Bopha . K. Bopha.

23/11/2022

RCS 112 :DISCRETE STRUCTURES -3

11/23/2022

GRAPHS in general

- The word graph is used in mathematics in two quite distinct senses.
- You probably met it for the first time in the context of graphs of functions;
  - ➢ the usual curves that we draw in the plane to represent the behavior of real-valued functions.
- In the context of combinatorics, graph has a quite different meaning, which is now explained.

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- In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc
  - One practical example: The link structure of a website could be represented by a directed graph
- Graph theory is also used to study molecules in chemistry and physics
- Graph theory is also widely used in sociology as a way, for example, to measure actors' prestige through the use of social network analysis software
- Likewise, graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or habitats) and the edges represent migration paths, or movement between the regions.
  - This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

study qns

- How can we lay cable at minimum cost to make every telephone reachable from every other?
- What is the fastest route from the national capital to each state capital?
- How can n jobs be filled by n people with maximum total utility?
- What is the maximum flow per unit time from source to sink in a network of pipes?

Edg means lines
Vertex means node

study qns

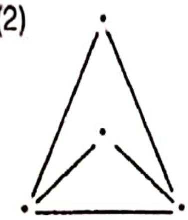
- How many layers does a computer chip need so that wires in the same layer don't cross?
- How can the season of a sports league be scheduled into the minimum number of weeks?
- In what order should a traveling salesman visit cities to minimize travel time?
- Can we color the regions of every map using four colors so that neighboring regions receive different colors?

❖These and many other practical problems involve graph theory.

- A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge and two vertices (not necessarily distinct) called its endpoints.
- Or
- A graph G consists of a set of objects V = {v_1, v_2, v_3, ...} called vertices (also called **points** or **nodes**) and other set E = {e_1, e_2, e_3, ...} whose elements are called edges (also called **lines** or **arcs**).

Here are some examples of graphs:

(1)

(2)

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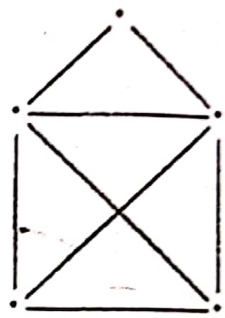


How to draw graph !!!

- In general, we visualize graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing the endpoints of the edge.
- Points where the lines in our pictures cross are not necessarily vertices of the graph:
- In the fourth example above, there are only five vertices, and two edges cross at a point where there is no vertex of the graph.
- If the vertex u is connected to the vertex v, then v is also connected to u, and hence this relation of being connected is a symmetric relation.

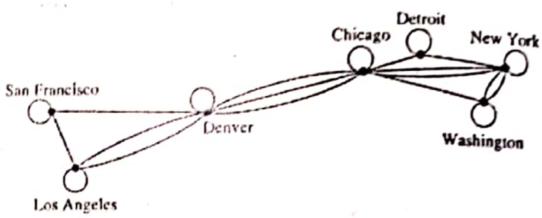
Definition

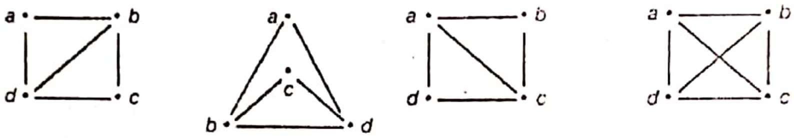
- A graph G = (V, E) consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.
- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph. OR
- Graphs with no loops and no multiple edges are then called simple graphs. Or A simple graph is a graph having no loops or multiple edges.
- A loop is an edge whose endpoints are equal.
- Multiple edges are edges having the same pair of endpoints.

Definition

- Graphs that may have multiple edges connecting the same vertices are called multigraphs.
- When there are m different edges associated to the same unordered pair of vertices {u, v}, we also say that {u, v} is an edge of multiplicity m.
- Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices, are called pseudo graphs.
- Two vertices u, v e V are connected in the graph if and only if u and v are related by the relation E. We write this as uEv.
- The graph is Finite if the set, V, of vertices is finite.
- We will only be concerned with finite graphs,

Definition…





(1) (2) (3) (4)

Undirected/directed graphs.

- For example, in a computer network, some links may operate in only one direction (such links are called single duplex lines). This may be the case if there is a large amount of traffic sent to some data centers, with little or no traffic going in the opposite direction.
- A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E.
- In other words, if each edge of the graph G has a direction then the graph is called **directed graph.**
- Each directed edge is associated with an ordered pair of vertices.
- The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.
- When we depict a directed graph with a line drawing, we use an arrow pointing from u to v to indicate the direction of an edge that starts at u and ends at v.

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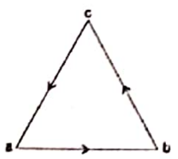
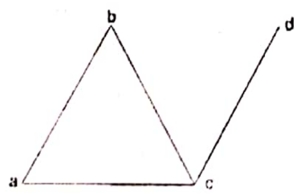
Undirected/directed graphs…

- A directed graph may contain loops, multiple directed edges, directed edges that connect vertices u and v in both directions;
- Note that we obtain a directed graph when we assign a direction to each edge in an undirected graph.
- When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.
- For some models we may need a graph where some edges are undirected, while others are directed.
- A graph with both directed and undirected edges is called a mixed graph.
- For example, a mixed graph might be used to model a computer network containing links that operate in both directions and other links that operate only in one direction.

