EMERGENT DYNAMICS FOR THE HYDRODYNAMIC CUCKER–SMALE SYSTEM IN A MOVING DOMAIN[*]

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Abstract. We study the emergent dynamics for the hydrodynamic Cucker–Smale system arising in the modeling of flocking dynamics in interacting many-body systems. Specifically, the initial value problem with a moving domain is considered to investigate the global existence and time-asymptotic behavior of classical solutions, provided that the initial mass density has bounded support and the initial data are in an appropriate Sobolev space. In order to show the emergent behavior of flocking, we make use of an appropriate Lyapunov functional that measures the total fluctuation in the velocity relative to the mean velocity. In our analysis, we present the local well-posedness of the smooth solutions via Lagrangian coordinates, and we extend to the global-in-time solutions by establishing the uniform flocking estimates.

1. Introduction. The purpose of this paper is to continue our study [14] on the large-time dynamics of the hydrodynamic Cucker–Smale (C-S) model that arises from the macroscopic description of flocking modeling. Consider an ensemble of many C-S flocking particles that occupy a finite region in $\mathbb{R}^d$. When the number of particles is sufficiently large and the system is in a close-to-flocking state, the dynamics of the ensemble can be described effectively by a fluid-type model, i.e., the hydrodynamic C-S model with a moving vacuum boundary. Let ρ and u be the local mass density and bulk velocity, respectively, of the ensemble, the support of which is denoted by a time-varying set $\Omega(t) := \{x \in \mathbb{R}^d \mid \rho(x,t) \neq 0\}$ for a given initially bounded open set $\Omega := \Omega(0)$. In this case, the macroscopic dynamics of the ensemble is governed by the initial value problem of the hydrodynamic C-S system:

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (x,t) \in \Omega(t) \times \mathbb{R}_+,\\
(1.1) \quad &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = -K\rho \int_{\Omega(t)} \psi(|x-y|)\rho(y,t)(u(x,t)-u(y,t))dy,\\
&(\rho, u)(x,0) = (\rho_0, u_0),
\end{aligned}
$$

where $\psi(|x-y|)$ is a smooth and positive function that represents the communication weight between C-S particles:

$$(1.2) \qquad \psi(|x-y|) := \frac{1}{(1+|x-y|^2)^\beta}, \quad \beta \geq 0.$$

System (1.1) is a monokinetic approximate model of the macroscopic description of a flocking model [16], where the continuum flocking group is close to a flocking state (see section 2.1 for a detailed discussion). The well-posedness issue for system (1.1) has been studied in several different settings. The authors [14] studied the global existence and the time-asymptotic behavior of classical solutions for a periodic domain $\mathbb{T}^d$. Ha, Huang, and Wang [13] reformulated system (1.1) into hyperbolic conservation laws with damping in a one-dimensional and all-to-all setting, before applying the variational approach [11, 12, 24, 25, 26], and they obtained a global entropic weak solution. Tadmor and Tan [22] studied the critical threshold phenomena on the global existence and finite-time breakdown of $\mathcal{C}^1$-solutions in a low-dimensional setting with $d \leq 2$. On the other hand, the mesoscopic-macroscopic coupled systems were proposed in [1, 2] to study the dynamics of C-S flocking particles interacting with the fluid.

Note that system (1.1), in the zero coupling limit $K \to 0$, reduces to the pressureless Euler system

$$(1.3) \qquad \begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0, \quad (x,t) \in \Omega(t) \times \mathbb{R}_+, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) &= 0, \end{aligned}$$

which has been studied extensively by the hyperbolic conservation laws community [3, 4, 5, 6, 7, 8, 11, 12, 17, 21, 24, 25, 26, 27] in the last decade.

In this paper, we establish the global well-posedness and time-asymptotic behavior of the smooth solutions to (1.1), provided the density $\rho(x,t)$ of the particles has initially bounded support and initial data $\rho(x,0)$ and $u(x,0)$ of density and velocity, respectively, belong to an appropriate Sobolev space; see section 3.1. It is well known that system (1.1) with $K = 0$, i.e., the pressureless Euler system (1.3), may fail to admit a global smooth solution even for small initial data. Specifically, a severe singularity such as a δ shock may occur in finite time. From a mathematical viewpoint, an interesting issue is whether the nonlocal flocking term in (1.1), i.e., $K > 0$, can prevent the formation of a singularity in finite time such that the solutions exist globally in time. Ha, Kang, and Kwon [14] constructed global-in-time classical solutions and demonstrated the time-asymptotic behavior of the solutions in the case of a spatial periodic domain, i.e., $\mathbb{T}^d$, $d \geq 1$, provided that the initial velocity data were sufficiently small in an appropriate Sobolev norm. By establishing the flocking estimate in terms of an appropriate Lyapunov functional that measures the total fluctuation in the velocities relative to the mean velocity, the velocity functions were shown to converge to the center of momentum, i.e., $u(x,t) \to \int(\rho u)/\int \rho$, as $t \to \infty$. This implies that the system exhibits time-asymptotic flocking. The key observation that the communication weight function $\psi(|x-y|)$ has a positive lower bound due to the compactness of the domain allows us to establish a uniform flocking estimate, by which the global-in-time smooth solution is constructed and the flocking behavior is justified. However, to our knowledge, our results, including the global well-posedness and the asymptotic behavior, do not extend to the case where the spatial domain is a whole space, at least in a straightforward manner. The main difficulty is the lack of the lower bound of the communication weight function. Due to this technical issue, the whole space problem seems to require a different treatment from the one for the periodic case.

In the current study, we consider the corresponding moving domain problem where the density ρ has initially bounded support, i.e., $\Omega = \operatorname{supp} \rho_0$ is a bounded open set in $\mathbb{R}^d$. It is natural to consider the physical situation where the ensemble of flocking agents has a finite size, i.e., it occupies a bounded region in the whole space $\mathbb{R}^d$, for which the problem (1.1) is considered. We prove that system (1.1) admits a global-in-time classical solution and that it exhibits the emergent behavior of flocking in the sense that the velocity converges to the center of the momentum as $t \to \infty$. This is a justification of the emergent flocking behavior in the framework of the moving domain problem for the hydrodynamic C-S model (1.1). In order to address the moving domain problem, we reformulate the problem in Lagrangian coordinates, where the system is transformed into an integro-differential equation for the Lagrangian velocity in a fixed domain Ω. Although the domain is fixed in this formulation, the corresponding communication weight function $\psi = \psi(t)$ depends on the time variable such that it may decrease in time. As explained for the periodic domain case, the uniform and positive lower bound for the communication function plays an important role in obtaining the decay estimate of a velocity fluctuation. To achieve this, we establish the flocking estimate in terms of a new Lyapunov functional to show that the velocity fluctuation relative to the mean velocity decays rapidly in the L^2-norm. Using this and the standard Sobolev interpolation also yields the velocity fluctuation decay in the $H^{s-\delta}$ norm, where $s > d/2 + 1$ and $\delta > 0$ is arbitrarily small, and we obtain the uniform estimate on the diameter of the support. Using this estimate and the structure of the communication weight function, we obtain the uniform lower bound for ψ. Next, using this and our careful energy estimates for the Lagrangian variables allow us to construct the global-in-time solution, and this justifies the flocking decay estimates for all times.

The rest of the paper is organized as follows. In section 2, we briefly discuss the heuristic derivation of the hydrodynamic C-S model (1.1) from the kinetic C-S model as a truncated moment system via the monokinetic ansatz, and we also present a new Lagrangian reformulation of (1.1) in terms of Lagrangian density and velocity. In section 3, we present a framework as well as the main results of a global existence and flocking estimate. In section 4, we present a detailed proof for Theorem 3.1. Finally, section 5 provides a summary of our main results and future directions.

