The flipped continued fraction

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The Flipped Continued Fraction

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Joint work with C. Kraaikamp and V. Masarotto

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The classical case

It is well-known that every real number x can be written as a finite (in case $x \in \mathbb{Q}$) or infinite (*regular*) *continued fraction expansion* (RCF) of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}},$$

where $a_0 \in \mathbb{Z}$ is such that $x - a_0 \in [0, 1)$, i.e. $a_0 = \lfloor x \rfloor$, and $a_n \in \mathbb{N}$ for $n \geq 1$.

The classical case

The *partial quotients* a_n are given by

$$a_n = a_n(x) = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor, \qquad \text{if } T^{n-1}(x) \neq 0,$$

where $T : [0,1) \to [0,1)$ is the continued fraction (or: Gauss) map, defined by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad \text{if } x \neq 0,$$

and $T(0) = 0$.

The classical case

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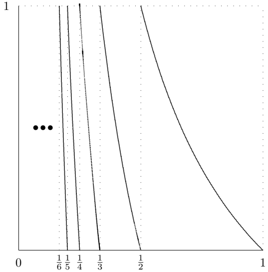


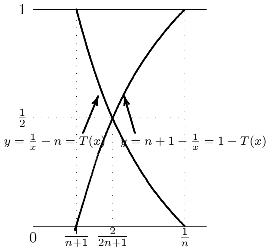
The continued fraction map T.

D-continued fraction or the Flipped CF

Let $D \subset [0,1]$ be a Borel measurable subset of the unit interval, then we define the map $T_D : [0,1) \to [0,1)$ by

$$T_D(x) := \begin{cases} \left\lfloor \frac{1}{x} \right\rfloor + 1 - \frac{1}{x}, & \text{if } x \in D \\ \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, & \text{if } x \in [0,1) \setminus D. \end{cases}$$

D-continued fraction



D-continued fraction

Setting $\varepsilon_1 = \varepsilon_1(x) = \begin{cases} -1, & \text{if } x \in D \\ +1, & \text{if } x \in [0,1) \setminus D, \end{cases}$ and

$$d_1 = d_1(x) = \begin{cases} \lfloor 1/x \rfloor + 1, & \text{if } x \in D \\ \\ \lfloor 1/x \rfloor, & \text{if } x \in [0,1) \setminus D, \end{cases}$$

it follows from definition of T_D that

$$T_D(x) = \varepsilon_1 \left(\frac{1}{x} - d_1 \right).$$

D-continued fraction

Setting for $n \geq 1$ for which $T_D^{n-1}(x) \neq 0$,

$$d_n = d_1(T_D^{n-1}(x)), \quad \varepsilon_n = \varepsilon_1(T_D^{n-1}(x)),$$

we find that

$$x = \frac{1}{d_1 + \varepsilon_1 T_D(x)} = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \cdots + \cfrac{\varepsilon_{n-1}}{d_n + \varepsilon_n T_D^n(x)}}}.$$

D-continued fraction

For each $n \geq 1$,

$$x = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \cdots + \cfrac{\varepsilon_{n-1}}{d_n + \varepsilon_n T_D^n(x)}}}.$$

- One needs to show that as $n \to \infty$, the above converges to

$$x = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \cdots + \cfrac{\varepsilon_{n-1}}{d_n + \ddots}}}.$$

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- The nth D-convergent of x is

$$\frac{p_n}{q_n} = \cfrac{1}{d_1 + \cfrac{\varepsilon_1}{d_2 + \cdots + \cfrac{\varepsilon_{n-1}}{d_n}}}.$$

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D-continued fraction

Using similar methods as in the regular CF, one can show that

- $$x = \frac{p_n + p_{n-1} T_D^n(x)\varepsilon_n}{q_n + q_{n-1} T_D^n(x)\varepsilon_n}.$$
- $\gcd(p_n, q_n) = 1.$
- $p_{n-1}q_n - p_n q_{n-1} = (-1)^n \prod_{k=1}^{n-1} \varepsilon_k = \pm 1.$
- $$x - \frac{p_n}{q_n} = \frac{(-1)^n (\prod_{k=1}^{n} \varepsilon_k) T_D^n(x)}{q_n(q_n + q_{n-1}\varepsilon_n T_D^n(x))}.$$

