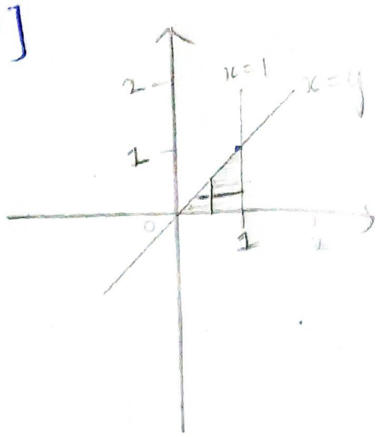
$\iint x^{m-1} y^{n} \, dx\, dy$

21BEC0445

1) Given limits are,

$y \le x \le 1$; $0 \le y \le 1$ [$\because$ Horizontal strip]

Now, by changing order of integration,



$0 \le y \le x$, $0 \le x \le 1$

Now,

$$= \int_0^1 \int_0^x x^2 . e^{xy} . dy . dx$$

$$= \int_0^1 [x . e^{xy}]_0^x . dx$$

$$= \int_0^1 (x . e^{x^2} - x . e^0) . dx$$

$$= \int_0^1 (x e^{x^2} - x) \, dx$$

$$= \int_0^1 x . e^{x^2} . dx - \int_0^1 x . dx$$

Solving $\int_0^1 x e^{x^2} . dx$

$$= \int_0^1 x . e^{x^2} dx$$

let $x^2 = t \Rightarrow$ $x=0, t=0,$ $x=1, t=1$

$2x \, dx = dt$

$dx = \frac{dt}{2x}$

Now,

$$= \int_0^1 \frac{e^t}{2} . dt$$

$$= e^t \Big|_0^1$$

$$= \frac{e-1}{2}$$

So,

$$= \frac{e-1}{2} - \frac{x^2}{2}\Big|_0^1$$

$$= \frac{e-1}{2} - [1/2]$$

$$= e - 3/2$$

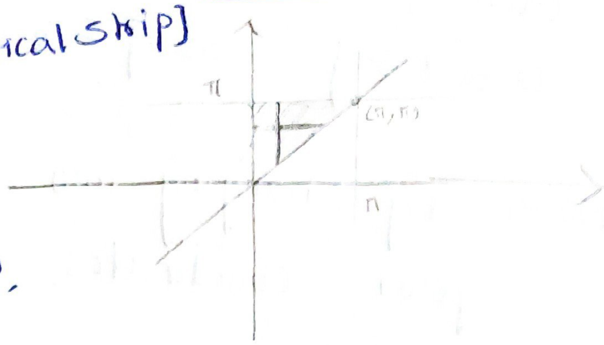
$$= \frac{2e-3}{2} \,//$$

$$= \frac{e}{2} - 1 \,//$$

21BECO445

(b) given limits,

$x \leq y \leq \pi \; ; \; 0 \leq x \leq \pi$ [Vertical strip]



Now, by changing the order of integration,

$0 \leq x \leq y \; ; \; 0 \leq y \leq \pi$

so,

$$= \int_0^{\pi} \int_0^{y} \frac{\sin y}{y} \cdot dx \cdot dy$$

$$= \int_0^{\pi} \cdot \left(\left[\frac{\sin y}{x} \cdot x \right]_0^y \right) dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} \cdot y \cdot dy$$

$$= \int_0^{\pi} \sin y \cdot dy$$

$$= \left[-\cos y \right]_0^{\pi}$$

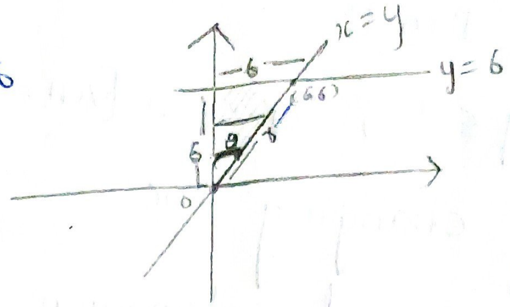
$$= -\cos \pi - (-\cos 0)$$

$$= -(-1) + 1$$

$$= 2 \;//$$

2) (a) Given,

$0 \le x \le y \ ; \ 0 \le y \le 6$



Now, by changing into polar coordinates,

$0 \le r \le 6\,\text{cosec}\,\theta$ $\left[\because \sin\theta = 6/r \quad r = 6/\sin\theta \quad r = 6\,\text{cosec}\,\theta \right]$