Notation. For any nonnegative integer k, $H^k := H^k(\Omega)$ denotes the kth order L^2 Sobolev space on Ω, and $\mathcal{C}^k(I; E)$ is the space of k times continuously differentiable functions from an interval $I \subset \mathbb{R}$ into a Banach space E. ∇^k denotes any partial derivative ∂^α with a multi-index α with $|\alpha| = k$.

2. Preliminaries. In this section, we briefly review how the hydrodynamic C-S model can be formally derived from the C-S flocking model [10] as part of a moment system in a near global flocking regime. We also provide a reformulation of system (1.1) in terms of Lagrangian variables.

2.1. Hydrodynamic C-S model. We provide a brief derivation of (1.1) from the C-S model based on previous studies [13, 14]. At the microscopic level, the C-S flocking model was first introduced [10] as an analytical modification of the Vicsek model [23]:

(2.1)
$$\frac{dx_i}{dt} = v_i, \quad i = 1, \cdots, N, \ \ t > 0,$$
$$\frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i),$$

where $x_i, v_i \in \mathbb{R}^d$ are the spatial position and velocity of the ith particle, respectively, and N is the number of C-S particles. Recall that the original Vicsek model [23] uses the finite-range interactions so that rigorous flocking estimates are still unknown in terms of initial data. In contrast, this C-S model use the mean-field interactions to allow rigorous flocking estimates. More precisely, it was introduced mainly to explain the flocking phenomena of self-propelled particles based on the following simple rules:

- A particle's acceleration is proportional to the weighted average of its neighboring particles' velocities;
- the communication weight ψ is a decreasing function of the distance between particles,

but without resorting to the first principles of physics. A typical type of communication weight function is given by $\psi(|x - y|) := (1 + |x - y|^2)^{-\beta}$, $\beta \geq 0$, in [10]. An interesting feature of the C-S model is that it exhibits a phase-like transition from disordered states to ordered states, depending on the spatial decay rate β of the communication weight ψ. Indeed, Cucker and Smale [10] derived sufficient conditions for global flocking in terms of the initial configuration and communication weight, and these results were further improved in [15, 16] (see [18, 19, 20] for generalized particle and kinetic C-S models). However, when the number of particles is sufficiently large, system (2.1) can be described effectively by the one-particle density function $f = f(x, \xi, t)$ at the spatial velocity position $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ at time $t > 0$, which has been obtained in the mean-field limit. In fact, using the BBGKY hierarchy from the particle C-S model [10], the following Vlasov-type equations for f were first introduced by Ha and Tadmor [16]:

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot (F[f] f) = 0, \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \ t > 0,$$

(2.2)
$$\text{where } F[f](x, \xi, t) = -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(\xi - \xi_*) f(y, \xi_*, t) d\xi_* dy.$$

A rigorous derivation of (2.2) was given in [15] by combining the particle-in-cell method with the measure-theoretic formulation, and the coupling between (2.2) and fluid models was also investigated [1, 2]. To obtain the hydrodynamic description of the model, macroscopic observables are introduced, such as the local mass, momentum, and energy densities; for a given $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$,

$$\rho(x, t) := \int_{\mathbb{R}^d} f d\xi, \qquad (\rho u)(x, t) := \int_{\mathbb{R}^d} \xi f d\xi,$$

$$(\rho E)(x, t) := \frac{1}{2}\rho|u|^2 + \rho e, \qquad \rho e := \frac{1}{2}\int_{\mathbb{R}^d} |\xi - u(x, t)|^2 f d\xi.$$

Then, these macroscopic observables satisfy a nonclosed system

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \ t > 0,$$

(2.3)
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = S^{(1)},$$

$$\partial_t(\rho E) + \nabla_x \cdot (\rho E u + P u + q) = S^{(2)},$$

where $P = (p_{ij})$ and $q = (q_1, \cdots, q_d)$ are the stress tensor and heat flow, respectively. Here,

(2.4) $$p_{ij} := \int_{\mathbb{R}^d} (\xi_i - u_i)(\xi_j - u_j) f d\xi, \quad q_i := \int_{\mathbb{R}^d} (\xi_i - u_i)|\xi - u|^2 f d\xi,$$

and the source terms are given by the following relations:

$$S^{(1)} := -K\rho \int_{\mathbb{R}^d} \psi(|x-y|)\Big(u(x,t)-u(y,t)\Big)\rho(y,t)dy,$$
$$S^{(2)} := -K\rho \int_{\mathbb{R}^d} \psi(|x-y|)\Big(E(x,t)+E(y,t)-u(x,t)\cdot u(y,t)\Big)\rho(y,t)dy.$$

Note that system (2.3) is not closed because we require the third velocity moment of f to calculate the heat flux q in (2.4). At present, there is no known suitable closure condition that is similar to the local Maxwellian ansatz for the Boltzmann equation. However, when the ensemble of C-S particles is almost in the flocking state (collisionless travelling state with a common velocity), a monokinetic ansatz for f can be employed:

$$(2.5) \qquad f(x,\xi,t) = \rho(x,t)\delta(\xi - u(x,t)).$$

Under the monokinetic ansatz assumption (2.5) for a kinetic density f, the internal energy, stress tensor, and heat flux vanish:

$$\rho e = 0, \quad p_{ij} = 0, \quad q_i = 0, \quad 1 \le i,j \le d,$$

and system (2.3) can be reduced to the pressureless Euler system with flocking dissipation:

$$(2.6) \quad \begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad x \in \mathbb{R}^d, \ \ t > 0, \\ &\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) = -K\rho \int_{\mathbb{R}^d} \psi(|x-y|)\Big(u(x,t)-u(y,t)\Big)\rho(y,t)dy. \end{aligned}$$

For a more detailed discussion of the derivation and its physicality, we refer the reader to our previous study [14]. Before we finish this subsection, we describe the conservation laws associated with (2.6).

LEMMA 2.1. *Let (ρ,u) be a sufficiently smooth solution to (2.6) that decays at infinity $|x| = \infty$ sufficiently fast. Then, the total mass and momentum are conserved:*

$$\int_{\mathbb{R}^d} \rho(x,t)dx = \int_{\mathbb{R}^d} \rho_0(x)dx, \qquad \int_{\mathbb{R}^d} (\rho u)(x,t)dx = \int_{\mathbb{R}^d} \rho_0(x)u_0(x)dx, \quad t \ge 0.$$

Proof. We directly integrate (2.6) and use the change of variables $x \leftrightarrow y$ to show that

$$\int_{\mathbb{R}^{2d}} \psi(|x-y|)\Big(u(x,t)-u(y,t)\Big)\rho(x,t)\rho(y,t)dydx$$
$$= \int_{\mathbb{R}^{2d}} \psi(|x-y|)\Big(u(y,t)-u(x,t)\Big)\rho(x,t)\rho(y,t)dydx$$
$$= -\int_{\mathbb{R}^{2d}} \psi(|x-y|)\Big(u(x,t)-u(y,t)\Big)\rho(x,t)\rho(y,t)dydx.$$

This yields

$$\int_{\mathbb{R}^{2d}} \psi(|x-y|)\Big(u(x,t)-u(y,t)\Big)\rho(x,t)\rho(y,t)dydx = 0. \qquad \square$$

2.2. Lagrangian formulation of (1.1). The initial value problem (1.1) posed in the introduction has a close relationship to the vacuum boundary problem of the fluid equations. In this section, we reformulate system (1.1) into an integro-differential

system using Lagrangian coordinates, where the computational domain is fixed as the initial domain. Recall that our goal is to verify the emergent flocking dynamics of system (1.1) in a long-term regime, which is equivalent to saying that the Lagrangian particle paths become parallel in physical space and the Lagrangian velocities converge to the common average velocity, which is a conserved quantity in (1.1).