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- it follows that

$$\left|x - \frac{p_n}{q_n}\right| = \left|\frac{T_D^n(x)}{q_n(q_n + q_{n-1}\varepsilon_n T_D^n(x))}\right| < \frac{1}{|q_n(q_n + q_{n-1}\varepsilon_n T_D^n(x))|} \to 0.$$

D-continued fraction

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Example- Folded α-expansion

In 1997, Marmi, Moussa and Yoccoz modified Nakada's α-expansions to the *folded* or *Japanese* continued fractions, with underlying map

$$\widetilde{T}_\alpha = \left|\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor_\alpha \right|, \qquad \text{for } 0 < x < \alpha, \quad x \neq 0; \qquad \widetilde{T}_\alpha(0) = 0,$$

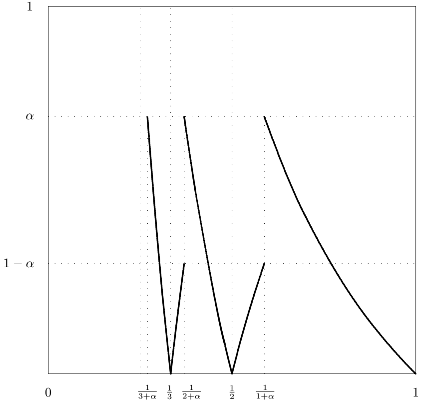
where $\lfloor x \rfloor_\alpha = \min\{p \in \mathbb{Z} : \ x < \alpha + p\}$.

Example- Folded α-expansion

Folded α-expansions can also be described as D-expansions with

$$D = \bigcup_{n=1}^{\infty} \Big(\frac{1}{n+1}, \frac{1}{n+\alpha} \Big];$$

Example- Folded α-expansion



Example- Backward CF

Let $D = [0,1)$, and let $x \in [0,1)$. In this case $[0,1) \setminus D = \emptyset$, so we always use the map

$$T_D = 1 + \left\lfloor \frac{1}{x} \right\rfloor - \frac{1}{x},$$

and we will get an expansion for x of the form

$$x = \cfrac{1}{d_1 + \cfrac{-1}{d_2 + \dots}} = [0; 1/d_1, -1/d_2, \dots];$$

Example- Backward CF

It is a classical result that every $x \in [0,1) \setminus \mathbb{Q}$ has a unique *backward* continued fraction expansion of the form

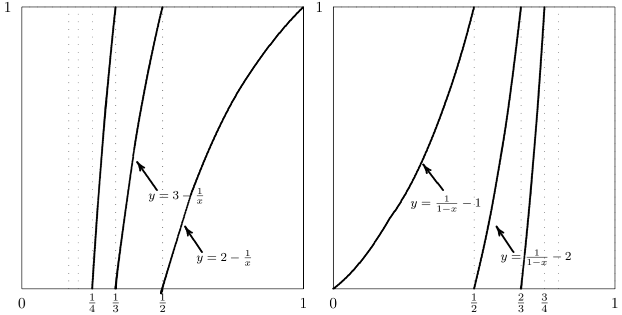
$$x = 1 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \dots}} = [0; -1/c_1, -1/c_2, \dots],$$

where the c_is are all integers greater than 1. This continued fraction is generated by the map

$$T_b(x) = \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor,$$

that we obtain from T_D via the isomorphism $\psi : x \mapsto 1 - x$, i.e., $\psi \circ T_b = T_D \circ \psi$

Example- Backward CF



Example- Odd and Even CF

Setting

$$D := D_{\text{odd}} = \bigcup_{n \text{ even}} \left[\frac{1}{n+1}, \frac{1}{n} \right),$$

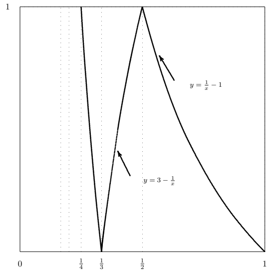
one easily finds that the D-expansion for every $x \in [0,1)$ only has odd partial quotients d_n. In case $D := D_{\text{even}} = D_{\text{odd}}^c$, the partial quotients are always even.

