Optimization Methods **Fall 2019**

Homework 3

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Notice

- The submission email is: **njuoptfall2019@163.com**.
- Please use the provided LaTeX file as a template. If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Equality Constrained Least-squares
Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m \times n}$ with $\mathbf{rank}\ A = n$, and $G \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\ G = p$.

a) Derive the Lagrange dual problem with Lagrange multiplier vector v.

b) Derive expressions for the primal solution $x^\star$ and the dual solution $v^\star$.

Solution. Write your solution here.

□

Problem 2: Support Vector Machines
Consider the following optimization problem

$$\text{minimize} \quad \sum_{i=1}^n \max\left(0, 1 - y_i(w^T x_i + b)\right) + \frac{\lambda}{2}\|w\|_2^2$$

where $x_i \in \mathbf{R}^d, y_i \in \mathbf{R}, i = 1, \cdots, n$ are given, and $w \in \mathbf{R}^d, b \in \mathbf{R}$ are the variables.

a) Derive an equivalent problem by introducing new variables $u_i, i = 1, \cdots, n$ and equality constraints

$$u_i = y_i(w^T x_i + b), i = 1, \cdots, n.$$

b) Derive the Lagrange dual problem of the above equivalent problem.

c) Give the Karush-Kuhn-Tucker conditions.

Hint: Let $\ell(x) = \max(0, 1 - x)$*. Its conjugate function* $\ell^*(y) = \sup_x (yx - \ell(x)) = \begin{cases} y, & -1 \le y \le 0 \\ \infty, & \textit{otherwise} \end{cases}$

Solution. Write your solution here.

□

Problem 3: Euclidean Projection onto the Simplex
Consider the following optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|y - x\|_2^2 \\ \text{subject to} & \mathbf{1}^T y = r \\ & y \succeq 0 \end{array}$$

where $r > 0$, $x \in \mathbb{R}^n$ is given, and $y \in \mathbf{R}^n$ is the variable. Give an algorithm to solve this problem and prove the correctness of your algorithm.
*Hint: Derive the Lagrangian of this problem and apply the Karush-Kuhn-Tucker conditions. If you need more hints, please read the following paper [**?**]*

Solution. Write your solution here.

□

Problem 4: Optimality Conditions
Consider the problem

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(2X) - \log\det(3X) \\ \text{subject to} & 2Xs = y \end{array}$$

with variable $X \in \mathbf{S}^n$ and domain $\mathbf{S}^n_{++}$. Here, $y \in \mathbf{R}^n$ and $s \in \mathbf{R}^n$ are given, with $s^T y = 1$.

a) Give the Lagrange and then derive the Karush-Kuhn-Tucker conditions.

b) Verify that the optimal solution is given by

$$X^\star = \frac{1}{2}\left(I + yy^T - \frac{ss^T}{s^T s}\right).$$

Solution. Write your solution here.

□

References

[1] Weiran Wang, and Miguel Á. Carreira-Peroiñán. Projection onto the probability simplex: An efficient algorithm with a simple proof, and an application. *arXiv:1309.1541*, 2013.