Now, we discuss the Lagrangian formulation of (1.1) loosely following [9]. First, we introduce the Lagrangian variables, which comprise the forward particle path $\eta = \eta(x,t)$ that represents the spatial position at time t of particles issued from $x \in \Omega$ at time 0, and the Lagrangian density q and velocity v evaluated along the particle path η. More specifically, for a fixed $x \in \Omega$, we consider particle-trajectory mapping

$$(2.7) \qquad \begin{cases} \dfrac{d\eta(x,t)}{dt} = u(\eta(x,t),t), & t > 0, \\ \eta(x,0) = x, \end{cases}$$

and we define

$$(2.8) \qquad q(x,t) := \rho(\eta(x,t),t), \quad v(x,t) := u(\eta(x,t),t).$$

Remark 2.1. For the particle-trajectory mapping $\eta(\cdot,t)$ of a smooth velocity field u, one can show, by a straightforward calculation, that

$$(2.9) \qquad \partial_t \det(\nabla\eta(x,t)) = \det(\nabla\eta(x,t))\,(\nabla \cdot u)|_{(\eta(x,t),t)},$$

where $\nabla\eta$ denotes a derivative of η with respect to x. Note that, since $\eta(\cdot,t) : \Omega \subset \mathbb{R}^d \to \Omega(t) \subset \mathbb{R}^d$ is a smooth mapping, $\nabla\eta$ is a $d \times d$ matrix whose (k,l) entry is $\partial_{x_l}\eta_k$, where η_k denotes the kth component of η. It follows from (2.9) and $\det(\nabla\eta(x,0)) = 1$ that $\det(\nabla\eta(\cdot,t)) \neq 0$ as long as $\exp\left(\int_0^t (\nabla \cdot u)|_{(\eta(x,\tau),\tau)}d\tau\right) < \infty$. This implies that $\eta(\cdot,t) : \Omega \to \Omega(t)$ is bijective. From this observation, one can consider the inverse mapping η^{-1} and its derivative $\nabla(\eta^{-1}) = (\nabla\eta)^{-1}$. Here and after, with no confusion, we shall denote the (j,i)-entry of $(\nabla\eta)^{-1}$ by $((\nabla\eta)^{-1})_{j,i}$.

LEMMA 2.2. *Let (ρ,u) be a sufficiently smooth solution to system* (1.1)–(1.2). *Then, the dynamics of the Lagrangian variables (η,q,v) defined by* (2.7) *and* (2.8) *are given by the following relations:*

(2.10)

$$\eta(x,t) = x + \int_0^t v(x,\tau)d\tau, \qquad x \in \Omega,\ t > 0,$$

$$q(x,t) = \rho_0(x)\,det(\nabla\eta(x,t))^{-1},$$

$$\partial_t v(x,t) = \int_\Omega \rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)(v(y,t) - v(x,t))dy, \quad v(x,0) = u_0(x).$$

Proof. Derivation of the equation for η. Note that η defined by (2.7) satisfies

$$\frac{d\eta(x,t)}{dt} = u(\eta(x,t),t) = v(x,t), \quad t > 0, \quad \eta(x,0) = x.$$

Then, we integrate the above equation to obtain

$$\eta(x,t) = x + \int_0^t v(x,\tau)d\tau \qquad x \in \Omega,\ \ t > 0.$$

Derivation of the equation for q. Consider the continuity equation in (1.1), which is evaluated on the Lagrangian path $(\eta(x,t),t) \in \mathbb{R}^d \times \mathbb{R}_+$:

$$(2.11) \qquad 0 = \partial_t \rho + \nabla \cdot (\rho u)\Big|_{(\eta(x,t),t)} = (\partial_t \rho + u \cdot \nabla \rho)\Big|_{(\eta(x,t),t)} + \rho \nabla \cdot u\Big|_{(\eta(x,t),t)}.$$

By a straightforward calculation together with (2.8) and (2.7), we have that

$$(\partial_t \rho + u \cdot \nabla \rho)\Big|_{(\eta(x,t),t)} = \partial_t q(x,t), \quad \text{and}$$

$$(2.12)$$

$$\nabla \cdot u\Big|_{(\eta(x,t),t)} = \sum_{i,j=1}^d \partial_i(\eta^{-1})_j \, \partial_j v_i = \sum_{i,j=1}^d ((\nabla\eta)^{-1})_{j,i} \, \partial_j v_i.$$

Here we recall that $\partial_i(\eta^{-1})_j = ((\nabla\eta)^{-1})_{j,i}$, where $\partial_i(\eta^{-1})_j$ is the i-partial derivative of the jth component of η^{-1}, and $((\nabla\eta)^{-1})_{j,i}$ denotes the (j,i) entry of $(\nabla\eta)^{-1}$. Then, we combine (2.11) and (2.12) to obtain

$$(2.13) \qquad \partial_t q + q \sum_{i,j=1}^d ((\nabla\eta)^{-1})_{j,i} \, \partial_j v_i = 0, \quad x \in \Omega, \ t > 0.$$

We also note that

$$\partial_t \det(\nabla\eta(x,t)) = \det(\nabla\eta(x,t)) \sum_{i,j=1}^n ((\nabla\eta)^{-1})_{j,i} \, \partial_j v_i.$$

Thus, (2.13) yields the desired relation for q:

$$\partial_t \Big(q(x,t)\det(\nabla\eta(x,t)) \Big) = 0.$$

We integrate the above relation in t and use $\nabla\eta(x,0) = I$ to find

$$q(x,t)\det(\nabla\eta(x,t)) = \rho_0(x)\det(\nabla\eta(x,0)) = \rho_0(x),$$

$$\text{i.e.,} \quad q(x,t) = \rho_0(x)\det(\nabla\eta(x,t))^{-1}.$$

Derivation of the equation for v. Note that the right-hand side of the momentum equation in (1.1) along the Lagrangian path can be rewritten as

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u)\Big|_{(\eta(x,t),t)} = \rho(\partial_t u + u \cdot \nabla u)\Big|_{(\eta(x,t),t)} = q(x,t)\partial_t v(x,t).$$

Hence, the momentum equation becomes

$$(2.14) \qquad q\partial_t v = q \int_\Omega \psi(|\eta(x,t) - \eta(y,t)|)q(y,t)(v(y,t) - v(x,t))\det(\nabla\eta(y,t))dy,$$

where $\Omega := \Omega(0)$, the support of ρ_0. Then, we substitute $q(x,t) = \rho_0(x) \times (\det(\nabla\eta(x,t)))^{-1}$ into $(2.14)_2$ to get

$$(2.15) \qquad \rho_0(x)\partial_t v(x,t) = \int_\Omega \psi(|\eta(x,t) - \eta(y,t)|)\rho_0(x)\rho_0(y)(v(y,t) - v(x,t))dy.$$

Since $\rho_0(x) > 0$ for all $x \in \Omega$, (2.15) can be reduced further by dividing by $\rho_0(x)$ to obtain

$$\partial_t v(x,t) = \int_\Omega \psi(|\eta(x,t) - \eta(y,t)|)\rho_0(y)(v(y,t) - v(x,t))dy. \qquad \square$$

Remark 2.2. Our Lagrangian formulation adopts the framework of Coutand–Shkoller [9] on the three-dimensional compressible Euler equations in physicalvacuum.

3. Description of our framework and main results. In this section, we briefly discuss the main assumptions used in this study and we present our main result on the global existence of smooth solutions and its emergent dynamics.

