Probability Midterm II

Math 321
10 April 2015

Name: $\underbrace{\hspace{8cm}}_{\text{by writing my name i swear by the honor code}}$

Course Instructor: Shirali Kadyrov

1. (*42 points*) Multiple Choice Questions. Circle your answers.

a. (*7 pts*) If the probability density function of a continuous random variable X is $f_X(x) = \frac{1}{3}x^2$ on $0 \le x \le 3$, then $P(0 \le X \le 2)$ is

A) $\frac{1}{9}$ B) $\frac{8}{9}$ C) $\frac{2}{3}$ D) 1

b. (*7 pts*) If the joint pdf of X, Y is given by $f_{X,Y}(x, y) = cxy^2$ on $0 \le x, y \le 2$ then c is

A) $\frac{3}{16}$ B) $\frac{1}{8}$ C) 4 D) any positive integer greater than 5

c. (*7 pts*) If continuous r.v. X is uniformly distributed over the interval $[2, 5]$, then the expected value of X is

A) 1 B) 3 C) $\frac{7}{2}$ D) $\frac{5}{2}$

d. (*7 pts*) Is it True or False that if $E(XY) = E(X)E(Y)$ then X and Y are independent?

A) True B) False

e. (*7 pts*) If the mgf of a r.v. X is $M_X(t) = e^{t^2/2}$ for $-\infty < t < \infty$ then the Var(X) is

A) $\frac{1}{2}$ B) ∞ C) $\frac{1}{3}$ D) 1

f. (*7 pts*) Which of the following is not a property of a binomial experiment?

A) the experiment consists of a sequence of n identical trials
B) each outcome can be referred to as a success or a failure
C) the probabilities of the two outcomes can change from one trial to the next
D) the trials are independent

2. (*21 points*) Suppose you roll three dice. Compute the following:

a. (*7 pts*) The expected value of the sum of the rolls.

X_i = # on the i-th die.

$X = X_1 + X_2 + X_3$

$E(X_i) = \sum_{k=1}^{6} k \cdot \frac{1}{6} = 3.5$

So, $E(X) = E(X_1) + E(X_2) + E(X_3)$

$= 10.5$

b. (*7 pts*) The expected value of the product of the rolls

From independence we get

$E(X_1 \cdot X_2 \cdot X_3) = E(X_1)E(X_2)E(X_3)$

$= 3.5^3$

c. (*7 pts*) The variance of the sum of the rolls.

From independence we have

$\mathrm{Var}\,X = \mathrm{Var}\,X_1 + \mathrm{Var}\,X_2 + \mathrm{Var}\,X_3$

$E(X_i^2) = \sum_{k=1}^{6} k^2 \cdot \frac{1}{6} = \frac{6 \cdot 7 \cdot 13}{6 \cdot 6} = \frac{91}{6}$

$\mathrm{Var}\,X_i = E(X_i^2) - E(X_i)^2 = \frac{91}{6} - 3.5^2$

So, $\mathrm{Var}\,X = 3 \cdot \left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right) = \frac{105}{12} = 8.75$

3. (*20 points*) The life time X (in hours) of an electronic component follows the distribution $f(x) = e^{-x}$ for $x > 0$. Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail.

a. (*10 pts*) Find the probability of a given single component failure within 3 hours.

$$P(X \leq 3) = \int_0^3 e^{-x} dx = -e^{-x} \Big|_0^3 = 1 - e^{-3}$$

b. (*10 pts*) Find the probability that the equipment will operate for at least 3 hours without failure.

Need to find probability that at most one component fails in 3 hours.

A = event no component fails in 3 hours

$$P(A) = P(X > 3)^3 = (e^{-3})^3 = e^{-9}.$$

B = exactly one fails in 3 hours.

$$P(B) = \binom{3}{1} (1 - e^{-3})^1 \cdot (e^{-3})^2$$

So, $\text{Ans} = P(A) + P(B) = e^{-9} + \binom{3}{1} \cdot (1 - e^{-3}) \cdot e^{-6}$

4. (*20 points*) The weekly demand for petrol, X (in thousands of litres), at a particular service station is a random variable with probability density function $f(x) = 2(1 - \frac{1}{x^2})$ for $1 \leq x \leq 2$. Suppose you have 1.5 thousand litres of petrol in the stocks.

a. (*10 pts*) What is the probability the stock 1.5 thousand litres will not be enough to meet the demand for 1 week?

$$P(X>1.5) = \int_{1.5}^{2} 2\left(1-\frac{1}{x^2}\right)dx =$$

$$= 2x + \frac{2}{x}\Big|_{1.5}^{2} = 2\left(0.5 + \frac{1}{2} - \frac{2}{3}\right)$$

$$= \frac{2}{3}$$

b. (*10 pts*) What is the expected value of the stock left over at the end of one week?

Y = stock left over at the end of one week.

$$Y = \begin{cases} 0 & X \geq 1.5 \\ 1.5 - X & 1 \leq X < 1.5 \end{cases}$$

So,

$$E(Y) = \int_{1.5}^{2} 0\, f(x)\,dx + \int_{1}^{1.5} (1.5-x)\, f(x)\,dx$$

$$= \int_{1}^{3/2} (3-2x)\left(1-\frac{1}{x^2}\right)dx = \int_{1}^{3/2}\left(3-2x-\frac{3}{x^2}+\frac{2}{x}\right)dx$$

$$= 3x - x^2 + \frac{3}{x} + 2\ln x\Big|_{1}^{3/2}$$

$$2\ln 3 - 3 \approx 0.06$$