Chapter 3: Random Variables

Section 3.2: Binomial and Hypergeometric Probabilities

3.2.1 The number of days, k, the stock rises is binomial with $n = 4$ and $p = 0.25$. The stock will be the same after four days if $k = 2$. The probability that $k = 2$ is $\binom{4}{2}(0.25)^2(0.75)^2 = 0.211$

3.2.2 Let k be the number of control rods properly inserted. The system fails if $k \le 4$. The probability of that occurrence is given by the binomial probability sum
$\sum_{k=0}^{4}\binom{10}{k}(0.8)^k(0.2)^{10-k} = 0.0064$

3.2.3 The probability of 12 female presidents is 0.23^{12}, which is approximately 1/50,000,000.

3.2.4 $1-\binom{6}{0}(0.153)^0(0.847)^6-\binom{6}{1}(0.153)^1(0.847)^5 = 0.231$

3.2.5 $1-\sum_{k=8}^{11}\binom{11}{k}(0.9)^k(0.1)^{11-k} = 0.0185$

3.2.6 The probability of k sightings is given by the binomial probability model with $n = 10{,}000$ and $p = 1/100{,}000$. The probability of at least one genuine sighting is the probability that $k \ge 1$. The probability of the complementary event, $k = 0$, is $(99{,}999/100{,}000)^{10{,}000} = 0.905$.
Thus, the probability that $k \ge 1$ is $1 - 0.905 = 0.095$.

3.2.7 For the two-engine plane, P(Flight lands safely) = P(One or two engines work properly)
$= \binom{2}{1}(0.6)^1(0.4)^1+\binom{2}{2}(0.6)^2(0.4)^0 = 0.84$
For the four-engine plane, P(Flight lands safely) = P(Two or more engines work properly)
$= \binom{4}{2}(0.6)^2(0.4)^2+\binom{4}{3}(0.6)^3(0.4)^1+\binom{4}{4}(0.6)^4(0.4)^0 = 0.8208$
The two-engine plane is a better choice.

3.2.8 Probabilities for the first system are binomial with $n = 50$ and $p = 0.05$. The probability that $k \ge 1$ is $1-(0.95)^{50} = 1 - 0.077 = 0.923$.
Probabilities for the second system are binomial with $n = 100$ and $p = 0.02$. The probability that $k \ge 1$ is $1-(0.98)^{100} = 1 - 0.133 = 0.867$
System 2 is superior from a bulb replacement perspective.

3.2.9 The number of 6's obtained in n tosses is binomial with $p = 1/6$. The first probability in question has $n = 6$. The probability that $k \geq 1$ is $1-(5/6)^6 = 1 - 0.33 = 0.67$.
For the second situation, $n = 12$. The probability that $k \geq 2$ one minus the probability that $k = 0$ or 1, which is $1-(5/6)^{12} - 12(1/6)(5/6)^{11} = 0.62$. Finally, take $n = 18$. The probability that $k \geq 3$ is one minus the probability that $k = 0$, 1, or 2, which is
$1-(5/6)^{18} - 18(1/6)(5/6)^{17} - 153(1/6)^2(5/6)^{16} = 0.60$.

3.2.10 The number of missile hits on the plane is binomial with $n = 6$ and $p = 0.2$. The probability that the plane will crash is the probability that $k \geq 2$. This event is the complement of the event that $k = 0$ or 1, so the probability is $1 - (0.8)^6 - 6(0.2)(0.8)^5 = 0.345$
The number of rocket hits on the plane is also binomial, but with $n = 10$ and $p = 0.05$.
The probability that the boat will be disabled is $P(k \geq 1)$, which is $1 - (0.95)^{10} = 0.401$

3.2.11 The number of girls is binomial with $n = 4$ and $p = 1/2$. The probability of two girls and two boys is $\binom{4}{2}(0.5)^4 = 0.375$. The probability of three and one is $2\binom{4}{3}(0.5)^4 = 0.5$, so the latter is more likely.

3.2.12 The number of recoveries if the drug is effective is binomial with $n = 12$ and $p = 1/2$. The drug will discredited if the number of recoveries is 6 or less. The probability of this is
$$\sum_{k=0}^{6}\binom{12}{k}(0.5)^{12} = 0.613.$$

3.2.13 The probability it takes k calls to get four drivers is $\binom{k-1}{3}0.80^4 0.20^{k-4}$. We seek the smallest number n so that $\sum_{k=4}^{n}\binom{k-1}{3}0.80^4 0.20^{k-4} \geq 0.95$. By trial and error, $n = 7$.

3.2.14 The probability of any shell hitting the bunker is 30/500 = 0.06. The probability of exactly k shells hitting the bunker is $p(k) = \binom{25}{k}(0.06)^k(0.94)^{25-k}$. The probability the bunker is destroyed is $1 - p(0) - p(1) - p(2) = 0.187$.

3.2.15 (1) The probability that any one of the seven measurements will be in the interval (1/2, 1) is 0.50. The probability that exactly three will fall in the interval is
$$\binom{7}{3}0.5^7 = 0.273$$

(2) The probability that any one of the seven measurements will be in the interval (3/4, 1) is 0.25. The probability that fewer than 3 will fall in the interval is
$$\sum_{k=0}^{2}\binom{7}{k}(0.25)^k(0.75)^{7-k} = 0.756$$

3.2.16 Use the methods of Example 3.2.3 for $p = 0.5$. Then the probabilities of Team A winning the series in 4, 5, 6, and 7 games are 0.0625, 0.125, 0.156, and 0.156, respectively. Since Team B has the same set of probabilities, the probability of the series ending in 4, 5, 6, and 7 games is double that for A, or 0.125, 0.250, 0.312, and 0.312, respectively. The "expected" frequencies are the number of years, 58, times the probability of each length. For example, we would "expect" 58(0.126) = 7.3 series of 4 games. The table below gives the comparison of observed and expected frequencies.

Number of games	Observed Number of years	Expected Number of years
4	12	58(0.125) = 7.3
5	10	58(0.250) = 14.5
6	12	58(0.312) = 18.1
7	24	58(0.312) = 18.1

Note that the model has equal expected frequencies for 6 and 7 length series, but the observed numbers are quite different. This model does not fit the data well.

3.2.17 By the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Let $x = p$ and $y = 1 - p$.

Then $1 = [p + (1-p)]^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

3.2.18 Any particular sequence having k_1 of Outcome 1 and k_2 of Outcome 2, must have $n - k_1 - k_2$ of Outcome 3. The probability of such a sequence is $p_1^{k_1} p_2^{k_2} (1 - p_2 - p_2)^{n-k_1-k_2}$.

The number of such sequences depends on the number of ways to choose the k_1 positions in the sequence for Outcome 1 and the k_2 positions for Outcome 2. The k_1 positions can be chosen in $\binom{n}{k_1}$ ways. For each such choice, the k_2 positions can be chosen in $\binom{n-k_1}{k_2}$ ways. Thus, $P(k_1$ of Outcome 1 and k_2 of Outcome 2) $= \binom{n}{k_1}\binom{n-k_1}{k_2} p_1^{k_1} p_2^{k_2} (1 - p_1 - p_2)^{n-k_1-k_2}$

$$= \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} p_1^{k_1} p_2^{k_2} (1-p_1-p_2)^{n-k_1-k_2}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} p_1^{k_1} p_2^{k_2} (1-p_1-p_2)^{n-k_1-k_2}$$

3.2.19 In the notation of Question 3.2.18, $p_1 = 0.5$ and $p_2 = 0.3$, with $n = 10$. Then the probability of 3 of Outcome 1 and 5 of Outcome 2 is $\frac{10!}{3!5!2!}(0.5)^3(0.3)^5(0.2)^2 = 0.031$

3.2.20 Use the hypergeometric model with $N = 12$, $n = 5$, $r = 4$, and $w = 12 - 4 = 8$. The probability that the committee will contain two accountants ($k = 2$) is $\dfrac{\binom{4}{2}\binom{8}{3}}{\binom{12}{5}} = 14/33$

3.2.21 "At least twice as many black bears as tan-colored" translates into spotting 4, 5, or 6 black bears. The probability is $\dfrac{\binom{6}{4}\binom{3}{2}}{\binom{9}{6}} + \dfrac{\binom{6}{5}\binom{3}{1}}{\binom{9}{6}} + \dfrac{\binom{6}{6}\binom{3}{0}}{\binom{9}{6}} = 64/84$

3.2.22 The probabilities are hypergeometric with $N = 4050$, $n = 65$, $r = 514$, and $w = 4050 - 514 = 3536$. The probability that k children have not been vaccinated is

$$\frac{\binom{514}{k}\binom{3536}{65-k}}{\binom{4050}{65}},\ k = 0, 1, 2, \ldots, 65$$

3.2.23 The probability that k nuclear missiles will be destroyed by the anti-ballistic missiles is hypergeometric with $N = 10$, $n = 7$, $r = 6$, and $w = 10 - 6 = 4$. The probability the Country B will be hit by at least one nuclear missile is one minus the probability that $k = 6$, or

$$1 - \frac{\binom{6}{6}\binom{4}{1}}{\binom{10}{7}} = 0.967$$

3.2.24 Let k be the number of questions chosen that Anne has studied. Then the probabilities for k are hypergeometric with $N = 10$, $n = 5$, $r = 8$, and $w = 10 - 8 = 2$. The probability of her getting at least four correct is the probability that $k = 4$ or 5, which is

$$\frac{\binom{8}{4}\binom{2}{1}}{\binom{10}{5}} + \frac{\binom{8}{5}\binom{2}{0}}{\binom{10}{5}} = \frac{140}{252} + \frac{56}{252} = 0.778$$

3.2.25 The probabilities for the number of men chosen are hypergeometric with $N = 18$, $n = 5$, $r = 8$, and $w = 10$. The event that both men and women are represented is the complement of the event that 0 or 5 men will be chosen, or $1 - \dfrac{\binom{8}{0}\binom{10}{5}}{\binom{18}{5}} + \dfrac{\binom{8}{5}\binom{10}{0}}{\binom{18}{5}} = 1 - \dfrac{252}{8568} + \dfrac{56}{8568} = 0.964$

3.2.26 The probability is hypergeometric with $N = 80$, $n = 10$, $r = 20$, and $w = 60$, and equals

$$\frac{\binom{20}{6}\binom{60}{4}}{\binom{80}{10}} = 0.0115$$

3.2.27 First, calculate the probability that exactly one real diamond is taken during the first three grabs. There are three possible positions in the sequence for the real diamond, so this probability is

$$\frac{3(10)(25)(24)}{(35)(34)(33)}.$$

The probability of a real diamond being taken on the fourth removal is 9/32. Thus, the desired probability is $\frac{3(10)(25)(24)}{(35)(34)(33)} \times \frac{9}{32} = \frac{162{,}000}{1{,}256{,}640} = 0.129.$

3.2.28 $$\frac{\binom{2}{2}\binom{8}{0} + \binom{6}{2}\binom{4}{0} + \binom{2}{2}\binom{8}{0}}{\binom{10}{2}} = \frac{1+15+1}{45} = \frac{17}{45} = 0.378$$

3.2.29 The k-th term of $(1 + \mu)^N = \binom{N}{k}\mu^k$

$$(1 + \mu)^r\,(1 + \mu)^{N-r} = \left(\sum_{i=1}^{r}\binom{r}{i}\mu^i\right)\left(\sum_{j=1}^{N-r}\binom{N-r}{j}\mu^j\right)$$

The k-th term of this product is $\sum_{i=1}^{r}\binom{r}{i}\binom{N-r}{k-i}\mu^k$

Equating coefficients gives $\binom{N}{k} = \sum_{i=1}^{k}\binom{r}{i}\binom{N-r}{k-i}$.

Dividing through by $\binom{N}{k}$ shows that the hypergeometric terms sum to 1.

3.2.30 $$\frac{\binom{r}{k+1}\binom{w}{n-k-1}}{\binom{N}{n}} \div \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}} = \binom{r}{k+1}\binom{w}{n-k-1} \div \binom{r}{k}\binom{w}{n-k}$$

$$= \binom{r}{k+1}\binom{w}{n-k-1} \div \binom{r}{k}\binom{w}{n-k}$$

$$= \frac{r!}{(k+1)!(r-k-1)!} \cdot \frac{w!}{(n-k-1)!(w-n+k+1)!} \cdot \frac{k!(r-k)!}{r!} \cdot \frac{(n-k)!(w-n+k)!}{w!}$$

$$= \frac{n-k}{(k+1)} \cdot \frac{r-k}{(w-n+k+1)}.$$

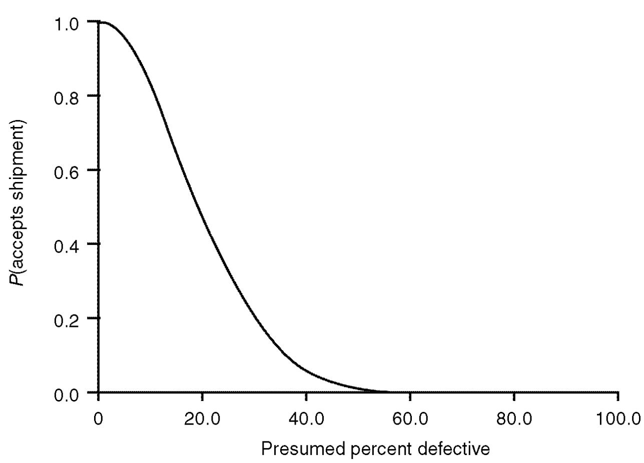
3.2.31 Let W_0, W_1 and W_2 be the events of drawing zero, one, or two white chips, respectively, from Urn I. Let A be the event of drawing a white chip from Urn II. Then
$P(\mathrm{a}) = P(A|W_0)P(W_0) + P(A|W_1)P(W_1) + P(A|W_2)P(W_2)$

$$= \frac{5}{11}\frac{\binom{5}{2}\binom{4}{0}}{\binom{9}{2}} + \frac{6}{11}\frac{\binom{5}{1}\binom{4}{1}}{\binom{9}{2}} + \frac{7}{11}\frac{\binom{5}{0}\binom{4}{2}}{\binom{9}{2}} = 53/99$$

3.2.32 For any value of r = number of defective items, the probability of accepting the sample is

$$p_r = \frac{\binom{r}{0}\binom{100-r}{10}}{\binom{100}{10}} + \frac{\binom{r}{1}\binom{100-r}{9}}{\binom{100}{10}}$$

Then the operating characteristic curve is the plot of the presumed percent defective versus the probability of accepting the shipment, or $100(r/100) = r$ on the x-axis and p_r on the y-axis. If there are 16 defective, you will accept the shipment approximately 50% of the time.



3.2.33 There are $\dfrac{r!}{r_1!r_2!r_3!}$ ways to divide the red chips into three groups of the given sizes. There are $\dfrac{(N-r)!}{(n_1-r_1)!(n_2-r_2)!(n_3-r_3)!}$ ways to divide the white chips into the three groups of the required sizes. The total number of ways to divide the N objects into groups of n_1, n_2, and n_3 objects is $\dfrac{N!}{n_1!n_2!n_3!}$. Thus, the desired probability is $\dfrac{\dfrac{r!}{r_1!r_2!r_3!}\dfrac{(N-r)!}{(n_1-r_1)!(n_2-r_2)!(n_3-r_3)!}}{\dfrac{N!}{n_1!n_2!n_3!}} = \dfrac{\binom{n_1}{r_1}\binom{n_2}{r_2}\binom{n_3}{r_3}}{\binom{N}{r}}$.

3.2.34 First, calculate the probability that the first group contains two disease carriers and the others have one each. The probability of this, according to Question 3.2.33, is $\dfrac{\binom{7}{2}\binom{7}{1}\binom{7}{1}}{\binom{21}{4}} = 49/285$.

The probability that either of the other two groups has 2 carriers and the others have one is the same. Thus, the probability that each group has at least one diseased member is
$3\dfrac{49}{285} = \dfrac{49}{95} = 0.516$. Then the probability that at least one group is disease free is
$1 - 0.516 = 0.484$.

3.2.35 There are $\binom{N}{n}$ total ways to choose the sample. There are $\binom{n_i}{k_i}$ ways to arrange for k_i of the n_i objects to be chosen, for each i. Using the multiplication rule shows that the probability of getting k_1 objects of the first kind, k_2 objects of the second kind, …, k_t objects of the t-th kind is
$\dfrac{\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_t}{k_t}}{\binom{N}{n}}$

3.2.36 In the notation of Question 3.2.33, let $n_1 = 5$, $n_2 = 4$, $n_3 = 4$, $n_4 = 3$, so $N = 16$. The sample size is given to be $n = 8$, and $k_1 = k_2 = k_3 = k_4 = 2$. Then the probability that each class has two

representatives is $\dfrac{\binom{5}{2}\binom{4}{2}\binom{4}{2}\binom{3}{2}}{\binom{16}{8}} = \dfrac{(10)(6)(6)(3)}{12{,}870} = \dfrac{1080}{12{,}870} = 0.084$.

