(1)$$

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Now let’s relate this to algorithms...

Basics of Algorithm Analysis

- Last class we discussed trade-offs between two algorithms
  - “How does algorithm 1’s time and space usage grow with increasing input size?”

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

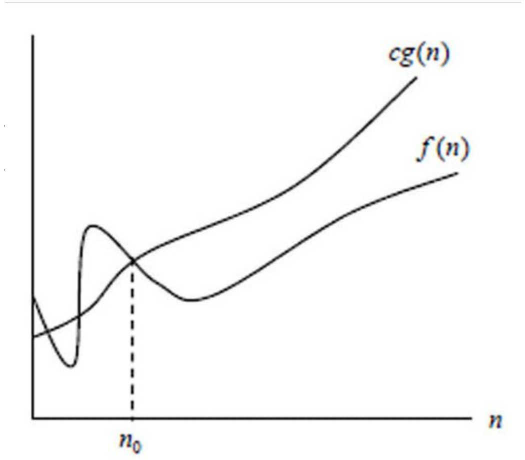
	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Big O

Big O notation expresses **Asymptotic Upper Bounds**

"If f(n) doesn't grow faster than g(n), up to a constant factor, for large enough n."

If $\exists\ n_0 \geq 0, c > 0 : f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0$ then $O(f(n)) = g(n$



Basics of Algorithm Analysis

- Q: Why do we simplify to only keep the dominant term in Big-O notation?
  - Worst case run time of $2n$ expressed as $O(n)$
  - Worst case run time of $n^2 + 3n$ expressed as $O(n^2)$
- A:
  - We'd like to classify run times at a coarse level of granularity so we can see similarities among different algorithms
  - Number of steps is arbitrary (language and machine dependant)

What is the Asymptotic Upper Bound of each of these functions? Write in Big-O notation

1. $f_2(x) = 15x^2 + 7x \log^3 x$
   $= O(x^2)$

2. $f_1(x) = 43x^2 \log^4 x + 12x^3 \log x + 52x \log x$
   $= O(x^3 \log x)$

3. $f_3(x) = 3x + 4 \log_5 x + 91x^2$
   $= O(x^2)$

4. $f_4(x) = 13 \cdot 3\text{^}(2x+9) + 4x^9$
   $= O(9\text{^}x)$

5. $f_5(x) = \sum$ (from $x=0$ to ∞) $1/2\text{^}x$
   $= O(1)$

Basics of Algorithm Analysis

- Sublinear time: O(logn) - algorithms where we can throw away a constant fraction of the input with each step of the algorithm
    - Binary search
    - Queries in a *balanced* BST
- Linear Time Algorithms: O(n) - run time is at most a constant factor times the input size
    - Process the input in a single pass spending constant time on each item
    - min/max algorithm, linear search
- O(nlogn): Sorting algorithms
- Quadratic time: $O(n^2)$ - nested loops
    - Processing all pairs in a list (every employer and every applicant
- Polynomial time: $O(n^k)$
    - Consider all subsets of size k
- Exponential time: $O(2^n)$
    - All subsets of a set of size n
- Factorial time: O(n!)
    - Number of ways to match n items with eachother

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Asymptotic Upper Bound of Common Operations

	Access	Search	Insertion	Deletion
Array				
Linked List				
Stack				
Queue				
BST				
AVL Tree				
Hash Map				