Undirected/directed graphs…

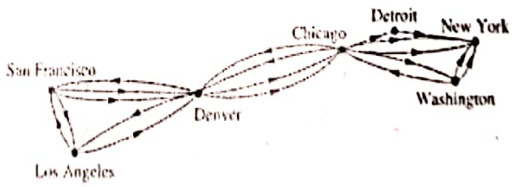




Terminology for the various types of graphs - summary

TABLE 1 Graph Terminology.			
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Example



- Although the terminology used to describe graphs may vary, three key questions can help us understand the structure of a graph:
- Are the edges of the graph undirected or directed (or both)?
- If the graph is undirected, are multiple edges present that connect the same pair of vertices?
- If the graph is directed, are multiple directed edges present?
- Are loops present?

Answering such questions helps us understand graphs. It is less important to remember the particular terminology used.

Graph Models

- Graphs are used in a wide variety of models.
- Niche Overlap Graphs in Ecology Graphs are used in many models involving the interaction of different species of animals.
- For instance, the competition between species in an ecosystem can be modeled using a niche overlap graph.
- Each species is represented by a vertex.
- An undirected edge connects two vertices if the two species represented by these vertices compete (that is, some of the food resources they use are the same).
- A niche overlap graph is a simple graph because no loops or multiple edges are needed in this model.

Graph Models...

- Acquaintanceship Graphs We can use graph models to represent various relationships between people.
  - For example, we can use a simple graph to represent whether two people know each other, that is, whether they are acquainted.
- Influence Graphs In studies of group behavior it is observed that certain people can influence the thinking of others.
  - A directed graph called an influence graph can be used to model this behaviour.
- The Web Graph The World Wide Web can be modeled as a directed graph where each Web page is represented by a vertex and where an edge starts at the Web page a and ends at the Web page b if there is a link on a pointing to b.

Basic Terminology

- Adjacent vertices, incident
  - Two vertices u and v in an undirected graph G are called adjacent (or neighbours) in G if u and v are endpoints of an edge of G.
- If e is associated with {u, v}, the edge e is called incident with the vertices u and v.
- The edge e is also said to connect u and v. The vertices u and v are called endpoints of an edge associated with {u, v}.

Basic Terminology

- Degree of a vertex denoted as deg(v).

❖ The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. Or

- The number of edges incident with a vertex v in a graph without loops is called the degree or valency of v

- Example 1
- What are the degrees of the vertices in the graphs G and H below?

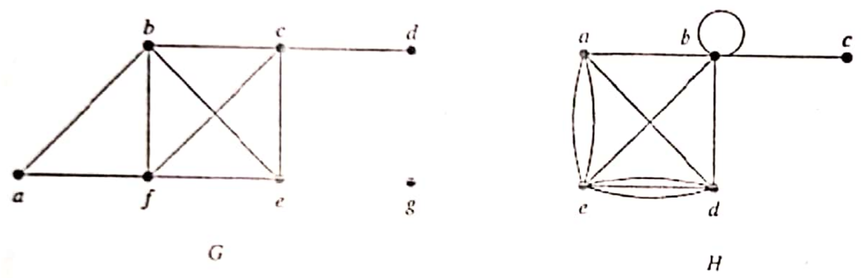
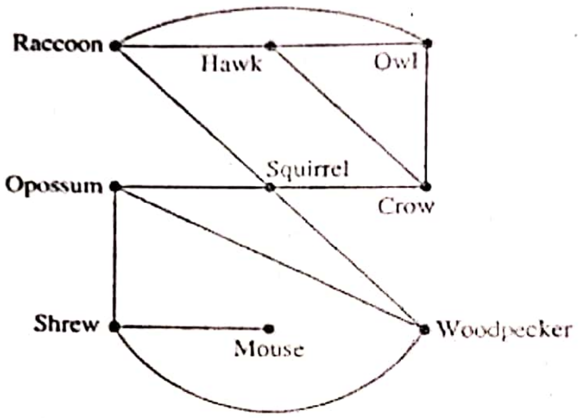


FIGURE 1 The Undirected Graphs G and H.

- Solution: In G, deg(a) = 2, deg(b) = deg(c) = deg(f) = 4, deg(d) = 1, deg(e) = 3, and deg(g) = 0.
- In H, deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, and deg(d) = 5.
- A vertex of degree zero is called isolated. Example vertex g in G
- Thus an isolated vertex is not adjacent to any vertex.
- A vertex is pendant if and only if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other vertex. Vertex d in graph G in Example 1 is pendant.

- Example 2
- What does the degree of a vertex in a niche overlap graph (introduced in Example above) represent? Which vertices in this graph are pendant and which are isolated



- Solution: There is an edge between two vertices in a niche overlap graph if and only if the two species represented by these vertices compete.
- Hence, the degree of a vertex in a niche overlap graph is the number of species in the ecosystem that compete with the species represented by this vertex.
- A vertex is pendant if the species competes with exactly one other species in the ecosystem.
- Finally, the vertex representing a species is isolated if this species does not compete with any other species in the ecosystem.
- The vertex representing a species is pendant if this species competes with only one other species.

