Optimization Methods **Fall 2019**

Homework 1

Instructor: Lijun Zhang *Name:* Chenxiao Gao, *StudentId:* 181220014

Notice

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Problem 1: Norms

A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom} f = \mathbb{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$

We use the notation $f(x) = \|x\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup\{z^T x | \|x\| \leq 1\}$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of the Euclidean norm (ℓ_2-norm) is the Euclidean norm, *i.e.*, prove that

$$\|z\|_{2*} = \sup\{z^T x | \|x\|_2 \leq 1\} = \|z\|_2$$

.

(*Hint:* Use Cauchy–Schwarz inequality.)

Solution.

a) We can prove that $f(x)$ satisfies the abovementioned properties:

1) randomly choose an vector x that satisfies $\|x\| \leq 1$, and it is trivial that $\|-x\| \leq 1$. If $z^T x \geq 0$, then we have $\|z\|_* = sup\{z^T x | \|x\| \leq 1\} \geq 0$, or else $\|z\|_* \geq z^T(-x) \geq 0$. Thus, $\|\cdot\|_*$is nonnegative.
2) based on 1), if $\|z\|_* = 0$ then we have $z^T x, z^T(-x) = -z^T x \leq 0$, which denotes that for all x $\in \mathbb{R}^n, z^T x = -z^T x = 0$, and $z^T = 0$. if $z^T = 0$, it is trivial that $\|z\|_* = 0$. Thus, $\|\cdot\|_*$ is definite.
3) for all $x \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$,
   if $t \geq 0$, then $\|tz\|_* = sup\{tz^T x | \|x\| \leq 1\} = tsup\{z^T x | \|x\| \leq 1\} = |t|\|z\|_*$(t is a constant), which means $\|\cdot\|_*$ is homogeneous.
   else if $t \leq 0$, then

$$\begin{aligned}
\|tz\|_* &= \|-t(-z)\|_* \\
&= sup\{(-t)(-z)^T(-x) | \|-x\| \leq 1\} \\
&= (-t)sup\{(-z)^T(-x) | \|-x\| \leq 1\} \\
&= |t|sup\{z^T x | \|-x\| \leq 1\} \\
&= |t|sup\{z^T x | \|x\| \leq 1\} \\
&= |t|\|z\|_*
\end{aligned}$$

   Therefore, $\|\cdot\|_*$ is homogeneous.

4) For all $x, y \in v$, $\|x+y\|_* = sup\{x^T z + y^T z | \|z\| \leq 1\} \leq sup\{x^T z | \|z\| \leq 1\} + sup\{y^T z | \|z\| \leq 1\} = \|x\|_* + \|y\|_*$, and the equality holds if and only if $x^T z$ and $y^T z$ achieves their maximum at the same point. Thus $\|\cdot\|$satisfies the triangle inequality.

b) From Cauchy–Schwarz inequality, we can show that $|z^T x|$ achieves maximum when $x = \frac{z}{\|z\|_2}$. Noticing that $\frac{z}{\|z\|_2} \leq 1$, we can conclude that$\|z\|_* = sup\{z^T x | \|x\| \leq 1\} = \frac{z^T z}{\|x\|_2} = \|z\|_2$, and the equality holds when $x = \frac{z}{\|z\|_2} or \frac{-z}{\|z\|_2}$, depending on the sign of $z^T x$. Q.E.D.

□

Problem 2: Affine and Convex Sets

Affine sets C_a and convex C_c sets are the sets satisfying the constraints below:

$$\begin{aligned} \theta x_1 + (1-\theta) x_2 \in C_a \\ \text{s.t. } x_1, x_2 \in C_a \end{aligned} \tag{1}$$

$$\begin{aligned} \theta x_1 + (1-\theta) x_2 \in C_c \\ \text{s.t. } x_1, x_2 \in C_c, 0 \geq \theta \leq 1 \end{aligned} \tag{2}$$

a) Is the set $\{\alpha \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$, where $p(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_k t^{k-1}$, affine?

b) Determine if each set below is convex.

1) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \leq 1\}$.
2) $\{(x, y) \in \mathbf{R}^2_{++} | x/y \geq 1\}$.
3) $\{(x, y) \in \mathbf{R}^2_{+} | xy \leq 1\}$.
4) $\{(x, y) \in \mathbf{R}^2_{+} | xy \geq 1\}$.
5) $\{(x, y) \in \mathbf{R}^2 | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$.

Solution.

a) Not necessarily.

proof:

Define $S = \{\alpha \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$for all α_1 and $\alpha_2 \in \mathbb{R}^n$

Consider $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 0, -2), t \in [-1, 1]$, we can see that both α_1 and α_2 belongs to S.

