5.4 多维随机变量函数的分布

本节研究的问题: 已知(X, Y)的分布, 求$Z = g(X, Y)$的分布.

如果$X_1, X_2, \cdots, X_n$是n维离散型随机变量, 那么$Z = g(X_1, X_2, \cdots, X_n)$为一维随机变量, 其分布列可以通过如下两步求得:

i) 对$X_1, X_2, \cdots, X_n$的各种取值, 计算Z对应的取值;

ii) 对相同的Z值, 合并其概率.

例5.9. 设(X, Y)的联合分布列为

$X \backslash Y$	0	1	2
0	1/4	1/6	1/8
1	1/4	1/8	1/12

求$Z_1 = X + Y, Z_2 = XY$的分布列.

解. 通过简单计算、合并可得Z_1和Z_2的分布列分别为:

Z_1	0	1	2	3
P	1/4	5/12	1/4	1/12

Z_2	0	1	2
P	19/24	1/8	1/12

.

□

对连续随机变量(X, Y), 其联合密度为$f(x, y)$, 如何求解$Z = g(X, Y)$的密度函数. 主要的解决思路是分布函数法, 即:

i) 求$Z = g(X, Y)$的分布函数

$$F_Z(z) = P(Z \le z) = P(g(x, y) \le z) = \int\int_{g(x,y)\le z} f(x, y)dxdy.$$

ii) 求Z的密度函数

$$f_Z(z) = F_Z^{'}(z).$$

5.4.1 极大极小分布

设随机变量X, Y相互独立, 其分布函数分别为$F_X(x), F_Y(y)$, 求$Z_1 = \max(X, Y)$, $Z_2 = \min(X, Y)$的分布函数. 求解Z_1的分布函数为

$$\begin{aligned} F_{Z_1}(z_1) &= P(Z_1 \le z_1) \\ &= P(\max(X, Y) \le z_1) = P(X \le z_1, Y \le z_1) \\ &= P(X \le z_1)P(Y \le z_1) = F_X(z_1)F_Y(z_1). \end{aligned}$$

进一步求解Z_2的分布函数为

$$\begin{aligned} F_{Z_2}(z_2) &= P(Z_2 \le z_2) \\ &= P(\min(X,Y) \le z_2) = 1 - P(\min(X,Y) > z_2) \\ &= 1 - P(X \ge z_2)P(Y > z_2) = 1 - (1 - F_X(z_2))(1 - F_Y(z_2)). \end{aligned}$$

上述结论可以进一步推广到n个独立的随机变量: 设$X_1, X_2, \cdots, X_n$为n个独立的随机变量, 其分布函数分别为$F_{X_i}(x)$, 则$Y = \max(X_1, X_2, \cdots, X_n)$, $Z = \min(X_1, X_2, \cdots, X_n)$的分布函数分别为

$$F_Y(y) = F_{X_1}(y)F_{X_2}(y)\cdots F_{X_n}(y), \quad F_Z(z) = 1 - (1 - F_{X_1}(z))(1 - F_{X_2}(z))\cdots(1 - F_{X_n}(z)).$$

特别地, 当$X_1, X_2, \cdots, X_n$ 独立同分布时, $F_Y(y) = (F_{X_1}(y))^n$, $F_Z(z) = 1 - (1 - F_{X_1}(z))^n$. 根据分布函数, 可以求得概率密度.

例5.10. 假设随机变量X与Y相互独立, 以及$X \sim e(\alpha)$, $Y \sim e(\beta)$, 求$Z_1 = \max(X,Y)$, $Z_2 = \min(X,Y)$的概率密度.

解. 由指数函数的定义可得随机变量X和Y的概率密度为

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \ge 0 \\ 0 & x \le 0 \end{cases} \qquad f_Y(y) = \begin{cases} \beta e^{-\beta y} & y \ge 0 \\ 0 & y \le 0 \end{cases}.$$

求解Z_1的分布函数为

$$F_{Z_1}(z_1) = F_X(z_1)F_Y(z_1) = \int_{-\infty}^{z_1} f_X(t)dt \int_{-\infty}^{z_1} f_Y(t)dt.$$

当$z_1 \le 0$时由$F_{Z_1}(z_1) = 0$; 当$z_1 > 0$时

$$F_{Z_1}(z_1) = \int_0^{z_1} f_X(t)dt \int_0^{z_1} f_Y(t)dt = \int_0^{z_1} \alpha e^{-\alpha t}dt \int_0^{z_1} \beta e^{-\beta y}dt = (1 - e^{-\alpha z_1})(1 - e^{-\beta z_1}).$$

两边对z_1求导可得其概率密度为

$$f_{Z_1}(z_1) = \begin{cases} \alpha e^{-\alpha z_1} + \beta e^{-\beta z_1} - (\alpha+\beta)e^{-(\alpha+\beta)z_1} & z_1 \ge 0 \\ 0 & z_1 < 0. \end{cases}$$