- The mouse is the only species represented by a pendant vertex, because the mouse competes only with the shrew and all other species compete with at least two other species.
- There are no isolated vertices in the graph in Figure above because every species in this ecosystem competes with at least one other species.

- The sum of the degrees of the vertices of a graph is twice the number of edges.
- Because each edge contributes twice to the sum of the degrees, once at each end
- In any graph the sum of the vertex degrees is even.

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

Example

- How many edges are there in a graph with 10 vertices each of degree six?

❖Solution: Because the sum of the degrees of the vertices is 6.10 = 60, it follows that 2e = 60. Therefore, e = 30.

DEFINITION

- When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.
- The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v).
- The initial vertex and terminal vertex of a loop are the same.

DEFINITION

- In a graph with directed edges the in-degree of a vertex v, denoted by deg-(v), is the number of edges with v as their terminal vertex.
- The out-degree of v, denoted by deg+(v), is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Example

- Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.

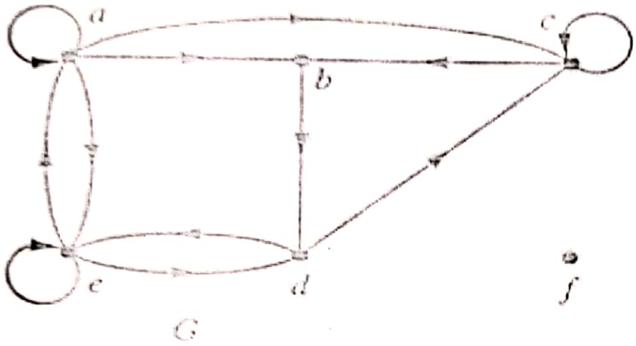


FIGURE 2 The Directed Graph G.

Solution: The in-degrees in G are $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, and $\deg^-(f) = 0$. The out-degrees are $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, and $\deg^+(f) = 0$. ◀

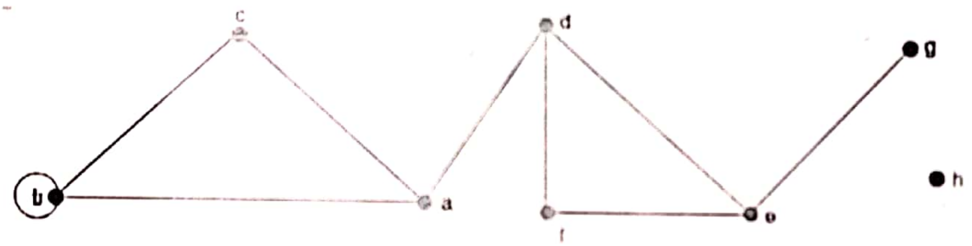
- Because each edge has an initial vertex and a terminal vertex, the sum of the indegrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same.
- Both of these sums are the number of edges in the graph.
- A vertex with zero in-degree is called a sink and a vertex with zero out-degree is called a source.
- try :Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

- Find the degree sequence of the following graph.



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handshaking lemma

- In any graph the sum of all the vertex- degrees is an even number.
- The degree sequence of a graph is the list of vertex degrees, usually written in non increasing order, as $d_1 \geq . . . \geq d_n$.

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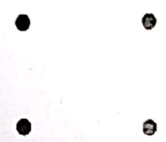
Some Special Simple Graphs

❖ Null graph

A null graph is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

- Null graph is denoted on n vertices by N_n



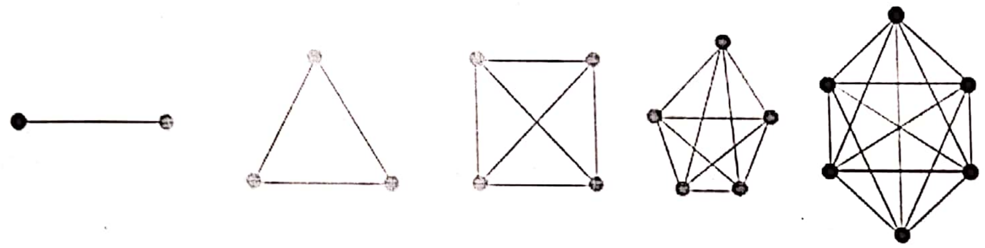
Some Special Simple Graphs

❖ Complete Graphs

- A simple graph G is said to be complete if every vertex in G is connected with every other vertex.
- i.e if G contains exactly one edge between each pair of distinct vertices.

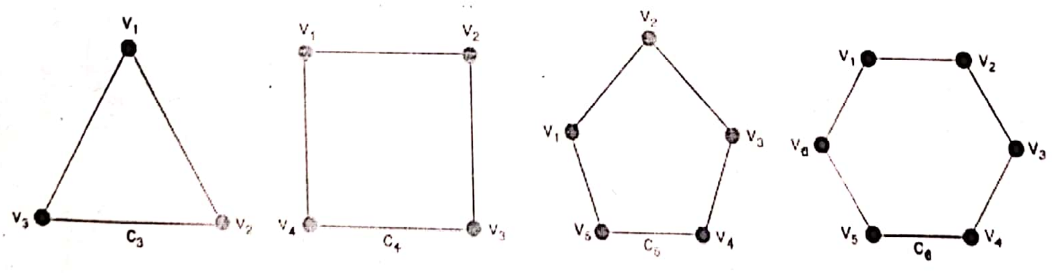
A comple graph is usually denoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges.

