Summary

We imagine two towns (Town 1 and Town 2) which each employ their populations in different industries, which are publicly supported through public investment. We are interested in the migration dynamics between the two towns as people seek better economic opportunities.

For a given town ($i \in \{1, 2\}$), each person is an employee of their town's industry, and takes home a certain salary S_i each month, based on how many total goods were produced in their town (Y_i is output produced). Taxes are taken from these salaries every month at some rate c_i (the rate may differ between towns), and 100% of the tax revenue is reinvested in the infrastructure. The health of the infrastructure H_i then impacts the productivity of that industry. At the end of the month, workers can choose whether to stay in their current town, or migrate and become an employee of the other town. This will change the population N_i of each town. Subsequently, the population change will effect the productivity of the industry in the next month, as well as the future health of the infrastructure, since the tax base will grow or shrink with population.

We are assuming this is a closed system, so any employees who choose to leave Town 1 will go to Town 2, and vice versa.

Variables

- P_i: population of workers in Town i (for simplicity, assume that every resident is also an employee of Industry i)
- Y_i: production output from Industry i
- S_i: salary of workers in Town i, a function of productive output Y_i, tax rate c, and corporate greed g (i.e. how much of the total revenue from production does the company take for its own profits)
- H_i: health of Industry i's infrastructure. For simplicity, we assume that productive output Y_i increases linearly with infrastructure health H_i.

When deciding whether to move, employees will compare their monthly salary S_i to the salary of their neighbors in the other town S_j. If they could make more money in the other town, they may decide to move. However, moving is costly and inconvenient, so the incentive to move must be greater than some inertial threshold m_i. We could have this threshold be the same for all employees, or we could try having a distribution of inertial values among the population.

Parameters

- $c = 0.10$: tax rate in Town i (a fraction between 0 and 1)
- $g = 0.15$: corporate greed for Industry i (a fraction between 0 and 1)
- $Y_0 = 4e^6$: minimum output produced when $P = P_0$
- $P_0 = 1000$: minimum number of employees needed
- $S_0 = 4000$: take-home salary at equilibrium number of employees in dollars
- $A = 4000000$
- $B = 100000$
- $C = 1$

- $H_0 = 0$: Infrastructure health equilibrium in dollars
- $\delta = 0.5$: Percentage rate of Infrastructure degradation
- $\gamma = 0.1$: Percentage weight of Infrastructure on output
- $r = 0.15$: Emigration inertia
- $D = 500$: Immigration inertia

System: Differential equations

$$Y = \gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0 \tag{1}$$

$$S = (1 - g - c)\frac{Y}{P} \tag{2}$$

$$\frac{dH}{dt} = -\delta' H + c' Y \tag{3}$$

$$\frac{dP^{\mathrm{S_u > S}}}{dt} = -r' \frac{S_u - S}{S_u} P_n \tag{4}$$

$$\frac{dP^{\mathrm{S_u < S}}}{dt} = \frac{S - S_u}{S_u} D' \tag{5}$$

System: Difference Equations

$$Y_n = \gamma(H_n - H_0) + \frac{A(P_n - P_0)}{B + C(P_n - P_0)} + Y_0 \tag{6}$$

$$S_n = (1 - g - c)\frac{Y_n}{P_n} \tag{7}$$

$$H_{n+1} = H_n - \delta H_n + c Y_n \tag{8}$$

$$P_{n+1}^{\mathrm{S_u > S_n}} = P_n - r \frac{S_u - S_n}{S_u} P_n \tag{9}$$

$$P_{n+1}^{\mathrm{S_u < S_n}} = P_n + \frac{S_n - S_u}{S_u} D \tag{10}$$

With everything factored in, the governing equations are:

$$Y = \gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0 \tag{11}$$

$$S = \frac{(1 - g - c)}{P}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{12}$$

$$\frac{dH}{dt} = -\delta' H + c'\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{13}$$

$$\frac{dP^{\mathrm{S_u > S}}}{dt} = -r' P + \frac{(1 - g - c) r'}{S_u}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{14}$$

$$\frac{dP^{\mathrm{S_u < S}}}{dt} = -D' + \frac{(1 - g - c) D'}{P S_u}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{15}$$

$\frac{dP}{dt} = -D + \frac{(1-g-c)D}{PS_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)$

Note that $\frac{dP}{dt}$ is a piecewise function. The two pieces are identical only when $S_u = S$. When that condition is satisfied, H and P are related by nullcline $\frac{dP}{dt} = 0$.