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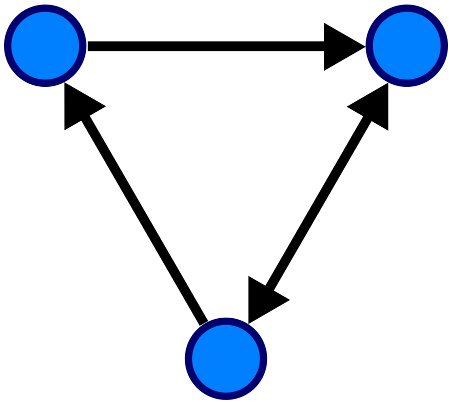
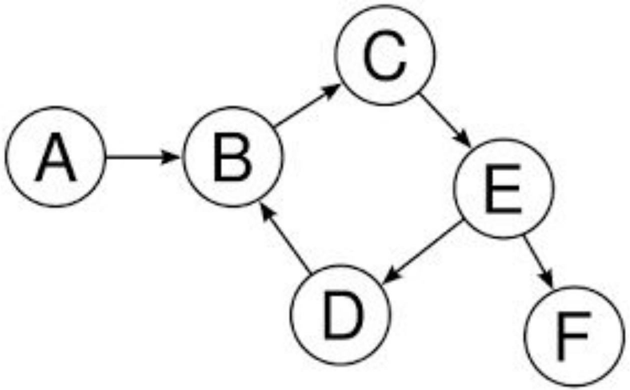
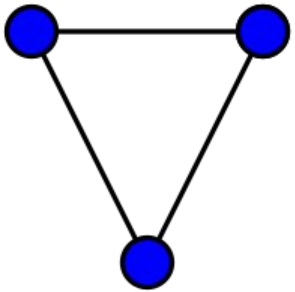
Graphs

- A way of representing relationships between pairs of objects
- Consist of ***Vertices (V)*** with pairwise connections between them ***Edges (E)***
- A ***Graph G*** is a set of vertices and edges **(V, E)**

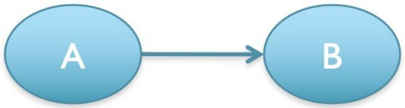






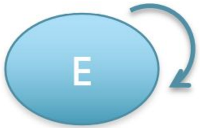
Edges

- An edge (u, v) connects vertices u and v
- Edges can be ***directed*** or ***undirected***
- An edge is said to be ***incident*** to a vertex if the vertex is one of the endpoints
- The ***degree*** of a vertex **deg(v)** is
  - the number of incident edges on v if in an undirected graph
  - the number of outgoing edges in a directed graph



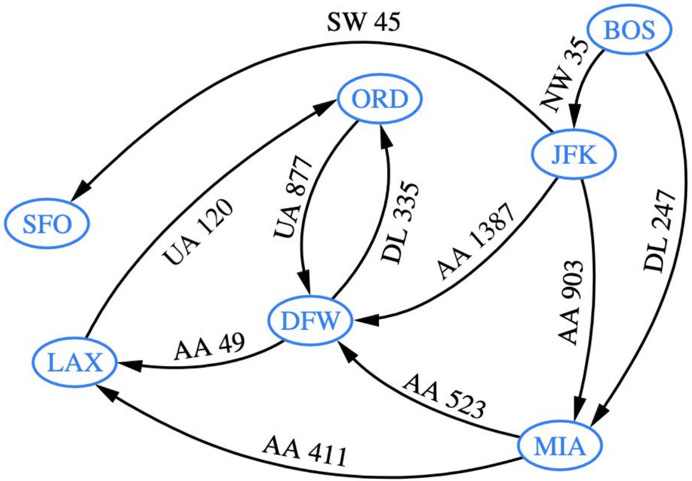
Directed Edge

Undirected Edge



Self Edge
(Unusual but usually allowed)

Directed vs Undirected Graphs



Example of a directed graph representing a flight network.

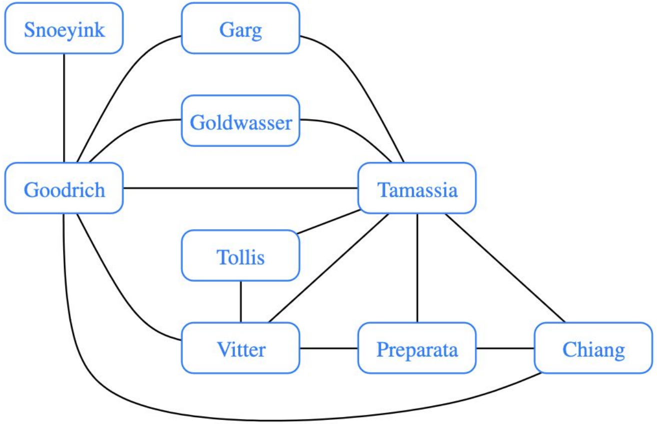
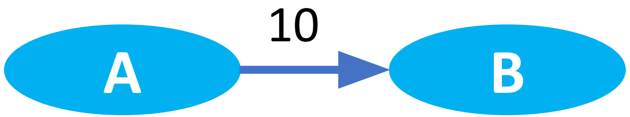


Figure 14.1: Graph of coauthorship among some authors.