- The graphs Kn for n = 1, 2, 3, 4, 5, 6 are show



CIRCLES

The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots\ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ..., $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



Some Special Simple Graphs...

❖Wheels

- We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for n≥ 3, and connect this new vertex to each of the n vertices in C_n, by new edges.

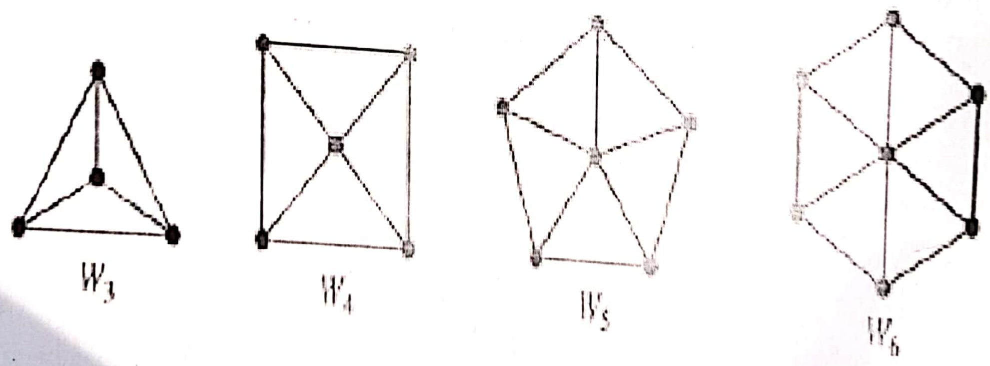


FIGURE 5 The Wheels W_3, W_4, W_5, and W_6.

❖**Regular graph**

- A graph in which all vertices are of equal degree, is called a **regular graph**.
- If the degree of each vertex is r, then the graph is called a regular graph of degree r.
- Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree n – 1.
- Also, note that, if G has n vertices and is regular of degree r, then G has $\left(\frac{1}{2}\right)$r n edges.

Some Special Simple Graphs...

❖Bipartite Graphs

- A simple graph G is called bipartite if its vertex set V can be partitioned into two disjointSets VI and V2 such that every edge in the graph connects a vertex in VI and a vertex in V2 (so that no edge in G connects either two vertices in VI or two vertices in V2).
- When this condition holds, we call the pair (V1, V2) a bipartition of the vertex set V of G.

Bipartite Graphs

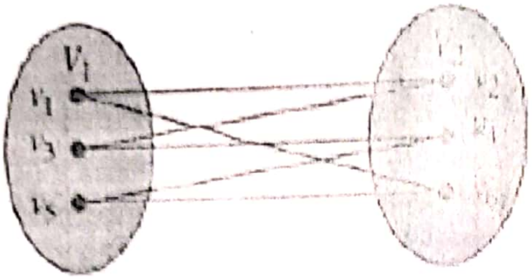


FIGURE 7 Showing That C_6 Is Bipartite.

C 6 is bipartite, as shown in Figure above, because its vertex set can be partitioned into the two sets VI = {v1, v3, v5} and V2 = {V2, V4, V6}, and every edge of C6 connects a vertex in VI and a Vertex in V2.

Bipartite Graphs

- K3 is not bipartite.
- To verify this, note that if we divide the vertex set of K3 into two disjoint sets, one of the two sets must contain two vertices.
- If the graph were bipartite, these two vertices could not be connected by an edge, but in K3 each vertex is connected to every other vertex by an edge.

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Bipartite Graphs

- Are the graphs G and H displayed in Figure 8 bipartite?

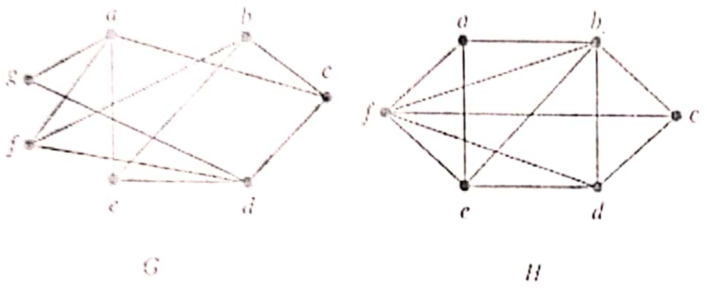


FIGURE 8 The Undirected Graphs G and H.

Bipartite Graphs

❖Solution:

- Graph G is bipartite because its vertex set is the union of two disjoint sets, {a, b, d} and {c, e, f, g}, and each edge connects a vertex in one of these subsets to a vertex in the other subset.
- (Note that for G to be bipartite it is not necessary that every vertex in {a, b, d} be adjacent to every vertex in {c, e, f, g}. For instance, band g are not adjacent.)
- Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset.
- (The reader should verify this by considering the vertices a, b, and f.)