Section 3.3: Discrete Random Variables

3.3.1 **(a)** Each outcome has probability 1/10

Outcome	X = larger no. drawn
1, 2	2
1, 3	3
1, 4	4
1, 5	5
2, 3	3
2, 4	4
2, 5	5
3, 4	4
3, 5	5
4, 5	5

Counting the number of each value of the larger of the two and multiplying by 1/10 gives the pdf:

k	$p_X(k)$
2	1/10
3	2/10
4	3/10
5	4/10

(b)

Outcome	X = larger no. drawn	V = sum of two nos.
1, 2	2	3
1, 3	3	4
1, 4	4	5
1, 5	5	6
2, 3	3	5
2, 4	4	6
2, 5	5	7
3, 4	4	7
3, 5	5	8
4, 5	5	9

k	$p_X(k)$
3	1/10
4	1/10
5	2/10
6	2/10
7	2/10
8	1/10
9	1/10

3.3.2 **(a)** There are $5 \times 5 = 25$ total outcomes. The set of outcomes leading to a maximum of k is $(X = k) = \{(j, k) | 1 \le j \le k - 1\} \cup \{(k, j) | 1 \le j \le k - 1\} \cup \{(k, k\}$, which has $2(k - 1) + 1 = 2k - 1$ elements. Thus, $p_X(k) = (2k - 1)/25$

(b)

Outcomes	V = sum of two nos.
(1, 1)	2
(1, 2) (2, 1)	3
(1, 3) (2, 2) (3, 1)	4
(1, 4) (2, 3) (3, 2) (4, 1)	5
(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)	6
(2, 5) (3, 4) (4, ,3) (5, 2)	7
(3, 5) (4, 4), (5, 3)	8
(4, 5) (5, 4)	9
(5, 5)	10

$p_V(k) = (k - 1)/25$ for $k = 1, 2, 3, 4, 5, 6$ and $p_V(k) = (11 - k)/25$ for $k > 6$

3.3.3 $p_X(k) = P(X = k) = P(X \le k) - P(X \le k - 1)$. But the event $(X \le k)$ occurs when all three dice are $\le k$ and that can occur in k^3 ways. Thus $P(X \le k) = k^3/216$.
Similarly, $P(X \le k - 1) = (k - 1)^3/216$. Thus $p_X(k) = k^3/216 - (k - 1)^3/216$.

3.3.4 $p_X(1) = 6/6^3 = 6/216 = 1/36$
$p_X(2) = 3(6)(5)/6^3 = 90/216 = 15/36$
$p_X(3) = (6)(5)(4)/6^3 = 120/216 = 20/36$

3.3.5

Outcomes	V = no. heads – no. tails
(H, H, H)	3
(H, H, T) (H, T, H) (T, H, H)	1
(T, T, H) (T, H, T) (T, H, H)	−1
(T, T, T)	−3

$p_X(3) = 1/8$, $p_X(1) = 3/8$, $p_X(-1) = 3/8$, $p_X(-3) = 1/8$

3.3.6

Outcomes	k	$p_X(k)$
(1, 1)	2	(1/6)(1/6) = 1/36
(2, 1)	3	(2/6)(1/6) = 2/36
(1, 3) (3, 1)	4	(1/6)(1/6) + (2/6)(1/6) = 3/36
(1, 4) (2, 3) (4, 1)	5	(1/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 4/36
(1, 5) (2, 4) (3, 3)	6	(1/6)(1/6) + (2/6)(1/6) + (2/6)(1/6) = 5/36
(1, 6) (2, 5) (3, 4) (4, 3)	7	(1/6)(1/6) +(2/6)(1/6)+ (2/6)(1/6)+ (1/6)(1/6) = 6/36
(2, 6) (3, 5) (4, 4)	8	(2/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 5/36
(1, 8) (3, 6) (4, 5)	9	(1/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 4/36
(2, 8) (4, 6)	10	(2/6)(1/6) + (1/6)(1/6) = 3/36
(3, 8)	11	(2/6)(1/6) = 2/36
(4, 8)	12	(1/6)(1/6) = 1/36

3.3.7 This is similar to Question 3.3.5. If there are k steps to the right (heads), then there are $4-k$ steps to the left (tails). The final position X is number of heads $-$ number of tails $= k-(4-k) = 2k-4$.

The probability of this is the binomial of getting k heads in 4 tosses = $\binom{4}{k}\frac{1}{16}$.

Thus, $p_X(2k-4) = \binom{4}{k}\frac{1}{16}$, $k = 0, 1, 2, 3, 4$

3.3.8 $p_X(2k-4) = \binom{4}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{4-k}$, $k = 0, 1, 2, 3, 4$

3.3.9 Consider the case $k = 0$ as an example. If you are on the left, with your friend on your immediate right, you can stand in positions 1, 2, 3, or 4. The remaining people can stand in 3! ways. Each of these must be multiplied by 2, since your friend could be the one on the left. The total number of permutations of the five people is 5!

Thus, $p_X(0) = (2)(4)(3!)/5! = 48/120 = 4/10$. In a similar manner

$p_X(1) = (2)(3)(3!)/5! = 36/120 = 3/10$

$p_X(2) = (2)(2)(3!)/5! = 24/120 = 2/10$

$p_X(3) = (2)(1)(3!)/5! = 12/120 = 1/10$

3.3.10 $p_{X_1}(k) = p_{X_2}(k) = \dfrac{\binom{2}{k}\binom{2}{2-k}}{\binom{4}{2}}$, $k = 0, 1, 2$. For $X_1 + X_2 = m$, let

$X_1 = k$ and $X_2 = m-k$, for $k = 0, 1, \ldots$, m. Then $p_{X_3}(m) = \sum_{k=0}^{m} p_{X_1}(k)p_{X_2}(m-k)$,

$m = 0, 1, 2, 3, 4$, or

m	$p_{X_3}(m)$
0	1/36
1	2/9
2	1/2
3	2/9
4	1/36

3.3.11 $P(2X+1=k) = P[X=(k-1)/2]$, so $p_{2X+1}(k) = p_X\left(\frac{k-1}{2}\right) = \binom{4}{\frac{k-1}{2}}\left(\frac{2}{3}\right)^{\frac{k-1}{2}}\left(\frac{1}{3}\right)^{4-\frac{k-1}{2}}$,

$k = 1, 3, 5, 7, 9$

3.3.12 $F_X(k) = P(X \le k) = k^3$, as explained in the solution to Question 3.3.3.

3.3.13 $F_X(k) = P(X \le k) = \sum_{j=0}^{k} P(X=j) = \sum_{j=0}^{k}\binom{4}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{4-j}$

3.3.14 $p_x(k) = F_X(k) - F_X(k-1) = \dfrac{k(k+1)}{42} - \dfrac{(k-1)k}{42} = \dfrac{k}{21}$

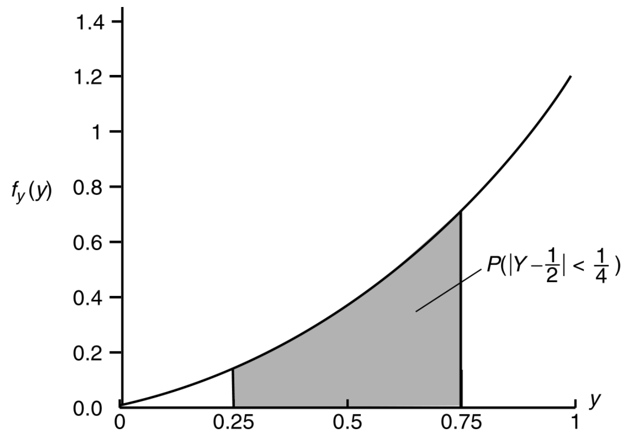
3.3.15 See the solution to Question 3.3.3.

Section 3.4: Continuous Random Variables

3.4.1 $P(0 \le Y \le 1/2) = \int_0^{1/2} 4y^3 dy = y^4\Big|_0^{1/2} = 1/16$

3.4.2 $P(3/4 \le Y \le 1) = \int_{3/4}^{1} \dfrac{2}{3} + \dfrac{2}{3}y\, dy = \dfrac{2y}{3} + \dfrac{y^2}{3}\Big|_{3/4}^{1} = 1 - \dfrac{11}{16} = \dfrac{5}{16}$

3.4.3 $P\left(|Y - 1/2| < 1/4\right) = P(1/4 < Y < 3/4) = \int_{1/4}^{3/4} \dfrac{3}{2}y^2\, dy = \dfrac{y^3}{2}\Big|_{1/4}^{3/4} = \dfrac{27}{128} - \dfrac{1}{128} = \dfrac{26}{128} = \dfrac{13}{64}$



3.4.4 $P(Y > 1) = \int_1^3 (1/9)y^2 dy = (1/27)\, y^3\Big|_1^3 = 1 - 1/27 = 26/27$

3.4.5 **(a)** $\int_{10}^{\infty} 0.2e^{-0.2y} dy = -e^{-0.2y}\Big|_{10}^{\infty} = e^{-2} = 0.135$

(b) If A = probability customer leaves on first trip, and B = probability customer leaves on second trip, then $P(a) = P(b) = 0.135$. In this notation,
$p_X(1) = P(a)P(B^C) + P(A^C)P(b) = 2(0.865)(0.135) = 0.23355$

3.4.6 Clearly the function given is non-negative and continuous. We must show that it integrates to 1.

$$\int_0^1 (n+2)(n+1)y^n(1-y)dy = \int_0^1 (n+2)(n+1)(y^n - y^{n+1})dy = (n+2)(n+1)\left(\frac{y^{n+1}}{n+1} - \frac{y^{n+2}}{n+2}\right)\Bigg|_0^1$$

$$= \left[(n+2)y^{n+1} - (n+1)y^{n+2}\right]\Big|_0^1 = 1$$

3.4.7 $F_Y(y) = P(Y \le y) = \int_0^y 4t^3 dt = t^4 \Big|_0^y = y^4$. Then $P(Y \le 1/2) = F_Y(1/2) = (1/2)^4 = 1/16$

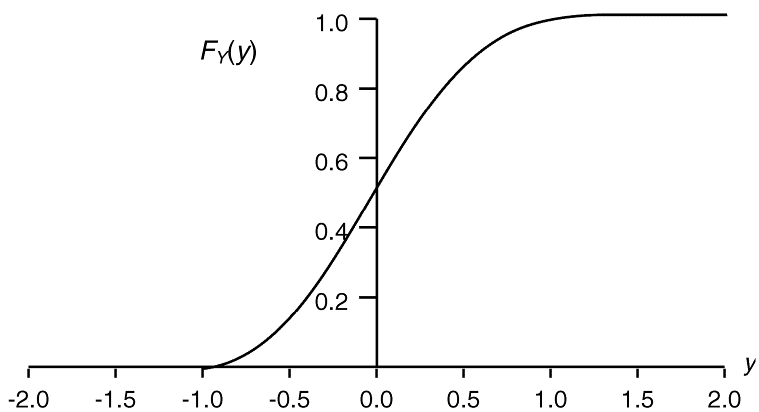
3.4.8 $F_Y(y) = P(Y \le y) = \int_0^y \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_0^y = 1 - e^{-\lambda y}$

3.4.9 For $y < -1$, $F_Y(y) = 0$

For $-1 \le y < 0$, $F_Y(y) = \int_{-1}^y (1+t)dt = \frac{1}{2} + y + \frac{1}{2}y^2$

For $0 \le y \le 1$, $F_Y(y) = \int_{-1}^y (1-|t|)dt = \frac{1}{2} + \int_0^y (1-t)dt = \frac{1}{2} + y - \frac{1}{2}y^2$

For $y > 1$, $F_Y(y) = 1$



3.4.10 (1) $P(1/2 < Y \le 3/4) = F_Y(3/4) - F_Y(1/2) = (3/4)^2 - (1/2)^2 = 0.3125$

(2) $f_Y(y) = \frac{d}{dy}F_Y = \frac{d}{dy}y^2 = 2y$, $0 \le y < 1$

$P(1/2 < Y \le 3/4) = \int_{1/2}^{3/4} 2y\,dy = y^2\Big|_{1/2}^{3/4} = 0.3125$

3.4.11 **(a)** $P(Y < 2) = F_Y(2)$, since F_Y is continuous over [0, 2]. Then $F_Y(2) = \ln 2 = 0.693$

(b) $P(2 < Y \le 2.5) = F_Y(2.5) - F_Y(2) = \ln 2.5 - \ln 2 = 0.223$

(c) The probability is the same as (b) since F_Y is continuous over [0, e]

(d) $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\ln y = \frac{1}{y}$, $1 \le y \le e$

3.4.12 First note that $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(4y^3 - 3y^4) = 12y^2 - 12y^3$, $0 \le y \le 1$.

Then $P(1/4 < Y \le 3/4) = \int_{1/4}^{3/4}(12y^2 - 12y^3)dy = (4y^3 - 3y^4)\Big|_{1/4}^{3/4} = 0.6875$.

3.4.13 $f_Y(y) = \dfrac{d}{dy}\dfrac{1}{12}(y^2 + y^3) = \dfrac{1}{6}y + \dfrac{1}{4}y^2, 0 \le y \le 2$

3.4.14 Integrating by parts, we find that $F_Y(y) = \int_0^y te^{-t}dt = -te^{-t} - e^{-t}\Big|_0^y = 1-(1+y)e^{-y}$.

3.4.15 $F'(y) = -1(1+e^{-y})^{-2}(-e^{-y}) = \dfrac{e^{-y}}{(1+e^{-y})^2} > 0$, so $F(y)$ is increasing. The other two assertions follow from the facts that $\lim\limits_{y\to -\infty} e^{-y} = \infty$ and $\lim\limits_{y\to \infty} e^{-y} = 0$.

3.4.16 $F_W(w) = P(W \le w) = P(2Y \le w) = P(Y \le w/2) = F_Y(w/2)$

$f_W(w) = \dfrac{d}{dw}F(w) = \dfrac{d}{dw}F_Y(w/2) = f_W(w/2)\cdot(w/2)' = \dfrac{1}{2}4(w/2)^3 = \dfrac{1}{4}w^3$

where $2(0) \le w \le 2(1)$ or $0 \le w \le 2$

3.4.17 $P(-a < Y < a) = P(-a < Y \le 0) + P(0 < Y < a)$

$= \int_{-a}^0 f_Y(y)dy + \int_0^a f_Y(y)dy = -\int_a^0 f_Y(-y)dy + \int_0^a f_Y(y)dy$

$= \int_0^a f_Y(y)dy + \int_0^a f_Y(y)dy = 2[F_Y(a) - F_Y(0)]$

But by the symmetry of f_Y, $F_Y(0) = 1/2$. Thus, $2[F_Y(a) - F_Y(0)] = 2[F_Y(a) - 1/2] = 2F_Y(a) - 1$

3.4.18 $F_Y(y) = \int_0^y (1/\lambda)e^{-t/\lambda}dt = 1 - e^{-t/\lambda}$, so

$h(y) = \dfrac{(1/\lambda)e^{-y/\lambda}}{1-(1-e^{-y/\lambda})} = 1/\lambda$

Since the hazard rate is constant, the item does not age. Its reliability does not decrease over time.

Section 3.5: Expected Values

3.5.1 $E(X) = -1(0.935) + 2(0.0514) + 18(0.0115) + 180(0.0016) + 1{,}300(1.35 \times 10^{-4})$
$+ 2{,}600(6.12 \times 10^{-6}) + 10{,}000(1.12 \times 10^{-7}) = -0.144668$

3.5.2 Let X be the winnings of betting on red in Monte Carlo. Then $E(X) = \dfrac{18}{37} - \dfrac{19}{37} = \dfrac{-1}{37}$.

Let X* be the winnings of betting on red in Las Vegas. Then $E(X) = \dfrac{18}{38} - \dfrac{20}{38} = \dfrac{-2}{38}$.

The amount bet, M, is the solution to the equation $M\left(\dfrac{-1}{37} - \dfrac{-2}{38}\right) = \$3{,}000$ or M is approximately equal to \$117, 167.

3.5.3 $E(X)$ = \$30,000(0.857375) + \$18,000(0.135375) + \$6,000(0.007125) + (−\$6,000)(0.000125) = \$28,200.00

3.5.4 Rule A: Expected value = $-5 + 0 \cdot \dfrac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} + 2 \cdot \dfrac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} + 10 \cdot \dfrac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = -49/15$

Rule B: Expected value = $-5 + 0 \cdot \dfrac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} + 1 \cdot \dfrac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} + 20 \cdot \dfrac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = -47/15$

Neither game is fair to the player, but Rule B has the better payoff.

3.5.5 P is the solution to the equation $\sum_{k=1}^{5}[kP(1-p_k) - 50{,}000p_k] = P\sum_{k=1}^{5}k(1-p_k) - 50{,}000\sum_{k=1}^{5}p_k$ $= 1000$, where p_k is the probability of death in year k, k = 1, 2, 3, 4, 5.
Since $\sum_{k=1}^{5}p_k = 0.00272$ and $\sum_{k=1}^{5}k(1-p_k) = 4.99164$, the equation becomes
$4.99164P - 50{,}000(0.00272) = 1000$, or $P = \$227.58$.

3.5.6 The random variable X is hypergeometric, where $r = 4$, $w = 96$, $n = 20$. Then $E(X) = \dfrac{4(20)}{4+96} = \dfrac{4}{5}$.

3.5.7 This is a hypergeometric problem where r = number of students needing vaccinations = 125 and w = number of students already vaccinated = 642 − 125 = 517. An absenteeism rate of 12% corresponds to a sample $n = (0.12)(642) \doteq 77$ missing students. The expected number of unvaccinated students who are absent when the physician visits is $\dfrac{125(77)}{125+517} \doteq 15$.

3.5.8 **(a)** $E(Y) = \int_0^1 y \cdot 3(1-y)^2\,dy = \int_0^1 3(y - 2y^2 + y^3)\,dy$

$= 3\left[\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right]_0^1 = \frac{1}{4}$

(b) $E(Y) = \int_0^\infty y \cdot 4ye^{-2y}\,dy = 4\left[-\frac{1}{2}y^2e^{-2y} - \frac{1}{2}ye^{-2y} - \frac{1}{4}e^{-2y}\right]_0^\infty = 1$

(c) $E(Y) = \int_0^1 y \cdot \left(\frac{3}{4}\right)dy + \int_2^3 y \cdot \left(\frac{1}{4}\right)dy = \left.\frac{3y^2}{8}\right|_0^1 + \left.\frac{y^2}{8}\right|_2^3 = 1$

(d) $E(Y) = \int_0^{\pi/2} y \cdot \sin y\,dy = (-y\cos y + \sin y)\Big|_0^{\pi/2} = 1$

3.5.9 $E(Y) = \int_0^3 y\left(\frac{1}{9}y^2\right)dy = \frac{1}{9}\int_0^3 y^3\,dy = \left.\frac{y^4}{36}\right|_0^3 = \frac{9}{4}$ years

3.5.10 $E(Y) = \int_a^b y \frac{1}{b-a} dy = \left.\frac{y^2}{2(b-a)}\right|_a^b = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b+a}{2}$. This simply says that a uniform bar will balance at its middle.