3.1. The framework. Next, we describe the conditions imposed on the communication weight ψ and initial data (ρ_0, u_0).

($\mathcal{A}1$) The communication weight function $\psi(r)$ satisfies

$$\psi(r) := \frac{1}{(1 + r^2)^\beta}, \quad \beta \in (0, \frac{1}{2}).$$

($\mathcal{A}2$) The initial density ρ_0 has a compact support Ω on the spatial domain $\mathbb{R}^d$ and it satisfies the positivity and unit mass conditions:

$$\inf_{x \in \Omega} \rho_0(x) > 0 \quad \text{and} \quad \int_\Omega \rho_0 dx = 1.$$

($\mathcal{A}3$) The initial data (ρ_0, u_0) satisfy the regularity

$$(\rho_0, u_0) \in H^s \times H^{s+1}, \quad s > \frac{d}{2} + 1,$$

where $H^s = H^s(\Omega)$ is the sth order L^2 Sobolev space on Ω.

Note that the regularity assumptions imposed on ψ and the initial data in ($\mathcal{A}1$) and ($\mathcal{A}2$) are reasonable because we are searching for classical $\mathcal{C}^1$-solutions. Note that the boundary of flocking groups forms a sharp vacuum boundary such that the initial mass and velocity densities have jump discontinuities across $\partial\Omega$.

3.2. Main results. We describe our main results under the framework given in the previous subsection. To achieve this, we first introduce a solution space $\mathfrak{Y}_k(T)$:

$$\mathfrak{Y}_k(T) := \{(q, v) : q \in C^0([0, T]; H^k) \cap C^1([0, T]; H^{k-1}),$$
$$v \in C^0([0, T]; H^{k+1}) \cap C^1([0, T]; H^k)\}.$$

Following Ha, Kang, and Kwon [14], we define the Lyapunov functional for the flocking estimate as follows:

$$(3.1) \qquad \mathcal{E}(t) := \int_{\Omega(t)} \rho |u(x, t) - m_c(t)|^2 dx, \quad m_c(t) := \int_{\Omega(t)} \rho u \, dx.$$

Note that $m_c(t)$ is the total momentum at time t, which is equal to the initial momentum $m_c(0)$ due to the momentum conservation (see Lemma 2.1), and the functional $\mathcal{E}$ measures the velocity variance around the expected mean velocity $m_c(0)$ over time. Thus, after we verify convergence to zero of $\mathcal{E}(t)$ as $t \to \infty$, we obtain the emergence of flocking for (1.1). Indeed, this Lyapunov functional was used previously to study the problem in the periodic domain [14]. However, for the moving boundary problem that we consider in the present study, it is much easier to work with the Lagrangian-variables–Lyapunov-functional $\mathcal{E}$ defined on the fixed domain Ω. Specifically, by the change of variable $\eta = \eta(x)$, we find that the Lyapunov functional can be expressed in terms of the Lagrangian variable v as follows:

$$(3.2) \quad \mathcal{E}(t) := \int_\Omega \rho_0(x) |v(x, t) - m_c(t)|^2 dx, \quad m_c(t) = m_c(0) = \int_\Omega \rho_0(x) v(x, t) dx,$$

where the ρ_0 are the initial data and $m_c(t)$ is time-independent. The decay estimates on $\mathcal{E}(t)$ will be derived in section 4. Now, we are ready to present our main result.

THEOREM 3.1. *Suppose that conditions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *hold. Then, there exists a positive constant* ε_0 *depending on* $\|\rho_0\|_{H^s}$ *such that if* $\|u_0\|_{H^{s+1}} < \varepsilon_0$, *then the initial value problem* (1.1) *has a unique global-in-time classical solution* (ρ, u), *which is given by*

$$(3.3) \qquad q(x,t) = \rho(\eta(x,t), t), \quad v(x,t) = u(\eta(x,t), t),$$

where $(q, v) \in \mathfrak{Y}_s(\infty)$ *and* $\eta \in C^1([0, \infty); H^{s+1})$. *Moreover, it satisfies the asymptotic flocking estimates*

$$\mathcal{E}(t) \leq e^{-\lambda t} \mathcal{E}(0) \quad \text{for all } t > 0,$$

where $\mathcal{E}$ *is the functional defined in* (3.1) *and* λ *depends on* $\text{diam}(\Omega)$, ε, *and* β.

Remark 3.1. 1. By the standard Sobolev embedding theorem, the solution $(q, v) \in \mathfrak{Y}_s(\infty)$, $s > \frac{d}{2} + 1$, as in Theorem 3.1, is a classical solution, i.e., $(q, v) \in \mathcal{C}^1(\Omega \times [0, \infty))$.

2. Expression (3.3) together with the inverse mapping $\eta^{-1}(\cdot, t) : \Omega(t) \to \Omega$ of the particle-trajectory mapping η defines the global classical solution (ρ, u). The invertibility of η has been discussed in Remark 2.1.

3. This theorem generalizes the previous result [14] on the periodic domain $\mathbb{T}^d$. For the one-dimensional case, the existence of entropic weak solutions was obtained previously [13] using the variational formulation. In a low-dimensional case with $d \leq 2$, more refined estimates are available in [22], where the critical threshold conditions on the global existence versus finite-time breakdown of $\mathcal{C}^1$-solutions are presented.

4. Proof of Theorem 3.1. In this section, we provide the proof of Theorem 3.1. Since the proof is rather long, we split the proof into two subsections. The first deals with a priori flocking estimates for the reformulated system. The second addresses the local existence and a priori estimates, which yield the global existence of classical solutions.

4.1. Flocking estimate. Next, we present the asymptotic flocking estimate for system (2.10) using a Lyapunov functional (3.2) in terms of Lagrangian variables, q and v.

LEMMA 4.1. *Let* $v = v(x, t)$ *be a smooth solution to* (2.10) *with the initial data* (ρ_0, u_0). *Then, we have the following:*

(i) $\dfrac{d}{dt} m_c(t) = 0, \quad t > 0.$

(ii) $\dfrac{d}{dt} \displaystyle\int_\Omega \rho_0(x) |v(x,t)|^2 dx = -\int_{\Omega^2} \rho_0(x) \rho_0(y) \psi(|\eta(x,t) - \eta(y,t)|) |v(y,t) - v(x,t)|^2 dy dx.$

(iii) $\mathcal{E}(t) \leq \mathcal{E}(0) \exp\left(-2 \displaystyle\int_0^t \psi_m(s) ds\right),$

where $\psi_m(s) := \displaystyle\inf_{x,y \in \Omega} \psi(|\eta(x,s) - \eta(y,s)|)$.

Proof. (i) We multiply $(2.10)_2$ by ρ_0 and integrate over Ω to obtain

$$\frac{d}{dt} \int_\Omega \rho_0(x) v(x,t) dx = \int_\Omega \int_\Omega \rho_0(x) \rho_0(y) \psi(|\eta(x,t) - \eta(y,t)|)(v(y,t) - v(x,t)) dy dx = 0.$$

Here, the last equality is due to the antisymmetry of the integrand in the transformation $x \iff y$.