同理可求得随机变量Z_2的分布函数和概率密度分别为

$$F_{Z_2}(z_2) = \begin{cases} 1 - e^{-(\alpha+\beta)z_2} & z_2 \ge 0 \\ 0 & z_2 < 0 \end{cases} \qquad f_{Z_2}(z_2) = \begin{cases} (\alpha+\beta)e^{-(\alpha+\beta)z_2} & z_2 \ge 0 \\ 0 & z_2 < 0. \end{cases}$$

□

5.4.2 和的分布$Z = X + Y$

引理5.1. 设二维连续随机变量(X,Y)的联合密度为$f(x,y)$, 则$Z = X + Y$的概率密度为

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx \quad \text{或} \quad f_Z(z) = \int_{-\infty}^{+\infty} f(z-y, y)dy.$$

解. 首先求解分布函数

$$
\begin{aligned}
F_Z(z) \quad &= P(Z \le z) = P(X+Y \le z) \\
&= \int\int_{x+y\le z} f(x,y)dxdy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z-x} f(x,y)dy \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z} f(x,u-x)du \qquad (\text{变量替换} u = y+x) \\
&= \int_{-\infty}^{z} (\int_{-\infty}^{+\infty} f(x,u-x)dx)du
\end{aligned}
$$

两边对z求导数可得$Z = X+Y$的概率密度为

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x,z-x)dx.$$

□

定理5.7 (卷积公式)**.** 如果连续随机变量X与Y独立, 则$Z = X+Y$的密度函数为

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy.$$

如果离散随机变量X与Y独立, 其分布列分别为$a_i = P(X = i)$ $(i = 0,1,\cdots)$和$b_j = P(Y = j)$ $(j = 0,1,\cdots)$, 则$Z = X+Y$的分布列为

$$P(Z = X+Y = k) = \sum_{i=0}^{k} a_i b_{k-i}.$$

定理5.8. 若随机变量$X \sim B(n_1,p)$和$Y \sim B(n_2,p)$独立, 则$Z = X+Y \sim B(n_1+n_2,p)$.

由此定理可推出: 如果$X_i \sim Ber(p) = B(1,p)$, 那么$\sum_{i=1}^{n} X_i \sim B(n,p)$.

Proof. 由卷积公式可得

$$
\begin{aligned}
P[Z=k] &= \sum_{i=0}^{k} P[X=i]P[Y=k-i] \\
&= \sum_{i=0}^{k} \binom{n_1}{i} p^i(1-p)^{n_1-i} \binom{n_2}{k-i} p^{k-i}(1-p)^{n_2-(k-i)} \\
&= p^k(1-p)^{n_1+n_2-k} \sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i} \\
&= \binom{n_1+n_2}{k} p^k(1-p)^{n_1+n_2-k}.
\end{aligned}
$$

□

定理5.9. 若随机变量$X \sim P(\lambda_1)$和$Y \sim P(\lambda_2)$相互独立, 则$Z = X+Y \sim P(\lambda_1+\lambda_2)$.

Proof. 由泊松分布的定义可知: 当$i \geq 0$和$j \geq 0$时, 有

$$P(X=i)=\frac{\lambda_1^i}{i!}e^{-\lambda_1} \qquad P(Y=j)=\frac{\lambda_2^j}{j!}e^{-\lambda_2}.$$

根据卷积公式有

$$\begin{aligned} P(Z=k) &= \sum_{i=0}^{k} P(X=i,Y=k-i)=\sum_{i=0}^{k} P(X=i)P(Y=k-i) \\ &= \sum_{i=0}^{k}\frac{\lambda_1^i}{i!}\frac{\lambda_2^{k-i}}{(k-i)!}e^{-(\lambda_1+\lambda_2)}=\frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda_1^i\lambda_2^{k-i}=\frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k \end{aligned}$$

□

定理5.10. 若随机变量$X \sim \mathcal{N}(\mu_1,\sigma_1^2)$和$Y \sim \mathcal{N}(\mu_2,\sigma_2^2)$相互独立, 则$X+Y \sim \mathcal{N}(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$.

Proof. 首先证明$Z=X-\mu_1+Y-\mu_2 \sim \mathcal{N}(0,\sigma_1^2+\sigma_2^2)$, 其中$X-\mu_1 \sim \mathcal{N}(0,\sigma_1^2)$和$Y-\mu_2 \sim \mathcal{N}(0,\sigma_2^2)$. 根据卷积公式有

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx=\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma_1^2}-\frac{(z-x)^2}{2\sigma_2^2}}dx \\ &= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{+\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(x-\frac{\sigma_1^2 z}{\sigma_1^2+\sigma_2^2}\right)^2-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}dx \\ &= \frac{e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}\sigma_1\sigma_2}\int_{-\infty}^{+\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(x-\frac{\sigma_1^2 z}{\sigma_1^2+\sigma_2^2}\right)^2}dx \\ &= \frac{e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}. \end{aligned}$$

由此可得$X-\mu_1+Y-\mu_2 \sim \mathcal{N}(0,\sigma_1^2+\sigma_2^2)$, 进一步证明$X+Y \sim \mathcal{N}(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$. □

习题. 若随机变量$X \sim e(\lambda_1)$和$Y \sim e(\lambda_2)$相互独立, 求$Z=X+Y$的分布函数和概率密度.