3.5.11 $E(Y) = \int_0^\infty y \cdot \lambda e^{-\lambda y} dy = \left.\left(-ye^{-\lambda y} - \frac{1}{\lambda}e^{-\lambda y}\right)\right|_0^\infty = \frac{1}{\lambda}$

3.5.12 Since $\frac{1}{y^2} \geq 0$, this function will be a pdf if its integral is 1, and $\int_1^\infty \frac{1}{y^2} dy = \left.-\frac{1}{y}\right|_1^\infty = 1$. However, what would be its expected value is $\int_1^\infty y \frac{1}{y^2} dy = \int_1^\infty \frac{1}{y} dy = \ln y\Big|_1^\infty$, but this last quantity is infinite.

3.5.13 Let X be the number of cars passing the emissions test. Then X is binomial with $n = 200$ and $p = 0.80$. Two formulas for $E(X)$ are:

(1) $E(X) = \sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{200} k\binom{200}{k}(0.80)^k (0.20)^{200-k}$

(2) $E(X) = np = 200(0.80) = 160$

3.5.14 The probability that an observation of Y lies in the interval (1/2, 1) is $\int_{1/2}^1 3y^2 dy = y^3\Big|_{1/2}^1 = \frac{7}{8}$

Then X is binomial with $n = 15$ and $p = 7/8$. $E(X) = 15(7/8) = 105/8$.

3.5.15 If birthdays are randomly distributed throughout the year, the city should expect revenue of (\$50)(74,806)(30/365) or \$307,421.92.

3.5.16 If we assume that the probability of bankruptcy due to fraud is 23/68, then we can expect 9(23/68) = 3.04, or roughly 3 of the 9 additional bankruptcies will be due to fraud.

3.5.17 For the experiment described, construct the table:

Sample	Larger of the two, k
1, 2	2
1, 3	3
1, 4	4
2, 3	3
2, 4	4
3, 4	4

Each of the six samples is equally likely to be drawn, so $p_X(2) = 1/6$, $p_X(3) = 2/6$, and $p_X(4) = 3/6$. Then $E(X) = 2(1/6) + 3(2/6) + 4(3/6) = 20/6 = 10/3$.

3.5.18

Outcome	X
HHH	6
HHT	2
HTH	4
HTT	1
THH	2
THT	0
TTH	1
TTT	0

From the table, we can calculate $p_X(0) = 1/4$, $p_X(1) = 1/4$, $p_X(2) = 1/4$, $p_X(4) = 1/8$, $p_X(6) = 1/8$. Then $E(X) = 0 \cdot (1/4) + 1 \cdot (1/4) + 2 \cdot (1/4) + 4 \cdot (1/8) + 6 \cdot (1/8) = 2$.

3.5.19 The "fair" ante is the expected value of X, which is

$$\sum_{k=1}^{9} 2^k\left(\frac{1}{2^k}\right) + \sum_{k=10}^{\infty} 1000\left(\frac{1}{2^k}\right) = 9 + \frac{1000}{2^{10}}\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right) = 9 + \frac{1000}{2^{10}}\frac{1}{1-\frac{1}{2}} = 9 + \frac{1000}{512} = \frac{5608}{512} = \$10.95$$

3.5.20 **(a)** $E(X) = \displaystyle\sum_{k=1}^{\infty} c^k\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty}\left(\frac{c}{2}\right)^k = \frac{c}{2}\sum_{k=0}^{\infty}\left(\frac{c}{2}\right)^k = \frac{c}{2-c}$

(b) $\displaystyle\sum_{k=1}^{\infty} \log 2^k\left(\frac{1}{2}\right)^k = \log 2\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$

To evaluate the sum requires a special technique: For a parameter t, $0 < t < 1$, note that

$\displaystyle\sum_{k=1}^{\infty} t^k = \frac{t}{1-t}$. Differentiate both sides of the equation with respect to t to obtain

$\displaystyle\sum_{k=1}^{\infty} kt^{k-1} = \frac{1}{(1-t)^2}$. Multiplying both sides by t gives the desired equation:

$\displaystyle\sum_{k=1}^{\infty} kt^k = \frac{t}{(1-t)^2}$. In the case of interest, $t = 1/2$, so $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = 2$, and $E(X) = 2 \cdot \log 2$.

3.5.21 $p_X(1) = \dfrac{6}{216} = \dfrac{1}{36}$

$p_X(2) = \dfrac{3(6)(5)}{216} = \dfrac{15}{36}$

$p_X(3) = \dfrac{6(5)(4)}{216} = \dfrac{20}{36}$

$E(X) = 1 \cdot \dfrac{1}{36} + 2 \cdot \dfrac{15}{36} + 3 \cdot \dfrac{20}{36} = \dfrac{91}{36}$

3.5.22 For the experiment described, construct the table

Sample	Absolute value of difference
1, 2	1
1, 3	2
1, 4	3
1, 5	4
2, 3	1
2, 4	2
2, 5	3
3, 4	1
3, 5	2
4, 5	1

If X denotes the absolute value of the difference, then from the table:
$p_X(1) = 4/10$, $p_X(2) = 3/10$, $p_X(3) = 2/10$, $p_X(4) = 1/10$
$E(X) = 1(4/10) + 2(3/10) + 3(2/10) + 4(1/10) = 2$

3.5.23 Let X be the length of the series. Then $p_X(k) = 2\binom{k-1}{3}\left(\frac{1}{2}\right)^k$, $k = 4, 5, 6, 7$.

$$E(X) = \sum_{k=4}^{7}(k)(2)\binom{k-1}{3}\left(\frac{1}{2}\right)^k = 4\left(\frac{2}{16}\right)+5\left(\frac{4}{16}\right)+6\left(\frac{5}{16}\right)+7\left(\frac{5}{16}\right)=\frac{93}{16} = 5.8125$$

3.5.24 Let X = number of drawings to obtain a white chip. Then

$p_X(k) = \frac{1}{k}\cdot\frac{1}{k+1}$, $k = 1, 2, \ldots$

$$E(X) = \sum_{k=1}^{\infty}k\left(\frac{1}{k(k+1)}\right)=\sum_{k=1}^{\infty}\frac{1}{k+1}.$$

For each n, let $T_n = \sum_{i=2^n}^{i=2^{n+1}}\frac{1}{i}$. Then $T_n \geq \frac{2^n}{2^{n+1}}=\frac{1}{2}$.

$\sum_{k=1}^{\infty}\frac{1}{k+1} \geq \sum_{n=1}^{\infty}T_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ This last sum is infinite, so $E(X)$ does not exist.

3.5.25 $$E(X) = \sum_{k=1}^{r}k\frac{\binom{r}{k}\binom{w}{n-k}}{\binom{r+w}{n}} = \sum_{k=1}^{r}k\frac{\frac{r!}{k!(r-k)!}\binom{w}{n-k}}{\frac{(r+w)!}{n!(r+w-n)!}}$$

Factor out the presumed value of $E(X) = rn/(r + w)$:

$$E(X) = \frac{rn}{r+w}\sum_{k=1}^{r}\frac{\frac{(r-1)!}{(k-1)!(r-k)!}\binom{w}{n-k}}{\frac{(r-1+w)!}{(n-1)!(r+w-n)!}} = \frac{rn}{r+w}\sum_{k=1}^{r}\frac{\binom{r-1}{k-1}\binom{w}{n-k}}{\binom{r-1+w}{n-1}}$$

Change the index of summation to begin at 0, which gives

$E(X) = \dfrac{rn}{r+w}\sum_{k=0}^{r-1}\dfrac{\binom{r-1}{k}\binom{w}{n-1-k}}{\binom{r-1+w}{n-1}}$. The terms of the summation are urn probabilities where there are $r-1$ red balls, w white balls, and a sample size of $n-1$ is drawn. Since these are the probabilities of a hypergeometric pdf, the sum is one. This leaves us with the desired equality $E(X) = \dfrac{rn}{r+w}$.

3.5.26 $E(X) = \sum_{j=1}^{\infty} jp_X(j) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} p_X(j) = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} p_X(j) = \sum_{k=1}^{\infty} P(X \geq k)$

3.5.27 **(a)** $0.5 = \int_0^m (\theta+1)y^\theta\, dy = y^{\theta+1}\Big|_0^m = m^{\theta+1}$, so $m = (0.5)^{\frac{1}{\theta+1}}$

(b) $0.5 = \int_0^m \left(y+\dfrac{1}{2}\right)dy = \left(\dfrac{y^2}{2}+\dfrac{y}{2}\right)\Big|_0^m = \dfrac{m^2}{2}+\dfrac{m}{2}$. Solving the quadratic equation $\dfrac{1}{2}(m^2+m-1) = 0$ gives $m = \dfrac{-1+\sqrt{5}}{2}$.

3.5.28 $E(3X-4) = 3E(X) - 4 = 3(10)(2/5) - 4 = 8$

3.5.29 \$100(12)(0.11) = \$132

3.5.30 $E(W) = \int_0^1 w\left(\dfrac{1}{\sqrt{w}}-1\right)dw = \left(\dfrac{2}{3}w^{3/2} - \dfrac{w^2}{2}\right)\Big|_0^1 = 1/6$

Also, $E(W) = E(Y^2) = \int_0^1 y^2[2(1-y)]\,dy = \left(\dfrac{2}{3}y^3 - \dfrac{2}{4}y^4\right)\Big|_0^1 = 1/6$.

3.5.31 $E(Q) = \int_0^\infty 2(1-e^{-2y})6e^{-6y}dy = 12\int_0^\infty (e^{-6y}-e^{-8y})dy$

$= 12\left[-\dfrac{1}{6}e^{-6y} + \dfrac{1}{8}e^{-8y}\right]_0^\infty = \dfrac{1}{2}$, or \$50,000

3.5.32 $E(\text{Volume}) = \int_0^1 5y^2 6y(1-y)dy = 30\int_0^1 (y^3-y^4)dy = 30\left[\dfrac{1}{4}y^4 - \dfrac{1}{5}y^5\right]_0^1 = 1.5 \text{ in}^3$

3.5.33 Class average = $E(g(Y)) = \int_0^{100} 10y^{1/2}\dfrac{1}{5000}(100-y)dy$

$= \dfrac{1}{500}\int_0^1 (100y^{1/2} - y^{3/2})dy = \dfrac{1}{500}\left[\dfrac{2}{3}100y^{3/2} - \dfrac{2}{5}y^{5/2}\right]_0^{100}$

= 53.3, so the professor's "curve" did not work.

3.5.34 $E(W) = \int_0^1 \left(y - \frac{2}{3}\right)^2 (2y)\, dy = 2\left(\frac{1}{4}y^4 - \frac{4}{9}y^3 + \frac{2}{9}y^2\right)\Bigg|_0^1 = 1/36$

3.5.35 The area of the triangle = $\frac{1}{4}y^2$, so E(Area) = $\int_6^{10} \frac{1}{4}y^2 \frac{1}{10-6}dy = \frac{1}{16}\frac{y^3}{3}\Bigg|_6^{10}$ = 16.33.

3.5.36 $1 = \sum_{i=1}^{n} ki = k\frac{n(n+1)}{2}$ implies $k = \frac{2}{n(n+1)}$

$E\left(\frac{1}{X}\right) = \sum_{i=1}^{n} \frac{1}{i}\frac{2}{n(n+1)}i$ = 2/(n + 1)

Section 3.6: The Variance

3.6.1 If sampling is done with replacement, X is binomial with n = 2 and p = 2/5. By Theorem 3.5.1, μ = 2(2/5) = 4/5.
$E(X^2)$ = 0 · (9/25) + 1· (12/25) + 4 · (4/25) = 28/25. Then Var(X) = 28/25 − $(4/5)^2$ = 12/25.

3.6.2 $\mu = \int_0^1 y\left(\frac{3}{4}\right)dy + \int_2^3 y\left(\frac{1}{4}\right)dy = 1$

$E(X^2) = \int_0^1 y^2\left(\frac{3}{4}\right)dy + \int_2^3 y^2\left(\frac{1}{4}\right)dy = \frac{11}{6}$

Var(X) = $\frac{11}{6} - 1 = \frac{5}{6}$

3.6.3 Since X is hypergeometric, $\mu = \frac{3(6)}{10} = \frac{9}{5}$

$E(X^2) = \sum_{k=0}^{3} k^2 \frac{\binom{6}{k}\binom{4}{3-k}}{\binom{10}{3}}$ = 0 · (4/120) + 1 · (36/120) + 4 · (60/12) + 9 · (20/120) =
456/120 = 38/10
Var(X) = 38/10 − $(9/5)^2$ = 28/50 = 0.56, and σ = 0.748

3.6.4 μ = 1/2. $E(Y^2) = \int_0^1 y^2(1)dy$ = 1/3. Var(Y) = 1/3 − $(1/2)^2$ = 1/12

3.6.5 $\mu = \int_0^1 y3(1-y)^2 dy = 3\int_0^1 (y - 2y^2 + y^3)dy$ = 1/4

$E(Y^2) = \int_0^1 y^2 3(1-y)^2 dy = 3\int_0^1 (y^2 - 2y^3 + y^4)dy$ = 1/10

Var(Y) = 1/10 − $(1/4)^2$ = 3/80

3.6.6 $\mu = \int_0^k y\frac{2y}{k^2}dy = \frac{2k}{3}$. $E(Y^2) = \int_0^k y^2\frac{2y}{k^2}dy = \frac{k^2}{2}$

Var(Y) = $\frac{k^2}{2} - \left(\frac{2k}{3}\right)^2 = \frac{k^2}{18}$. Var($Y$) = 2 implies $\frac{k^2}{18} = 2$ or $k = 6$.

3.6.7 $f_Y(y) = \begin{cases} 1-y, & 0 \le y \le 1 \\ 1/2, & 2 \le y \le 3 \\ 0, & \text{elsewhere} \end{cases}$

$\mu = \int_0^1 y(1-y)dy + \int_2^3 y\left(\frac{1}{2}\right)dy = 17/12$

$E(Y^2) = \int_0^1 y^2(1-y)dy + \int_2^3 y^2\left(\frac{1}{2}\right)dy = 13/4$

$\sigma = \sqrt{13/4 - (17/12)^2} = \sqrt{179}/12 = 1.115$

3.6.8 **(a)** $\int_1^\infty \frac{2}{y^3}dy = \left.\frac{-1}{y^2}\right|_1^\infty = 1$

(b) $E(Y) = \int_1^\infty y\frac{2}{y^3}dy = \left.\frac{-2}{y}\right|_1^\infty = 2$

(c) $E(Y^2) = \int_1^\infty y^2\frac{2}{y^3}dy = 2\ln y\,\Big|_1^\infty$, which is infinite.

3.6.9 Let Y = Frankie's selection. Johnny wants to choose k so that $E[(Y-k)^2]$ is minimized. The minimum occurs when $k = E(Y) = (a+b)/2$ (see Question 3.6.13).

3.6.10 $E(Y) = \int_0^1 y(5y^4)dy = 5\int_0^1 y^5 dy = \left.\frac{5}{6}y^6\right|_0^1 = \frac{5}{6}$

$E(Y^2) = \int_0^1 y^2(5y^4)dy = 5\int_0^1 y^6 dy = \left.\frac{5}{7}y^7\right|_0^1 = \frac{5}{7}$

Var(Y) = $E(Y^2) - E(Y)^2 = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36} = \frac{5}{252}$

3.6.11 Using integration by parts, we find that

$E(Y^2) = \int_0^\infty y^2\lambda e^{-\lambda y}dy = -y^2e^{-\lambda y}\Big|_0^\infty + \int_0^\infty 2ye^{-\lambda y}dy = 0 + \int_0^\infty 2ye^{-\lambda y}dy,$

The right hand term is $2\int_0^\infty ye^{-\lambda y}dy = \frac{2}{\lambda}\int_0^\infty y\lambda e^{-\lambda y}dy = \frac{2}{\lambda}E(Y) = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$.

Then Var(Y) = $E(Y^2) - E(Y)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

3.6.12 For the given Y, $E(Y) = 1/2$ and $\text{Var}(Y) = 1/4$. Then

$$P\left(Y > E(Y) + 2\sqrt{\text{Var}(Y)}\right) = P\left(Y > \frac{1}{2} + 2\sqrt{\frac{1}{4}}\right) = P\left(Y > \frac{3}{2}\right)$$

$$= \int_{3/2}^{\infty} 2e^{-2y}\,dy = 1 - \int_{\infty}^{3/2} 2e^{-2y}\,dy = 1 - (1 - e^{-2(3/2)}) = e^{-3}$$

$= 0.0498$

3.6.13 $E[(X - a)^2] = E[((X - \mu) + (\mu - a))^2]$
$= E[(X - \mu)^2] + E[(\mu - a)^2] + 2(\mu - a)E(X - \mu)$
$= \text{Var}(X) + (\mu - a)^2$, since $E(X - \mu) = 0$. This is minimized when $a = \mu$, so the minimum of $g(\mathbf{a})$ $= \text{Var}(X)$.

3.6.14 $\text{Var}(-5Y + 12) = (-5)^2\text{Var}(Y) = 25(3/80) = 15/16$

3.6.15 $\text{Var}\left(\frac{5}{9}(Y - 32)\right) = \left(\frac{5}{9}\right)^2 \text{Var}(Y)$, by Theorem 3.6.2. So $\sigma\left(\frac{5}{9}(Y - 32)\right) = \left(\frac{5}{9}\right)\sigma(Y) = \frac{5}{9}(15.7) =$ 8.7°C.

