Unit 3
Numbers

Introduction

This unit is all about numbers. You use numbers in many different ways every day; for example, you might use them to tell the time, look for a particular page in a book, find the price of an item that you want to buy, check your bank balance or make a measurement. You probably don't think about the numbers themselves, and the interesting properties that they have, but these properties have fascinated many people for thousands of years. This unit will give you just a glimpse of the many properties of numbers.

In particular, you will learn about some properties of prime numbers, and about different types of numbers, such as rational and irrational numbers, and how they differ from each other.

> 'Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.'
>
> Paul Erdős (1913–1996), Hungarian mathematician.
>
> Paul Erdős was one of the most unusual and prolific mathematicians in history. He travelled constantly, living out of a suitcase, and collaborated with other mathematicians wherever he went.

Numbers are of course an important part of mathematics, and it is essential that you are able to work with them confidently and perform calculations with them, both by hand and by using your calculator. These skills will underpin much of your later work in the module, so this unit also gives you an opportunity to revise and practise some of your skills in working with numbers, and to learn some new number skills. If you are a little rusty on some of the basic number skills, such as adding fractions, then you may find that you need more detail than is provided in this unit. If so, then you should find it helpful to consult Maths Help via the link on the module website.

In the final section of the unit, you will look at how numbers in the context of *ratio* are useful in all sorts of everyday situations. In particular, you will learn about *aspect ratio*, which provides a way to describe the shapes of rectangles. Many forms of media involve rectangular shapes; for example, computer and television screens, photographs, printed pages and video pictures are all usually rectangular. Aspect ratio is important in determining, for example, how well different shapes of rectangular picture fit on different shapes of rectangular screen.

The calculator section of the MU123 Guide is needed for two of the activities in this unit. Activities 27 and 35, on pages 173 and 187, respectively, are in the MU123 Guide. If you do not have the MU123 Guide to hand when you reach these activities, then you can omit them and return to them later.

1 Natural numbers

As you saw in Unit 1, the **integers** are the numbers

$$\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots.$$

This section is about the **positive integers**,

$$1,\ 2,\ 3,\ \ldots,$$

which are also known as the **natural numbers**.

The natural numbers are the first numbers that people learn about, and you might think of them as rather uninteresting. In fact, they have many intriguing properties that continue to fascinate mathematicians, and are an important part of the branch of mathematics known as number theory. Their properties also have many important applications in the real world.

1.1 Multiples

When you use a cash machine to withdraw money, it gives you options for the amount of money that it will dispense. The options might be

$$£20,\quad £30,\quad £40,\quad £50,\quad £100,\quad £200.$$

These amounts are all *multiples* of £10. Most UK cash machines dispense only multiples of £10, because they contain only £10 and £20 notes.

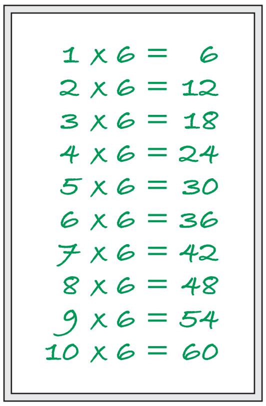


Figure 1 The first ten multiples of six are the answers in the six times table

In general, a **multiple** of a natural number is the result of multiplying it by a natural number. For example, the multiples of 6 are 6, 12, 18, 24, 30, and so on, because

$$1 \times 6 = 6,$$
$$2 \times 6 = 12,$$
$$3 \times 6 = 18,$$
and so on.

Another way to think of the multiples of 6 is that they are the numbers into which 6 divides exactly. For example, 324 is a multiple of 6 because

$$324 \div 6 = 54, \text{ and 54 is a whole number,}$$

but 472 is a not a multiple of 6 because

$$472 \div 6 = 78.666\ldots, \text{ which is not a whole number.}$$

The numbers $0, -6, -12, \ldots$ can also be considered to be multiples of 6. However, this section is all about *positive* integers; so, for example, 'the first four multiples of 6' means the first four positive multiples: $6, 12, 18$ and 24.

Activity 1 Multiples of natural numbers

(a) Write down the first five multiples of 7.

(b) The tickets for an event cost £11 each, and all the ticket money is put in a cash box that is initially empty. After the event, the cash box is found to contain £4183. Is this a correct amount?

Common multiples

Look at these lists of the first few multiples of 6 and 8:

multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, . . . ,
multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, ...