例5.11. 设随机变量$X \sim U(0,1)$和$Y \sim U(0,1)$相互独立, 求$Z=X+Y$的概率密度.

解. 由卷积公式可得

$$f_Z(z)=\int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx.$$

由$X \sim U(0,1)$和$Y \sim U(0,1)$可知: 当$x \in [0,1]$时有$f_X(x)=1$; 当$z-x \in [0,1]$时有$f_Y(z-x)=1$, 即积分区域为$\{x \in [0,1], z-x \in [0,1]\}$. 由此可得:

- 当 $z \leq 0$ 或 $z \geq 2$ 时, 有$f_Z(z)=0$;
- 当 $z \in (0,1)$ 时, 有$f_Z(z)=\int_0^z 1dz=z$;
- 当 $x \in [1,2)$ 时, 有$f_Z(z)=\int_{z-1}^1 dx=2-z$.

□

例5.12. 设随机变量$X \sim U(0,1)$和$Y \sim e(1)$相互独立, 求$Z = X + Y$的概率密度.

解. 由卷积公式有

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx.$$

由于$X \sim U(0,1)$和$Y \sim e(1)$可知$f_X(x)f_Y(z-x) \neq 0$的区域为$\{x \in [0,1], z \geq x\}$. 因此有:

- 当$z \leq 0$时有$f_Z(z) = 0$;
- 当$0 \leq z \leq 1$时有$f_Z(z) = \int_0^z e^{-(z-x)} dx = e^{-z}(e^z - 1) = 1 - e^z$;
- 当$z \geq 1$时, $f_Z(z) = \int_0^1 e^{-(z-x)} dx = e^{-z}(e^1 - 1) = (e-1)e^{-z}$.

□

5.4.3 随机变量的乘/除法分布

设二维连续连续随机变量(X, Y)的联合概率密度为$f(x, y)$, 则随机变量$Z = XY$的概率密度为

$$f_{XY}(z) = \int_{-\infty}^{+\infty} \frac{1}{|x|} f(x, \frac{z}{x}) dx.$$

随机变量$Z = Y/X$的概率密度为

$$f_{Y/X}(z) = \int_{-\infty}^{+\infty} |x| f(x, xz) dx.$$

这里给出除法$Z = Y/X$概率密度的详细证明(同理证明$Z = XY$). 考虑分布函数

$$\begin{aligned}
F_{Y/X}(z) &= P(Y/X \leq z) = \int\int_{y/x \leq z} f(x, y) dx dy \\
&= \int\int_{x<0, y \geq zx} f(x, y) dx dy + \int\int_{x>0, y \leq zx} f(x, y) dx dy \\
&= \int_{-\infty}^{0} dx \int_{zx}^{+\infty} f(x, y) dy + \int_{0}^{+\infty} dx \int_{-\infty}^{xz} f(x, y) dy.
\end{aligned}$$

变量替换$t = y/x$有

$$\begin{aligned}
F_{Y/X}(z) &= \int_{-\infty}^{0} dx \int_{z}^{-\infty} x f(x, tx) dt + \int_{0}^{+\infty} dx \int_{-\infty}^{z} x f(x, tx) dt \\
&= \int_{-\infty}^{0} \int_{-\infty}^{z} (-x) f(x, tx) dt dx + \int_{0}^{+\infty} \int_{-\infty}^{z} x f(x, tx) dt dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{z} |x| f(x, tx) dt dx = \int_{-\infty}^{z} dt \int_{-\infty}^{+\infty} |x| f(x, tx) dx.
\end{aligned}$$

最后求导得到概率密度.

5.4.4 随机变量的联合分布函数

随机变量(X,Y)的联合概率密度为$f(x,y)$, 设(X,Y)的函数

$$U=g_1(X,Y)\quad V=g_2(X,Y)$$

如何求(U,V)的联合分布, 有如下结论:

定理5.11. *若$U=g_1(X,Y)$和$V=g_2(X,Y)$有连续偏导, 且存在反函数*

$$x=x(u,v)\quad y=y(u,v),$$

则(U,V)的联合密度为

$$f_{UV}(u,v)=f_{XY}(x(u,v),y(u,v))|J|$$

其中J为变换的雅可比行列式, 即$|J|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\frac{\partial(u,v)}{\partial x\partial y}\right|^{-1}=\begin{vmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{vmatrix}^{-1}$.