Example- CF without a particular digit

Fix a positive integer ℓ, and suppose that we want an expansion in which the digit ℓ never appears, that is $a_n \neq \ell$ for all $n \geq 1$. Now just take $D = (\frac{1}{l+1}, \frac{1}{l}]$ in order to get an expansion with no digits equal to ℓ.

Example- CF without a particular digit

An expansion with no digit equals 3.



From regular CF to flipped CF- Singularizations

Let a, b be positive integers, $\varepsilon = \pm 1$, and let $\xi \in [0, 1)$. A *singularization* is based on the identity

$$a + \cfrac{\varepsilon}{1 + \cfrac{1}{b + \xi}} = a + \varepsilon + \frac{-\varepsilon}{b + 1 + \xi}.$$

Singularizations

To see the effect of a singularization on a continued fraction expansion, let $x \in [0,1)$, with continued fraction expansion

$$x = [a_0; \varepsilon_0/a_1, \varepsilon_1/a_2, \varepsilon_2/a_3, \dots].$$

and suppose that for some $n \geq 0$ one has

$$a_{n+1} = 1;\ \varepsilon_{n+1} = +1,\ a_n + \varepsilon_n \neq 0$$

Singularization then changes the above continued fraction expansion into

$$[a_0; \varepsilon_0/a_1, \dots, \varepsilon_{n-1}/(a_n + \varepsilon_n), -\varepsilon_n/(a_{n+2} + 1), \dots].$$

Insertions

An insertion is based upon the identity

$$a + \frac{1}{b+\xi} = a + 1 + \cfrac{-1}{1 + \cfrac{1}{b-1+\xi}},$$

where $\xi \in [0,1)$ and a, b are positive integers with $b \geq 2$.

let $x \in [0,1)$, with continued fraction expansion

$$x = [a_0; \varepsilon_0/a_1, \varepsilon_1/a_2, \varepsilon_2/a_3, \ldots].$$

Suppose that for some $n \geq 0$ one has

$$a_{n+1} > 1; \ \varepsilon_n = 1.$$

An insertion 'between' a_n and a_{n+1} will change the above CF into

$$[a_0; \varepsilon_0/a_1, \ldots, \varepsilon_{n-1}/(a_n + 1), -1/1, 1/(a_{n+1} - 1), \ldots].$$

Insertions/Singularizations

Every time we insert between a_n and a_{n+1} we decrease a_{n+1} by 1, i.e. the new $(n+2)$th digit equals $a_{n+1} - 1$. This implies that for every n we can insert between a_n and a_{n+1} at most $(a_{n+1} - 1)$ times.
On the other hand, suppose that $a_{n+1} = 1$ and that we singularize it. Then both a_n and a_{n+2} will be increased by 1, so we can singularize at most one out of two consecutive digits

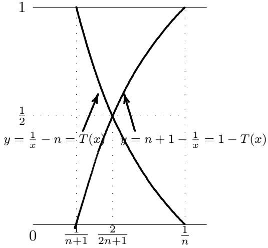
From Regular CF to Flipped CF

For $n \in \mathbb{N}$, let $x \in I_n := \left(\frac{1}{n+1}, \frac{1}{n}\right]$, so that the RCF-expansion of x looks like

$$x = \cfrac{1}{n + \cfrac{1}{\ddots}},$$

and suppose that $x \in I_n \cap D \neq \emptyset$.

From regular to flipped



From Regular CF to Flipped CF via insertions

Suppose $x \in \left(\frac{1}{n+1}, \frac{2}{2n+1}\right] \cap D$, then the regular CF is

- $$x = \cfrac{1}{n + \cfrac{1}{1 + \cfrac{1}{a_3 + \xi}}} = [0; n, 1, a_3, \ldots].$$
- Singularizing the second digit, equal to 1, in the previous expansion we find
  $$x = \cfrac{1}{n + 1 + \cfrac{-1}{a_3 + 1 + \xi}} = [0; 1/(n+1), -1/(a_3+1), \ldots].$$

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From Regular CF to Flipped CF via insertions

We now look at the D-CF of x.

- We have

$$T_D(x) = n + 1 - \frac{1}{x} = 1 - T(x) = 1 - \frac{1}{1 + \frac{1}{a_3 + \xi}} = \frac{1}{a_3 + 1 + \xi},$$

- Thus the D-expansion of x is

$$x = \frac{1}{n + 1 + \frac{-1}{a_3 + 1 + \xi}}$$

- T_D acts as a singularization on $\left(\frac{1}{n+1}, \frac{2}{2n+1}\right] \cap D$.