Notice that the number 24 appears in both lists. We say that it is a *common* multiple of 6 and 8. There are other common multiples of 6 and 8: the number 48 also appears in both lists, and if the lists were extended, then you would see that other numbers, for example 72, 96 and 120, also appear in both lists. In fact there are infinitely many common multiples of 6 and 8. The common multiple 24 is special, however, as it is the smallest. We say that it is the *lowest common multiple* of 6 and 8.

These ideas are summarised in the box below.

A **common multiple** of two or more numbers is a number that is a multiple of all of them. The **lowest common multiple** (**LCM**) of two or more numbers is the smallest number that is a multiple of all of them. An alternative name for lowest common multiple is **least common multiple**.

In the previous box, 'number' means 'natural number'. We often use the word 'number' in this way when we are discussing the natural numbers.

Activity 2 Finding lowest common multiples

(a) Write down the first four multiples of each of the numbers 2, 3, 4, 6 and 8. Use your answers to find the lowest common multiple of each of the following pairs of numbers.

(i) 2 and 3 (ii) 4 and 6 (iii) 4 and 8

(b) Find the lowest common multiple of 2, 3 and 8.

Activity 3 Sharing chocolates

(a) A confectionery manufacturer wants to produce a box of chocolates that can be shared evenly among the people in any group of four or fewer people. What is the smallest number of chocolates that can be in the box?

(b) If the manufacturer wanted the box to contain more chocolates than this, what would be the next suitable number of chocolates?

1.2 Factors

A natural number that divides exactly into a second natural number is called a **factor** or **divisor** of the second number. For example, 2 is a factor of 10, since 2 divides exactly into 10.

Factors are closely related to multiples, since '2 is a factor of 10' means the same as '10 is a multiple of 2'. Another way of saying the same thing is '10 is **divisible** by 2'.

Every natural number greater than 1 has at least two factors, itself and 1, but most numbers have more factors than this. For example, the number 10 has four factors: 1, 2, 5 and 10.

The factors of a number can be arranged into **factor pairs**, where the two factors in each pair multiply together to give the number. For example, the factor pairs of 10 are

$$\begin{aligned} &1, 10 \quad (\text{since } 1 \times 10 = 10),\\ &2, 5 \quad\;\; (\text{since } 2 \times 5 = 10). \end{aligned}$$

You can use the idea of factor pairs to help you find all the factors of a number. Here is the strategy – you might like to think about why it works.

Strategy To find the factors of a number

- Try the numbers $1, 2, 3, 4, \ldots$ in turn. Whenever you find a factor, write down the other factor in the factor pair.
- Stop when you get a factor pair that you have already.

The following example illustrates this strategy.

Example 1 Finding the factors of a number

Find all the factors of 28.

Solution

The first factor pair of a number is always 1 and the number.

The first factor pair is 1, 28.

Try 2.

The next factor pair is 2, 14.

Try 3: it's not a factor. Try 4.

The next factor pair is 4, 7.

Try 5: it's not a factor. Try 6: it's not a factor. Try 7: this gives the factor pair 7, 4, which is already found, so stop. To finish, list the factors in increasing order.

The factors of 28 are 1, 2, 4, 7, 14 and 28.

Activity 4 Finding the factors of a number

Find all the factors of the following numbers.

(a) 20 (b) 24 (c) 45

Because the factors of a number form pairs, most numbers have an even number of factors. The only exceptions are the square numbers, each of which has an odd number of factors. Remember that a *square number* is the result of multiplying a whole number by itself. For example, 25 is a square number because $25 = 5 \times 5$. Square numbers have an odd number of factors because one of the factors of a square number pairs with itself. For example, the square number 25 has three factors: the factor pair 1, 25, and the factor 5, which pairs with itself.

The tests in the box below can be useful when you are trying to find the factors of a number. They give you a quick way to tell whether a given number is divisible by 2, 3, 5 or 9.

Divisibility tests

A number is divisible by

- 2 if it ends in 0, 2, 4, 6 or 8
- 3 if its digits add up to a multiple of 3
- 5 if it ends in 0 or 5
- 9 if its digits add up to a multiple of 9.

If a number does not satisfy a test above, then it is not divisible by the specified number.

You may see an explanation of why the tests for divisibility by 3 and 9 work if you go on to take further mathematics modules.

Activity 5 Testing for divisibility

(a) Is 621 divisible by 3?

(b) Is 273 divisible by 9?

Common factors

Look at these lists of the factors of 12 and 18:

factors of 12: 1, 2, 3, 4, 6, 12;
factors of 18: 1, 2, 3, 6, 9, 18.