Now, $\tan\theta = 1$

$\theta = 45^{\circ}$

So, θ limits are, $\pi/4 \le \theta \le \pi/2$

Now,

$$= \int_{\pi/4}^{\pi/2} \int_{0}^{6\,\text{cosec}\,\theta} r\cos\theta \cdot r\, dr \cdot d\theta \qquad [x = r\cos\theta]$$

$$= \int_{\pi/4}^{\pi/2} \int_{0}^{6\,\text{cosec}\,\theta} r^2 \cdot \cos\theta \cdot dr \cdot d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\left(\frac{r^3}{3} \right)_0^{6\,\text{cosec}\,\theta} \cdot \cos\theta \right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{(6^3)\,\text{cosec}^3\theta}{3} \cdot \cos\theta \right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} 72\,\text{cosec}^3\theta \cdot \cos\theta \cdot d\theta = \int_{\pi/4}^{\pi/2} 72 \cdot \text{cosec}^2\theta \cdot \cot\theta \cdot d\theta$$

Now,

(21BEC0445)

let $\cot\theta = t$ | $\theta = \pi/4 \rightarrow t = 1$

$-\operatorname{cosec}^2\theta . d\theta = dt$ | $\theta = \pi/2 \rightarrow t = 0$

$d\theta = \frac{-dt}{\operatorname{cosec}^2\theta}$

Now,

$$= \int_1^0 -72.t.dt$$

$$= \int_0^1 72.t.dt$$

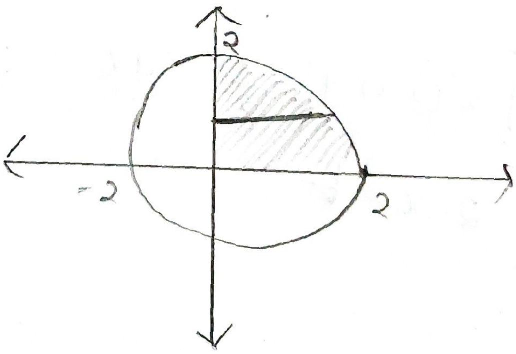
$$= 72.\int_0^1 t.dt$$

$$= 72.\left(\frac{t^2}{2}\right)_0^1$$

$$= \frac{72}{2} = 36 \;//$$

(b) given, $0 \le x \le \sqrt{4-y^2}$; $0 \le y \le 2$

Now, by changing to polar co-ordinates



$0 \le r \le 2$; $0 \le \theta \le \pi/2$

$$= \int_0^{\pi/2}\int_0^2 r^2 . r\, dr\, d\theta \quad (\because \text{as, for circle, } x = r\cos\theta,\ y = r\sin\theta)$$

$$= \int_0^{\pi/2}\int_0^2 r^3 . dr . d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^2 . d\theta$$

$$= \int_0^{\pi/2} 4 . d\theta$$

$$= [4\theta]_0^{\pi/2} = 2\pi \;//$$

position of a

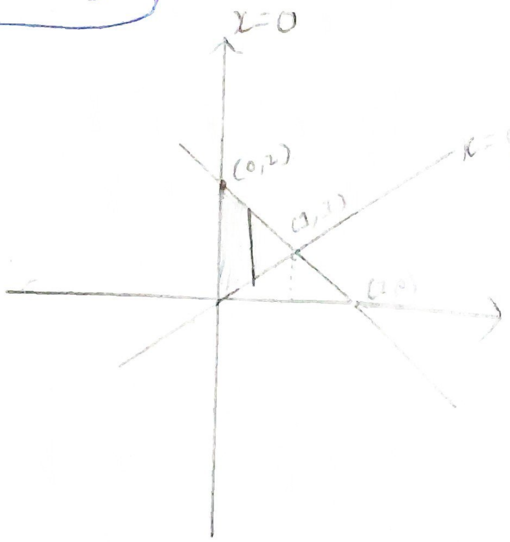
21BECOHUS

(3) Here,

$0 \le z \le x^2+y^2$

$x \le y \le 2-x$

$0 \le x \le 1$



Now,

volume is,

$$= \int_0^1 \int_x^{2-x} \int_0^{x^2+y^2} dz\, dy\, dx$$

$$= \int_0^1 \int_x^{2-x} (z)_0^{x^2+y^2} \cdot dy\, dx$$

$$= \int_0^1 \int_x^{2-x} x^2+y^2 \cdot dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{2-x} \cdot dx$$