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From Regular CF to Flipped CF via insertions

Suppose $x \in \left(\frac{2}{2n+1}, \frac{1}{n}\right] \cap D$.

- RCF-expansion of x is given by

$$x = \cfrac{1}{n + \cfrac{1}{a_2 + \xi}},$$

where $\xi \in [0,1]$ and with $a_2 \geq 2$ because $T(x) \leq 1/2$.

- An insertion after the first partial quotient yields

$$x = \cfrac{1}{n + 1 + \cfrac{-1}{1 + \cfrac{1}{a_2 - 1 + \xi}}}.$$

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From Regular CF to Flipped CF via insertions

Since $x \in (1/(n+1), 1/n] \cap D$, the D-expansion of x is given by

$$x = \frac{1}{n + 1 + -1(T_D(x))}.$$

- Computing $T_D(x)$ we find

$$T_D(x) = 1 - T(x) = 1 - \frac{1}{a_2 + \xi} = \frac{1}{1 + \cfrac{1}{a_2 - 1 + \xi}},$$

- so that the D-expansion of x is

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- Thus we see that T_D acts as an insertion on $\left(\frac{2}{2n+1}, \frac{1}{n}\right] \cap D$.

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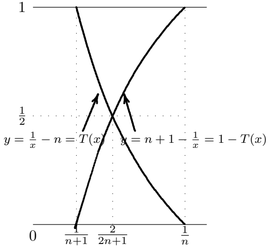
From regular to flipped



Theorem: Let x be a real irrational number with RCF-expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}},$$

and with *tails* $t_n = [0; 1/a_{n+1}, 1/a_{n+2}, \dots]$. Let D be a measurable subset of $[0,1)$. Then the following algorithm yields the D-expansion of x:

(1) Let $m := \inf\{m \in \mathbb{N} \cup \{\infty\} : \ t_m \in D \text{ and } \varepsilon_m = 1\}$. In case $m = \infty$, the RCF-expansion of x is also the D-expansion of x. In case $m \in \mathbb{N}$:

- (i) If $a_{m+2} = 1$, singularize the digit a_{m+2} in order to get

$$x = [a_0; \dots, 1/(a_{m+1}+1), -1/a_{m+3}, \dots].$$

- (ii) If $a_{m+2} \neq 1$, insert $-1/1$ after a_{m+1} to get

$$x = [a_0; \dots, 1/a_{m+1} + 1, -1/1, 1/(a_{m+2} - 1), \dots].$$

(2) Replace the RCF-expansion of x with the continued fraction obtained in $[(1)]$, and let t_n denote the new tails. Repeat the above procedure.

Quadratic Irrationals

A number x is called *quadratic irrational* if it is a root of a polynomial $ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$, $a \neq 0$, and $b^2 - 4ac$ not a perfect square (i.e., if x is an irrational root of a quadratic equation).

Quadratic irrationals and regular CF

Theorem: A number x is a quadratic irrational number if and only if x has an eventually periodic regular continued fraction expansion.

we say that a D-expansion of x is *purely periodic* of period-length m, if the initial block of m partial quotients is repeated throughout the expansion, that is, if $a_{km+1} = a_1, \ldots, a_{(k+1)m} = a_m$, and $\varepsilon_{km+1} = \varepsilon_1, \ldots, \varepsilon_{(k+1)m} = \varepsilon_m$ for every $k \geq 1$. The notation for such a continued fraction is

$$x = [a_0; \varepsilon_0/\overline{a_1, \varepsilon_1/a_2, \ldots, \varepsilon_{m-1}/a_m, \varepsilon_m}].$$

An *(eventually) periodic* continued fraction consists of an *initial block* of length $n \geq 0$ followed by a *repeating block* of length m and it is written as

$$x = [a_0; \varepsilon_0/a_1, \varepsilon_1/a_2, \ldots, \varepsilon_{n-1}/a_n, \varepsilon_n/\overline{a_{n+1}, \ldots, \varepsilon_{n+m-1}/a_{n+m}, \varepsilon_{n+m}}].$$

Quadratic irrationals and flipped CF

Theorem: Let D be a measurable subset in the unit interval. Then a number x is a quadratic irrational number if and only if x has an eventually periodic D-expansion.