The numbers that appear in both lists are called the *common* factors of 12 and 18. So the common factors of 12 and 18 are 1, 2, 3 and 6. The largest common factor of 12 and 18 is 6, and this is usually called the *highest* common factor. These ideas are summarised in the box below.

A **common factor** of two or more numbers is a number that is a factor of all of them. The **highest common factor** (**HCF**) of two or more numbers is the *largest* number that is a factor of all of them. An alternative name for highest common factor is **greatest common divisor** (**GCD**).

Activity 6 Finding highest common factors

Use the solution to Activity 4, and the lists of factors of 12 and 18 given above, to list the common factors of each of the following sets of numbers. Hence write down the highest common factor of each set of numbers.

(a) 20 and 24

(b) 12 and 24

(c) 18, 20 and 24

Highest common factors can be useful when you are cancelling fractions. When you want to cancel a fraction down to its simplest form, you need to divide top and bottom by the highest common factor of the numerator and denominator. For example, consider the fraction $\frac{24}{30}$. Dividing top and bottom by 6, the highest common factor of 24 and 30, gives $\frac{4}{5}$. Because 6 is the *highest* common factor of 24 and 30, the numerator and denominator of the simplified fraction have no common factors, other than 1, so the fraction is in its simplest form.

It doesn't matter if you don't immediately spot the highest common factor of the numerator and denominator of a fraction – you can always cancel the fraction down in stages, in the way shown in Unit 1.

There is a quicker way to find highest common factors than the method that you have seen in this subsection. It involves *prime numbers*, which you will learn about in the next subsection.

1.3 Prime numbers

You have seen that every natural number greater than 1 has at least two factors, itself and 1. A natural number that has *exactly* two factors is called a **prime number**, or just a **prime**. For example, 3 is a prime number since its only factors are 1 and 3, but 4 is not a prime number because its factors are 1, 2 and 4. The number 1 is *not* a prime number, as it has only one factor, namely 1. Here are the first 25 prime numbers.

The prime numbers under 100

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

The prime numbers are the 'building blocks' of all the natural numbers, in a sense that you will learn about in the next subsection.

There is a simple algorithm for finding all the prime numbers up to a certain number, which is attributed to the Greek mathematician Eratosthenes (c. 276 BCE – c. 197 BCE), and known as the *Sieve of Eratosthenes*. Eratosthenes was a librarian at the famous library at Alexandria.

Eratosthenes was a man of many talents: he was a mathematician, geographer, historian and literary critic. He was the first person to make a good measurement of the circumference of the Earth, and he came up with the idea of the leap year, which stops the calendar drifting out of step with the seasons.

The algorithm for finding prime numbers works well provided that the certain number is not too large, and modified forms of it are still used by mathematicians today. The module website has a link to a website demonstrating the Sieve of Eratosthenes.

Activity 7 Last digits of prime numbers

Every prime number except 2 and 5 ends in 1, 3, 7 or 9. Can you explain why?

Prime numbers play a central role in both abstract and real-world mathematics. In the real world, prime numbers are often used in encryption systems that are used to protect confidential information when it is transmitted electronically. Encryption involves turning such information into an unrecognisable form so that it cannot be understood until it is turned back into its original form.

Mathematicians have proved many theorems about prime numbers, and they are still working to prove many conjectures. In the next activity you are asked to investigate some properties of prime numbers. This activity will give you a taste of the area of mathematics known as number theory.

Activity 8 Investigating prime numbers

(a) The following table lists all the odd prime numbers under 30. (Of course, the only *even* prime number is 2!) Complete the second row to give the remainder when each prime number is divided by 4.

Prime number	3	5	7	11	13	17	19	23	29
Remainder	3	1	3						

(b) Some prime numbers can be written as the sum of two square numbers: for example, $29 = 25 + 4$. Complete the following table with ticks and crosses to indicate whether each prime number can be written as the sum of two square numbers.

Prime number	3	5	7	11	13	17	19	23	29
Sum of two squares?									✓

(c) By comparing the completed tables from parts (a) and (b), make a conjecture about which odd prime numbers can be written as the sum of two square numbers. Try to obtain more evidence for your conjecture by considering one or two slightly larger prime numbers.

There are infinitely many prime numbers – this was proved more than 2000 years ago by the Greek mathematician Euclid. You will learn more about Euclid in Unit 8. Despite the fact that there are infinitely many prime numbers, it is difficult to identify very large prime numbers.