$$= \int_0^1 \left(x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right) \cdot dx$$

$$= \int_0^1 \left(2x^2 - x^3 + \frac{(2-x)^3}{3} - \frac{4x^3}{3}\right) dx$$

$$= \int_0^1 \left(2x^2 + \frac{8 - x^3 - 12x + 6x^2}{3} - \frac{7x^3}{3}\right) dx$$

$$= \int_0^1 \left(\frac{6x^2 + 8 - x^3 - 12x + 6x^2 - 7x^3}{3}\right) dx$$

$$= \int_0^1 \left(\frac{12x^2 + 8 - 8x^3 - 12x}{3}\right) dx$$

$$= \int_0^1 \left(4x^2 + \frac{8}{3} - \frac{8}{3}x^3 - 4x\right) dx$$

$$= \left(\frac{4}{3}x^3 + \frac{8}{3}x - \frac{2}{3}x^4 - 2x\right]_0^1 = \frac{4}{3} + \frac{8}{3} - \frac{2}{3} - 2 = \frac{10}{3} - 2 = \frac{4}{3}$$

Q1BECOUHS

1) given

$0 \leq z \leq x$;

$0 \leq x \leq \sqrt{1-y^2}$;

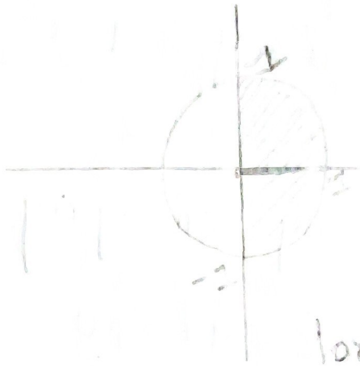
$-1 \leq y \leq 1$

by changing to polar co-ordinates

$0 \leq z \leq r\cos\theta$

$-\pi/2 \leq \theta \leq \pi/2$

$0 \leq r \leq 1$



for θ and r graph

NOW,

$$= \int_0^1 \int_{-\pi/2}^{\pi/2} \int_0^{r\cos\theta} r^2 \cdot dz\, d\theta (r dr)$$

$$= \int_0^1 \int_{-\pi/2}^{\pi/2} r^2 (z)_0^{r\cos\theta}\, d\theta (r dr)$$

$$= \int_0^1 \int_{-\pi/2}^{\pi/2} r^3 \cdot \cos\theta \cdot d\theta \cdot (r dr)$$

$$= \int_0^1 \left(r^3 \left(\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\right) r \cdot dr\right)$$

$$= \int_0^1 r^4 (\sin\theta)_{-\pi/2}^{\pi/2}\, dr$$

$$= \int_0^1 2r^4 \cdot dr$$

$$= 2\left[\frac{r^5}{5}\right]_0^1$$

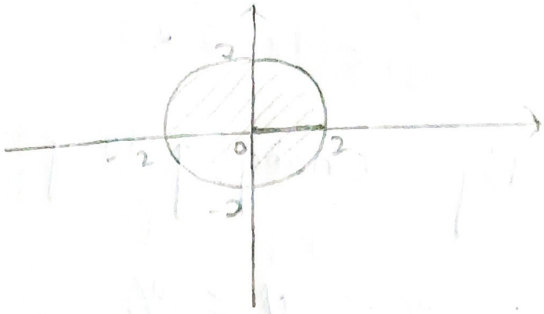
$$= 2/5 \;//$$

22BECOM45

(b)

$$0 \le z \le 4-y$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$
$$-2 \le x \le 2$$

Now by changing to polar coordinates,



$$0 \le z \le 4 - r\sin\theta$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

So,

$$= \int_0^{2\pi}\int_0^{2}\int_0^{4-r\sin\theta} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(z)_0^{4-r\sin\theta} \, dr \cdot d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (4r - r^2\sin\theta) \, dr \cdot d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^3}{3}\sin\theta\right)_0^2 \cdot d\theta$$

$$= \int_0^{2\pi} 8 - \frac{8}{3}\sin\theta \cdot d\theta$$

$$= \left(8\theta + \frac{8}{3}\cos\theta\right)_0^{2\pi}$$

$$= 16\pi + \frac{8}{3} - \left(0 + 8/3\right)$$

$$= 16\pi \,//$$

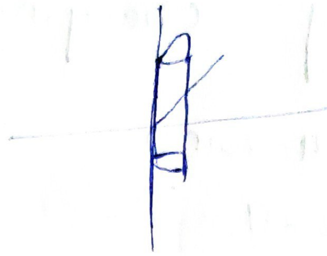
(5)