Weighted Graphs

Edges have weights/costs



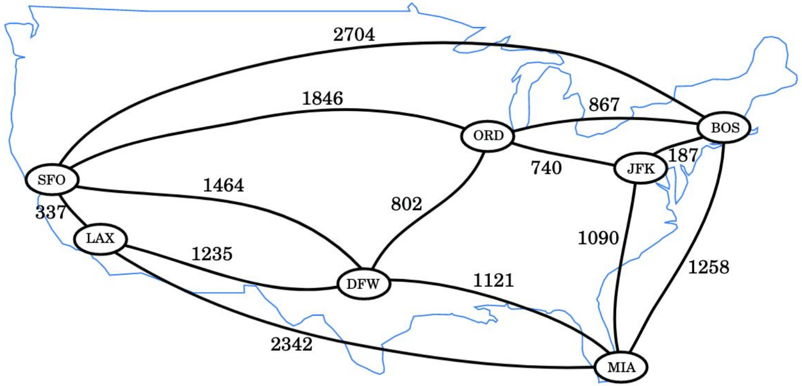
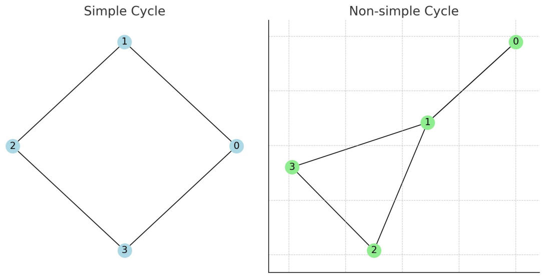


Figure 14.14: A weighted graph whose vertices represent major U.S. airports and whose edge weights represent distances in miles. This graph has a path from JFK to LAX of total weight 2,777 (going through ORD and DFW). This is the minimum-weight path in the graph from JFK to LAX.

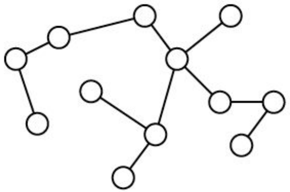
More terminology



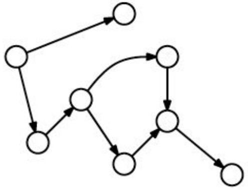
- A *path* in a graph is a sequence of vertices $\langle v_0, \ldots, v_k \rangle$ such that (v_{i-1}, v_i) is an edge for $i = 1, \ldots, k$
- The *length* of a path is the number of edges, k.
- A *cycle* is a path containing at least one edge and for which $v_0 = v_k$
- A cycle is simple if its edges and vertices (except for v_0 and v_k) are distinct.

More terminology

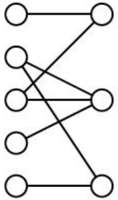
- An *acyclic* graph contains no simple cycles
- An acyclic connected graph is a *tree*
- The vertices of a *bipartite* graph can be partitioned into two disjoint subsets, V_1 and V_2 such that all edges have one endpoint in V_1 and the other one in V_2



(Free) Tree



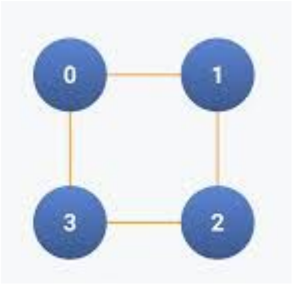
DAG

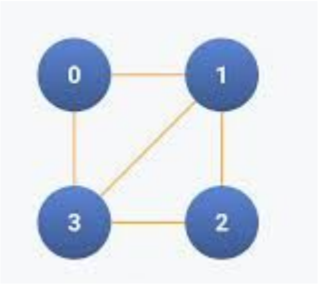


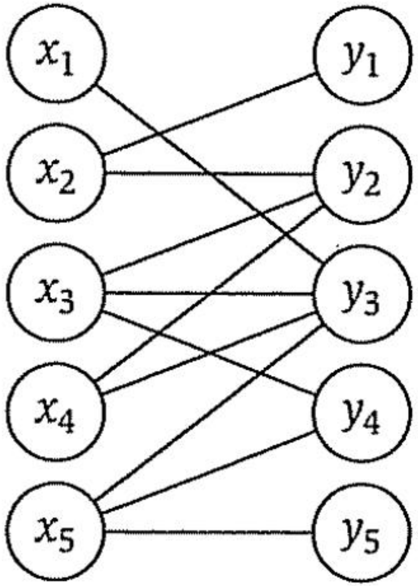
Bipartite graph

Bipartite Graphs

Is this graph bipartite?