3.6.16 (1) $E[(W - \mu)/\sigma] = (1/\sigma)E[(W - \mu)] = 0$
(2) $\text{Var}[(W - \mu)/\sigma] = (1/\sigma^2)\text{Var}[(W - \mu)] = (1/\sigma^2)\sigma^2 = 1$

3.6.17 **(a)** $f_Y(y) = \dfrac{1}{b-a} f_U\left(\dfrac{y-a}{b-a}\right) = \dfrac{1}{b-a}$, $(b - a)(0) + a \le y \le (b - a)(1) + a$, or

$f_Y(y) = \dfrac{1}{b-a}$, $a \le y \le b$, which is the uniform pdf over $[a, b]$

(b) $\text{Var}(Y) = \text{Var}[(b - a)U + a] = (b - a)^2\,\text{Var}(U) = (b - a)^2/12$

3.6.18 $E(Y) = 5.5$ and $\text{Var}(Y) = 0.75$.

$$E(W_1) = 0.2281 + (0.9948)E(Y) + E(E_1) = 0.2281 + (0.9948)(5.5) + 0 = 5.6995$$

$$E(W_2) = -0.0748 + (1.0024)E(Y) + E(E_2) = -0.0748 + (1.0024)(5.5) + 0 = 5.4384$$

$$\text{Var}(W_1) = (0.9948)^2\,\text{Var}(Y) + \text{Var}(E_1) = (0.9948)^2(0.75) + 0.0427 = 0.7849$$

$$\text{Var}(W_2) = (1.0024)^2\,\text{Var}(Y) + \text{Var}(E_2) = (1.0024)^2(0.75) + 0.0159 = 0.7695$$

The above two equalities follow from the second corollary to Theorem 3.9.5. So the second procedure is better, since the mean of W_2 is closer to the true mean, and it has smaller variance.

3.6.19 $E(Y^r) = \displaystyle\int_0^2 y^r \frac{1}{2}\,dy = \frac{1}{2}\frac{y^{r+1}}{r+1}\bigg|_0^2 = \frac{2^r}{r+1}$

$E[(Y - 1)^6] = \displaystyle\sum_{j=0}^{6}\binom{6}{j}E(Y^j)(-1)^{6-j} = \sum_{j=0}^{6}\binom{6}{j}\frac{2^r}{r+1}(-1)^{6-j}$

$= (1)(1) + (-6)(1) + 15(4/3) + (-20)(2) + (15)(16/5) + (-6)(32/6) + (1)(64/7) = 1/7$

3.6.20 For the given f_Y, $\mu = 1$ and $\sigma = 1$.

$$\gamma_1 = \frac{E\left[(Y-1)^3\right]}{1} = \sum_{j=0}^{3}\binom{3}{j}E(Y^j)(-1)^{3-j} = \sum_{j=0}^{3}\binom{3}{j}(j!)(-1)^{3-j}$$

(1)(1)(−1) + (3)(1)(1) + (3)(2)(−1) + (1)(6)(1) = 2

3.6.21 For the uniform random variable U, $E(U) = 1/2$ and $\text{Var}(U) = 1/12$. Also the k-th moment of U is $E(U^k) = \int_0^1 u^k\,du = 1/(k+1)$. Then the coefficient of kurtosis is

$$\gamma_2 = \frac{E[(U-\mu)^4]}{\sigma^4} = \frac{E[(U-\mu)^4]}{\text{Var}^2(U)} = \frac{E[(U-1/2)^4]}{(1/12)^2} = (144)\sum_{k=0}^{4}\binom{4}{k}\frac{1}{k+1}\left(-\frac{1}{2}\right)^{4-k}$$

$= (144)(1/80) = 9/5$

3.6.22 $10 = E\left[(W-2)^3\right] = \sum_{j=0}^{3}\binom{3}{j}E(W^j)(-2)^{3-j} = (1)(1)(-8) + (3)(2)(4) + (3)E(W^2)(-2) + (1)(4)(1)$

This would imply that $E(W^2) = 5/3$. In that case, $\text{Var}(W) = 5/3 - (2)^2 < 0$, which is not possible.

3.6.23 Let $E(X) = \mu$; let σ be the standard deviation of X. Then $E(aX + b) = a\mu + b$. Also, $\text{Var}(aX + b) = a^2\sigma^2$, so the standard deviation of $aX + b = a\sigma$. Then

$$\gamma_1 = \frac{E\left[\left((aX+b)-(a\mu+b)\right)^3\right]}{(a\sigma)^3}$$

$$= \frac{a^3E\left[(X-\mu)^3\right]}{a^3\sigma^3} = \frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \gamma_1(X)$$

The demonstration for γ_2 is similar.

3.6.24 **(a)** Question 3.4.6 established that Y is a pdf for any positive integer n. As a corollary, we know that $1 = \int_0^1 (n+2)(n+1y^n(1-y)dy$ or equivalently, for any positive integer n,

$$\int_0^1 y^n(1-y)dy = \frac{1}{(n+2)(n+1)}$$

Then $E(Y^2) = \int_0^1 y^n(n+2)(n+1)y^n(1-y)dy = (n+2)(n+1)\int_0^1 y^{n+2}(1-y)dy$

$= \dfrac{(n+2)(n+1)}{(n+4)(n+3)}$. By a similar argument, $E(Y) = \dfrac{(n+2)(n+1)}{(n+4)(n+3)} = \dfrac{(n+1)}{(n+3)}$.

Thus, $\text{Var}(Y) = E(Y^2) - E(Y)^2 = \dfrac{(n+2)(n+1)}{(n+4)(n+3)} - \dfrac{(n+1)^2}{(n+3)^2} = \dfrac{2(n+1)}{(n+4)(n+3)^2}$

(b) $E(Y^k) = \int_0^1 y^k(n+2)(n+1)y^n(1-y)dy = (n+2)(n+1)\int_0^1 y^{n+k}(1-y)dy$

$$= \frac{(n+2)(n+1)}{(n+k+2)(n+k+1)}$$

3.6.25 **(a)** $1 = \int_1^{\infty} cy^{-6}dy = c\left[\frac{y^{-5}}{-5}\right]_1^{\infty} = c\frac{1}{5}$, so $c = 5$.

(b) $E(Y^r) = 5\int_1^{\infty} y^r y^{-6}dy = 5\left[\frac{y^{r-5}}{r-5}\right]_1^{\infty}$. For this last expression to be finite, r must be < 5.

The highest integral moment is $r = 4$.

Section 3.7: Joint Densities

3.7.1 $1 = \sum_{x,y} p(x, y) = c\sum_{x,y} xy = c[(1)(1) + (2)(1) + (2)(2) + (3)(1)] = 10c$, so $c = 1/10$

3.7.2 $1 = \int_0^1\int_0^1 c(x^2 + y^2)dxdy = c\int_0^1\int_0^1 x^2dxdy + c\int_0^1\int_0^1 y^2dydx$

$= c\int_0^1 \left[\frac{x^3}{3}\right]_0^1 dy + c\int_0^1 \left[\frac{y^3}{3}\right]_0^1 dx = c\int_0^1 \frac{1}{3}dy + c\int_0^1 \frac{1}{3}dx = \frac{2}{3}c$, $c = 3/2$.

3.7.3 $1 = \int_0^1\int_0^y c(x+y)dxdy = c\int_0^1\left[\frac{x^2}{2} + xy\right]_0^y dy = c\int_0^1 \frac{3y^2}{2}dy = c\left[\frac{y^3}{2}\right]_0^1 = \frac{c}{2}$, so $c = 2$.

3.7.4 $1 = c\int_0^1\left(\int_0^y xydx\right)dy = c\int_0^1\left(\left[\frac{x^2y}{2}\right]_0^y\right)dy = c\int_0^1 \frac{y^3}{2}dy$

$c\left[\frac{y^4}{8}\right]_0^1 = c\left(\frac{1}{8}\right)$, so $c = 8$

3.7.5 $P(X = x, Y = y) = \dfrac{\binom{3}{x}\binom{2}{y}\binom{4}{3-x-y}}{\binom{9}{3}}$, $0 \le x \le 3$, $0 \le y \le 2$, $x + y \le 3$

3.7.6 $P(X = x, Y = y) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{44}{4-x-y}}{\binom{52}{4}}$, $0 \le x \le 4$, $0 \le y \le 4$, $x + y \le 4$

3.7.7 $P(X > Y) = p_{X,Y}(1, 0) + p_{X,Y}(2, 0) + p_{X,Y}(2, 1)$
$= 6/50 + 4/50 + 3/50 = 13/50$

3.7.8

Outcome	X	Y
H H H	1	3
H H T	0	2
H T H	1	2
H T T	0	1
T H H	1	2
T H T	0	1
T T H	1	1
T T T	0	0

(x, y)	$p_{X,Y}(x, y)$
(0, 0)	1/8
(0, 1)	2/8
(0, 2)	1/8
(0, 3)	0
(1, 0)	0
(1, 1)	1/8
(1, 2)	2/8
(1, 3)	1/8

3.7.9

		Number of 2's, X		
		0	1	2
Number of 3's, Y	0	16/36	8/36	1/36
	1	8/36	2/36	0
	2	1/36	0	0

From the matrix above, we calculate

$p_Z(0) = p_{X,Y}(0, 0) = 16/36$
$p_Z(1) = p_{X,Y}(0, 1) + p_{X,Y}(1, 0) = 2(8/36) = 16/36$
$p_Z(2) = p_{X,Y}(0, 2) + p_{X,Y}(2, 0) + p_{X,Y}(1, 1) = 4/36$

3.7.10 **(a)** $1 = \int_0^1 \int_0^1 c\, dxdy = \text{c, so } c = 1$ **(b)** $P(0 < X < 1/2, 0 < Y < 1/4) = \int_0^{1/4} \int_0^{1/2} 1\, dxdy = 1/8$

3.7.11 $P(Y < 3X) = \int_0^\infty \int_x^{3x} 2e^{-(x+y)} dydx = \int_0^\infty e^{-x}\left(\int_x^{3x} 2e^{-y} dy\right) dx$

$= 2\int_0^\infty e^{-x}\left(\left[-e^{-y}\right]_x^{3x}\right) dx = 2\int_0^\infty e^{-x}\left[e^{-x} - e^{-3x}\right] dx$

$= 2\int_0^\infty \left[e^{-2x} - e^{-4x}\right] dx = 2\left[-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x}\right]_0^\infty = \frac{1}{2}$

3.7.12 The density is the bivariate uniform over a circle of radius 2. The area of the circle is $\pi(2)^2 = 4\pi$. Thus, $f_{X,Y}(x, y) = 1/4\pi$ for $x^2 + y^2 \le 4$.

3.7.13 $P(X < 2Y) = \int_0^1 \int_{x/2}^1 (x+y)dydx$

$= \int_0^1 \int_{x/2}^1 x\, dydx + \int_0^1 \int_{x/2}^1 y\, dydx$

$= \int_0^1 \left[x - \frac{x^2}{2}\right]dx + \int_0^1 \left[\frac{1}{2} - \frac{x^2}{8}\right]dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^1 + \left[\frac{x}{2} - \frac{x^3}{24}\right]_0^1 = \frac{19}{24}$

3.7.14 The probability of an observation falling into the interval (0, 1/3) is $\int_0^{1/3} 2t\, dt = 1/9$. The probability of an observation falling into the interval (1/3, 2/3) is $\int_{1/3}^{2/3} 2t\, dt = 1/3$. Assume without any loss of generality that the five observations are done in order. To calculate $p_{X,Y}(1, 2)$, note that there are $\binom{5}{1}$ places where the observation in (0, 1/3) could occur, and $\binom{4}{2}$ choices for the location of the observations in (1/3, 2/3). Then $p_{X,Y}(1, 2) = \binom{5}{1}\binom{4}{2}(1/9)^1(1/3)^2(5/9)^2$ = 750/6561.

3.7.15 The set where $y > h/2$ is a triangle with height $h/2$ and base $b/2$. Its area is $bh/8$. Thus the area of the set where $y < h/2$ is $bh/2 - bh/8 = 3bh/8$. The probability that a randomly chosen point will fall in the lower half of the triangle is $(3bh/8)/(bh/2) = 3/4$.

3.7.16 $p_X(x) = \frac{\binom{3}{x}}{\binom{9}{3}} \sum_{y=0}^{\min(2,3-x)} \binom{2}{y}\binom{4}{3-x-y} = \frac{\binom{3}{x}\binom{6}{3-x}}{\binom{9}{3}}, x = 0,1,2,3$

3.7.17 From the solution to Question 3.7.8, $p_X(x) = 1/8 + 2/8 + 1/8 = 1/2$, $x = 0, 1$,
$p_Y(0) = 1/8$, $p_Y(1) = 3/8$, $p_Y(2) = 3/8$, $p_Y(3) = 1/8$.

3.7.18 Let X_1 be the number in the upper quarter; X_2, the number in the middle half. From Question 3.2.18, we know that $P(X_1 = 2, X_2 = 2) = \frac{6!}{2!2!2!}(0.25)^2(0.50)^2(0.25)^2 = 0.088$. The simplest way to deal with the marginal probability is to recognize that the probability of belonging to the middle half is binomial with $n = 6$ and $p = .05$. This probability is $\binom{6}{2}(0.5)^2(0.5)^4 = 0.234$.

3.7.19 **(a)** $f_X(x) = \int_0^1 \frac{1}{2}dy = \frac{y}{2}\Big|_0^1 = \frac{1}{2}$, $0 \le x \le 2$

$f_Y(y) = \int_0^2 \frac{1}{2}dx = \frac{x}{2}\Big|_0^2 = 1$, $0 \le y \le 1$

(b) $f_X(x) = \int_0^1 \frac{3}{2}y^2\,dy = \frac{1}{2}y^3\Big|_0^1 = 1/2$, $0 \le x \le 2$

$f_Y(y) = \int_0^2 \frac{3}{2}y^2\,dx = \frac{3}{2}y^2x\Big|_0^2 = 3y^2$, $0 \le y \le 1$

(c) $f_X(x) = \int_0^1 \frac{2}{3}(x+2y)dy = \frac{2}{3}\left(xy + y^2\right)\Big|_0^1 = \frac{2}{3}(x+1)$, $0 \le x \le 1$

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)dx = \frac{2}{3}\left(\frac{x^2}{2} + 2xy\right)\Big|_0^1 = \frac{4}{3}y + \frac{1}{3}$, $0 \le y \le 1$

(d) $f_X(x) = c\int_0^1 (x+y)dy = c\left(xy + \frac{y^2}{2}\right)\Big|_0^1 = c\left(x + \frac{1}{2}\right)$, $0 \le x \le 1$

In order for the above to be a density, $1 = \int_0^1 c\left(x + \frac{1}{2}\right)dx = c\left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^1 = c$, so

$f_X(x) = x + \frac{1}{2}$, $0 \le x \le 1$

$f_Y(y) = y + \frac{1}{2}$, $0 \le y \le 1$, by symmetry of the joint pdf

(e) $f_X(x) = \int_0^1 4xy\,dy = 2xy^2\Big|_0^1 = 2x$, $0 \le x \le 1$

$f_Y(y) = 2y$, $0 \le y \le 1$, by the symmetry of the joint pdf

(f) $f_X(x) = \int_0^\infty xye^{-(x+y)}dy = xe^{-x}\int_0^\infty ye^{-y}dy$

$= xe^{-x}(-ye^{-y} - e^{-y})\Big|_0^\infty = xe^{-x}$, $0 \le x$

$f_Y(y) = ye^{-y}$, $0 \le y$, by symmetry of the joint pdf

(g) $f_X(x) = \int_0^\infty ye^{-xy-y}dy = \int_0^\infty ye^{-(x+1)y}dy$

Integrating by parts gives

$\left(-\frac{y}{x+1}e^{-(x+1)y} - \left(\frac{1}{x+1}\right)^2 e^{-(x+1)y}\right)\Big|_0^\infty = \left(\frac{1}{x+1}\right)^2$, $0 < x$

$f_Y(y) = \int_0^\infty ye^{-xy-y}dx = \int_0^\infty ye^{-y}e^{-xy}dx = ye^{-y}\left(-\frac{1}{y}\right)e^{-xy}\Big|_0^\infty = e^{-y}$, where $0 \le y$.

3.7.20 **(a)** $f_X(x) = \int_x^2 \frac{1}{2}dy = \left.\frac{y}{2}\right|_x^2 = 1 - \frac{x}{2}$, $0 \le x \le 2$

$f_Y(y) = \int_0^y \frac{1}{2}dx = \frac{1}{2}y$, $0 \le y \le 2$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x \frac{1}{x}dy = \frac{1}{x}y \Big|_0^x = 1$, $0 \le x \le 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 \frac{1}{x}dx = \ln x \Big|_y^1 = -\ln y$, $0 \le y \le 1$

(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^{1-x} 6x\,dy = 6xy \Big|_0^{1-x} = 6x(1-x)$, $0 \le x \le 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{1-y} 6x\,dx = 3x^2\Big|_0^{1-y} = 3(1-y)^2$, $0 \le y \le 1$

3.7.21 $f_X(x) = \int_0^{1-x} 6(1-x-y)dy = 6\left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} = 6\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]$

$= 3 - 6x + 3x^2$, $0 \le x \le 1$

3.7.22 $f_Y(y) = \int_0^y 2e^{-x}e^{-y}dx = -2e^{-x}e^{-y}\Big|_0^y = 2e^{-y} - 2e^{-2y}$, $0 \le y$

3.7.23 $p_X(x) = \sum_{y=0}^{4-x} \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{3}\right)^y\left(\frac{1}{6}\right)^{4-x-y}$

$= \frac{4!}{x!(4-x)!}\left(\frac{1}{2}\right)^x \sum_{y=0}^{4-x} \frac{(4-x)!}{y![(4-x)-y]!}\left(\frac{1}{3}\right)^y\left(\frac{1}{6}\right)^{4-x-y} = \frac{4!}{x!(4-x)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{3}+\frac{1}{6}\right)^{4-x}$

$= \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$

Thus, X is binomial with $n = 4$ and $p = 1/2$. Similarly, Y is binomial with $n = 4$ and $p = 1/3$.

3.7.24 **(a)** Consider any outcome with x of the first kind, y of the second, and (necessarily) $n - x - y$ of the third kind. The probability of such an outcome is $p_1^x p_2^y\left(1 - p_1 - p_2\right)^{n-x-y}$. All we need to know now is how many outcomes there are with x of the first kind, y of the second, and $n - x - y$ of the third kind. This question is resolved by the number of ways to choose the places for these three kinds of outcomes, which by Theorem 2.6.2 is $\frac{n!}{x!y!(n-x-y)!}$.

