# CS340 Analysis of Algorithms

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| **Handout:** | **2** | **Professor:** | **Dianna Xu** |
| **Title:** | **Dijkstra's Shortest Path** | **E-mail:** | `dxu@cs.brynmawr.edu` |
| **Date:** | | **URL:** | **http://cs.brynmawr.edu/cs340** |

Sample description, pseudo code and proof of correctness for Dijkstra's shortest path algorithm. Please refer to lecture notes for details of the algorithm design and time analysis. Recall that given a directed graph $G=(V,E)$, with associated edge weights $w(u,v)$ for each edge $(u,v) \in E$, and a source vertex $s \in V$, we want to compute the shortest path from $s$ to all vertices in $V$.

## 1 Description

The key idea of Dijstra's is to maintain an estimate of the shortest path length `d[v]` for each vertex $v$ from $s$, stored in an array `d` indexed by the vertices. The algorithm updates these estimates as it processes more and more vertices, a process known as relaxation. Every iteration, the algorithm will select the unvisited vertex $u$ with the smallest known distance from $s$, i.e. `d[u]` is minimum, mark it visited, and update the estimate of all of its neighbors, by comparing the existing estimate `d[v]` of a neighbor $v$ (note that `d[v]` currently stores the best known shortest path from $s$ to $v$ without going through $u$) with the distance of going from $s$ through $u$ to $v$ and update if necessary. A priority queue keyed by each `d[u]` is used to keep track of which vertex to process next.

## 2 Pseudocode

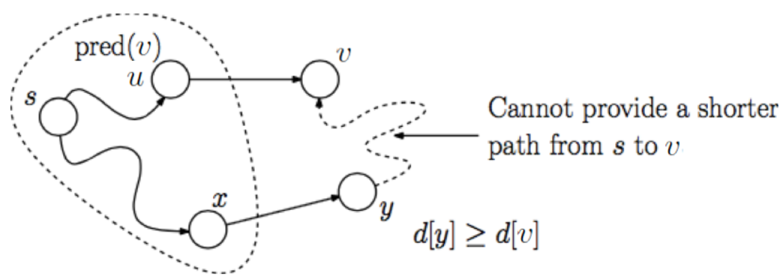
```
Function Dijkstra(G=(V,E),s)
    for each u∈V do
        d[u] = ∞
    end
    d[s]= 0
    Q = priority queue of all vertices u keyed by d[u]
    while Q is not empty do
        u = extractMin from Q
        for each v∈ Adj[u] do
            // if going through u to v is shorter
            if d[u] + w(u, v) < d[v] then
                d[v] = d[u] + w(u, v)
                decrease v's key value in Q to d[v]
            end
        end
    end
```

## 3 Correctness Proof

Termination is easily argued because the set $Q$ is finite and we delete at least one vertex in each iteration of the `while` loop. The inner `for` loop also teriminates because $G$ is finite.

We prove the optimality of `Dijstra's` by showing that the estimates are computed correctly for all processessed (visited) vertices. Let $\delta(s,v)$ denote the length of the true shortest path from $s$ to $v$.
**Lemma**: $d[v] = \delta(s,v)$ , $\forall v \in S$, where $S$ is the visited/processed set by Dijkstra's
**Proof**: by induction.

- Base case: $|S| = 1$, $S$ consists of only the source vertex $s$. We set $d[s]$ to 0 and $\delta(s,s) = 0$.
- IH: assume that $d[v] = \delta(s,v)$ , $\forall v \in S$, where $|S| = k$
- Want to show: lemma holds for $|S| = k+1$, that is, the $d[v]$ of the next vertex added to $S$ is also computed correctly.
  Let $v$ be the next vertex added to $S$, along with edge $(u,v)$. Note that $u$ must be a vertex already in $S$, because otherwise the shortest path will be disconnected. We argue that the true shortest path from $s$ to $v$ must be $d[u] + w(u,v)$. Suppose this is not true, and let us consider any other $s \longrightarrow v$ path $P$. Note that no matter which edges $P$ uses, because $v \notin S$, $P$ must have an edge that goes across the cut from $S$ to $V \setminus S$. Let $(x,y)$ be the first edge taken by $P$ where $x \in S$ and $y \in V \setminus S$. Note that it may be that $x = s$ and/or $y = v$, but $u$ and $x$ must be distinct.

  By the IH, $u$ and $x$ are both correctly processed and therefore $d[u] = \delta(s,u)$ and $d[x] = \delta(s,x)$ and in addtion, because relaxations were applied when processing $u$ and $x$, $d[y] = d[x] + w(x,y)$ and $d[v] = d[u] + w(u,v)$. By construction, because Dijkstra's selected $u$ and not $y$ as the next vertex to process, we must have $d[v] \leq d[y]$. Therefore, we have

  $$\delta(s,v) \leq d[v] \leq d[y] = \delta(s,y)$$

  With a $d[y]$ already greater than $d[v]$, and we know that $G$ does not contain any edges with negative weights, there is no way for us to construct a $P$ that connects $s$ to $y$ then to $v$ that can achieve a total length shorter than $d[v]$. Thus we have our contradiction. ■