Basic Graph Combinatorics

For an undirected graph:

- $|E| = 0 \leq m \leq \binom{n}{2}$
- $= \frac{n(n-1)}{2} \in O(n^2)$
- $\sum_{v \in V} \deg(v) = 2m$

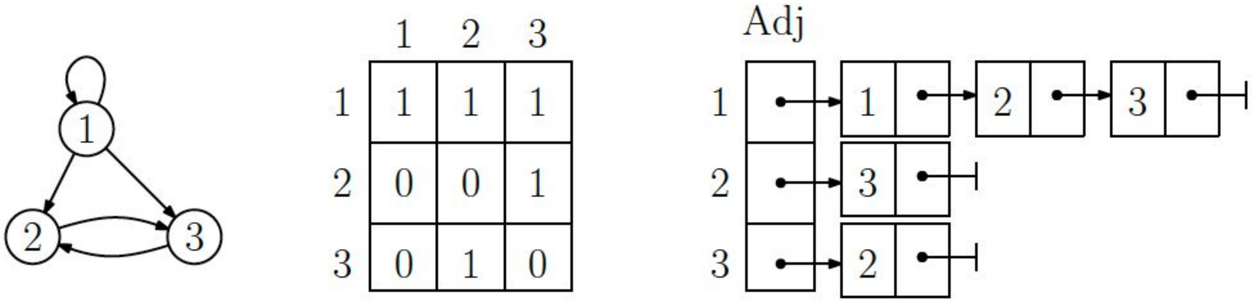
Basic Graph Combinatorics

For a directed graph:

- $|E| = 0 \leq m \leq n^2 - n$
- $\in O(n^2)$
- $\sum_{v \in V} \text{indeg}(v) = m$
- $\sum_{v \in V} \text{outdeg}(v) = m$

Representing a graph

- Adjacency matrix: An $n \times n$ matrix defined for $1 \leq$ (u, v) $\leq n$: A [u , v] $= 1$ if $(u, v) \in E$
- Adjacency list: An array of pointers where for $1 \leq v \leq n$, $Adj[v]$ points to a list containing the vertices that are adjacent to v



	1	2	3
1	1	1	1
2	0	0	1
3	0	1	0

Adjacency matrix

Adjacency list

Representing a graph - Adjacency List

Runtime Complexity: (In terms of V and E

- addVertex:
  - O(V)
- addEdge:
  - O(E)
- removeVertex:
  - O(V*E)
- removeEdge:
  - O(E)

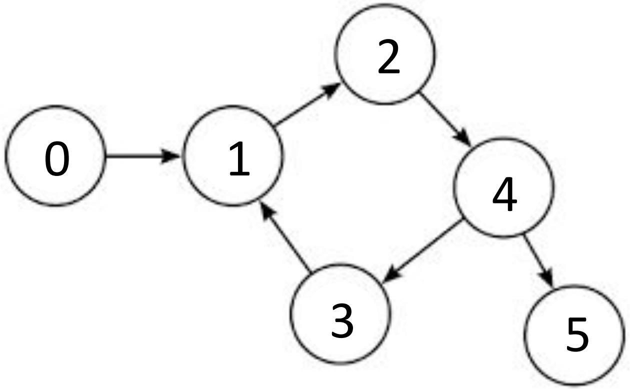
Representing a graph

Adjacency Matrix -

each index in the array is another array

Maintains an VxV matrix

where each slot (i,j) represents an outgoing edge from i to j



	1				
		1			
				1	
	1				
			1		1

Representing a graph - Adjacency Matrix

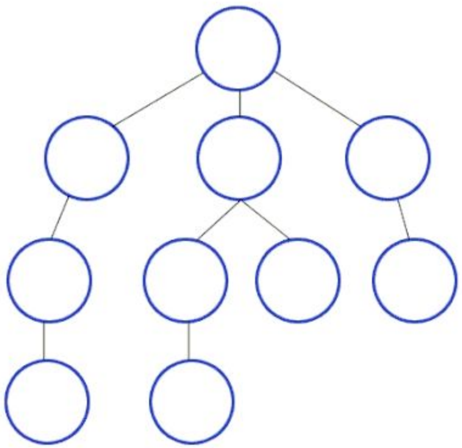
Runtime Complexity: (In terms of V and E)