(b) The proof is an adaptation of the solution to Question 3.7.23.

3.7.25 **(a)** $S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

(b) $F_{X,Y}(1, 2) = P(X \le 1, Y \le 2) = P(\{(H, 1), (H, 2), (T, 1), (T, 2)\}) = 4/12 = 1/3$

3.7.26 $F_{X,Y}(1, 2) = \sum_{i=0}^{1}\sum_{j=0}^{2} p_{X,Y}(i, j)$

$$= \frac{\binom{5}{0}\binom{4}{1}\binom{3}{3}}{\binom{12}{4}} + \frac{\binom{5}{0}\binom{4}{2}\binom{3}{2}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{0}\binom{3}{3}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{1}\binom{3}{2}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{2}\binom{3}{1}}{\binom{12}{4}}$$

$$= \frac{4+18+5+60+90}{495} = \frac{177}{495} = 0.358$$

3.7.27 **(a)** $F_{X,Y}(u, v) = \int_0^u\int_0^v \frac{3}{2}y^2 dy\, dx = \int_0^u\left[\frac{1}{2}y^3\Big|_0^v\right]dx = \int_0^u \frac{1}{2}v^3 dx = \frac{1}{2}uv^3$

(b) $F_{X,Y}(u, v) = \int_0^u\int_0^v \frac{2}{3}(x+2y)dy\, dx = \int_0^u\left[\frac{2}{3}(xy+y^2)\Big|_0^v\right]dx = \int_0^u \frac{2}{3}(vx+v^2dx) = \frac{1}{3}u^2v + \frac{2}{3}uv^2$

(c) $F_{X,Y}(u, v) = \int_0^u\int_0^v 4xydy\, dx = \int_0^u x\left[2y^2\Big|_0^v\right]dx = 2v^2\int_0^u xdx = u^2v^2$

3.7.28 **(a)** For $0 \le u \le v \le 2$, $F_{X,Y}(u,v) = \int_0^u\int_x^v \frac{1}{2}dy\, dx = \int_0^u \frac{y}{2}\Big|_x^v dx = \frac{1}{2}\int_0^u (v-x)dx = \frac{1}{4}(2uv-u^2)$

(b) $F_{X,Y}(u, v) = \int_0^v\int_y^u \frac{1}{x}dy\, dx = \int_0^v\left[\ln x\Big|_y^u\right]dy = \int_0^v \ln u - \ln y dy = v\ln u - v\ln v + v$

(c) Case I: $v \le 1 - u$

$F_{X,Y}(u, v) = \int_0^v\int_0^u 6xdx\, dy = \int_0^v\left[3x^2\Big|_0^u\right]dy = \int_0^v 3u^2dy = 3u^2v$

Case II: $v > 1 - u$

$F_{X,Y}(u, v) = \int_0^u\int_0^v 6xdy\, dx = \int_{1-v}^u\int_{1-x}^v 6xdy\, dx$

$= 3u^2v - [3u^2v - 3u^2 + 2u^3 - 3(1-v)^2v + 3(1-v)^2 - 2(1-v)^3]$

$= 3u^2 - 2u^3 + 3(1-v)^2v - 3(1-v)^2 + 2(1-v)^3 = 3u^2 - 2u^3 - (1-v)^3$

3.7.29 By Theorem 3.7.3, $f_{X,Y} = \frac{\partial^2}{\partial x\partial y}F_{X,Y} = \frac{\partial^2}{\partial x\partial y}(xy) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}xy\right)$

$= \frac{\partial}{\partial x}(x) = 1$, $0 \le x \le 1$, $0 \le y \le 1$.

The graph of $f_{X,Y}$ is a square plate of height one over the unit square.

3.7.30 By Theorem, 3.7.3, $f_{X,Y} = \frac{\partial^2}{\partial x\partial y}F_{X,Y} = \frac{\partial^2}{\partial x\partial y}[(1-e^{-\lambda y})(1-e^{-\lambda x})]$

$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}[(1-e^{-\lambda y})(1-e^{-\lambda x})] = \frac{\partial}{\partial x}[(\lambda e^{-\lambda y})(1-e^{-\lambda x})]$

$= \lambda e^{-\lambda y}\lambda e^{-\lambda x}$, $x \ge 0$, $y \ge 0$

3.7.31 First note that $1 = F_{X,Y}(1, 1) = k[4(1^2)(1^2) + 5(1)(1^4)] = 9k$, so $k = 1/9$.

Then $f_{X,Y} = \dfrac{\partial^2}{\partial x \partial y} F_{X,Y} = \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{4}{9}x^2y^2 + \dfrac{5}{9}xy^4\right)$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{4}{9}x^2y^2 + \frac{5}{9}xy^4\right) = \frac{\partial}{\partial x}\left(\frac{8}{9}x^2y + \frac{20}{9}xy^3\right) = \frac{16}{9}xy + \frac{20}{9}y^3$$

$$P(0 < X < 1/2,\ 1/2 < Y < 1) = \int_0^{1/2}\int_{1/2}^1 \left(\frac{16}{9}xy + \frac{20}{9}y^3\right) dy dx$$

$$= \int_0^{1/2} \frac{8}{9}xy^2 + \frac{5}{9}y^4\Big|_{1/2}^1 dx = \int_0^{1/2}\left(\frac{2}{3}x + \frac{25}{48}\right)dx = \frac{1}{3}x^2 + \frac{25}{48}x\ \Big|_0^{1/2} = 11/32$$

3.7.32 $P(a < X \le b, Y \le d) = F_{X,Y}(b, d) - F_{X,Y}(a, d)$
$P(a < X \le b, Y \le c) = F_{X,Y}(b, c) - F_{X,Y}(a, c)$
$P(a < X \le b, c < Y \le d) = P(a < X \le b, Y \le d) - P(a < X \le b, Y \le c)$
$= (F_{X,Y}(b, d) - F_{X,Y}(a, d)) - (F_{X,Y}(b, c) - F_{X,Y}(a, c)) = F_{X,Y}(b, d) - F_{X,Y}(a, d) - F_{X,Y}(b, c) + F_{X,Y}(a, c)$

3.7.33 $P(X_1 \ge 1050, X_2 \ge 1050, X_3 \ge 1050, X_4 \ge 1050)$

$$= \int_{1050}^{\infty}\int_{1050}^{\infty}\int_{1050}^{\infty}\int_{1050}^{\infty}\prod_{i=1}^{4}\frac{1}{1000}e^{-x_i/1000}dx_1dx_2dx_3dx_4$$

$$= \left(\int_{1050}^{\infty}\frac{1}{1000}e^{-x/1000}dx\right)^4 = (e^{-1.05})^4 = 0.015$$

3.7.34 **(a)** $p_{X,Y,Z}(x, y, z) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{4}{z}\binom{40}{6-x-y-z}}{\binom{52}{6}}$ where $0 \le x, y, z \le 4$, $x + y + z \le 6$

(b) $p_{X,Y}(x, y) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{44}{6-x-y}}{\binom{52}{6}}$ where $0 \le x, y \le 4$, $x + y \le 6$

$p_{X,Z}(x, z) = \dfrac{\binom{4}{x}\binom{4}{z}\binom{44}{6-x-z}}{\binom{52}{6}}$ where $0 \le x, z \le 4$, $x + z \le 6$

3.7.35 $p_{X,Y}(0, 1) = \displaystyle\sum_{z=0}^{2} p_{X,Y,Z}(0,1,z) = \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\sum_{z=0}^{2}\frac{1}{z!(2-z)!}\left(\frac{1}{6}\right)^z\left(\frac{1}{4}\right)^{2-z}$

$$= \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\left(\frac{1}{2}\right)\sum_{z=0}^{2}\frac{2!}{z!(2-z)!}\left(\frac{1}{6}\right)^z\left(\frac{1}{4}\right)^{2-z} = \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\left(\frac{1}{2}\right)\left(\frac{1}{6}+\frac{1}{4}\right)^2 = \frac{25}{576}$$

3.7.36 **(a)** $f_{X,Y}(x, y) = \int_0^\infty f_{X,Y,Z}(x,y,z)dz = \int_0^\infty (x+y)e^{-z}dz$

$= (x+y)\left[-e^{-z}\right]_0^\infty = (x + y),\ 0 \le x, y \le 1$

(b) $f_{Y,Z}(y, z) = \int_0^1 f_{X,Y,Z}(x,y,z)dx = \int_0^1 (x+y)e^{-z}dx = e^{-z}\left[\frac{x^2}{2}+xy\right]_0^1 = \left(\frac{1}{2}+y\right)e^{-z}$,

$0 \le y \le 1,\ z \ge 0$

(c) $f_Z(z) = \int_0^1 f_{Y,Z}(y,z)dy = \int_0^1\left(\frac{1}{2}+y\right)e^{-z}dy = e^{-z}\left[\frac{y}{2}+\frac{y^2}{2}\right]_0^1 = e^{-z},\ z \ge 0$

3.7.37 $f_{W,X}(w, x) = \int_0^1\int_0^1 f_{W,X,Y,Z}(w,x,y,z)dydz = \int_0^1\int_0^1 16wxyz\ dydz = \int_0^1\left[8wxy^2z\right]_0^1 dz = \int_0^1 [8wxz]dz$

$= \left[4wxz^2\right]_0^1 = 4wx,\ 0 < w, x < 1$

$P(0 < W < 1/2,\ 1/2 < X < 1) = \int_0^{1/2}\int_{1/2}^1 4wx\ dxdw =$

$\int_0^{1/2} 2w\left[x^2\right]_{1/2}^1 dx = \int_0^{1/2}\frac{3}{2}w\ dw = \frac{3}{4}w^2\Big|_0^{1/2} = \frac{3}{16}$

3.7.38 We must show that $p_{X,Y}(j, k) = p_X(j)p_Y(k)$. But for any pair (j, k), $p_{X,Y}(j, k) = 1/36 = (1/6)(1/6) = p_X(j)p_Y(k)$.

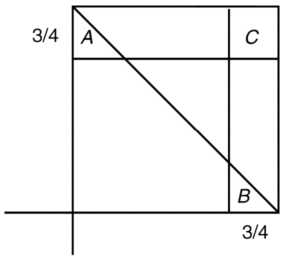
3.7.39 The marginal pdfs for $f_{X,Y}$ are $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$ (Hint: see the solution to 3.7.19**(f)**). Their product is $f_{X,Y}$, so X and Y are independent. The probability that one component fails to last 1000 hours is $1 - e^{-1000\lambda}$. Because of independence of the two components, the probability that two components both fail is the square of that, or $(1 - e^{-1000\lambda})^2$.

3.7.40 **(a)** $p_{X,Y}(x, y) = \begin{cases} \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10} & y = x \\ \frac{1}{4}\cdot\frac{1}{5} = \frac{1}{20} & y \ne x \end{cases}$

(b) $p_X(x) = 1/4$, since each ball in Urn I is equally likely to be drawn
$p_Y(y) = 1/10 + 3(1/20) = 1/4$

(c) $P(X = 1, Y = 1) = 1/10$, but $P(X = 1)P(Y = 1) = 1/16$

3.7.41 First, note $k = 2$. Then, 2 times area of $A = P(Y \ge 3/4)$. Also, 2 times area of $B = P(X \ge 3/4)$. The square C is the set $(X \ge 3/4) \cap (Y \ge 3/4)$. However, C is in the region where the density is 0. Thus, $P((X \ge 3/4) \cap (Y \ge 3/4))$ is zero, but the product $P(X \ge 3/4)P(Y \ge 3/4)$ is not zero.



3.7.42 $f_X(x) = \int_0^1 \frac{2}{3}(x+2y)dy = \frac{2}{3}(x+1)$

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)dx = \frac{2}{3}\left(2y+\frac{1}{2}\right)$

But $\frac{2}{3}(x+1)\frac{2}{3}\left(2y+\frac{1}{2}\right) \neq \frac{2}{3}(x+2y)$.

3.7.43 $P(Y < X) = \int_0^1\int_0^x f_{X,Y}(x, y)\, dydx = \int_0^1\int_0^x (2x)(3y^2)\, dydx = \int_0^1 2x^4 dx = \frac{2}{5}$

3.7.44 $F_X(x) = \int_0^x \frac{t}{2}dt = \frac{x^2}{4}$. $F_Y(y) = \int_0^y 2t\, dt = y^2$

$F_{X,Y}(x, y) = F_X(x)F_Y(y) = \frac{x^2y^2}{4}$, $0 \le x \le 2$, $0 \le y \le 1$

3.7.45 $P\left(\frac{Y}{X} > 2\right) = P(Y > 2X) = \int_0^1\int_0^{y/2} (2x)(1)dxdy = \int_0^1 \left[x^2\right]_0^{y/2} dy = \left.\frac{y^3}{12}\right|_0^1 = \frac{1}{12}$

3.7.46 $P(a < X < b, c < Y < d) = \int_a^b\int_c^d xye^{-(x+y)}dydx = \int_a^b xe^{-x}\left(\int_c^d ye^{-y}dy\right)dx$

$= \int_a^b xe^{-x}dx\int_c^d ye^{-y}dy = P(a < X < b)\, P(c < Y < d)$

3.7.47 Take $a = c = 0$, $b = d = 1/2$. Then $P(0 < X < 1/2, 0 < Y < 1/2) = \int_0^{1/2}\int_0^{1/2}(2x + y - 2xy)dydx$ $= 5/32$.

$f_X(x) = \int_0^1 (2x + y - 2xy)dy = x + 1/2$, so $P(0 < X < 1/2) = \int_0^{1/2}\left(x+\frac{1}{2}\right)dx = \frac{3}{8}$

$f_Y(y) = \int_0^1 (2x + y - 2xy)dx = 1$, so $P(0 < X < 1/2) = 1/2$. But, $5/32 \neq (3/8)(1/2)$

3.7.48 We proceed by showing that the events $g(X)\ \varepsilon\ A$ and $h(Y)\ \varepsilon B$ are independent, for sets of real numbers, A and B. Note that $P(g(X)\ \varepsilon A$ and $h(Y)\ \varepsilon B) = P(X\ \varepsilon\ g^{-1}(\mathrm{a})$ and $Y\ \varepsilon\ g^{-1}(\mathrm{b}))$.
Since X and Y are independent, $P(X\ \varepsilon\ g^{-1}(\mathrm{a})$ and $Y\ \varepsilon\ g^{-1}(\mathrm{b})) = P(X\ \varepsilon\ g^{-1}(\mathrm{a}))P(Y\ \varepsilon\ g^{-1}(\mathrm{b})) =$
$P(g(X)\ \varepsilon\ A)P(h(Y)\ \varepsilon\ B)$

3.7.49 Let K be the region of the plane where $f_{X,Y} \neq 0$. If K is not a rectangle with sides parallel to the coordinate axes, there exists a rectangle $A = \{(x, y) | \, a \le x \le b, c \le y \le d\}$ with $A \cap K = \varnothing$, but for $A_1 = \{(x, y) | \, a \le x \le b, \text{all } y\}$ and $A_2 = \{(x, y) | \text{ all } x, c \le y \le d\}$, $A_1 \cap K \neq \varnothing$ and $A_2 \cap K \neq \varnothing$. Then $P(\mathbf{a}) = 0$, but $P(A_1) \neq 0$ and $P(A_2) \neq 0$. However, $A = A_1 \cap A_2$, so $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$.

3.7.50 $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots x_n) = \prod_{j=1}^{n}(1/\lambda)e^{-x_j/\lambda} = (1/\lambda)^n e^{-\frac{1}{\lambda}\sum_{j=1}^{n}x_j}$

3.7.51 **(a)** $P(X_1 < 1/2) = \int_0^{1/2} 4x^3 dx = x^4\Big|_0^{1/2} = 1/16$

(b) This asks for the probability of exactly one success in a binomial experiment with $n = 4$ and $p = 1/16$, so the probability is $\binom{4}{1}(1/16)^1(15/16)^3 = 0.206$.

(c) $f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4) = \prod_{j=1}^{4}4x_j^3 = 256(x_1x_2x_3x_4)^3,\ 0 \le x_1,x_2,x_3,x_4 \le 1$

(d) $F_{X_2,X_3}(x_2,x_3) = \int_0^{x_3}\int_0^{x_2}(4s^3)(4t^3)ds\,dt = \int_0^{x_2}4s^2 ds\int_0^{x_3}4t^3 dt = x_2^4x_3^4$, $0 \le x_2, x_3 \le 1$.

3.7.52 $P(X_1 < 1/2, X_2 > 1/2, X_3 < 1/2, X_4 > 1/2, \ldots, X_{2k} > 1/2)$
$= P(X_1 < 1/2)P(X_2 > 1/2)P(X_3 < 1/2)P(X_4 > 1/2), \ldots, P(X_{2k} > 1/2)$ because the X_i are independent. Since the X_i are uniform over the unit interval, $P(X_i < 1/2) = P(X_i > 1/2) = 1/2$. Thus the desired probability is $(1/2)^{2k}$.

Section 3.8: Transforming and Combining Random Variables

3.8.1 **(a)** $p_{X+Y}(w) = \sum_{\text{all } x} p_X(x)p_Y(w-x)$. Since $p_X(x) = 0$ for negative x, we can take the lower limit of the sum to be 0. Since $p_Y(w-x) = 0$ for $w - x < 0$, or $x >$ w, we can take the upper limit of the sum to be w. Then we obtain

$$p_{X+Y}(w) = \sum_{k=0}^{w} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{w-k}}{(w-k)!} = e^{-(\lambda+\mu)}\sum_{k=0}^{w}\frac{1}{k!(w-k)!}\lambda^k\mu^{w-k}$$

$$= e^{-(\lambda+\mu)}\frac{1}{w!}\sum_{k=0}^{w}\frac{w!}{k!(w-k)!}\lambda^k\mu^{w-k} = e^{-(\lambda+\mu)}\frac{1}{w!}(\lambda+\mu)^w,\ w = 0, 1, 2, \ldots$$

This pdf has the same form as the ones for X and Y, but with parameter $\lambda + \mu$.