That equation is as follows:

$$\begin{aligned}
S_u &= (1-g-c)\left(\frac{Y}{P}\right) \\
S_u &= \frac{(1-g-c}{P}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
\frac{-(1-g-c)\gamma}{P}H &= \frac{(1-g-c}{P}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) - S_u
\end{aligned}$$

Which comes to

$$H = \frac{S_u P}{(1-g-c)\gamma} - \frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{16}$$

Finding Nullclines

Let's find the nullcline for $\frac{dH}{dt} = 0$. Here's the algebra:

$$\begin{aligned}
0 &= -\delta' H_n + c' Y_n \\
0 &= -\delta' H_n + c'\left(\gamma(H_n-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
\delta' H - c'\gamma H &= +c'\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
H(\delta' - c'\gamma) &= +c'\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)
\end{aligned}$$

Now for $\frac{dP}{dt}^{\mathrm{S_u>S}} = 0$. Here is the algebra:

$$\begin{aligned}
0 &= -r'P\frac{S_u - S}{S_u} \\
0 &= -r'P\left(\frac{S_u - (1-g-c)\frac{Y}{P}}{S_u}\right) \\
0 &= -r'P\left(\frac{S_u - \frac{(1-g-c)}{P}\left(\gamma(H_n-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)}{S_u}\right) \\
0 &= S_u - \frac{(1-g-c)}{P}\left(\gamma(H_n-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
\frac{\gamma(1-g-c)}{P}H &= S_u - \frac{(1-g-c)}{P}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)
\end{aligned}$$

So the nullcline for $\frac{dP}{dt}^{\mathrm{S_u>S}} = 0$ is:

$$H = \frac{S_u P}{\gamma(1-g-c)} - \frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{17}$$

Finally, for $\frac{dP}{dt}^{\mathrm{S_u<S}} = 0$. Here is the algebra:

$$\begin{aligned}
0 &= \frac{S - S_u}{S_u} D' \\
0 &= \frac{(1-g-c)\frac{Y}{P} - S}{S_u} D' \\
0 &= \frac{\frac{(1-g-c)}{P}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) - S_u}{S_u} D' \\
0 &= \frac{(1-g-c)}{P}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) - S_u \\
0 &= (1-g-c)\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) - S_u P \\
-\gamma(1-g-c)H &= (1-g-c)\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) - S_u P \\
H &= -\frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) + \frac{S_u P}{\gamma(1-g-c)}
\end{aligned}$$

Which is the same as

$$H = \frac{S_u P}{\gamma(1-g-c)} - \frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{18}$$

In summary For $\frac{dH}{dt} = 0$, the nullcline is

$$H = \frac{c'}{\delta' - c'\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{19}$$

and for $\frac{dP}{dt} = 0$, the nullcline is

$$H = \frac{S_u P}{\gamma(1-g-c)} - \frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{20}$$

Note that this second equation is the same whether we use $\frac{dP}{dt}^{\mathrm{S_u>S}} = 0$ or $\frac{dP}{dt}^{\mathrm{S_u<S}} = 0$.

Finding Fixed Points

We set the nullcines equal to each other and solve for P:

$$\frac{c'}{\delta' - c'\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) = \frac{S_u P}{\gamma(1-g-c)} - \frac{1}{\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)$$

A lot of algebra ensues. The following parameter groupings help us stay sane:

$$
\begin{aligned}
m &= \frac{S_u}{\gamma(1-g-c)} \\
n &= \left(\frac{1}{\gamma} + \frac{c'}{\delta' - c'\gamma}\right) \\
a &= Y_0 - \gamma H_0 \\
b &= \frac{m}{n} = \frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)} \\
b^2 &= \frac{S_u^2(\delta' - c'\gamma)^2}{\delta'^2(1-g-c)^2} \\
ab &= \frac{S_u(\delta' - c'\gamma)(Y_0 - \gamma H_0)}{\delta'(1-g-c)} \\
a^2 &= (Y_0 - \gamma H_0)^2
\end{aligned}
$$

Setting

$$A_0 = bC \tag{21}$$

$$B_0 = bB - bCP_0 - A - aC \tag{22}$$

$$C_0 = AP_0 - aB + aCP_0 \tag{23}$$

we find our fixed points at:

$$P = \frac{-B_0 \pm \sqrt{B_0^2 - 4A_0C_0}}{2A_0} \tag{24}$$

Note the values of B_0^2 and $4A_0C_0$:

$$B_0^2 = bB^2 + b^2C^2P_0^2 + A^2 + a^2C^2 - 2b^2BCP_0 - 2bAB - 2abBC + 2bACP_0 + 2abC^2P_0 + 2aAC \tag{25}$$

$$4A_0C_0 = 4bACP_0 - 4acBC - 4abC^2P_0 \tag{26}$$

Therefore the discriminant is

$$B_0^2 - 4A_0C_0 = bB^2 + b^2C^2P_0^2 + A^2 + a^2C^2 - 2b^2BCP_0 - 2bAB + 2abBC - 2bACP_0 + 6abC^2P_0 + 2aAC \tag{27}$$