(ii) To obtain the energy dissipation estimate, we use $(2.10)_3$ and the same transformation trick $x \iff y$ to obtain

$$\frac{d}{dt} \int_{\Omega} \rho_0(x)|v(x,t)|^2 dx$$

$$= 2 \int_{\Omega} \rho_0(x)v(x,t) \cdot \partial_t v(x,t) dx$$

$$= 2 \int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)v(x,t) \cdot (v(y,t) - v(x,t))dydx$$

$$= -\int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)|v(y,t) - v(x,t)|^2 dydx$$

$$\leq 0.$$

(iii) Due to the conservation of $m_c(t)$ in (i), we can see that

$$\mathcal{E}(t) := \int_{\Omega} \rho_0(x)|v(x,t) - m_c(0)|^2 dx.$$

Using straightforward calculations, we obtain

$$\frac{d}{dt}\mathcal{E}(t) = 2 \int_{\Omega} \rho_0(x)(v(x,t) - m_c(0)) \cdot \partial_t v(x,t) dx$$

$$= 2 \int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)v(x,t) \cdot (v(y,t) - v(x,t))dydx$$

$$- m_c(0) \cdot \int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)(v(y,t) - v(x,t))dydx$$

$$= -\int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)|v(y,t) - v(x,t)|^2 dydx$$

$$\leq -\psi_m(t) \int_{\Omega^2} \rho_0(x)\rho_0(y)|v(y,t) - v(x,t)|^2 dydx$$

$$= -\psi_m(t)\Big[\int_{\Omega^2} \rho_0(x)\rho_0(y)(|v(y,t) - m_c(t)|^2 + |v(x,t) - m_c(t)|^2)dydx$$

$$- 2 \int_{\Omega^2} \rho_0(x)\rho_0(y)(u(x,t) - m_c(t)) \cdot (u(y,t) - m_c(t))dydx\Big]$$

$$= -2\psi_m(t)\mathcal{E}(t),$$

where $\psi_m(t) := \inf_{x,y\in\Omega} \psi(|\eta(x,t) - \eta(y,t)|)$ and we use the fact that

$$\int_{\Omega^2} \rho_0(x)\rho_0(y)\psi(|\eta(x,t) - \eta(y,t)|)(v(y,t) - v(x,t))dydx = 0,$$
$$\int_{\Omega^2} \rho_0(x)\rho_0(y)(u(x,t) - m_c(t)) \cdot (u(y,t) - m_c(t))dydx\Big)dydx = 0.$$

Here, the last equality is due to the unit mass $\int_{\Omega} \rho_0 dx = 1$ and the definition of m_c. □

Remark 4.1. (i) Note that Lemma 4.1(iii) implies that

$$\int_0^{\infty} \psi_m(t)dt = \infty \qquad \Longrightarrow \qquad \lim_{t\to\infty} \mathcal{E}(t) = 0.$$

(ii) In Lemma 4.3, for our global solution (ρ, u), we will show that there exists a positive constant $\psi_L > 0$ such that

$$\inf_{t \geq 0} \psi_m(t) \geq \psi_L,$$

which implies the flocking estimate.

4.2. Global existence of classical solutions. In this subsection, we show the global existence of a classical solution to (1.1) by establishing the local existence of classical solutions and a priori estimates.

4.2.1. Local existence. Next, we show the local existence of classical solutions. To construct the approximate solutions $\{(\eta^n, v^n)\}$, we consider the approximation iteration scheme. More specifically, we take the initial data for the zeroth approximation, i.e.,

$$v^0(x, t) := u_0, \quad (x, t) \in \Omega \times \mathbb{R}_+,$$

and when the nth approximation (η^n, v^n), $n \geq 1$, is given, the $(n+1)$th approximation (η^{n+1}, v^{n+1}) is defined as a solution to the following linear system:

$$
\begin{aligned}
(4.1) \quad & \eta^{n+1}(x, t) = x + \int_0^t v^n(x, s)ds, \\
& \partial_t v^{n+1}(x, t) = \int_\Omega \rho_0(y)\psi(|\eta^{n+1}(x, t) - \eta^{n+1}(y, t)|)(v^n(y, t) - v^n(x, t))dy,
\end{aligned}
$$

subject to the initial data

$$v^{n+1}(x, 0) = u_0(x).$$

PROPOSITION 4.2 (local-in-time existence). *Suppose that assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$ hold. Then, for any positive constants $M_1 < M_2$, there is a positive constant T_0, which depends only on M_1 and M_2, such that if $\|u_0\|_{H^{s+1}} < M_1$, then the initial value problem (2.10) has a unique solution $v \in C^0([0, T_0]; H^{s+1}) \cap C^1([0, T_0]; H^s)$ that satisfies*

$$(4.2) \qquad \sup_{0 \leq t \leq T_0} \|v(t)\|_{H^{s+1}} < M_2.$$

Proof. A large part of the proof is quite similar to that for the transport equation, so we provide some details on new features and we only present an outline for standard parts of the proof. By straightforward calculations for the first equation of (4.1), we obtain the uniform H^s bound

$$(4.3) \qquad \|\eta^n\|_{L^\infty(0, T; H^s)} \leq C(\Omega) + T\|v^{n-1}\|_{L^\infty(0, T; H^s)}.$$

Then we use the integral form of the second equation of (4.1),

$$v^{n+1}(x, t) = u_0(x) + \int_0^t \int_\Omega \rho_0(y)\psi(|\eta^{n+1}(x, \tau) - \eta^{n+1}(y, \tau)|)(v^n(y, \tau) - v^n(x, \tau))dyd\tau,$$

to evaluate the H^s norm of v^{n+1} in a straightforward fashion. Here we have used the standard Moser-type inequalities and uniform bound (4.3). This together with an inductive argument yields the uniform estimates for $\|v^n\|_{H^{s+1}}$. Specifically, we show

that there are positive constants T_0 and $M_2 > M_1$, which depend only on the initial data and Ω, such that

$$\|v^n\|_{L^\infty(0,T_0;H^{s+1})} < M_2 \quad \text{for all } n \in \mathbb{N} .$$

Then, the difference equations for $(\eta^{n+1} - \eta^n)$ and $(v^{n+1} - v^n)$ are considered to show the convergence. By a standard energy method together with calculus inequalities as in the previous estimates, we can show that the sequences $\{\eta^n\}$ and $\{v^n\}$ are Cauchy sequences in $L^\infty(0,T_0;H^{s+1})$. Then we use functional analytic arguments and the structure of (4.1) to show the desired regularity. Since the proofs of the convergence and regularity are similar to those for the transport equations, we omit the details here. For more details of the arguments specialized for the pressureless Euler equations, we refer the reader to [14].   □

4.2.2. A priori estimates. Now, we present the uniform *a priori* estimates of $\|v(t)\|_{H^{s+1}}$ for the initial value problem (2.10). We use these estimates to construct the global-in-time solution in the next subsection. To derive the a priori estimates, we first show that the communication weight function $\psi(|\eta(x,t) - \eta(y,t)|)$ in (2.10) has a positive uniform lower bound provided that $\|v(t)\|_{H^{s+1}}$ remains bounded by some constant.

LEMMA 4.3 (A priori uniform bound). *Suppose that the conditions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *hold. For any given* $T > 0$, *if the solution* v *of* (2.10) *satisfies* $\|v\|_{L^\infty(0,T;H^{s+1})} \leq \varepsilon_1$ *for some constant* $\varepsilon_1 > 0$, *then there exists a positive constant* ψ_L *independent of* t *such that*

$$\psi_m(t) := \inf_{x,y\in\Omega} \psi(|\eta(x,t) - \eta(y,t)|) \geq \psi_L, \quad 0 \leq t \leq T.$$

Proof. First, we apply the method of characteristics to the continuity equation $(1.1)_1$ to obtain

$$\rho(\eta(x,t),t) = \rho_0(x) \exp\Big(-\int_0^t \nabla_x \cdot u(\eta(x,s),s)ds \Big).$$