(b) $p_{X+Y}(w) = \sum_{\text{all } x} p_X(x)p_Y(w-x)$. The lower limit of the sum is 1.

For this pdf, we must have $w - k \ge 1$ so the upper limit of the sum is w– 1. Then

$$p_{X+Y}(w) = \sum_{k=1}^{w-1}(1-p)^{k-1}p(1-p)^{w-k-1}p = (1-p)^{w-2}p^2\sum_{k=1}^{w-1}1 = (w-1)(1-p)^{w-2}p^2,\ w = 2, 3, 4, \ldots$$

The pdf for $X + Y$ does not have the same form as those for X and Y, but Section 4.5 will show that they all belong to the same family—the negative binomial.

3.8.2 $f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx = \int_0^w (xe^{-x})(e^{-(w-x)})dx = e^{-w}\int_0^w x\,dx = \frac{w^2}{2}e^{-w},\ w \geq 0$

3.8.3 First suppose that $0 \leq w \leq 1$. As in the previous problem the upper limit of the integral is w, and $f_{X+Y}(w) = \int_0^w (1)(1)dx = w$. Now consider the case $1 \leq w \leq 2$. Here, the first integrand vanishes unless x is ≤ 1. Also, the second pdf is 0 unless $w - x \leq 1$ or $x \geq w - 1$. Then $f_{X+Y}(w)$ $= \int_{w-1}^{1} (1)(1)dx = 2 - w$.

In summary, $f_{X+Y}(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2-w & 1 \leq w \leq 2 \end{cases}$

3.8.4 Consider the continuous case. It suffices to show that $F_{V,X+Y} = F_V F_{X+Y}$. $F_{V,X+Y}(v,w)$ $= P(V \leq v, X + Y \leq w) =$

$$\int_{-\infty}^{v}\int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_V(v)f_X(x)f_Y(y)dydxdv = \int_{-\infty}^{v} f_V(v)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_X(x)f_Y(y)dydx\right)dv$$

$= F_V(v)F_{X+Y}(w)$

3.8.5 $F_W(w) = P(W \leq w) = P(Y^2 \leq w) = P\left(Y \leq \sqrt{w}\right) = f = F_Y\left(\sqrt{w}\right)$

Now differentiate both sides to obtain $f_W(w) = \frac{d}{dw}F_W(w) = \frac{d}{dw}F_Y\left(\sqrt{w}\right) = \frac{1}{2\sqrt{w}}f_Y\left(\sqrt{w}\right)$.

3.8.6 From Question 3.8.5, $f_W(w) = \frac{1}{2\sqrt{w}}f_Y\left(\sqrt{w}\right)$. Since $f_Y\left(\sqrt{w}\right) = 1$, $f_W(w) = \frac{1}{2\sqrt{w}}$, $0 \leq w \leq 1$.

3.8.7 From Question 3.8.5,

$f_W(w) = \frac{1}{2\sqrt{w}}f_Y\left(\sqrt{w}\right)$. Thus $f_W(w) = \frac{1}{2\sqrt{w}}6\sqrt{w}\left(1-\sqrt{w}\right) = 3\left(1-\sqrt{w}\right)$ where $0 \leq w \leq 1$.

3.8.8 From Question 3.8.5

$$f_{Y^2}(u) = \frac{1}{2\sqrt{u}}f_Y\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}a\left(\sqrt{u}\right)^2 e^{-b\left(\sqrt{u}\right)^2} = \frac{1}{2}a\sqrt{u}e^{-bu}.$$

Then $f_W(w) = \frac{2}{m}f_u(\frac{2}{m}w) = \frac{2}{m}\frac{1}{2}a\sqrt{\frac{2}{m}w}e^{-b(2w/m)} = \frac{\sqrt{2}}{m^{3/2}}a\sqrt{w}e^{-b(2w/m)}$, $0 \leq w$.

3.8.9 **(a)** Let $W = XY$. Then $f_W(w) = \int_{-\infty}^{\infty}\frac{1}{|x|}f_X(x)f_Y(w/x)dx$

Since $f_Y(w/x) \neq 0$ when $0 \leq w/x \leq 1$, then we need only consider $w \leq x$.

Similarly, $f_X(x) \neq 0$ implies $x \leq 1$. Thus the integral becomes $\int_w^1 \frac{1}{x}dx = \ln x\Big|_w^1 = -\ln w$, $0 \leq w \leq 1$.

(b) Again let $W = XY$. Since the range of integration here is the same as in Part **(a)**, we can write $f_V(v) = \int_{-\infty}^{\infty}\frac{1}{|x|}f_X(x)f_Y(w/x)dx = \int_w^1 \frac{1}{x}2(x)2(w/x)dx = 4w\int_w^1\frac{1}{x}dx = -4w\ln w,\ 0 \leq w \leq 1$.

3.8.10 **(a)** Let $W = Y/X$. Then $f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)dx$

Since $f_X(x) = 0$ for $x < 0$, the lower limit of the integral is 0. Since $f_Y(wx) = 0$ for $wx > 1$, we must have $wx \le 1$ or $x \le 1/w$.

Case I: $0 \le w \le 1$: In this case $1/w > 1$, so the upper limit of the integral is 1.

$F_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^1 x(1)(1)dx = 1/2$

Case II: $w > 1$: In this case $1/w \le 1$, so the upper limit of the integral is $1/w$.

$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^{1/w} x(1)(1)dx = \frac{1}{2w^2}$

(b) Case I: $0 \le w \le 1$: The limits of the integral are 0 and 1.

$F_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^1 x(2x)(2wx)dx = w$

Case II: $w > 1$: The limits of the integral are 0 and $1/w$.

$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^{1/w} x(2x)(2xw)dx = \frac{1}{w^3}$

3.8.11 Let $W = Y/X$. Then $f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)dx = \int_0^{\infty} x(xe^{-x})e^{-wx}dx = \int_0^{\infty} x^2 e^{-(1+w)x}dx$

$= \frac{1}{1+w}\left(\int_0^{\infty} x^2(1+w)e^{-(1+w)x}dx\right)$

Let V be the exponential random variable with parameter $1 + w$. Then the quantity in parentheses above is $E(V^2)$.

But $E(V^2) = \text{Var}(V) + E^2(V) = \frac{1}{(1+w)^2} + \frac{1}{(1+w)^2} = \frac{2}{(1+w)^2}$ (See Question 3.6.11)

Thus, $f_W(w) = \frac{1}{1+w}\left(\frac{2}{(1+w)^2}\right) = \frac{2}{(1+w)^3}$, $0 \le w$.

Section 3.9: Further Properties of the Mean and Variance

3.9.1 Let X_i be the number from the i-th draw, $i = 1, \ldots, r$. Then for each i, $E(X_i) = \frac{1+2+\ldots+n}{n} = \frac{n+1}{2}$. The sum of the numbers drawn is $\sum_{i=1}^{r} X_i$, so the expected value of the sum is $\sum_{i=1}^{r} E(X_i) = \frac{r(n+1)}{2}$.

3.9.2 $f_{X,Y}(x, y) = \lambda^2 e^{-\lambda(x+y)} = \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda y}\right)$ implies that $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$.

Then $E(X + Y) = E(X) + E(Y) = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$

3.9.3 From Question 3.7.19(c), $f_X(x) = \frac{2}{3}(x+1)$, $0 \le x \le 1$, so $E(X) = \int_0^1 x\frac{2}{3}(x+1)dx$

$= \frac{2}{3}\int_0^1 (x^2+x)dx = \frac{5}{9}$

Also, $f_Y(y) = \frac{4}{3}y + \frac{1}{3}$, $0 \le y \le 1$, so $E(Y) = \int_0^1 y\left(\frac{4}{3}y+\frac{1}{3}\right)dy = \int_0^1\left(\frac{4}{3}y^2+\frac{1}{3}y\right)dy = \frac{11}{8}$.

Then $E(X + Y) = E(X) + E(Y) = \frac{5}{9}+\frac{11}{18}=\frac{7}{6}$.

3.9.4 Let $X_i = 1$ if a shot with the first gun is a bull's eye and 0 otherwise, $i = 1, \ldots, 10$. $E(X_i) = 0.30$. Let $V_i = 1$ if a shot with the second gun is a bull's-eye and 0 otherwise, $i = 1, \ldots, 10$. $E(V_i) = 0.40$.

Cathie's score is $4\sum_{i=1}^{10} X_i + 6\sum_{i=1}^{10} V_i$, and her expected score is $E\left(4\sum_{i=1}^{10} X_i + 6\sum_{i=1}^{10} V_i\right)$

$= 4\sum_{i=1}^{10} E(X_i) + 6\sum_{i=1}^{10} E(V_i) = 4(10)(0.30) + 6(10)(0.40) = 36.$

3.9.5 $\mu = E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) = \sum_{i=1}^{n} a_i \mu = \mu\sum_{i=1}^{n} a_i$, so the given equality occurs if and only if

$\sum_{i=1}^{n} a_i = 1.$

3.9.6 Let X_i be the daily closing price of the stock on day i. The daily expected gain is $E(X_i) = (1/8)p - (1/8)q = (1/8)(p - q)$. After n days the expected gain is $(n/8)(p - q)$.

3.9.7 **(a)** $E(X_i)$ is the probability that the i-th ball drawn is red, $1 \le i \le n$. Draw the balls in order without replacement, but do not note the colors. Then look at the i-th ball first. The probability that it is red is surely independent of when it is drawn. Thus, all of these expected values are the same and each equals $r/(r + w)$.

(b) Let X be the number of red balls drawn. Then $X = \sum_{i=1}^{n} X_i$ and $E(X) = \sum_{i=1}^{n} E(X_i) = nr/(r + w)$.

3.9.8 Let X_1 = number showing on face 1; X_2 = number showing on face 2. Since X_1 and X_2 are independent, $E(X_1X_2) = E(X_1)E(X_2) = (3.5)(3.5) = 12.25$.

3.9.9 First note that $1 = \int_{10}^{20}\int_{10}^{20} k(x+y)dydx = k\cdot 3000$, so $k = \frac{1}{3000}$.

If $\frac{1}{R} = \frac{1}{X}+\frac{1}{Y}$, then $R = \frac{XY}{X+Y}$.

$E(R) = \frac{1}{3000}\int_{10}^{20}\int_{10}^{20}\frac{xy}{x+y}(x+y)dydx = \frac{1}{3000}\int_{10}^{20}\int_{10}^{20} xydydx = 7.5.$

3.9.10 From Question 3.8.5, $f_{X^2}(w) = \frac{1}{2\sqrt{w}}$, so $E(X^2) = \int_0^1 w\frac{1}{2\sqrt{w}}dw = \frac{1}{2}\int_0^1 \sqrt{w}dw = \frac{1}{3}$, with a similar result holding for Y^2. Then $E(X^2 + Y^2) = 2/3$.

3.9.11 The area of the triangle is the random variable $W = \frac{1}{2}XY$. Then

$$E\left(\frac{1}{2}XY\right) = \frac{1}{2}E(XY) = \frac{1}{2}E(X)E(Y) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

3.9.12 The Y_i are independent for $i = 1, 2, \ldots, n$. Thus,

$$E\left(\sqrt[n]{Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n}\right) = E\left(\sqrt[n]{Y_1}\right)\cdot E\left(\sqrt[n]{Y_2}\right)\cdot \ldots E\left(\sqrt[n]{Y_n}\right)$$

The Y_i all have the same uniform pdf, so it suffices to calculate $E\left(\sqrt[n]{Y_1}\right)$, which is

$\int_0^1 \sqrt[n]{y}\cdot 1\, dy = \frac{n}{n+1}$. Thus, the expected value of the geometric mean is $\left(\frac{n}{n+1}\right)^n$.

Note that the *arithmetic* mean is constant at 1/2 and does not depend on the sample size.

3.9.13

x	y	$f_{X,Y}$	xy	$xyf_{X,Y}$
1	1	1/36	1	1/36
1	2	1/36	2	2/36
1	3	1/36	3	3/36
1	4	1/36	4	4/36
1	5	1/36	5	5/36
1	6	1/36	6	6/36
2	2	2/36	4	8/36
2	3	1/36	6	6/36
2	4	1/36	8	8/36
2	5	1/36	10	10/36
2	6	1/36	12	12/36
3	3	3/36	9	27/36
3	4	1/36	12	12/36
3	5	1/36	15	15/36
3	6	1/36	18	18/36
4	4	4/36	16	64/36
4	5	1/36	20	20/36
4	6	1/36	24	24/36
5	5	5/36	25	125/36
5	6	1/36	30	30/36
6	6	6/36	36	216/36

$E(XY)$ is the sum of the last column = $\frac{616}{36}$. Clearly $E(X) = 7/2$.

$E(Y) = 1\frac{1}{36} + 2\frac{2}{36} + 3\frac{5}{36} + 4\frac{7}{36} + 5\frac{9}{36} + 6\frac{11}{36} = \frac{161}{36}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{616}{36} - \frac{7}{2}\cdot\frac{161}{36} = \frac{105}{72}$

3.9.14 $\text{Cov}(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b)E(cY + d)$

$= E(acXY + adX + bcY + bd) - [aE(X) + b][cE(Y) + d]$

$= acE(XY) + adE(X) + bcE(Y) + bd - acE(X)E(Y) - adE(X) - bcE(Y) - bd$

$= ac[E(XY) - E(X)E(Y)] = ac\text{Cov}(X, Y)$

3.9.15 $\int_0^{2\pi} \cos x dx = \int_0^{2\pi} \sin x dx = \int_0^{2\pi} (\cos x)(\sin x)\, dx = 0$, so $E(X) = E(Y) = E(XY) = 0$.

Then Cov(X, Y) = 0. But X and Y are functionally dependent, $Y = \sqrt{1-X^2}$, so they are probabilistically dependent.

3.9.16 $E(XY) = \int_0^1 y\int_{-y}^{y} x\, dxdy = \int_0^1 y\left[\frac{x^2}{2}\right]_{-y}^{y} dy = \int_0^1 y\cdot 0\, dy = 0$

$E(X) = \int_0^1\int_{-y}^{y} x\, dxdy = 0$, so Cov($X$, Y) = 0. However, X and Y are dependent since

$P(-1/2 < x < 1/2, 0 < Y < 1/2) = P(0 < Y < 1/2) \neq P(-1/2 < x < 1/2)P(0 < Y < 1/2)$

3.9.17 The random variables are independent and have the same exponential pdf, so Var(X + Y) = Var(X) + Var(Y). By Question 3.6.11, Var(X) = Var(Y) = $\frac{1}{\lambda^2}$, so Var(X + Y) = $\frac{2}{\lambda^2}$.

3.9.18 From Question 3.9.3, we have $E(X + Y) = E(X) + E(Y) = 5/9 + 11/18 = 21/18 = 7/6$.

$E[(X + Y)^2] = \int_0^1\int_0^1 (x+y)^2\frac{2}{3}(x+2y)dxdy = \frac{2}{3}\int_0^1\int_0^1 (x^2+2xy+y^2)(x+2y)dxdy$

$\frac{2}{3}\int_0^1\int_0^1 (x^3+2x^2y+xy^2+2x^2y+4xy^2+2y^3)dxdy$

$= \frac{2}{3}\int_0^1\int_0^1 (x^3+4x^2y+5xy^2+2y^3)dxdy = \frac{2}{3}\int_0^1 \left(\frac{1}{4}x^4+\frac{4}{3}x^3y+\frac{5}{2}x^2y^2+2xy^3\right)\Bigg|_0^1 dy$

$= \frac{2}{3}\int_0^1 \left(\frac{1}{4}+\frac{4}{3}y+\frac{5}{2}y^2+2y^3\right)dy = \frac{2}{3}\left(\frac{1}{4}+\frac{4}{6}+\frac{5}{6}+\frac{2}{4}\right) = \frac{3}{2}$

Then $\mathrm{Var}(X + Y) = E[(X + Y)^2] - E(X + Y)^2 = \frac{3}{2} - \left(\frac{7}{6}\right)^2 = \frac{5}{36}$.

3.9.19 First note that $E\left[\left(\sqrt{Y_1Y_2}\right)^2\right] = E[Y_1Y_2] = E(Y_1)E(Y_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$, since the Y_i are independent, for i = 1, 2.

$E\left(\sqrt{Y_1Y_2}\right) = E\left(\sqrt{Y_1}\right)E\left(\sqrt{Y_2}\right) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$, since $E(Y_i) = \int_0^1 \sqrt{y}\cdot 1\, dy = \frac{2}{3}$, i = 1, 2.

Then Var $\left(\sqrt{Y_1Y_2}\right) = \frac{1}{4} - \left(\frac{4}{9}\right)^2 = \frac{17}{324}$.

3.9.20 $E(W) = E(4X + 6Y) = 4E(X) + 6E(Y) = 4np_X + 6mp_Y$
$\mathrm{Var}(W) = \mathrm{Var}(4X + 6Y) = 16\mathrm{Var}(X) + 36\mathrm{Var}(Y) = 16np_X(1 - p_X) + 36mp_Y(1 - p_Y)$

3.9.21 Let U_i be the number of calls during the i-th hour in the normal nine hour work day. Then $U = U_1 + U_2 + \ldots + U_9$ is the number of calls during this nine hour period. $E(U) = 9(7) = 63$. For a Poisson random variable, the variance is equal to the mean, so $\mathrm{Var}(U) = 9(7) = 63$. Similarly, if V is the number of calls during the off hours, $E(V) = \mathrm{Var}(V) = 15(4) = 60$.
Let the total cost be the random variable $W = 50U + 60V$.
Then $E(W) = E(50U + 60V) = 50E(U) + 60E(V) = 50(63) + 60(60) = 6750$;
$\mathrm{Var}(W) = \mathrm{Var}(50U + 60V) = 50^2\mathrm{Var}(U) + 60^2\mathrm{Var}(V) = 50^2(63) + 60^2(60) = 373{,}500$.