Bipartite Graphs

- THEOREM
- A simple graph is bipartite if and only if it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

❖Example

- Using the Theorem above determine whether the graphs G and H above are bipartite.
- Solution: We first consider the graph G. We will try to assign one of two colours, say red and blue, to each vertex in G so that no edge in G connects a red vertex and a blue vertex.
- Without loss of generality we begin by arbitrarily assigning red to a.
- Then, we must assign blue to c, e, f, and g, because each of these vertices is adjacent to a.
- To avoid having an edge with two blue endpoints, we must assign red to all the vertices adjacent to either c, e, f, or g.
- This means that we must assign red to both band d (and means that a must be assigned red, which it already has been). We have now assigned colours to all vertices, with a, b, and d red and c, e, f, and g blue.
- Checking all edges, we see that every edge connects a red vertex and a blue vertex. Hence, by
- Theorem 4 the graph G is bipartite.

- Next, we will try to assign either red or blue to each vertex in H so that no edge in H connects a red vertex and a blue vertex.
- Without loss of generality we arbitrarily assign red to a.
- Then, we must assign blue to b, e, and f, because each is adjacent to a.
- But this is not possible because e and f are adjacent, so both cannot be assigned blue.
- This argument shows that we cannot assign one of two colours to each of the vertices of H so that no adjacent vertices are assigned the same colour.
- It follows by Theorem 4 that H is not bipartite.
- Theorem 4 is an example of a result in the part of graph theory known as graph colourings.
- Graph colourings is an important part of graph theory with important applications.

❖Bipartite Graphs and Matching

- Bipartite graphs can be used to model many types of applications that involve matching the elements of one set to elements of another
- Example Job Assignments, marriages etc

- END

RCS 112

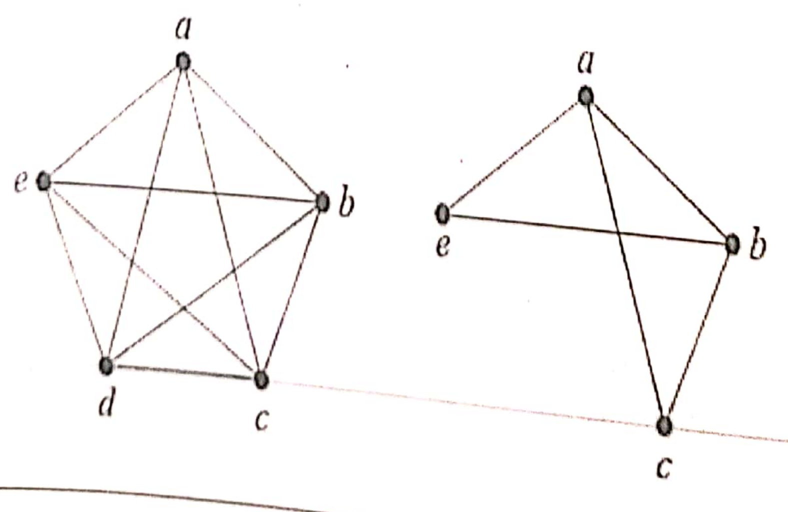
L2

SUBGRAPH

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

Given a set of vertices of a graph, we can form a sub graph of this graph with these vertices and the edges of the graph that connect them.

Let $G = (V, E)$ be a simple graph. The subgraph induced by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.



- The graph G shown in Figure above is a subgraph of K5.
- If we add the edge connecting c and e to G, we obtain the subgraph induced by W = {a, b, c, e}.

REMOVING OR ADDING EDGES OF A GRAPH

- Given a graph G = (V ,E) and an edge e ∈ E, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by G − e, has the same vertex set V as G. Its edge set is E − e.
- Hence, G − e = (V ,E − {e}).
- We can also add an edge e to a graph to produce a new larger graph when this edge connects two vertices already in G.
- We denote by G + e the new graph produced by adding a new edge e, connecting two previously nonincident vertices, to the graph G Hence, G + e = (V ,E ∪ {e}).
- The vertex set of G + e is the same as the vertex set of G and the edge set is the union of the edge set of G and the set {e}.

REMOVING VERTICES FROM A GRAPH

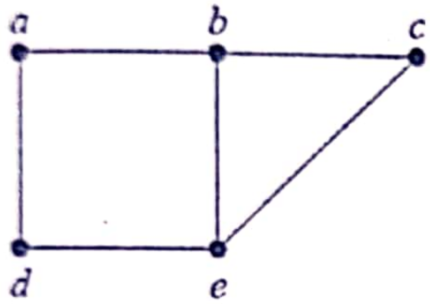
- When we remove a vertex v and all edges incident to it from G = (V ,E), we produce a subgraph, denoted by G – v.
- Observe that G – v = (V – v,E'), where E' is the set of edges of G not incident to v.
- Similarly, if V' is a subset of V, then the graph G – V ' is the subgraph (V – V' ,E'), where E' is the set of edges of G not incident to a vertex in V'

GRAPH UNIONS

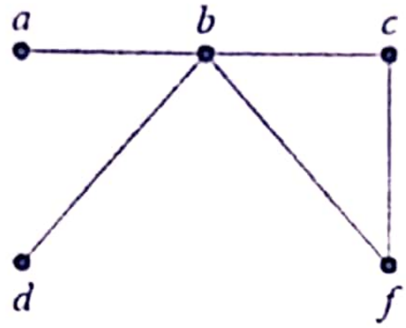
The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example

