ASSIGNMENT-8

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1 QUESTION No-2.26(Matrices)

Using elementary transformations, find the inverse of each of the matrices:

1) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

2 Solution

1) Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.0.1}$$

The augmented matrix $[A|I]$ is as given below:-

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.0.2}$$

We apply the elementary row operations on $[A|I]$ as follows :-

$$[A|I] = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.0.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.0.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right) \tag{2.0.5}$$

By performing elementary transformations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|A]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\therefore \mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.0.6}$$

2) QR decomposition of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Let α and β be the column vectors of given matrix $\mathbf{A}$,

$$\alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.7}$$

we can express these as

$$\alpha = k_1 \mathbf{u_1} \tag{2.0.8}$$

$$\beta = r_1 \mathbf{u_1} + k_2 \mathbf{u_2} \tag{2.0.9}$$

Where,

$$k_1 = \|\alpha\| \tag{2.0.10}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{2.0.11}$$

$$r_1 = \frac{\mathbf{u_1}^T \beta}{\|\mathbf{u_1}\|^2} \tag{2.0.12}$$

$$\mathbf{u_2} = \frac{\beta - r_1 \mathbf{u_1}}{\|\beta - r_1 \mathbf{u_1}\|} \tag{2.0.13}$$

$$k_2 = \mathbf{u_2}^T \beta \tag{2.0.14}$$

From (2.0.8) and (2.0.9) we can write:

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{A} = \mathbf{QR} \tag{2.0.16}$$

From the above equation we can see that $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an Orthogonal matrix

$$\mathbf{Q^T Q} = \mathbf{I} \tag{2.0.17}$$

Now by using (2.0.10) to (2.0.14) we get:

$$\implies k_1 = \left\|\begin{pmatrix}2\\1\end{pmatrix}\right\| \tag{2.0.18}$$

$$\therefore k_1 = \sqrt{5} \tag{2.0.19}$$

$$\implies \mathbf{u_1} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} \tag{2.0.20}$$

$$\therefore \mathbf{u_1} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.0.21}$$

$$\implies r_1 = \frac{\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}{1} \tag{2.0.22}$$

$$\therefore r_1 = \frac{3}{\sqrt{5}} \tag{2.0.23}$$

$$\implies \mathbf{u_2} = \frac{\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{\sqrt{5}}\begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{\sqrt{5}}\begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}\right\|} \tag{2.0.24}$$

$$\therefore \mathbf{u_2} = \begin{pmatrix}\frac{-1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix} \tag{2.0.25}$$

$$\implies k_2 = \begin{pmatrix}\frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{2.0.26}$$

$$\therefore k_2 = \frac{1}{\sqrt{5}} \tag{2.0.27}$$

From equations (2.0.15) and (2.0.16) the obtained **QR** decomposition is

$$\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\\\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \frac{3}{\sqrt{5}}\\0 & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.0.28}$$