3.9.22 $L = \sum_{i=1}^{50} B_i + \sum_{i=1}^{49} M_i$, where B_i is the length of the i-th brick, and M_i is the thickness of the i-th mortar separation. Assume all of the B_i and M_i are independent. By Theorem 3.9.5, $\text{Var}(L) = 50\text{Var}(B_1) + 49\text{Var}(M_1) = 50\left(\frac{1}{32}\right)^2 + 49\left(\frac{1}{16}\right)^2 = 0.240$. Thus, the standard deviation of L is 0.490.

3.9.23 Let R_i be the resistance of the i-th resistor, $1 \le i \le 6$. Assume the R_i are independent and each has standard deviation σ. Then the variance of the circuit resistance is $\text{Var}\left(\sum_{i=1}^{6} R_i\right) = 6\sigma^2$. The circuit must have $6\sigma^2 \le (0.4)^2$ or $\sigma \le 0.163$.

3.9.24 Let p be the probability the gambler wins a hand. Let T_k be his winnings on the k-th hand. Then $E(T_k) = kp$. Also, $E\left(T_k^2\right) = k^2p$, so $\text{Var}(T_k) = k^2p - (kp)^2 = k^2(p - p^2)$.

The total winnings $T = \sum_{k=1}^{n} T_k$, so $E(T) = \sum_{k=1}^{n} kp = \frac{n(n+1)}{2} p$.

$\text{Var}(T) = \sum_{k=1}^{n} k^2(p - p^2) = \frac{n(n+1)(2n+1)}{6}(p - p^2)$

Section 3.10: Order Statistics

3.10.1 $P(Y_3' < 5) = \int_0^5 f_{Y_3'}(y)dy = \int_0^5 \frac{4!}{(3-1)!(4-3)!}\frac{y}{10}^{3-1}\left(1 - \frac{y}{10}\right)^{4-3}\frac{1}{10}dy$

$= \frac{12}{10^4}\int_0^5 y^2(10 - y)dy = \frac{12}{10^4}\left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_0^5 = \frac{12}{10^4}\left[\frac{10}{3}5^3 - \frac{1}{4}5^4\right] = 5/16$

3.10.2 First find F_Y: $F_Y(y) = \int_0^y 3t^2 dt = y^3$. Then $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75)$.

But $P(Y_5' < 0.75) = \int_0^{0.75} \frac{6!}{(5-1)!(6-5)!}(y^3)^{5-1}(1 - y^3)^{6-5} 3y^2 dy$

$= \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1 - y^3)3y^2 dy = \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1 - y^3)3y^2 dy$

$= \int_0^{0.75} 90(y^{14})(1 - y^3)dy = 90\left[\frac{y^{15}}{15} - \frac{y^{18}}{18}\right]_0^{0.75} = 0.052,$

so $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75) = 1 - 0.052 = 0.948$.

3.10.3 $P(Y_2' > y_{60}) = 1 - P(Y_2' < y_{60}) = 1 - P(Y_1 < y_{60}, Y_2 < y_{60})$
$= 1 - P(Y_1 < y_{60})P(Y_2 < y_{60}) = 1 - (0.60)(0.60) = 0.64$

3.10.4 The complement of the event is $P(Y_1' > 0.6) \cup P(Y_5' < 0.6)$.

These are disjoint events, so their probability is $P(Y_1' > 0.6) + P(Y_5' < 0.6)$.

But $P(Y_1' > 0.6) = P(Y_1, Y_2, Y_3, Y_4, Y_5 > 0.6) = [P(Y > 0.6)]^5 = \left(\int_{0.6}^{1} 2y\,dy\right)^5 = (0.64)^5 = 0.107$

Also, $P(Y_5' < 0.6) = P(Y_1, Y_2, Y_3, Y_4, Y_5 < 0.6) = [P(Y < 0.6)]^5 = \left(\int_{.0}^{0.6} 2y\,dy\right)^5 = (0.36)^5 = 0.006$

The desired probability is $1 - 0.107 - 0.006 = 0.887$.

3.10.5 $P(Y_1' > m) = P(Y_1, \ldots, Y_n > m) = \left(\dfrac{1}{2}\right)^n$

$P(Y_n' > m) = 1 - P(Y_n' < m) = 1 - P(Y_1, \ldots, Y_n < m)$

$= 1 - P(Y_1 < m) \cdot \ldots \cdot P(Y_n < m) = 1 - \left(\dfrac{1}{2}\right)^n$

If $n \geq 2$, the latter probability is greater.

3.10.6 $P(Y_{\min} < 0.2) = \int_0^{0.2} n f_Y(y)[1 - F_Y(y)]^{n-1}\,dy$ by Theorem 3.10.1.

Since $F_Y(y) = 1 - e^{-y}$, $\int_0^{0.2} n f_Y(y)[1 - F_Y(y)]^{n-1}\,dy = \int_0^{0.2} n e^{-y}[1 - (1 - e^{-y})]^{n-1}$

$= \int_0^{0.2} n e^{-ny}\,dy = -e^{-ny}\Big|_0^{0.2} = 1 - e^{-0.2n}$

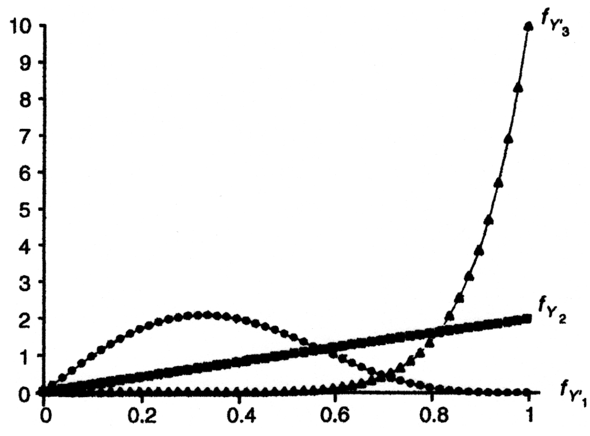
But $(1 - e^{-0.2n}) > 0.9$ if $e^{-0.2n} < 0.1$, which is equivalent to $n > -\dfrac{1}{0.2}\ln 0.1 = 11.513$. The smallest n satisfying this inequality is 12.

3.10.7 $P(0.6 < Y_4' < 0.7) = F_{Y_4'}(0.7) - F_{Y_4'}(0.6) = \int_{0.6}^{0.7} \dfrac{6!}{(4-1)!(6-4)!} y^{4-1}(1-y)^{6-4}(1)\,dy$ (by Theorem 3.10.2) $= \int_{0.6}^{0.7} 60y^3(1-y)^2\,dy = \int_{0.6}^{0.7} 60(y^3 - 2y^4 + y^5)\,dy = (15y^4 - 24y^5 + 10y^6)\Big|_{0.6}^{0.7}$

$= 0.74431 - 0.54432 = 0.19999$

3.10.8 First note that $F_Y(y) = \int_0^y 2t\,dt = y^2$. Then by Theorem 3.10.2,

$f_{Y_1'}(y) = 5(2y)(1-y^2)^{5-1} = 10y(1-y^2)^4$. By this same result, $f_{Y_5'}(y) = 5(2y)(y^2)^{5-1} = 10y^9$.



3.10.9 $P(Y_{\min} > 20) = P(Y_1 > 20, Y_2 > 20, \ldots, Y_n > 20) = P(Y_1 > 20)P(Y_2 > 20) \ldots P(Y_n > 20)$
$= [P(Y > 20)]^n$. But 20 is the median of Y, so $P(Y > 20) = 1/2$. Thus, $P(Y_{\min} > 20) = (1/2)^n$.

3.10.10 $P(Y_{\min} = Y_n) = P(Y_n < Y_1, Y_n < Y_2, \ldots, Y_n < Y_{n-1}) = P(Y_n < Y_1)P(Y_n < Y_2) \ldots P(Y_n < Y_{n-1}) = \left(\frac{1}{2}\right)^n$

3.10.11 The graphed pdf is the function $f_Y(y) = 2y$, so $F_Y(y) = y^2$
Then $f_{Y_4'}(y) = 20y^6(1 - y^2)2y = 40y^7(1 - y^2)$ and $F_{Y_4'}(y) = 5y^8 - 4y^{10}$.
$P(Y_4' > 0.75) = 1 - F_{Y_4'}(0.75) = 1 - 0.275 = 0.725$
The probability that none of the schools will have fewer than 10% of their students bused is
$P(Y_{\min} > 0.1) = 1 - F_{Y_{\min}}(0.1) = 1 - \int_0^{0.1} 10y(1-y^2)^4\,dy = 1 - \left[-(1-y^2)^5\right]_0^{0.1} = 0.951$ (see Question 3.10.8).

3.10.12 Using the solution to Question 3.10.6, we can, in a similar manner, establish that $f_{Y_1'}(y) = n\lambda e^{-n\lambda y}$. The mean of such an exponential random variable is the inverse of its parameter, or $1/n\lambda$.

3.10.13 If $Y_1, Y_2, \ldots Y_n$ is a random sample from the uniform distribution over [0, 1], then by Theorem 3.10.2, the quantity $\frac{n!}{(i-1)!(n-i)!}\left[F_Y(y)\right]^{i-1}\left[1 - F_Y(y)\right]^{n-i} f_y(y)$

$= \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}$ is the pdf of the i-th order statistic.

Thus, $1 = \int_0^1 \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}\,dy = \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^{i-1}(1-y)^{n-i}\,dy$

or, equivalently, $\int_0^1 y^{i-1}(1-y)^{n-i}\,dy = \frac{(i-1)!(n-i)!}{n!}$.

3.10.14 $E(Y_i') = \int_0^1 y \cdot \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}\,dy = \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^i(1-y)^{n-i}\,dy$

$= \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^{(i+1)-1}(1-y)^{(n+1)-(i+1)}\,dy = \frac{n!}{(i-1)!(n-i)!}\frac{[(i+1)-1]![(n+1)-(i+1)]!}{(n+1)!}$

where this last equality comes from the result in Question 3.10.13.

Thus, $E(Y_i') = \frac{n!}{(i-1)!(n-i)!}\frac{i!(n-1)!}{(n+i)!} = \frac{i}{(n+1)}$

3.10.15 This question translates to asking for the probability that a random sample of three independent uniform random variables on [0, 1] has range $R \le 1/2$. Example 3.10.6 establishes that $F_R(r) = 3r^2 - 2r^3$. The desired probability is $F_R(1/2) = 3(1/2)^2 - 2(1/2)^3 = 0.5$.

3.10.16 This question requires finding the probability that a random sample of three independent exponential random variables on [0, 10] has range $R \le 2$. From Equation 3.10.5, we find the joint pdf of $Y_{\min}$ and $Y_{\max}$ to be $3[F_Y(v) - F_Y(u)]f_Y(u)f_Y(v) = 3[(1 - e^{-v}) - (1 - e^{-u})]e^{-u}e^{-v}$
$= 3(e^{-2u-v} - e^{-u-2v}),\ u \le v$.

$P(R \le 2)$ is obtained by integrating the joint pdf over a strip such as pictured in Figure 3.10.4, but the strip is infinite in extent. Thus,

$P(R \le 2) = \int_0^\infty \int_u^{u+2} 3(e^{-2u-v} - e^{-u-2v})dvdu$. The inner integral is

$$\int_u^{u+2} (e^{-2u-v} - e^{-u-2v})dv = e^{-2u}(-e^{-v})\Big|_u^{u+2} - e^{-u}\left(-\frac{1}{2}e^{-2v}\right)\Big|_u^{u+2}$$

$$= e^{-2u}(e^{-u} - e^{-u-2}) - \frac{1}{2}e^{-u}(e^{-2u} - e^{-2u-4}) = e^{-3u}\left(\frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4}\right)$$

Then $P(R \le 2) = 3\left(\frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4}\right)\int_0^\infty e^{-3u}du = \frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4} = 0.374$.

Section 3.11: Conditional Densities

3.11.1 $p_X(x) = \frac{x+1+x\cdot 1}{21} + \frac{x+2+x\cdot 2}{21} = \frac{3+5x}{21}$, $x = 1, 2$

$$p_{Y|x}(y) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{x+y+xy}{3+5x}, \ y = 1, 2$$

3.11.2 The probability that $X = x$ and $Y = y$ is the probability of y 4's on the first two rolls and $x - y$ rolls on the last four rolls. These events are independent, so

$$p_{X,Y}(x, y) = \binom{2}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y} \quad \text{for } y \le x$$

Then $p_{Y|x}(y) = \dfrac{p_{X,Y}(x,y)}{p_X(x)} = \dfrac{\binom{2}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}}{\binom{6}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{6-x}} = \dfrac{\binom{2}{y}\binom{4}{x-y}}{\binom{6}{x}}$,

$0 \le y \le \min(2, x)$, which we recognize as a hypergeometric distribution.

3.11.3 $p_{Y|x}(y) = \dfrac{p_{X,Y}(x,y)}{p_X(x)} = \dfrac{\binom{8}{x}\binom{6}{y}\binom{4}{3-x-y}}{\binom{18}{3}} \div \dfrac{\binom{8}{x}\binom{10}{3-x}}{\binom{18}{3}} = \dfrac{\binom{6}{y}\binom{4}{3-x-y}}{\binom{10}{3-x}}$, with $0 \le y \le 3 - x$

3.11.4 $P(X = 2|Y = 2) = \dfrac{P(X=2, Y=2)}{P(Y=2)}$

$$= \frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} \div \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{4}{2}\binom{48}{3}} = 0.015$$

3.11.5 **(a)** $1/k = \sum_{x=1}^{3}\sum_{y=1}^{3}(x+y) = 36$, so $k = 1/36$

(b) $p_X(x) = \frac{1}{36}\sum_{y=1}^{3}(x+y) = \frac{1}{36}(3x+6)$

$$p_{Y|x}(1) = \frac{p_{X,Y}(x,1)}{p_X(x)} = \frac{\frac{1}{36}(x+1)}{\frac{1}{36}(3x+6)} = \frac{x+1}{3x+6},\ x = 1, 2, 3$$

3.11.6 **(a)** $p_{X,Y}(x, y) = p_{Y|x}(y)p_X(x) = \binom{x}{y}\left(\frac{1}{2}\right)^x\left(\frac{1}{3}\right),\ y \le x$

(b) $p_Y(0) = \left(\frac{1}{3}\right)\sum_{x=1}^{3}\binom{x}{0}\left(\frac{1}{2}\right)^x = \frac{7}{24}$

$p_Y(1) = \left(\frac{1}{3}\right)\sum_{x=1}^{3}\binom{x}{1}\left(\frac{1}{2}\right)^x = \frac{11}{24}$

$p_Y(2) = \left(\frac{1}{3}\right)\sum_{x=2}^{3}\binom{x}{2}\left(\frac{1}{2}\right)^x = \frac{5}{24}$

$p_Y(3) = \left(\frac{1}{3}\right)\binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{24}$

3.11.7 $p_Z(z) = \frac{1\cdot 1+1\cdot z+1\cdot z}{54}+\frac{1\cdot 2+1\cdot z+2\cdot z}{54}+\frac{2\cdot 1+2\cdot z+1\cdot z}{54}+\frac{2\cdot 2+2\cdot z+2\cdot z}{54}$

$= \frac{9+12z}{54},\ z = 1, 2$

Then $p_{X,Y|z}(x, y) = \frac{xy+xz+yz}{9+12z},\ x = 1, 2 \quad y = 1, 2 \quad z = 1, 2$

3.11.8 $p_{W,X}(1, 1) = P(\{(1, 1, 1)\}) = 3/54$
$p_{W,X}(2, 2) = 33/54$; $P(W = 2, X = 2) = P(X = 2)$. But $P(X = 2) = P(Z = 2)$ by symmetry, and from Question 3.11.7, this probability is 33/54. Then $p_{W,X}(2, 1) = 1 - 3/54 - 33/54 = 18/54$
Finally, $p_{W|1}(1) = (3/54)/(21/54) = 1/7$; $p_{W|1}(2) = (18/54)/(21/54) = 6/7$; and
$p_{W|2}(2) = (33/54)/(33/54) = 1$

3.11.9 $p_{X|x+y=n}(x) = \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k, Y=n-k)}{P(X+Y=n)}$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}} = \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

but the right hand term is a binomial probability with parameters n and $\lambda/(\lambda+\mu)$.

3.11.10 Let U be the number of errors made by Compositor A in the 100 pages. Then the pdf of U is Poisson with parameter $\lambda = 200$. Similarly, if V is the number of errors made by Compositor B, then the pdf for V is Poisson with $\mu = 300$. From the previous question, $P(U \le 259 | U + V = 520)$ is binomial with parameter $n = 520$ and $p = 200/(200 + 300) = 2/5$.

The desired probability is $\displaystyle\sum_{k=0}^{259}\binom{520}{k}\left(\frac{2}{5}\right)^k\left(\frac{3}{5}\right)^{520-k}$

3.11.11 $P(X > s + t | X > t) = \dfrac{P(X > s+t \text{ and } X > t)}{P(X > t)} = \dfrac{P(X > s+t)}{P(X > t)}$

$$= \frac{(1/\lambda)\int_{s+t}^{\infty} e^{-x/\lambda}dx}{(1/\lambda)\int_{t}^{\infty} e^{-x/\lambda}dx} = \frac{-(1/\lambda)e^{-x/\lambda}\Big|_{s+t}^{\infty}}{-(1/\lambda)e^{-x/\lambda}\Big|_{t}^{\infty}} = \frac{(1/\lambda)e^{-(s+t)/\lambda}}{(1/\lambda)e^{-t/\lambda}} = e^{-s/\lambda} = \int_s^{\infty}(1/\lambda)e^{-x/\lambda}dx = P(X > s)$$

3.11.12 **(a)** $f_X(x) = \int_x^{\infty} 2e^{-x}e^{-y}dy = 2e^{-2x}$, $x > 0$, so $P(X < 1) = \int_0^1 2e^{-2x}dx = 1 - e^{-2} = 1 - 0.135 = 0.865$

Also, $P(X < 1, Y < 1) = \int_0^1\int_0^x 2e^{-(x+y)}dydx = \int_0^1 2e^{-x}\left[-e^{-y}\right]_0^x dx = \int_0^1(2e^{-x} - 2e^{-2x})dx$

$= -2e^{-x} + e^{-2x}\Big|_0^1 = 0.400$. Then the conditional probability is $\dfrac{0.400}{0.865} = 0.462$

(b) $P(Y < 1 | X = 1) = 0$, since the joint pdf is defined with y always larger than x.