However, for $\alpha_* = \theta\alpha_1 + (1-\theta)\alpha_2$, the absolute value of the polynomial $p(t) = 1 + (\theta - 2)t^2$ is not necessarily below 1.

Thus, the set S is not necessarily affine.

b) Define S as the following sets.

1) Convex.

proof:

For all $(x_1, y_1) and (x_2, y_2) \in \mathbb{R}^2_{++}$, we have $x_1 \leq y_1, x_2 \leq y_2$, so $\theta x_1 + (1-\theta)x_2 \leq \theta y_1 + (1-\theta) y_2$.

$\because \theta, (1-\theta), y_1, y_2 \geq 0$

$\therefore \frac{\theta x_1 + (1-\theta) x_2}{\theta y_1 + (1-\theta) y_2} \leq 1$

Therefore the set S is convex.

2) Convex.

proof:

For all $(x_1, y_1) and (x_2, y_2) \in \mathbb{R}^2_{++}$, we have $x_1 \geq y_1, x_2 \geq y_2$, so $\theta x_1 + (1-\theta)x_2 \geq \theta y_1 + (1-\theta) y_2$.

$\because \theta, (1-\theta), y_1, y_2 \geq 0$

$\therefore \frac{\theta x_1 + (1-\theta) x_2}{\theta y_1 + (1-\theta) y_2} \geq 1$

Fherefore the set S is convex.

3) Not convex.
   proof:
   Assign $(x_1, y_1) = (2, 1/2), (x_2, y_2) = (2, 1/2), \theta = 1/2$
   $\therefore (\theta x_1 + (1-\theta)x_2)(\theta y_1 + (1-\theta)y_2) = (5/4)^2 = 25/16 > 1$
   Thus, the set S is not convex.
4) Convex.
   proof:
   $\because x_1y_1 \geq 1, x_2y_2 \geq 1$
   $\therefore x_1y_2 + x_2y_1 \geq \frac{x_1}{x_2} + \frac{x_2}{x_1} \geq 2$
   $\therefore (\theta x_1 + (1-\theta)x_2)(\theta y_1 + (1-\theta)y_2) = (x_2y_2) + (\theta - \theta^2)(2 - x_1y_2 - x_2y_1)$
   $\because (\theta - \theta^2) \leq 0, 2 - x_1y_2 - x_2y_1 \leq 0$
   $\therefore (\theta x_1 + (1-\theta)x_2)(\theta y_1 + (1-\theta)y_2) \geq 1$
   Therefore the set S is convex.
5) Not Convex.
   proof:
   Assign $(x_1, y_1) = (0, 0), (x_2, y_2) = (1, \frac{e-e^{-1}}{e+e^{-1}}), \theta = 1/2$
   Towever, $\frac{1}{2}y_1 + \frac{1}{2}y_2 \neq tanh(\frac{1}{2}x_1 + \frac{1}{2}x_2)$
   Therefore, the set S is not convex.

□

Problem 3: Examples

a) Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n | x^\top Ax + b^\top x + c \leq 0\}, \tag{3}$$

   with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

   1) Show that C is convex if $A \succeq 0$.
   2) Is the following statement true? The intersection of C and the hyperplane defined by $g^\top x + h = 0$ is convex if $A + \lambda gg^\top \succeq 0$ for some $\lambda \in \mathbb{R}$.

b) The polar of $C \subseteq \mathbb{R}^n$ is defined as the set

$$C^\circ = \{y \in \mathbb{R}^n | y^\top x \leq 1 \text{ for all } x \in C\}$$

   1) Show that C° is convex.
   2) What is a polar of a polyhedra?
   3) What is the polar of the unit ball for a norm $|| \cdot ||$?
   4) Show that if C is closed and convex, with $0 \in C$, then $(C^\circ)^\circ = C$

Solution.

a) 1) proof:
   C is convex
   $\Leftrightarrow$for any $x_1 \in C$, $x_2 \in C$, and $\theta \in [0, 1]$, $\theta x_1 + (1-\theta)x_2 \in C$
   $\Leftrightarrow (\theta x_1 + (1-\theta)x_2)^T A(\theta x_1 + (1-\theta)x_2) + b^T(\theta x_1 + (1-\theta)x_2) + c \leq 0$
   $\Leftrightarrow \theta^2 x_1^T Ax_1 + (1-\theta)^2 x_2^T Ax_2 + \theta(1-\theta)x_1^T Ax_2 + \theta(1-\theta)x_2^T Ax_1 + b^T(\theta x_1 + (1-\theta)x_2) + c \leq 0$
   (considering the shapes of $x_1^T Ax_2$ and $x_2^T Ax_1$ are both 1*1, $x_1^T Ax_2$ actually equals to $x_2^T Ax_1$)
   $\Leftrightarrow \theta^2 x_1^T Ax_1 + (1-\theta)^2 x_2^T Ax_2 + 2\theta(1-\theta)x_1^T Ax_2 + b^T(\theta x_1 + (1-\theta)x_2) + c \leq 0$