- Find the union of the graphs G1 and G2 shown in Figure below



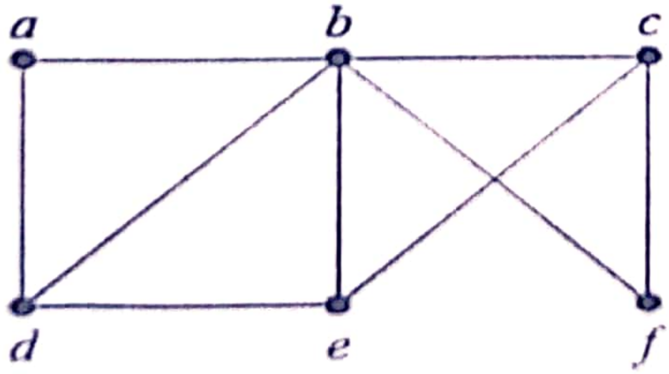
G_1



G_2

Soln

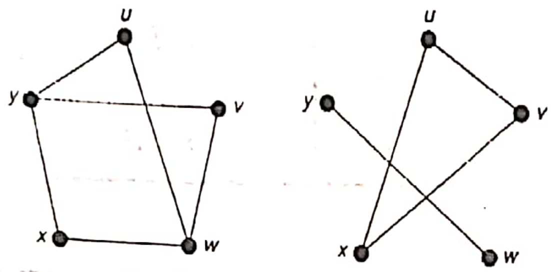
- The vertex set of the union G1 ∪ G2 is the union of the two vertex sets, namely, {a, b, c, d, e, f }.
- The edge set of the union is the union of the two edge sets



$G_1 \cup G_2$

The complement of a simple graph

- If G is a simple graph with vertex-set V(G), its complement $\overline{G}$ is the simple graph with vertex-set V(G) in which two vertices are adjacent if and only if they are not adjacent in G. Note that the complement of $\overline{G}$ is G



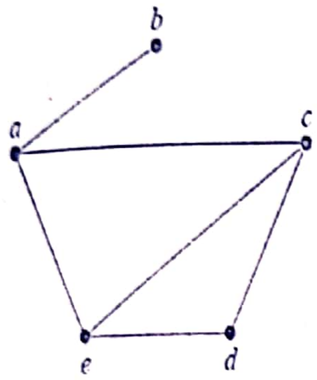
Representing Graphs and Graph Isomorphism

❖ Representing Graphs

- One way to represent a graph without multiple edges is to list all the edges of this graph.
- Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

Example

- Use adjacency lists to describe the simple graph below



A Simple Graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	*b, c, e*
b	*a*
c	*a, d, e*
d	*c, e*
e	*a, c, d*

Example

- Represent the directed graph shown in Figure below by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

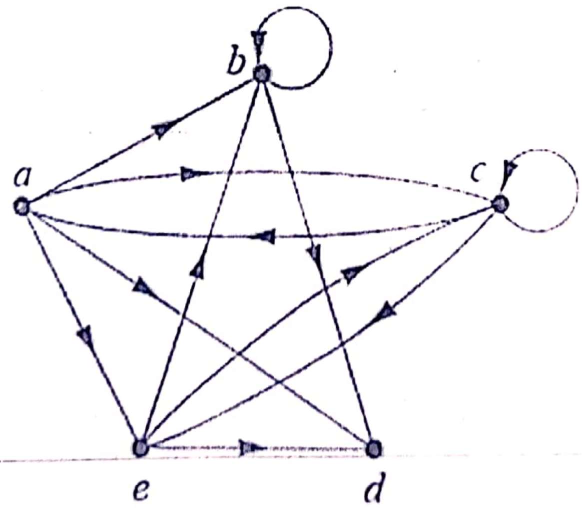


TABLE 2. An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

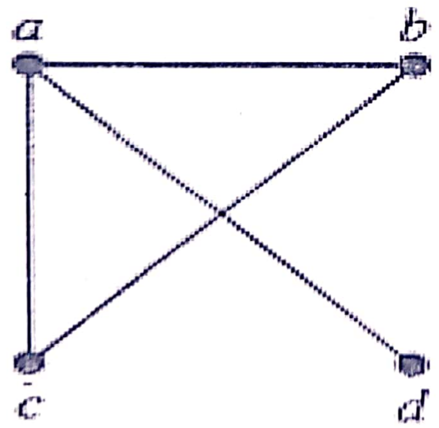
Adjacency Matrices

- Carrying out graph algorithms using the representation of graphs by lists of edges, or by adjacency lists, can be cumbersome if there are many edges in the graph.
- To simplify computation, graphs can be represented using matrices.
- Two types of matrices commonly used to represent graphs will be presented here.
- One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges
- Suppose that G = (V, E) is a simple graph where |V| = n. Suppose that the vertices of G are listed arbitrarily as v1, v2. . . v n .