When we expand all that out, we get...

$$
\begin{aligned}
B_0^2 - 4A_0C_0 = {} & B^2\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + C^2P_0^2\left(\frac{S_u^2(\delta' - c'\gamma)^2}{\delta'^2(1-g-c)^2}\right) \\
& + A^2 \\
& + C^2(Y_0 - \gamma H_0)^2 \\
& - 2BCP_0\left(\frac{S_u^2(\delta' - c'\gamma)^2}{\delta'^2(1-g-c)^2}\right) \\
& - 2AB\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + 2BC\left(\frac{S_u(\delta' - c'\gamma)(Y_0 - \gamma H_0)}{\delta'(1-g-c)}\right) \\
& - 2ACP_0\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + 6C^2P_0\left(\frac{S_u(\delta' - c'\gamma)(Y_0 - \gamma H_0)}{\delta'(1-g-c)}\right) \\
& + 2AC(Y_0 - \gamma H_0)
\end{aligned}
$$

$\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right)$ With C=1, that is:

$$
\begin{aligned}
B_0^2 - 4A_0C_0 = {} & B^2\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + P_0^2\left(\frac{S_u^2(\delta' - c'\gamma)^2}{\delta'^2(1-g-c)^2}\right) \\
& + A^2 \\
& + (Y_0 - \gamma H_0)^2 \\
& - 2BP_0\left(\frac{S_u^2(\delta' - c'\gamma)^2}{\delta'^2(1-g-c)^2}\right) \\
& - 2AB\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + 2B\left(\frac{S_u(\delta' - c'\gamma)(Y_0 - \gamma H_0)}{\delta'(1-g-c)}\right) \\
& - 2AP_0\left(\frac{S_u(\delta' - c'\gamma)}{\delta'(1-g-c)}\right) \\
& + 6P_0\left(\frac{S_u(\delta' - c'\gamma)(Y_0 - \gamma H_0)}{\delta'(1-g-c)}\right) \\
& + 2A(Y_0 - \gamma H_0)
\end{aligned}
$$

So, in total, the fixed point P is:

$$P = \frac{-(B\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right) - CP_0\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right) - A - C(Y_0-\gamma H_0)) + \sqrt{B_0^2-4A_0C_0}}{2C\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right)}$$

With $C=1$, that is:

$$P = \frac{-\left(B\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right) - P_0\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right) - A - (Y_0-\gamma H_0)\right) + \sqrt{B_0^2-4A_0C_0}}{2\left(\frac{S_u(\delta'-c'\gamma)}{\delta'(1-g-c)}\right)} \tag{28}$$

Jacobian

To simplify our notation, we define:

$$F(H,P) = \frac{dH}{dt} \tag{29}$$

$$G_i(H,P) = \frac{dP}{dt} \tag{30}$$

where $i=1$ is the equation we use for the case where $S_u > S_n$ (that is Equation 4), and $i=2$ is the equation we use for the case where $S_u < S_n$ (that is Equation 5). Then our Jacobian is:

$$J_i = \begin{bmatrix} \frac{dF}{dH} & \frac{dF}{dP} \\ \frac{dG_i}{dH} & \frac{dG_i}{dP} \end{bmatrix} \tag{31}$$

Note that this really gives us two Jacobians, J_1 and J_2, depending on which function $G_i(H,P)$ we are using.

The following equations give our partial derivatives.

$$\begin{aligned}
\frac{dF}{dH} &= -\delta' + c'\gamma \\
\frac{dF}{dP} &= \frac{c'AB}{(B+C(P-P_0))^2} \\
\frac{dG_1}{dH} &= \frac{-r\gamma(1-g-c)}{S_uP} \\
\frac{dG_1}{dP} &= -r' + \frac{ABr'(1-g-c)}{S_u(B+C(P-P_0))^2} \\
\frac{dG_2}{dH} &= \frac{(1-g-c)D'\gamma}{S_uP} \\
\frac{dG_2}{dP} &= -\frac{1}{P^2}\left(\frac{(1-g-c)D'}{S_u}\right)\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P0)} + Y_0\right) \\
&\quad + \frac{-CP^2+BP_0+2CP_0P-CP_0^2}{(P(B+C(P-P_0)))^2}\left(\frac{D'A(1-g-c)}{S_u}\right)
\end{aligned}$$

Why stop there?