Thousands of prime number enthusiasts throughout the world are interested in breaking the record for the largest known prime number. Most of them search for Mersenne primes – primes of the form $2^n - 1$ for some natural number n. For example, 7 and 31 are Mersenne primes because $7 = 2^3 - 1$ and $31 = 2^5 - 1$. Mersenne primes are named after the French mathematician Marin Mersenne (1588–1648), who investigated them. It can be proved that if $2^n - 1$ is prime, then n must be prime.

At the time of writing, fifty-one Mersenne primes are known. The latest one was discovered in December 2018. Before that, in September 2008, the discoverer of the first Mersenne prime with more than ten million digits won a \$100 000 prize. These prime numbers (and 15 other large Mersenne primes) were found using software provided by the *Great Internet Mersenne Prime Search*, a scheme in which enthusiasts can download free software that uses the spare processing power of their computers.

At the time of writing, a \$3000 prize is available for each new Mersenne prime discovered, and there is a \$150 000 prize for the first Mersenne prime with more than 100 million digits. The module website has a link to a website where you can find out the latest news about prime numbers.

Figure 2 Marin Mersenne

It would be easy to find prime numbers if they formed some sort of regular pattern, but they do not. The Swiss mathematician Leonhard Euler (1707–1783) wrote: ‘Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the mind will never penetrate.’

1.4 Prime factors

A natural number greater than 1 that is not a prime number is called a **composite number**. The first few composite numbers are

$$4, 6, 8, 9, 10, 12, 14, 15, \ldots .$$

Unlike a prime number, a composite number can be written as a product of two factors, neither of which is 1. For example,

$$360 = 20 \times 18.$$

Composite numbers can often be written as products of even more factors. For example, the number 360 can be broken down into a product of more factors in the following way. The factors 20 and 18 are themselves composite, so they can also be written as products of two factors, neither of which is 1. For example,

$$20 = 4 \times 5 \quad \text{and} \quad 18 = 3 \times 6.$$

So 360 can be written as a product of *four* factors, none of which is 1:

$$360 = 4 \times 5 \times 3 \times 6.$$

The process here can be set out as a **factor tree**, as shown in Figure 3.

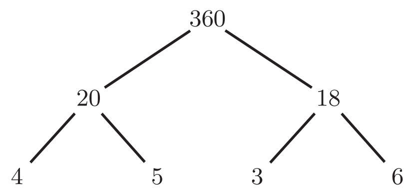


Figure 3 A factor tree for 360

You can continue the process until all the numbers at the ends of the tree are prime numbers. The result is shown in Figure 4, with the prime numbers circled.

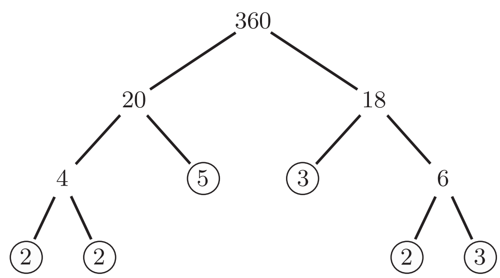


Figure 4 A completed factor tree for 360

It follows from Figure 4 that

$$360 = 2 \times 2 \times 5 \times 3 \times 2 \times 3.$$

Writing the factors in increasing order gives

$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5.$$

And writing $2 \times 2 \times 2$ as 2^3 and 3×3 as 3^2 gives the simpler form

$$360 = 2^3 \times 3^2 \times 5.$$

Here the number 360 is written as a product of prime factors. You can use a process similar to the one above to write any composite number as a product of prime factors.

The process of writing a natural number as a product of factors greater than 1 (whether prime or not) is called **factorisation**. The factorisation above was started by writing $360 = 20 \times 18$, but it could also have been started by writing $360 = 10 \times 36$, or $360 = 9 \times 40$, for example. And there are different ways to proceed with the numbers further down the factor tree, too. Figure 5 shows a different factor tree for 360.

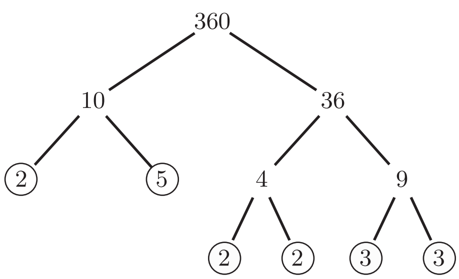


Figure 5 Another factor tree for 360

This second factor tree gives the same result:

$$360 = 2^3 \times 3^2 \times 5.$$

Activity 9 Using a factor tree

(a) Use a factor tree to write the number 300 as a product of prime factors.

(b) Repeat part (a) using a different factor tree.

In Activity 9 you should have obtained the same answer in parts (a) and (b). That is, you should have obtained the same prime factors, and the same number of each of the prime factors.