- The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the n x n zero-one matrix with 1 as its $(i, j)^{th}$ entry when v_i and V_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is A = $[a_{ij}]$' then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

- Example Use an adjacency matrix to represent the graph



- Solution: We order the vertices as a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

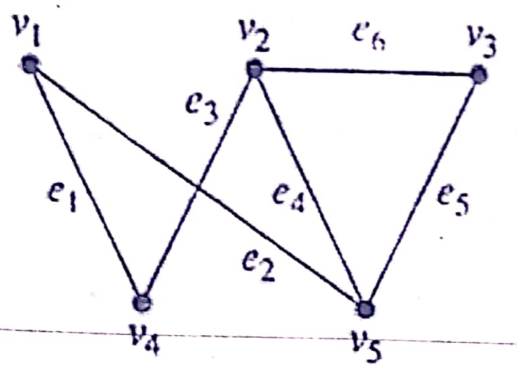
- The adjacency matrix of a simple graph is symmetric, that is, aij = a ji, because both of these entries are 1 when Vj and Vi are adjacent, and both are 0 otherwise.
- Furthermore, because a simple graph has no loops, each entry aij i = 1, 2, 3. . . n, is o.

Incidence Matrices

Another common way to represent graphs is to use incidence matrices. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

- Example; Represent the graph shown in Figure below with an incidence matrix.



Solution:

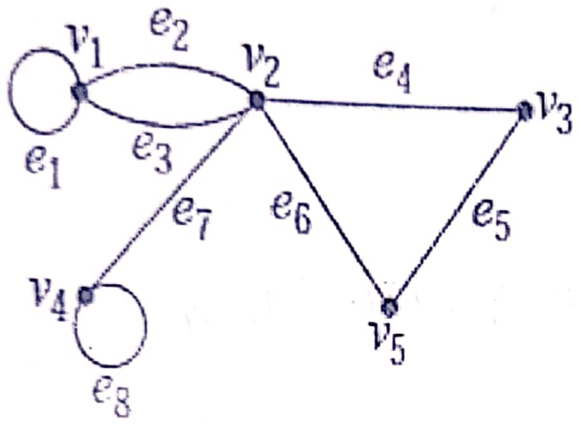
- The incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} .$$

Incidence Matrices

- Incidence matrices can also be used to represent multiple edges and loops.
- `Multiple edges are represented in the incidence matrix using columns with identical entries, because these edges are incident with the same pair of vertices.
- Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

- Example -Represent the pseudograph shown in Figure below using an incidence matrix.



Solution: The incidence matrix for this graph is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Work to be done Individual

- How to find the **INCIDENCE MATRIX and ADJACENCY MATRIX OF A DIGRAPH**

Individual work
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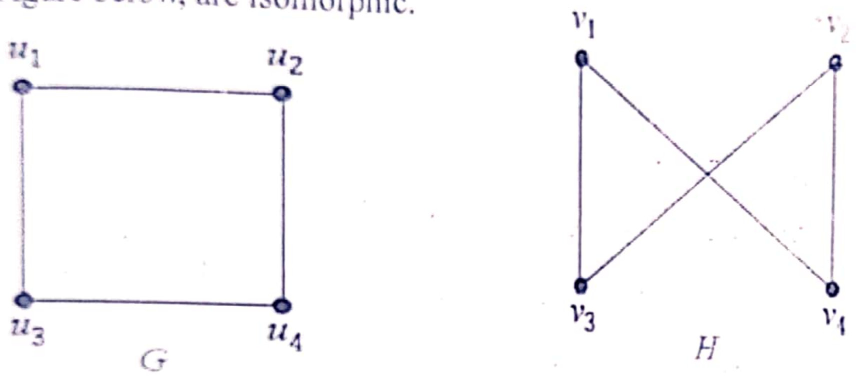
Isomorphism of Graphs

- Different compounds can have the same molecular formula but can differ in structure.
- Such compounds can be represented by graphs that cannot be drawn in the same way.

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

- Example Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure below, are isomorphic.

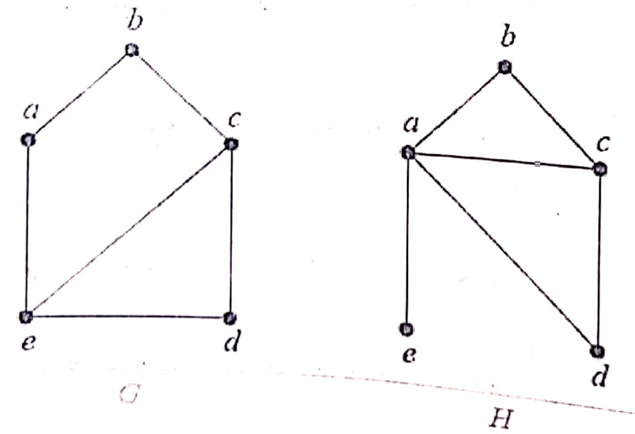


- *Solution:* The function f with $f(u1) = v1$, $f(u2) = v4$, $f(u3) = v3$, and $f(u4) = v2$ is a one-to- one correspondence between V and W.
- To see that this correspondence preserves adjacency, note that adjacent vertices in G are $u1$ and $u2$, $u1$ and $u3$, $u2$ and $u4$, and $u3$ and $u4$, and each of the pairs $f(u1) = v1$ and $f(u2) = v4$, $f(u1) = v1$ and $f(u3) = v3$, $f(u2) = v4$ and $f(u4) = v2$, and $f(u3) = v3$ and $f(u4) = v2$ consists of two adjacent vertices in H.