Note that since $v(x,t) = u(\eta(x,t),t)$ and $v \in L^\infty(0,T;H^{s+1})$, the above relation implies that

$$\rho_0(x) = 0 \;\Rightarrow\; \rho(\eta(x,t),t) = 0, \quad 0 < t \leq T.$$

For the desired uniform lower bound, we show that the diameter of the spatial support $\Omega(t)$ for the local density ρ is uniformly bounded. To achieve this, we use a bootstrapping argument. First, we show that the growth of $\Omega(t)$ is at most sublinear over time, and we then use this crude estimate and the flocking estimate to obtain the uniform bound estimate for $\Omega(t)$. Suppose that

$$(4.4) \qquad\qquad\qquad \|v\|_{L^\infty(0,T;H^{s+1})} \leq \varepsilon_1.$$

Step A (a sublinear growth estimate). It follows from (4.4) and the equation for η that

$$|\eta(x,t) - \eta(y,t)| = \Big|x - y + \int_0^t (v(x,s) - v(y,s))ds\Big| \leq |x - y| + 2t\|v\|_{L^\infty(\Omega\times[0,T])}$$
$$\leq \operatorname{diam}(\Omega) + 2\varepsilon_1 T \quad \text{for } x,y \in \Omega, \ t \in [0,T).$$

This yields

$$\text{diam}(\Omega(t)) \leq \text{diam}(\Omega) + 2\varepsilon_1 t, \quad t \geq 0.$$

Thus

(4.5)
$$\begin{aligned}
\psi_m(t) &= \inf_{x,y \in \Omega} (1 + |\eta(x,t) - \eta(y,t)|^2)^{-\beta} \\
&\geq (1 + 4(\text{diam}(\Omega) + \varepsilon_1 t)^2)^{-\beta} \\
&\geq C(1 + \varepsilon_1 t)^{-2\beta},
\end{aligned}$$

where $C > 0$ depends on $\text{diam}(\Omega)$.

Step B (a uniform bound estimate). For a uniform bound estimate, we use the flocking estimate obtained in Lemma 4.1, i.e., we now apply (4.5) to (iii) in Lemma 4.1 to yield

$$\begin{aligned}
\int_\Omega \rho_0(x)|v(x,t) - m_c(t)|^2 dx &\leq \mathcal{E}(0) \exp\Big(-2 \int_0^t \psi_m(s)ds \Big) \\
&\leq \mathcal{E}(0) \exp\Big(-C \int_0^t (1 + \varepsilon_1 s)^{-2\beta} ds \Big) \\
&\leq C \exp\Big(-\frac{(1 + \varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big).
\end{aligned}$$

Since $\inf_{x \in \Omega} \rho_0(x) > 0$, we obtain the L^2 estimate

(4.6)
$$\|v - m_c\|_{L^2} \leq C \exp\Big(-\frac{2(1 + \varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big).$$

Then, by the Gagliardo–Nirenberg inequality and (4.6), we obtain the higher order decay estimate

(4.7)
$$\begin{aligned}
\|v - m_c\|_{H^s} &\leq C\|v - m_c\|_{L^2}^{1-\alpha} \|v - m_c\|_{H^{s+1}}^\alpha \\
&\leq C(\|v\|_{H^{s+1}} + \|m_c\|_{L^2})^\alpha \|v - m_c\|_{L^2}^{1-\alpha} \leq C\|v - m_c\|_{L^2}^{1-\alpha} \\
&\leq C \exp\Big(-C \frac{(1 + \varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big),
\end{aligned}$$

where $\alpha := \frac{s}{s+1} \in (0,1)$. Thus, by the Sobolev inequality, we obtain

(4.8)
$$\|v - m_c\|_{L^\infty} \leq C \exp\Big(-C \frac{(1 + \varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big).$$

This yields that for any $x, y \in \Omega$,

$$\begin{aligned}
|\eta(x,t) - \eta(y,t)| &\leq |x - y| + \int_0^t (\|v(x,s) - m_c\|_{L^\infty} + \|v(y,s) - m_c\|_{L^\infty}) ds \\
&\leq \text{diam}(\Omega) + C \int_0^\infty \exp\Big(-C \frac{(1 + \varepsilon_1 s)^{1-2\beta}}{\varepsilon_1} \Big) ds \\
&= \text{diam}(\Omega) + C(\beta)\varepsilon_1^{\frac{2\beta}{1-2\beta}},
\end{aligned}$$

where the last equality is true thanks to the condition that $0 < \beta < \frac{1}{2}$. Note that the condition $0 < \beta < \frac{1}{2}$ is crucial since $C(\beta) \sim (1 - 2\beta)^{-1} \to \infty$ as $\beta \to \frac{1}{2}-$. This

together with the structure of ψ implies that ψ_m has a uniform positive lower bound in time, i.e., $\psi_m(t) \geq \psi_L > 0$ for $0 < t \leq T$, where

$$(4.9) \qquad \psi_L := (1 + (C + C(\beta)\varepsilon_1^{\frac{2\beta}{1-2\beta}})^2)^{-\beta}.$$

This completes the proof. □

Now, we use Lemma 4.3 to obtain the uniform a priori estimates for $\|v(t)\|_{H^{s+1}}$.

PROPOSITION 4.4 (A priori uniform bound). *Suppose that conditions $(\mathcal{A}1)$–$(\mathcal{A}3)$ hold. For any given $T > 0$, there exists a constant $\varepsilon_1 > 0$ such that if the solution v of (2.10) satisfies $\|v\|_{L^\infty(0,T;H^{s+1})} \leq \varepsilon_1$, we have*

$$\|v(t)\|_{H^{s+1}} \leq C_0 \|u_0\|_{H^{s+1}} + \frac{\varepsilon_1}{2} \quad for \ t \in [0, T],$$

where $C_0 := C(1 + \|\rho_0\|_{H^s})^{1/2}$ for some generic constant $C > 0$.

Proof. Since the proof is lengthy, we split it into several steps, as follows.

Step A (zeroth-order estimate). We multiply $(2.10)_3$ by v and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 = -\int_{\Omega^2} \rho_0(y)\psi(|\eta(x) - \eta(y)|)|v(x)|^2 dxdy$$
$$+ \int_{\Omega^2} \rho_0(y)\psi(|\eta(x) - \eta(y)|)v(y) \cdot v(x)dxdy$$
$$=: \mathcal{I}_1 + \mathcal{I}_2.$$

We use Lemma 4.3 and $\|\rho_0\|_{L^1} = 1$ to obtain

$$\mathcal{I}_1 \leq -\psi_L \|v\|_{L^2}^2.$$

We note that the positive constant ψ_L defined in (4.9) is bounded below uniformly from zero for any small $\varepsilon_1 > 0$. We use Young's inequality and the energy dissipation estimate (ii) in Lemma 4.1 to estimate

$$\mathcal{I}_2 \leq \|\psi\|_{L^\infty} \int_{\Omega^2} \rho_0(y)|v(y)||v(x)|dxdy$$
$$\leq \|\psi\|_{L^\infty}\left(\delta \int_{\Omega^2} \rho_0(y)|v(x)|^2 dxdy + \frac{1}{4\delta}\int_{\Omega^2} \rho_0(y)|v(y)|^2 dxdy\right)$$
$$\leq C_1\delta\|\rho_0\|_{L^1}\|v\|_{L^2}^2 + C\int_\Omega \rho_0(y)|u_0(y)|^2 dy$$
$$\leq C_1\delta\|v\|_{L^2}^2 + C\|\rho_0\|_{L^\infty}\|u_0\|_{L^2}^2.$$

We choose a value of $\delta > 0$ so that $\psi_L - C_1\delta =: C_2 > 0$, and by combining the estimates above, we obtain

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq -C_2\|v\|_{L^2}^2 + C\|\rho_0\|_{L^\infty}\|u_0\|_{L^2}^2.$$