(a) given

$-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}$ (sphere of eq $x^2+y^2+z^2=4$)

$0 \le y \le \sqrt{2-x^2}$

$0 \le x \le 2$



By changing to polar,

$-\pi/2 \le \phi \le \pi/2$

$0 \le \theta \le \pi/2$

$0 \le r \le 2$

So,

$$= \int_{r=0}^{2} \int_{\theta=0}^{\pi/2} \int_{\phi=-\pi/2}^{\pi/2} r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{\pi/2} \left(2\int_{0}^{\pi/2} r^2 . \sin\phi \, d\phi \right) d\theta . dr$$ (∵ definite integral property)

$$= \int_{r=0}^{2} \int_{\theta=0}^{\pi/2} r^2 . d\theta . dr$$

$$= \int_{r=0}^{2} [r^2 . \theta]_0^{\pi/2} . dr$$

$$= \int_{r=0}^{2} \frac{r^2\pi}{2} \, dr$$

$$= \left(\frac{r^3\pi}{6} \right)_0^2 = \frac{8\pi}{6} \; //$$

21BEC0445

6) $\int_0^1 \frac{x}{\sqrt{1-x^5}} \cdot dx$

Sol: to convert into β or γ function form, & simplify.

$= \int_0^1 x(1-x^5)^{-1/2} \cdot dx$

let $x^5 = t \Rightarrow x = t^{1/5}$ | $x=0 \rightarrow t=0$, $x=1 \rightarrow t=1$

$5x^4 dx = dt$

$dx = \frac{dt}{5x^4} = \frac{dt}{5t^{4/5}}$

$= \int_0^1 t^{1/5}(1-t)^{-1/2} \cdot \frac{dt}{5t^{4/5}}$ $\left(\begin{array}{l} m-1 = -3/5 \Rightarrow m = 2/5 \\ n-1 = -1/2 \Rightarrow n = 1/2 \end{array} \right)$

$= \frac{\int_0^1 t^{-3/5}(1-t)^{-1/2} \cdot dt}{5} = \frac{\beta(2/5, 1/2)}{5}$ //

Now, $\beta(2/5, 1/2) = \frac{\Gamma(2/5) \cdot \Gamma(1/2)}{\Gamma(9/10)}$

21BEC0045
AKHIL.DONTHULA

7) $\int_0^{\pi} \frac{1}{\sqrt{\sin\theta}} \cdot d\theta \Rightarrow 2\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} \, d\theta$

NOW,

let $\sin\theta = t$

$\cos\theta \cdot d\theta = dt$

$d\theta = \frac{dt}{\sqrt{1-\sin^2\theta}}$

$d\theta = \frac{dt}{\sqrt{1-t^2}}$

$\theta = 0, t = 0$

$\theta = \frac{\pi}{2}, t = 1$

NOW,

$= 2\int_0^{1} \cdot t^{-1/2}(1-t^2)^{-1/2} \cdot dt$

let $t^2 = x$

$t = \sqrt{x}$

$2t \cdot dt = dx$

$dt = dx/2\sqrt{x}$

$t = 0, x = 0$

$t = 1, x = 1$

NOW,

$= \cancel{2}\int_0^1 \cdot x^{-1/4}(1-x)^{-1/2} \cdot \frac{dx}{\cancel{2}\sqrt{x}}$

$= \int_0^1 x^{-1/4}(1-x)^{-1/2} \cdot \frac{dx}{\sqrt{x}} \Rightarrow \int_0^1 x^{-3/4}(1-x)^{-1/2} \cdot dx$

NOW comparing with β function.