- addVertex:
  - O(V^2) if we need to expand
- addEdge:
  - O(1)
- removeVertex:
  - O(V^2)
- removeEdge:
  - O(1)

Graph Traversals

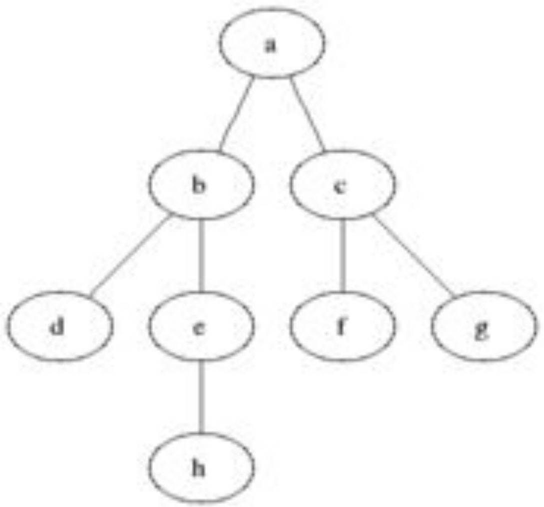
Depth First Search (DFS)

- start at root node and explore as far as possible along each branch
- Pre-order, in-order, and post-order traversals are DFS



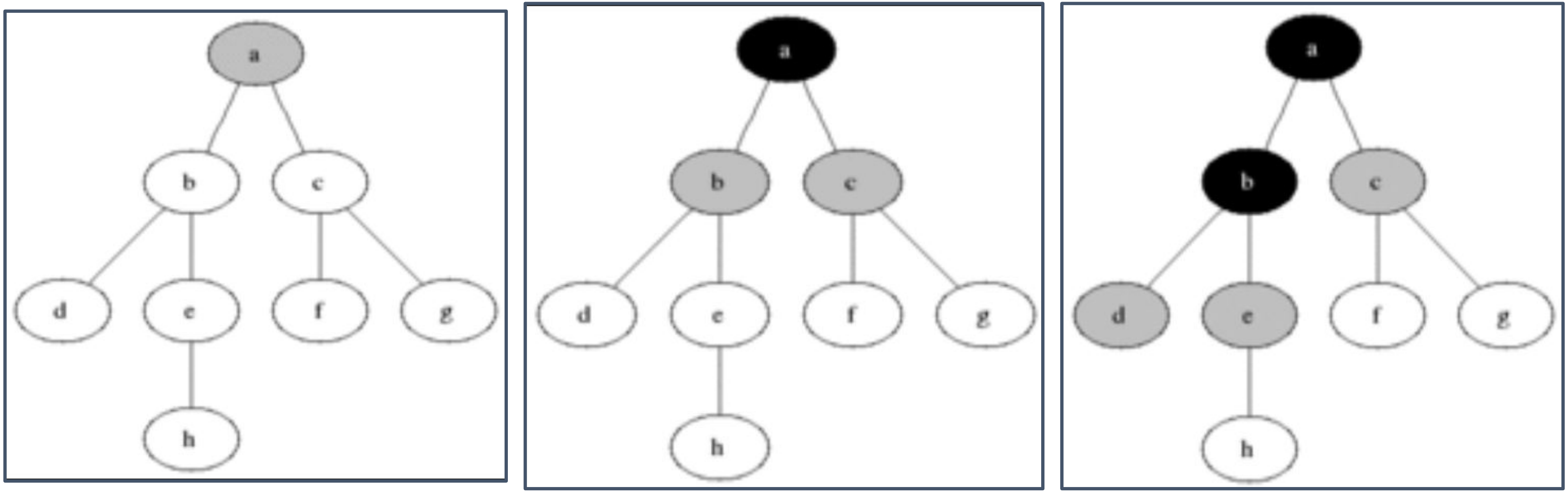
Breadth First Search (BFS)