System 2.0: Differential equations

$$Y = \gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0 \tag{32}$$
$$S = (1 - g - c)\frac{Y}{P} \tag{33}$$
$$W = \alpha * S_n + \beta H \tag{34}$$
$$\frac{dH}{dt} = -\delta' H + c' Y \tag{35}$$
$$\frac{dP}{dt}^{\mathrm{W_u > W}} = -r' \frac{W_u - W}{W_u} P_n \tag{36}$$
$$\frac{dP}{dt}^{\mathrm{W_u < W}} = \frac{W - W_u}{W_u} D' \tag{37}$$

With everything factored in, the governing equations are:

$$Y = \gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0 \tag{38}$$
$$S = \frac{(1 - g - c)}{P}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{39}$$
$$W = \frac{\alpha(1 - g - c)}{P}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) + \beta H \tag{40}$$
$$\frac{dH}{dt} = -\delta' H + c'\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) \tag{41}$$
$$\frac{dP}{dt}^{\mathrm{W_u > W}} = -r' P + \frac{r'\alpha(1 - g - c)}{W_u}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) + \frac{r'\beta H}{W_u} \tag{42}$$
$$\frac{dP}{dt}^{\mathrm{W_u < W}} = \frac{D'\alpha(1 - g - c)}{PW_u}\left(\gamma(H - H_0) + \frac{A(P - P_0)}{B + C(P - P_0)} + Y_0\right) + \frac{D'\beta H}{W_u} - D' \tag{43}$$

Nullclines for System 2

The nullcline for $\frac{dH}{dt}$ is the same as before:

$$H = \frac{c'}{\delta' - c'\gamma}\left(-\gamma H_0 + \frac{A(P-P_0)}{B + C(P-P_0)} + Y_0\right) \tag{44}$$

But we have to solve $\frac{dP}{dt}^{W_u > W}$ again. That algebra looks like this:

$$\begin{aligned}
0 &= -r'P + \frac{r'\alpha(1-g-c)}{W_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) + \frac{r'\beta H}{W_u} \\
0 &= -P + \frac{\alpha(1-g-c)}{W_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) + \frac{\beta H}{W_u} \\
-\frac{\alpha\gamma(1-g-c)H}{W_u} - \frac{\beta H}{W_u} &= -P + \frac{\alpha(1-g-c)}{W_u}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
H\left[-\frac{\alpha\gamma(1-g-c)}{W_u} - \frac{\beta}{W_u}\right] &= -P + \frac{\alpha(1-g-c)}{W_u}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)
\end{aligned}$$

If we set $M = \frac{-\alpha\gamma(1-g-c)-\beta}{W_u}$, then the nullcline for $\frac{dP}{dt}^{W_u > W}$ is:

$$H = -\frac{P}{M} + \frac{\alpha(1-g-c)}{W_u M}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{45}$$

Additionally, in this case, we cannot expect the nullcline to be the same for both halves of the piecewise function. So we must also find the nullcline for $\frac{dP}{dt}^{W_u < W}$:

$$\begin{aligned}
0 &= -D' + \frac{D'\beta H}{W_u} + \frac{D'\alpha(1-g-c)}{PW_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
0 &= -1 + \frac{\beta H}{W_u} + \frac{\alpha(1-g-c)}{PW_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
0 &= -P + \frac{\beta HP}{W_u} + \frac{\alpha(1-g-c)}{W_u}\left(\gamma(H-H_0) + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
-\frac{\alpha\gamma(1-g-c)H}{W_u} - \frac{\beta HP}{W_u} &= -P + \frac{\alpha(1-g-c)}{W_u}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \\
H\left[-\frac{\alpha\gamma(1-g-c)}{W_u} - \frac{\beta P}{W_u}\right] &= -P + \frac{\alpha(1-g-c)}{W_u}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right)
\end{aligned}$$

If we set $N(P) = \frac{-\alpha\gamma(1-g-c)-\beta P}{W_u}$, then the nullcline is:

$$H = -\frac{P}{N(P)} + \frac{\alpha(1-g-c)}{W_u N(P)}\left(-\gamma H_0 + \frac{A(P-P_0)}{B+C(P-P_0)} + Y_0\right) \tag{46}$$

Importantly, Equation 45 is no longer equivalent to Equation 46. This makes it more complicated to calculate fixed points and stability.