By Gronwall's inequality, we obtain

$$(4.10) \qquad \|v\|_{L^2}^2 \leq C(1 + \|\rho_0\|_{L^\infty})\|u_0\|_{L^2}^2 \leq C(1 + \|\rho_0\|_{H^s})\|u_0\|_{L^2}^2.$$

Step B (higher-order estimates). For any k with $1 \leq k \leq s+1$, we apply ∇_x^k to (2.10) to obtain

$$\partial_t \nabla_x^k v = - \int_\Omega \rho_0(y) \psi(\eta(x), \eta(y)) \nabla_x^k v(x) dy$$

$$+ \int_\Omega \rho_0(y) \nabla_x^k \psi(\eta(x), \eta(y))(v(y) - v(x)) dy$$

$$- \sum_{1 \leq l \leq k-1, \ k \geq 2} \binom{k-1}{l} \int_\Omega \rho_0(y) \nabla_x^l \psi(\eta(x), \eta(y)) \nabla_x^{k-l} v(x) dy.$$

We multiply the above equation by $\nabla^k v$ and integrate it over Ω to obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x^k v\|_{L^2}^2$$

$$= - \int_{\Omega^2} \rho_0(y) \psi(\eta(x), \eta(y)) |\nabla_x^k v(x)|^2 dx dy$$

$$+ \int_{\Omega^2} \rho_0(y) \nabla_x^k \psi(\eta(x), \eta(y))(v(y) - v(x)) \cdot \nabla_x^k v(x) dx dy$$

$$- \sum_{1 \leq l \leq k-1, \ k \geq 2} \binom{k-1}{l} \int_{\Omega^2} \rho_0(y) \nabla_x^l \psi(\eta(x), \eta(y)) \nabla_x^{k-l} v(x) \cdot \nabla_x^k v(x) dx dy$$

$$=: \sum_{k=1}^3 \mathcal{I}_k,$$

where the last term $\mathcal{I}_3$ above appears only when $k \geq 2$.

Similarly to Step A, we use Lemma 4.3 and the unit mass assumption to obtain

(4.11) $$\mathcal{I}_1 \leq -\psi_L \|\nabla_x^k v\|_{L^2}^2.$$

For $\mathcal{I}_2$, we first use (4.8), the boundedness of Ω, and the smoothness of ψ to get

$$\mathcal{I}_2 \leq \|\rho_0\|_{L^\infty} \int_{\Omega^2} (|v(y) - m_c| + |m_c - v(x)|) |\nabla_x^k \psi(\eta(x), \eta(y))| |\nabla_x^k v(x)| dx dy$$

$$\leq C \|\rho_0\|_{H^s} \|v - m_c\|_{L^\infty} \left(\int_{\Omega^2} |\nabla_x^k \psi(\eta(x), \eta(y))|^2 dx dy \right)^{1/2} \left(\int_\Omega |\nabla_x^k v(x)|^2 dx \right)^{1/2}$$

$$\leq C \|\rho_0\|_{H^s} \exp\left(-C \frac{(1 + \varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \right) \|\psi\|_{C^k} \|\nabla_x \eta\|_{H^{k-1}(\Omega)}^a \|\nabla_x^k v\|_{L^2(\Omega)},$$

where $a \geq 1$ is some constant depending on k.
It follows from $(2.10)_1$ that

$$\nabla_x \eta(x, t) = I + \int_0^t \nabla_x v(x, \tau) d\tau \quad \text{and} \quad \nabla_x^l \eta(x, t) = \int_0^t \nabla_x^l v(x, \tau) d\tau \quad \text{for } l \geq 2.$$

By this, we have, for $1 \leq l \leq s+1$,

(4.12) $$\|\nabla_x^l \eta\|_{L^2} \leq |\Omega| + \int_0^t \|\nabla_x^l v\|_{L^2} d\tau \leq C + \int_0^t \|v\|_{H^{s+1}} d\tau \leq C(1 + \varepsilon_1 t).$$

Thus for $1 \leq k \leq s+1$, we have

$$\|\nabla_x \eta\|_{H^{k-1}} \leq \|\nabla_x \eta\|_{H^s} \leq C(1 + \varepsilon_1 t).$$

Using this, we estimate

$$\mathcal{I}_2 \leq C\|\rho_0\|_{H^s} \exp\Big(-C\frac{(1+\varepsilon_1 t)^{1-2\beta}}{\varepsilon_1}\Big)(1+\varepsilon_1 t)^a \|\nabla_x^k v\|_{L^2}$$

(4.13)

$$= C\|\rho_0\|_{H^s} \underbrace{\exp\Big(-C\frac{(1+\varepsilon_1 t)^{1-2\beta}}{\varepsilon_1}\Big)\frac{(1+\varepsilon_1 t)^a}{\varepsilon_1}}_{=:h(\varepsilon_1, t)} \varepsilon_1 \|\nabla_x^k v\|_{L^2}.$$

Here, we notice that $h(\varepsilon_1, t)$ defined above is a continuous function of $(\varepsilon_1, t) \in \mathbb{R}_+^2$ and satisfies

(4.14)

$$h(\varepsilon_1, t) \to 0 \quad (i) \text{ as } \varepsilon_1 \to 0 \text{ for fixed } t > 0 \quad \text{or} \quad (ii) \text{ as } t \to \infty \text{ for fixed } \varepsilon_1 > 0.$$

This implies that $h(\varepsilon_1, t)$ is uniformly bounded in time $t > 0$, and that for any given $\delta > 0$, one can choose $\varepsilon_1 > 0$ such that $h(\varepsilon_1, t) \leq \delta$ for all $t > 0$.

We now choose $\delta > 0$ so small that

(4.15)

$$C\|\rho_0\|_{H^s}^2 \delta^2 < \frac{\psi_L}{4},$$

where C is the constant that appeared in the last line of (4.13). For such small δ, by our previous observation, we can choose $\varepsilon_1 > 0$ so that $h(\varepsilon_1, t) \leq \delta$ for all $t > 0$. Using this and Young's inequality for $\mathcal{I}_2$ as in (4.13), we have

$$\mathcal{I}_2 \leq C\|\rho_0\|_{H^s} \delta \varepsilon_1 \|\nabla_x^k v\|_{L^2} \leq C\|\rho_0\|_{H^s}^2 \delta^2 \|\nabla_x^k v\|_{L^2}^2 + \frac{\varepsilon_1^2}{8^{k+1}} \leq \frac{\psi_L}{4}\|\nabla_x^k v\|_{L^2}^2 + \frac{\varepsilon_1^2}{8^{k+1}}.$$

We combine this with (4.11) to get

(4.16)

$$\mathcal{I}_1 + \mathcal{I}_2 \leq -\frac{3\psi_L}{4}\|\nabla_x^k v\|_{L^2}^2 + \frac{\varepsilon_1^2}{8^{k+1}}.$$

For $\mathcal{I}_3$, we use the commutator estimate and the boundedness of Ω to get

$$\mathcal{I}_3 \leq C\|\rho_0\|_{L^\infty}\Big(\int_{\Omega^2}\Big|\sum_{1\leq l\leq k-1,\ k\geq 2}\nabla_x^l\psi(\eta(x),\eta(y))\nabla_x^{k-l}v(x)\Big|^2 dxdy\Big)^{1/2}$$

$$\Big(\int_\Omega |\nabla_x^k v(x)|^2 dx\Big)^{1/2}$$

(4.17)

$$\leq C\|\rho_0\|_{L^\infty}\Big(\|\nabla_x^{k-1}\psi(\eta(\cdot),\eta(\cdot))\|_{L^2(\Omega^2)}\|\nabla_x v\|_{L^\infty(\Omega)}$$

$$+ \|\nabla_x\psi(\eta(\cdot),\eta(\cdot))\|_{L^\infty(\Omega^2)}\|\nabla_x^{k-1}v\|_{L^2(\Omega)}\Big)\|\nabla_x^k v\|_{L^2}$$

$$\leq C\|\rho_0\|_{L^\infty}\Big(\|\psi\|_{C^\infty}\|\nabla_x\eta\|_{H^{k-2}(\Omega)}^a\|\nabla_x v\|_{L^\infty(\Omega)}$$

$$+ \|\psi\|_{C^\infty}\|\nabla_x\eta\|_{L^\infty(\Omega)}\|\nabla_x^{k-1}v\|_{L^2(\Omega)}\Big)\|\nabla_x^k v\|_{L^2},$$

where $a \geq 1$ is some constant depending on k.