- Starts at the root and explores all nodes at the present “depth” before moving to nodes on the next level
- Extra memory is usually required to keep track of the nodes that have not yet been explored

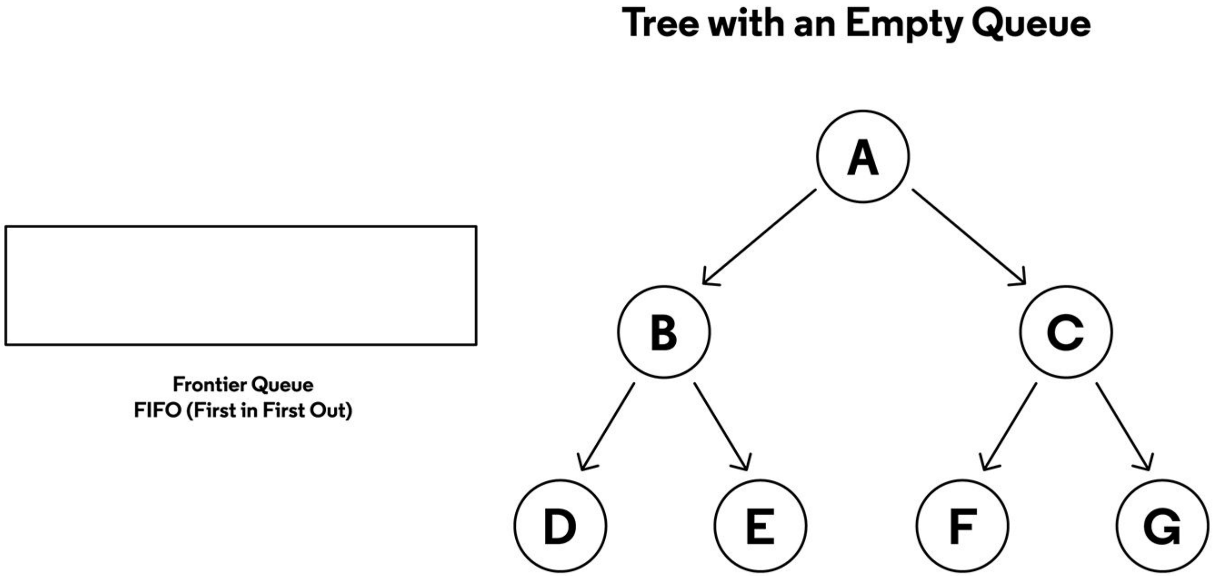


Breadth-First Traversal

pseudo-code?



Breadth First Search (BFS)



https://www.codecademy.com/article/tree-traversal

Runtime Analysis of BFS

$|V| = n, |E| = m$

1. Initialization:
   a. $O(n)$

2. Traversal
   a. While-loop: we visit each vertex once
   b. Nested for-loop: we visit each child of that vertex
      i. Number of iterations depends on the *degree*

$$
\begin{aligned}
T(n) &= n + \Sigma_{u \in V} (\deg(u) + 1) \\
&= n + (\Sigma_{u \in V} \deg(u)) + n = 2n + \Sigma_{u \in V} \deg(u) \\
&= 2n + 2m \\
&= O(n + m)
\end{aligned}
$$

Summary

1. Know the following:
    a. Exponent, logarithm, and summation rules and how to simplify equations
    b. Growth rate of classes of algorithms (logn grows slower than n which grows slower than n^2...)
    c. Asymptotic Upper Bound of functions (given a function write in Big-O notation)
    d. Data structures - asymptotic upper bound of basic operations
    e. Graphs - know terminology, basic rules, how to represent them, and traversal algorithms

2. Upcoming deadlines:
    a. Lab 0 due tomorrow
    b. HW1 due Sep 15th (next Monday)
    c. Continue reading textbook

3. Next class: more basics of algorithmic analysis
    a. **Quiz** on basic math rules, complexity, and DS review