$\because b^T x_1 \leq -c - x_1^T A x_1, b^T x_1 \leq -c - x_2^T A x_2$
$\therefore P = \theta^2 x_1^T A x_1 + (1-\theta)^2 x_2^T A x_2 + 2\theta(1-\theta) x_1^T A x_2 + b^T(\theta x_1 + (1-\theta) x_2) + c$
$\leq \theta^2 x_1^T A x_1 + (1-\theta)^2 x_2^T A x_2 + 2\theta(1-\theta) x_1^T A x_2 + \theta(-c - x_1^T A x_1) + (1-\theta)(-c - x_2^T A x_2) + c$
$= \theta(\theta-1) x_1^T A x_1 + \theta(\theta-1) x_2^T A x_2 + 2\theta(1-\theta) x_1^T A x_2$
$= \theta(1-\theta)(2x_1^T A x_2 - x_1^T A x_1 - x_2^T A x_2)$
$= \theta(1-\theta)(x_1^T A(x_2 - x_1) + (x_1 - x_2)^T A x_2)$
$= \theta(1-\theta)(-(x_1 - x_2)^T A(x_1 - x_2))$
$\because A \in \mathbb{S}_+^n$
$\therefore \theta(1-\theta)(-(x_1 - x_2)^T A(x_1 - x_2)) \leq 0$
$\therefore P \leq 0$
This means the original proposition 'C is convex' is true.
Q.E.D.

2) the statement is true.
proof:
Let D stands for the hyperplane defined by $g^T x + h = 0$
First, it is trivial that any convex combination of $x_1, x_2 \in C \cup D$ also belongs to D, for hyperplanes are always convex.
So we only need to prove that the convex combination belongs to C.
For any $x_1 \in C \cup D$and $x_2 \in C \cup D$, we have $g^T x_1 + h = g^T x_2 + h = 0$
$\because g^T x_1 + h = g^T x_2 + h = 0$
$\therefore x_1^T g = x_2^T g = 0$
$\therefore x_1^T g g^T x_1 = x_2^T g g^T x_2 = 0$
$\therefore (x_1 - x_2)^T g g^T (x_1 - x_2) = 0$
$\because A + \lambda g g^T \in \mathbb{S}_+^n$
$\therefore (x_1 - x_2)^T (A + g g^T)(x_1 - x_2) \geq 0$
$\because (x_1 - x_2)^T g g^T (x_1 - x_2) = 0$
$\therefore (x_1 - x_2)^T A (x_1 - x_2) = (x_1 - x_2)^T (A + g g^T)(x_1 - x_2) - (x_1 - x_2)^T g g^T (x_1 - x_2) \geq 0$
This means the polynomial P defined in 1) satisfies $P \leq 0$
Therefore, the set C is convex, and $\theta x_1 + (1-\theta) x_2 \in C \cup D$
Q.E.D.

b) 1) proof:
For any $y_1, y_2 \in C^\circ$, we have $y_1^T x \leq 1$(for all $x \in C$) and $y_2^T x \leq 1$(for all $x \in C$).
$\therefore$ For all $x \in C$and $\theta \in [0,1]$, $(\theta y_1 + (1-\theta) y_2)^T x = \theta y_1^T x + (1-\theta) y_2^T x \leq \theta + (1-\theta) = 1$
$\therefore \theta y_1 + (1-\theta) y_2 \in C^\circ$
Therefore C° is convex.

2) solution:
Suppose P is a bounded polyhedron. (I am not sure whether polyhedra are all bounded, but here I suppose they are)
Therefore, P can be represented by $conv\{v_1, v_2, ...v_k\}$, for k finite, and any element from P can be represented by $\theta_1 v_1 + ... + \theta_k v_k$, where $\theta_i \geq 0$ and $1^T \theta = 1$
For $x \in P^\circ$, the inner product of x and $\theta_1 v_1 + ... + \theta_k v_k$ is below 1 no matter how θ varies.
So we let x satisfies $x^T v_i \leq 1, i = 1, 2...k$, so that $x^T(\theta_1 v_1 + ... + \theta_k v_k) = \theta_1 x^T v_1 + ... + \theta_k x^T v_k \leq \theta_1 + \theta_2 + ... + \theta_k = 1$
In the meantime, the set $\{x \in \mathbb{R}^n | x^T v_i \leq 1, i = 1, 2...k\}$ is actually a polygedron.
So, the polar of a polygedron is a polyhedron.