(c) $f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2e^{-(x+y)}}{2e^{-2x}} = e^x e^{-y}$, $x < y$

(d) $E(Y | x) = \int_x^{\infty} ye^x e^{-y}dy = e^x\int_x^{\infty} ye^{-y}dy = e^x[-e^{-y}(y+1)]_x^{\infty} = e^x[e^{-x}(x+1)] = x+1$

3.11.13 $f_X(x) = \int_0^1 (x + y)dy = xy + \dfrac{y^2}{2}\Bigg|_0^1 = x + \dfrac{1}{2}$, $0 \le x \le 1$

$f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{x+y}{x+\frac{1}{2}}$, $0 \le y \le 1$

3.11.14 $f_X(x) = \int_0^{1-x} 2\,dy = 2y\Big|_0^{1-x} = 2(1 - x)$, $0 \le x \le 1$

$f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2}{2(1-x)} = \dfrac{1}{1-x}$, $0 \le y \le 1 - x$

For each x, the conditional pdf does not depend on y, so it is a uniform pdf.

3.11.15 $f_{X,Y}(x, y) = f_{Y|x}(y)f_X(x) = \left(\dfrac{2y+4x}{1+4x}\right)\dfrac{1}{3}(1+4x) = \dfrac{1}{3}(2y+4x)$

$f_Y(y) = \int_0^1\dfrac{1}{3}(2y+4x)dx = \dfrac{1}{3}(2xy+2x^2)\Big|_0^1 = \dfrac{1}{3}(2y+2)$, with $0 \le y \le 1$

3.11.16 (a) $f_X(x) = \int_0^1 \frac{2}{5}(2x+3y)dy = \frac{2}{5}\left(2xy + \frac{3}{2}y^2\right)\Big|_0^1 = \frac{4}{5}x + \frac{3}{5}$, with $0 \le x \le 1$

(b) $f_{Y|x}(y) = \dfrac{\frac{2}{5}(2x+3y)}{\frac{4}{5}x+\frac{3}{5}} = \dfrac{4x+6y}{4x+3}$, $0 \le y \le 1$

(c) $f_{Y|\frac{1}{2}}(y) = \frac{1}{5}(2+6y)$

$P(1/4 \le Y \le 3/4) = \int_{1/4}^{3/4} \frac{1}{5}(2+6y)dy = 10/20 = 1/2$

3.11.17 $f_Y(y) = \int_0^y 2\,dx = 2y$

$f_{X|y}(x) = \dfrac{2}{2y} = \dfrac{1}{y}$, $0 < x < y$

$f_{X|\frac{3}{4}}(x) = \dfrac{1}{\frac{3}{4}} = \dfrac{4}{3}$, $0 < x < 3/4$

$P(0 < X < ½ | Y = 3/4) = \int_0^{1/2} \frac{4}{3}dx = \frac{2}{3}$

3.11.18 $f_Y(y) = \int_0^y \frac{xy}{2}dx = \frac{x^2y}{4}\Big|_0^y = \frac{y^3}{4}$, $0 < y < 2$

$f_{X|y}(x) = \dfrac{xy}{2}\Big/\dfrac{y^3}{4} = \dfrac{2x}{y^2}$, $0 \le x < y \le 2$

$f_{X|\frac{3}{2}}(x) = \dfrac{2x}{(3/2)^2} = \dfrac{8}{9}x$, $0 \le x < 3/2$

$P(X < 1 | Y = 3/2) = \int_0^1 \frac{8}{9}x\,dx = \frac{4}{9}x^2\Big|_0^1 = \frac{4}{9}$

3.11.19 $f_{X_4,X_5}(x_4,x_5) = \int_0^1\int_0^1\int_0^1 32x_1x_2x_3x_4x_5\,dx_1dx_2dx_3 = 4x_4x_5$, $0 < x_4, x_5 < 1$

$f_{X_1,X_2,X_3|x_4,x_5}(x_1,x_2,x_3) = \dfrac{32x_1x_2x_3x_4x_5}{4x_4x_5} = 8x_1x_2x_3$, $0 < x_1, x_2, x_3 < 1$

Note: the five random variables are independent, so the conditional pdfs are just the marginal pdfs.

3.11.20 (a) $f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}\left(x^2y + \frac{xy^2}{4}\right)\Big|_0^2 = \frac{6}{7}(2x^2 + x)$

(b) $P(X > 2Y) = \int_0^1 \int_0^{\frac{1}{2}x} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx$

$= \int_0^1 \left[\frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\right)\right]_0^{\frac{1}{2}x} dx = \int_0^1 \frac{6}{7}\left(\frac{9}{16}x^3\right) dx = \frac{27}{224}$

(c) $P(Y > 1 \mid X > 1/2) = \frac{P(X > 1/2, Y > 1)}{P(X > 1/2)}$

First calculate the numerator: $P(X > 1/2, Y > 1) = \int_{1/2}^1 \int_1^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = \frac{55}{112}$

We know f_X from part **(a)** so the denominator is $P(X > 1/2) = \int_{1/2}^1 \frac{6}{7}(2x^2 + x)dx = \frac{23}{28}$

The conditional probability requested is $\frac{55}{112} \Big/ \frac{23}{28} = \frac{55}{92}$.

Section 3.12: Moment-Generating Functions

3.12.1 Let X be a random variable with $p_X(k) = 1/n$, for $k = 0, 1, 2, \ldots, n - 1$, and 0 otherwise.

$M_X(t) = E(e^{tX}) = \sum_{k=0}^{n-1} e^{tk} p_X(k) = \sum_{k=0}^{n-1} e^{tk} \frac{1}{n} = \frac{1}{n}\sum_{k=0}^{n-1} (e^t)^k = \frac{1 - e^{nt}}{n(1 - e^t)}$.

(Recall that $1 + r + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}$).

3.12.2 $f_X(-3) = 6/10$; $f_X(5) = 4/10$.
$M_X(t) = E(e^{tX}) = e^{-3t}(6/10) + e^{5t}(4/10)$

3.12.3 For the given binomial random variable,

$E(e^{tX}) = M_X(t) = \left(1 - \frac{1}{3} + \frac{1}{3}e^t\right)^{10}$. Set $t = 3$ to obtain $E(e^{3X}) = \frac{1}{3^{10}}(2 + e^3)^{10}$

3.12.4 $M_X(t) = \sum_{k=0}^{\infty} e^{tk}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3e^t}{4}\right)^k$

$= \frac{1}{4}\frac{1}{1 - \frac{3e^t}{4}} = \frac{1}{4 - 3e^t}$, $0 < e^t < 4/3$

3.12.5 **(a)** Normal with $\mu = 0$ and $\sigma^2 = 12$ **(b)** Exponential with $\lambda = 2$
(c) Binomial with $n = 4$ and $p = ½$ **(d)** Geometric with $p = 0.3$

3.12.6 $M_Y(t) = E(e^{tY}) = \int_0^1 e^{ty}\, y\, dy + \int_1^2 e^{ty}(2-y)\, dy$

$$= \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_0^1 + \frac{2}{t}e^{ty}\Big|_1^2 - \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_1^2$$

$$= \left(\frac{1}{t} - \frac{1}{t^2}\right)e^{t} - \left(-\frac{1}{t^2}\right) + \frac{2}{t}e^{2t} - \frac{2}{t}e^{t} - \left(\frac{2}{t} - \frac{1}{t^2}\right)e^{2t} + \left(\frac{1}{t} - \frac{1}{t^2}\right)e^{t}$$

$$= \frac{1}{t^2} + \frac{1}{t^2}e^{2t} - \frac{2}{t^2}e^{t} = \frac{1}{t^2}(e^t - 1)^2$$

3.12.7 $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t - 1)}$

3.12.8 $M_Y(t) = E(e^{tY}) = \int_0^{\infty} e^{ty}\, ye^{-y} dy = \int_0^{\infty} ye^{-y(1-t)} dy = \frac{1}{1-t}\int_0^{\infty} y(1-t)e^{-y(1-t)} dy$

$$= \left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2},$$

since the integral is the mean of an exponential pdf with parameter $(1 - t)$, which is $\frac{1}{1-t}$.

3.12.9 $M_Y^{(1)}(t) = \frac{d}{dt}e^{t^2/2} = te^{t^2/2}$

$M_Y^{(2)}(t) = \frac{d}{dt}te^{t^2/2} = t(te^{t^2/2}) + e^{t^2/2} = (t^2+1)e^{t^2/2}$

$M_Y^{(3)}(t) = \frac{d}{dt}(t^2+1)e^{t^2/2} = (t^2+1)te^{t^2/2} + 2te^{t^2/2}$, and $E(Y^3) = M_Y^{(3)}(0) = 0$

3.12.10 From Example 3.12.3, $M_Y(t) = \frac{\lambda}{(\lambda - t)}$ and $M_Y^{(1)}(t) = \frac{\lambda}{(\lambda - t)^2}$.

Successive differentiation gives $M_Y^{(4)}(t) = \frac{(4!)\lambda}{(\lambda - t)^5}$. Then $E(Y^4) = M_Y^{(4)}(0) = \frac{(4!)\lambda}{\lambda^5} = \frac{24}{\lambda^4}$.

3.12.11 $M_Y^{(1)}(t) = \frac{d}{dt}e^{at + b^2t^2/2} = (a + b^2t)e^{at + b^2t^2/2}$, so $M_Y^{(1)}(0) = a$

$M_Y^{(2)}(t) = (a + b^2t)^2 e^{at+b^2t^2/2} + b^2 e^{at+b^2t^2/2}$, so $M_Y^{(2)}(0) = a^2 + b^2$.

Then $\mathrm{Var}(Y) = (a^2 + b^2) - a^2 = b^2$

3.12.12 Successive differentiation of $M_Y(t)$ gives

$M_Y^{(4)}(t) = \alpha^4 k(k+1)(k+2)(k+3)(1 - \alpha t)^{-k-4}$. Thus, $E(Y^4) = M_Y^{(4)}(0) = \alpha^4 k(k+1)(k+2)(k+3)$

3.12.13 The moment generating function of Y is that of a normal variable with mean $\mu = -1$ and variance $\sigma^2 = 8$. Then $E(Y^2) = \mathrm{Var(Y)} + \mu^2 = 8 + 1 = 9$.

3.12.14 $M_Y^{(1)}(t) = \dfrac{d}{dt}(1-t/\lambda)^{-r} = (-r)(1-t/\lambda)^{-r-1}(-\lambda) = \lambda r(1-t/\lambda)^{-r-1}$

$$M_Y^{(2)}(t) = \frac{d}{dt}\lambda r(1-t/\lambda)^{-r-1} = (-r-1)\lambda r(1-t/\lambda)^{-r-2}(-\lambda) = \lambda^2 r(r+1)(1-t/\lambda)^{-r-2}$$

Continuing in this manner yields $M_Y^{(k)}(t) = \lambda^r \dfrac{(r+k-1)!}{(r-1)!}(1-t/\lambda)^{-r-k}$.

Then $E(Y^k) = M_Y^{(k)}(0) = \lambda^r \dfrac{(r+k-1)!}{(r-1)!}$.

3.12.15 $M_Y(t) = \displaystyle\int_a^b e^{ty}\frac{1}{b-a}dy = \frac{1}{(b-a)t}e^{ty}\Big|_a^b = \frac{1}{(b-a)t}(e^{tb}-e^{at})$ for $t \neq 0$

$$M_Y^{(1)}(t) = \frac{1}{(b-a)}\left[\frac{be^{tb}-ae^{at}}{t} - \frac{e^{tb}-e^{at}}{t^2}\right]$$

$E(Y) = \lim\limits_{t\to 0} M_Y^{(1)}(t) = \dfrac{1}{(b-a)}\lim\limits_{t\to 0}\left[\dfrac{be^{tb}-ae^{at}}{t} - \dfrac{e^{tb}-e^{at}}{t^2}\right]$. Applying L'Hospital's rule gives

$$E(Y) = \frac{1}{(b-a)}\left[(b^2-a^2) - \frac{b^2-a^2}{2}\right] = \frac{(a+b)}{2}$$

3.12.16 $M_Y^{(1)}(t) = \dfrac{(1-t^2)2e^{2t}-(-2t)e^{2t}}{(1-t^2)^2} = 2\dfrac{(1+t-t^2)e^{2t}}{(1-t^2)^2}$, so $E(Y) = M_Y^{(1)}(0) = 2$.

$$M_Y^{(2)}(t) = \frac{2(1-t^2)^2[(1-2t)e^{2t}+2(1+t-t^2)e^{2t}] - 2(1-t^2)(-2t)2(1+t-t^2)e^{2t}}{(1-t^2)^4}$$

so $M_Y^{(2)}(0) = 6$. Thus $\text{Var}(Y) = E(Y^2) - \mu^2 = 6 - 4 = 2$.

3.12.17 Let $Y = \dfrac{1}{\lambda}V$, where $f_V(y) = ye^{-y}$, $y \geq 0$. Question 3.12.8 establishes that $M_V(t) = \dfrac{1}{(1-t)^2}$. By Theorem 3.12.3(a), $M_Y(t) = M_V(t/\lambda) = 1(1-t/\lambda)^2$.

3.12.18 $M_{Y_1+Y_2+Y_3}(t) = M_{Y_1}(t)M_{Y_2}(t)M_{Y_3}(t) = \left(\dfrac{1}{(1-t/\lambda)^2}\right)^3 = \dfrac{1}{(1-t/\lambda)^6}$

3.12.19 (a) Let X and Y be two Poisson variables with parameters λ and μ, respectively.
Then $M_X(t) = e^{-\lambda+\lambda e^t}$ and $M_Y(t) = e^{-\mu+\mu e^t}$.
$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{-\lambda+\lambda e^t}e^{-\mu+\mu e^t} = e^{-(\lambda+\mu)+(\lambda+\mu)e^t}$.
This last expression is that of a Poisson variable with parameter $\lambda + \mu$, which is then the distribution of $X + Y$.

(b) Let X and Y be two exponential variables, with parameters λ and μ, respectively.

Then $M_X(t) = \dfrac{\lambda}{(\lambda - t)}$ and $M_Y(t) = \dfrac{\mu}{(\mu - t)}$.

$M_{X+Y}(t) = M_X(t)M_Y(t) = \dfrac{\lambda}{(\lambda - t)}\dfrac{\mu}{(\mu - t)}$.

This last expression is not that of an exponential variable, and the distribution of $X + Y$ is not exponential.

(c) Let X and Y be two normal variables, with parameters μ_1, σ_1^2 and μ_2, σ_2^2 respectively.

Then $M_X(t) = e^{\mu_1 t + \sigma_1^2 t^2/2}$ and $M_Y(t) = e^{\mu_2 t + \sigma_2^2 t^2/2}$.

$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\mu_1 t + \sigma_1^2 t^2/2} e^{\mu_2 t + \sigma_2^2 t^2/2} = e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$.

This last expression is that of a normal variable with parameters $\mu_1 + \mu_2$ and σ_1^2 and σ_2^2, which is then the distribution of $X + Y$.

3.12.20 From the moment-generating function of X, we know that it is binomial with $n = 5$ and $p = 3/4$. Then $P(X \le 2) = (1/4)^5 + 5(3/4)(1/4)^4 + 10(3/4)^2(1/4)^3 = 0.104$

3.12.21 Let $S = \sum_{i=1}^{n} Y_i$. Then $M_s(t) = \prod_{i=1}^{n} M_{Y_i}(t) = \left(e^{\mu t + \sigma^2 t^2/2}\right)^n = e^{n\mu t + n\sigma^2 t^2/2}$.

$M_{\bar{Y}}(t) = M_{S/n}(t) = M_S(t/n) = e^{\mu t + (\sigma^2/n)t^2/2}$. Thus $\bar{Y}$ is normal with mean μ and variance σ^2/n.

3.12.22 From the moment-generating function of W, we know that $W = X + Y$, where X is Poisson with parameter 3, and Y is binomial with parameters $n = 4$ and $p = 1/3$. Also, X and Y are independent.
Then $P(W \le 1) = p_X(0)p_Y(0) + p_X(0)p_Y(1) + p_X(1)p_Y(0)$
$= (e^{-3})(2/3)^4 + (e^{-3})4(1/3)(2/3)^3 + (3e^{-3})(2/3)^4 = 0.059$

3.12.23 (a) $M_W(t) = M_{3X}(t) = M_X(3t) = e^{-\lambda + \lambda e^{3t}}$. This last term is not the moment-generating function of a Poisson random variable, so W is not Poisson.

(b) $M_W(t) = M_{3X+1}(t) = e^t M_X(3t) = e^t e^{-\lambda + \lambda e^{3t}}$. This last term is not the moment-generating function of a Poisson random variable, so W is not Poisson.

3.12.24 (a) $M_W(t) = M_{3Y}(t) = M_Y(3t) = e^{\mu(3t) + \sigma^2(3t)^2/2} = e^{(3\mu)t + 9\sigma^2 t^2/2}$.
This last term is the moment-generating function of a normal random variable with mean 3μ and variance $9\sigma^2$, which is then the distribution of W.

(b) $M_W(t) = M_{3Y+1}(t) = e^t M_Y(3t) = e^t e^{\mu(3t) + \sigma^2(3t)^2/2} = e^{(3\mu+1)t + 9\sigma^2 t^2/2}$. This last term is the moment-generating function of a normal random variable with mean $3\mu + 1$ and variance $9\sigma^2$, which is then the distribution of W.