- The proof is based on the result for the regular CF, together with the fact that a D-expansion is obtained from the regular CF by singularizations and insertions.

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Invariant measures

Theorem: Suppose D is a countable union of disjoint intervals, then T_D admits at most a finite number of ergodic exact T_D-invariant measures absolutely continuous with respect to Lebesgue measure.

- The proof of this result relies on a Theorem by Rychlik where he characterized ergodic measures of a certain family of piecewise continuous maps.

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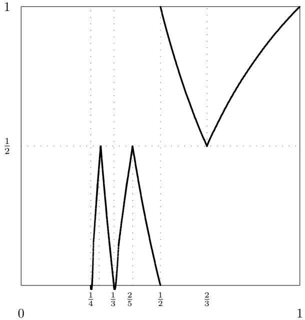
Some examples

Let $D = (\frac{2}{3}, 1) \cup \bigcup_{n=2}^{\infty} \left(\frac{1}{n+1}, \frac{2}{2n+1} \right]$



Some examples

There are two ergodic components absolutely continuous with respect to Lebesgue measure. One is finite with support $[0, 1/2]$ (this continued fraction is in fact the folded nearest integer continued fraction), and the other is σ-finite with support $(1/2, 1)$ (this one is in essence Ito's mediant map)

Suppose $D = \bigcup_{i=0}^{\infty} \left(\frac{1}{n_i+1}, \frac{1}{n_i} \right]$, where $(n_i)_{i \geq 0}$ is a sequence of positive integers.

- $[0,1)$ is the only T_D forward invariant set. Hence, T_D admits a unique ergodic invariant measure equivalent to Lebesgue measure on $[0,1)$. Furthermore, it is finite if and only if D doesn't contain 1, and σ-finite infinite if $1 \in D$.

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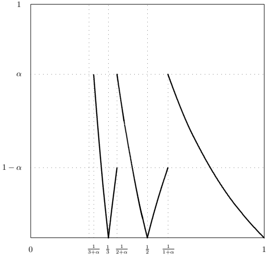
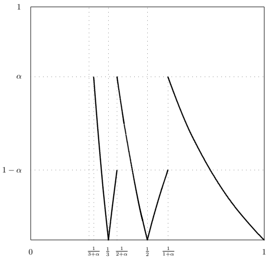
Some examples

Let $\alpha \in (0,1)$, and suppose $D = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n+\alpha} \right]$



Some examples

T_D has one ergodic component absolutely continuous with respect to Lebesgue measure, which is finite and with support the interval $[0, \max\{\alpha, 1-\alpha\})$.



- Finding the density of an invariant measure is in general an extremely hard problem.
- To get an idea of the density, we use Birkhoff's Ergodic Theorem: for a measurable set A, and for a.e. x

$$\mu(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_A(T^i(x)),$$

where 1_A is the characteristic function of A.
- We make histograms by counting the number of times that the orbit of a point lies in a particular interval.

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Simulations of invariant densities

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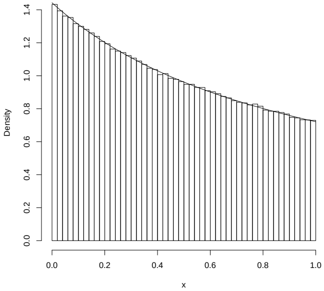
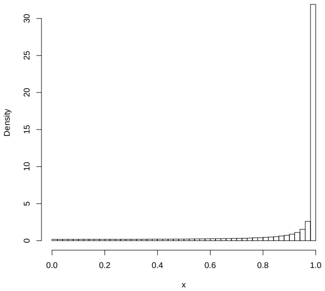
Invariant Density

$D = [1/4, 1/3)$



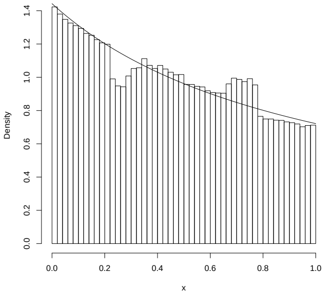
Invariant Density

$D = [1/2, 1)$



Invariant Density

$D = [0.3, 0.45)$



The End