We here notice that two terms $\|\nabla_x v\|_{L^\infty(\Omega)}$ and $\|\nabla_x^{k-1}v\|_{L^2(\Omega)}$ above can be written as

$$\|\nabla_x v\|_{L^\infty(\Omega)} = \|\nabla_x(v - m_c)\|_{L^\infty(\Omega)},$$

$$\|\nabla_x^{k-1}v\|_{L^2(\Omega)} = \|\nabla_x^{k-1}(v - m_c)\|_{L^2(\Omega)},$$

where the last equality is due to $2 \leq k \leq s+1$ in (4.17). This and Sobolev's inequality yield

$$\mathcal{I}_3 \leq C\|\rho_0\|_{L^\infty} \Big(\|\nabla_x \eta\|_{H^{s-1}}^a + \|\nabla_x \eta\|_{H^s} \Big) \|v - m_c\|_{H^s} \|\nabla_x^k v\|_{L^2}.$$

We use (4.7) and (4.12) to have

$$\mathcal{I}_3 \leq C\|\rho_0\|_{L^\infty} (1 + \varepsilon_1 t)^a \exp\Big(- C\frac{(1+\varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big) \|\nabla_x^k v\|_{L^2}$$

$$= C\|\rho_0\|_{H^s} \exp\Big(- C\frac{(1+\varepsilon_1 t)^{1-2\beta}}{\varepsilon_1} \Big) \frac{(1+\varepsilon_1 t)^a}{\varepsilon_1} \varepsilon_1 \|\nabla_x^k v\|_{L^2}.$$

Using the property of (4.14) similarly as in the estimate for $\mathcal{I}_2$, we choose δ such as in (4.15) so that

$$\mathcal{I}_3 \leq C\|\rho_0\|_{H^s} \delta \varepsilon_1 \|\nabla_x^k v\|_{L^2}$$

$$\leq C\|\rho_0\|_{H^s} \delta^2 \|\nabla_x^k v\|_{L^2}^2 + \frac{\varepsilon_1^2}{8^{k+1}}$$

$$\leq \frac{\psi_L}{4} \|\nabla_x^k v\|_{L^2}^2 + \frac{\varepsilon_1^2}{8^{k+1}}.$$

Therefore, we combine this and (4.16) to have

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x^k v\|_{L^2}^2 \leq -\frac{\psi_L}{2}\|\nabla_x^k v\|_{L^2}^2 + \frac{1}{4}\frac{\varepsilon_1^2}{8^k}.$$

This implies, by Gronwall's inequality, that

$$\|\nabla^k v\|_{L^2}^2 \leq \|\nabla^k u_0\|_{L^2}^2 + \frac{1}{2}\frac{\varepsilon_1^2}{8^k}.$$

Finally, we combine the above estimates for $1 \leq k \leq s+1$ together with (4.10) to obtain

$$\|v\|_{H^{s+1}}^2 \leq C(1 + \|\rho_0\|_{H^s})\|u_0\|_{H^{s+1}}^2 + \frac{\varepsilon_1^2}{4},$$

which yields

$$\|v\|_{H^{s+1}} \leq C(1 + \|\rho_0\|_{H^s})^{1/2}\|u_0\|_{H^{s+1}} + \frac{\varepsilon_1}{2}.$$

This completes the proof. □

4.3. Proof of Theorem 3.1. We are now ready to provide a proof of Theorem 3.1.

Part A (existence). First, we set a positive constant $\varepsilon_0 := \frac{\varepsilon_1}{4C_0}$, where $\varepsilon_1 > 0$ and $C_0 > 0$ are constants as in Proposition 4.4.

For any initial data $u_0 \in H^{s+1}$ satisfying $\|u_0\|_{H^{s+1}} \leq \varepsilon_0$, we define the maximal existence time $T_{\max} \geq 0$ of the system (2.10) by

(4.18) $$T_{\max} := \sup\{t \geq 0 \; : \; \sup_{0 \leq \tau \leq t} \|v(\tau)\|_{H^{s+1}} \leq \varepsilon_1\}.$$

By Proposition 4.2 and with $M_1 = \varepsilon_0, M_2 = \varepsilon_1$, there exists $T_0 = T_0(\varepsilon_0, \varepsilon_1) > 0$. This implies that $T_{\max}$ in (4.18) is well-defined and $T_{\max} > 0$.

If we assume $T_{\max} < \infty$, then we can use the continuation argument to have

$$(4.19) \qquad \sup_{0 \leq \tau \leq T_{\max}} \|v(\tau)\|_{H^{s+1}} = \varepsilon_1.$$

On the other hand, we use Proposition 4.4 to obtain

$$\sup_{0 \leq \tau \leq T_{\max}} \|v(\tau)\|_{H^{s+1}} \leq C_0 \|u_0\|_{H^{s+1}} + \frac{\varepsilon_1}{2} \leq C_0 \varepsilon_0 + \frac{\varepsilon_1}{2} = \frac{3\varepsilon_1}{4},$$

which is a contradiction to (4.19). Hence, we can conclude that $T_{\max} = \infty$. This completes the proof of the global-in-time existence.

Part B (asymptotic behavior). Since we have demonstrated the global-in-time existence of the classical solutions, the computations in Lemma 4.1 are completely justified. Also, if we revisit Lemma 4.3, then $\psi_m(t) = \min_{x,y \in \Omega} \psi(|\eta(x,t) - \eta(y,t)|) > \psi_L$ for some constant $\psi_L > 0$ that is independent of $t > 0$. Therefore, the time-global classical solutions verify the flocking estimate

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-2\psi_L t}, \quad t \geq 0.$$

This completes the proof.   □

5. Conclusion. In this paper, we investigated the emergent dynamics of flocking for the hydrodynamic Cucker–Smale (HCS) system in the moving domain framework. Specifically, when the initial data are compactly supported in $\mathbb{R}^d$, we constructed the time-global solution and showed that the velocity converges to the center of the momentum as $t \to \infty$ by establishing the uniform flocking estimate. In previous studies, a global existence theory for the HCS model was considered only in some restricted cases. For example, the authors [14] studied the global existence and flocking behavior of classical solutions in periodic domains. Ha, Huang, and Wang [13] recently investigated the HCS model with all-to-all couplings $\psi \equiv 1$ in a one-dimensional setting, where the model was reduced to the pressureless Euler system with damping under suitable normalized conditions. When the flocking force is turned off ($K = 0$), the HCS model becomes the pressureless Euler equations that appear in the modeling of galaxy formation in the homogeneous universe [27]. It is known that the pressureless Euler equations with smooth initial data may develop delta shocks. Regarding this issue, our result asserts that the flocking force acts as a regularizing mechanism to prevent the formation of singularities such as delta shocks and vacuum in finite time. The smallness assumption on the velocity, the decay rate $\beta \in (0, 1/2)$, and the structure of the communication weight function have been crucially used in our analysis. It is interesting to see if these conditions can be relaxed and/or the work can be extended in a different framework. These issues need to be considered in a future work.

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