$m-1 = -3/4$

$m = 1/4$

$n-1 = -1/2$

$n = 1/2$

So, $\int_0^1 x^{-3/4}(1-x)^{-1/2} \cdot dx = \beta\left(1/4, 1/2\right)$ //

8) $\int_0^1 x^m (1-x^n)^p \cdot dx$

NOW, $x^n = t \Rightarrow x = t^{1/n}$

$n x^{n-1} dx = dt$ $\quad x=0 \to t=0$

$x=1 \to t=1$

NOW,

$$= \int_0^1 t^{m/n} (1-t)^p \cdot \frac{dt}{n\, t^{\left(\frac{n-1}{n}\right)}} \; dt$$

$$= \frac{1}{n} \int_0^1 t^{\left(\frac{m+n-1}{n}\right)} (1-t)^p \, dt$$

By comparing with β function,

$$= \frac{\beta\left(\frac{m+n-1}{n}+1,\ p+1\right)}{n}$$

ii) NOW evaluating $\int_0^1 \frac{1}{\sqrt{1-x^n}} \cdot dx$

$m=0$, so,

value will be $\dfrac{\beta\left(\frac{n-1}{n}\ 3/2\right)}{n}$ //

$e^{-a^2x^2} = t$ $\quad a^2x^2 = t$,

9) given $\int_0^\infty x^n e^{-a^2x^2} \cdot dx$

let $\Rightarrow a^2x^2 = t$; $2a^2x \cdot dx = dt$

$x = \frac{\sqrt{t}}{a}$ $\quad \boxed{dx = \frac{dt}{2a^2x} = \frac{dt}{2a\sqrt{t}}}$

so,

$x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

$$= \int_0^\infty \frac{t^{n/2}}{a^n} \cdot e^{-t} \cdot \frac{dt}{2a\sqrt{t}}$$

$$= \int_0^\infty \frac{t^{n/2} \cdot e^{-t} dt}{2a^{n+1} \cdot \sqrt{t}}$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty t^{\frac{n-1}{2}} \cdot e^{-t} \cdot dt$$ → comparing with gamma function

$$= \frac{1}{2a^{n+1}} \cdot \Gamma\left(\frac{n+1}{2}\right) // \quad \left[\because \frac{n-1}{2} = x-1, \quad x = \frac{n+1}{2}\right]$$

ii) NOW, evaluating $\int_0^\infty e^{-a^2x^2} \cdot dx$

let $\Rightarrow a^2x^2 = t$

$x = \frac{\sqrt{t}}{a}$; $2a^2x \cdot dx = dt$

$dt = \frac{dt}{2a^2x} = \frac{dt}{2a\sqrt{t}}$

NOW,

$$= \int_0^\infty e^{-t} \cdot \frac{dt}{2a\sqrt{t}}$$

$$= \frac{1}{2a} \int_0^\infty t^{-1/2} e^{-t} dt$$

→ NOW comparing with Γ function

$n - 1 = -1/2 \Rightarrow n = 1/2$ $\quad \left[\because \Gamma(1/2) = \sqrt{\pi}\right]$

so, $= \frac{1}{2a} \Gamma(1/2) = \frac{1}{2a}(\sqrt{\pi}) = \sqrt{\pi}/2a$ //

(10) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $\Rightarrow$ 21BECO1045

$\frac{x}{a}=u$, $\frac{y}{b}=v$, $\frac{z}{c}=w \Rightarrow u+v+w=1$

$x=au$, $y=bv$, $z=cw$

$$J\left(\frac{x,y,z}{u,v,w}\right)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

Now,

$$\iiint_V f(x,y,z).dx\,dy\,dz = \iiint f(u,v,w).J\left(\frac{x,y,z}{u,v,w}\right)du\,dv\,dw$$

So, $\int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc.\,dw.\,dv.\,du$

$= \int_0^1 \int_0^{1-u} (abc(1-u-v))\,dv.\,du$

$= \int_0^1 abc\left(v-uv-\frac{v^2}{2}\right)_0^{1-u} du = abc\int_0^1 (1-u)-u(1-u)-\frac{v^2}{2}.\,du$

21BECO4HS

$= abc \int_0^1 (1-u)^2 - \frac{(1-u)^2}{2} \cdot du$

$= abc \int_0^1 \frac{(1-u)^2}{2} \cdot du$

$= \frac{abc}{2} \int_0^1 u^0 \cdot (1-u)^2 \cdot du$

by comparing with β function

$m-1=0$

$m=1$; $n-1=2$

$n=3$

Now,

$= \frac{abc}{2} \beta(1,3)$

$= \frac{abc}{2} \cdot \frac{\Gamma(1) \cdot \Gamma(3)}{\Gamma(4)}$

$= \frac{abc}{2} \cdot \frac{2!}{3!}$ $\quad \left(\because \Gamma(1) = 1 \right.$ $\left. \Gamma(n+1) = n! \right)$

$= \frac{abc}{2} \times \frac{2}{6}$

$= \frac{abc}{6}$ //