Determining whether Two Simple Graphs are Isomorphic

- Isomorphic simple graphs
  - Must have the same number of vertices, because there is a one-to-one correspondence between the sets of vertices of the graphs.
  - Must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges.
  - They must have the same degree sequence i.e the degrees of the vertices must be the same.

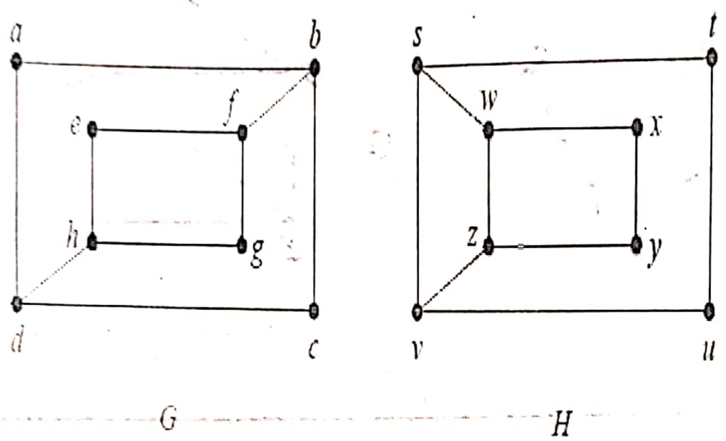
- Show that the graphs displayed in Figure below are not isomorphic.



- *Solution:* Both G and H have five vertices and six edges.
- However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one.
- It follows that G and H are not isomorphic.

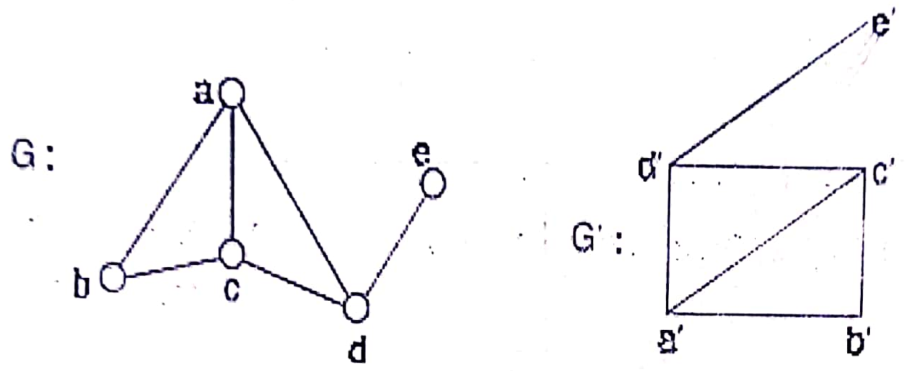
- The number of vertices, the number of edges, and the number of vertices of each degree are all invariants under isomorphism.
- If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic.
- However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic.
- There are no useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.

- Example Determine whether the graphs shown in Figure below are isomorphic



- The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three.
- Because these invariants all agree, it is still conceivable that these graphs are isomorphic.
- However, G and H are not isomorphic.
- To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H.
- However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

Try -Show that the following graphs are isomorphic



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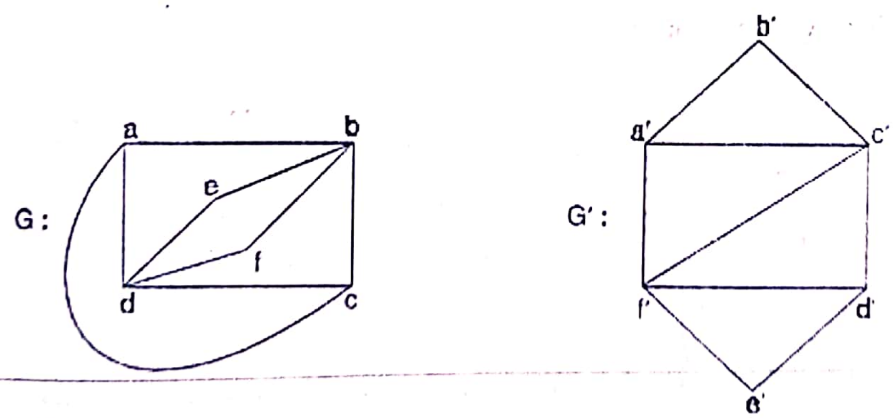
Try -Are the 2-graphs given below isomorphic ? Give a reason.



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Solution. Let us enumerate the degree of the vertices

Vertices of degree 4 : $b - f'$

$d - c'$

Vertices of degree 3 : $a - a'$

$c - d'$

Vertices of degree 2 : $e - b'$

$f - e'$

Now the vertices of degree 3, in G are a and c and they are adjacent in G′, while these are a' and d' which are not adjacent in G′.

Hence the 2-graphs are not isomorphic.

- The best practical general purpose software for isomorphism testing, called NAUTY, can be used to determine whether two graphs with as many as 100 vertices are isomorphic in less than a second on a modern PC

APPLICATIONS OF GRAPH ISOMORPHISMS

- Graph isomorphism, and functions that are almost graph isomorphism, arise in applications of graph theory to chemistry and to the design of electronic circuits, and other areas including bioinformatics and computer vision.

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