3) solution:
According to the definition of dual norm $\|\cdot\|_*$: $\|z\|_* = sup\{z^T x | \|x\| \leq 1\}$,
the polar of the unit ball for $\|\cdot\|$ is $\{y | \|y\|_* \leq 1\}$

4) proof:
We prove $(C^\circ)^\circ = C$ by 2 steps.
$(\supseteq)$:
Suppose $x \in C$.
$\therefore$ for all $y \in C^\circ$, $y^T x \le 1$
$\because y^T x = (y^T x)^T = x^T y$
$\therefore$ for all $y \in C^\circ$, $x^T y \le 1$
$\therefore x \in (C^\circ)^\circ$
$\therefore (C^\circ)^\circ \supseteq C$.
$(\subseteq)$:
Suppose there exists at least one element $x \in (C^\circ)^\circ$ but $x \notin C$.
According to the Separating Hyperplane Theorem, there exists a seperating hyperplane $a^T x = b$ between the non-intersecting sets C and $(C^\circ)^\circ - C$.
By adjusting the value of a, we can scale down the value of b to 1, so we simply assume the hyperplane here is $a^T x = 1$
$\because 0 \in C$ and C is a closed set
$\therefore$ for all $z \in C, a^T z \le 1$; for all $w \in (C^\circ)^\circ, a^T w > 1$
$\therefore a \in C^\circ$
However, the supposition we made before indicates that there exists one element $x \in (C^\circ)^\circ - C$, and for x we have $a^T x > 1$, and this contradicts with $x \in (C^\circ)^\circ$
Therefore $(C^\circ)^\circ - C = \emptyset$.
So $(C^\circ)^\circ \subseteq C$
In conclusion, $(C^\circ)^\circ = C$.

□

Problem 4: Operations That Preserve Convexity

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax + b}{c^\top x + d}, \psi(y) = \frac{Ey + f}{g^\top y + h}, \tag{4}$$

with domains **dom** $\phi = \{x | c^\top x + d > 0\}$, **dom** $\psi = \{y | g^\top y + h > 0\}$. We associate with ϕ and ψ the matrices

$$\begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}, \begin{bmatrix} E & f \\ g^\top & h \end{bmatrix}, \tag{5}$$

respectively.

Now, consider the composition Γ of ϕ and ψ, *i.e.*, $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}\Gamma = \{x \in \mathbf{dom}\ \phi | \phi(x) \in \mathbf{dom}\ \psi\}. \tag{6}$$

Show that Γ is linear-fractional, and that the matrix associate with it is the product

$$\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}. \tag{7}$$

Solution.

$$\begin{aligned} \because \Gamma(x) &= \psi(\phi(x)) \\ &= \frac{E\frac{Ax+b}{c^T x+d} + f}{g^T \frac{Ax+b}{c^T x+d} + h} \\ &= \frac{(EA + fc^T)x + Eb + fd}{(g^T A + hc^T)x + g^T b + hd} \end{aligned} \tag{8}$$

So apparently Γ is linear-fractional and it can be associated with the matrix

$$\begin{bmatrix} EA + fc^T & Eb + fd \\ g^T A & hc^T \end{bmatrix}, \tag{9}$$

which is exactly the product

$$\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}. \tag{10}$$

Q.E.D.

□

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following

1) K^* is indeed a convex cone.

2) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

Solution.

1) proof:

$\because$ for any $y_1, y_2 \in K^\star$, we have $x^T y_1 \geq 0, x^T y_2 \geq 0$ (for any $x \in K$).

$\therefore$ for any $x \in K$ and $\theta_1, \theta_2 \in [0, +\infty]$, $x^T(\theta_1 y_1 + \theta_2 y_2) = \theta_1 x^T y_1 + \theta_2 x^T y_2 \geq 0$.

$\therefore \theta_1 x_1 + \theta_2 x_2 \in K^\star$

Therefore, $K^\star$ is indeed a cone.

Q.E.D.

2) proof:

For any $y \in K_2^\star$, we always have $x^T y \geq 0$ where x is arbitrarily chosen from K_2.

Considering $K_1 \subseteq K_2$, $x^T y \geq 0$ still holds when x is from K_1.

So for any $y \in K_2^\star$, we always have $x^T y \geq 0$ where x is arbitrarily chosen from K_2, which means $y \in K_1^\star$

Therefore, $K_2^\star \subseteq K_1^\star$

Q.E.D.

□