In fact, no matter how you factorise a composite number into a product of prime factors, you will always obtain the same answer (except that you can usually change the order of the factors – for example, you could write $15 = 3 \times 5$ or $15 = 5 \times 3$). It can be proved that this is true for *every* composite number, an important result known as the *fundamental theorem of arithmetic*.

The fundamental theorem of arithmetic

Every natural number greater than 1 can be written as a product of prime numbers in just one way (except that the order of the primes in the product can be changed).

Figure 6 Carl Friedrich Gauss in 1803

A version of the fundamental theorem of arithmetic appears in Euclid's *Elements* (c. 300 BCE). The first rigorous proof was given by the German mathematician Carl Friedrich Gauss (1777 – 1855) in his famous treatise *Disquisitiones Arithmeticae*, published in 1801.

For a natural number that is prime, the 'product' is just the number itself.

The **prime factorisation** of a natural number is the product of prime factors that is equal to it. Here are the prime factorisations of the numbers from 2 to 10:

$$\begin{array}{lll} 2 = 2, & 3 = 3, & 4 = 2^2, \\ 5 = 5, & 6 = 2 \times 3, & 7 = 7, \\ 8 = 2^3, & 9 = 3^2, & 10 = 2 \times 5. \end{array}$$

The fundamental theorem of arithmetic is the reason why the prime numbers can be thought of as the building blocks of the natural numbers.

When you want to write a number as a product of prime factors, it is sometimes helpful to be systematic. Start by writing the number as

$$\text{the smallest possible prime} \times \text{a number},$$

and do the same with each composite number in the factor tree. If you do this for the number 252, then you obtain the factor tree in Figure 7. So $252 = 2^2 \times 3^2 \times 7$.

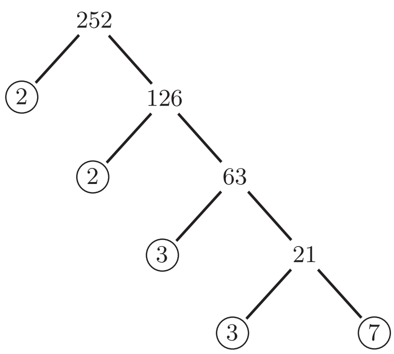


Figure 7 A systematic factor tree for 252

With this systematic method, at each level of the factor tree you get a prime factor and a composite factor, until the final level when you get two prime factors. At each stage 'the smallest possible prime' is the same as, or bigger than, the previous prime factor.

You don't need to set out the working as a factor tree – you might prefer to set it out like this:

$$\begin{aligned}252 &= 2 \times 126\\ &= 2 \times 2 \times 63\\ &= 2 \times 2 \times 3 \times 21\\ &= 2 \times 2 \times 3 \times 3 \times 7\\ &= 2^2 \times 3^2 \times 7.\end{aligned}$$

Activity 10 Factorising numbers into products of primes

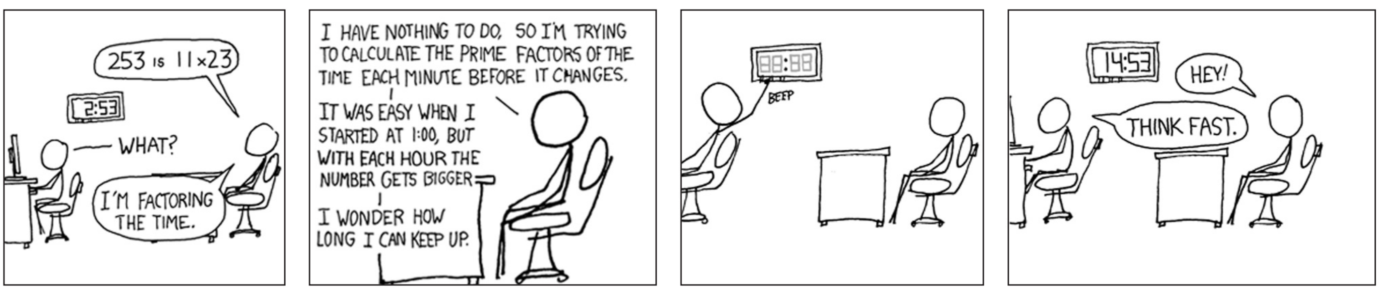
Write each of the following numbers as a product of prime factors.

(a) 72 (b) 855 (c) 1000 (d) 847

The method suggested above for finding the prime factorisation of a number works well if the prime factors are fairly small, but it is time-consuming if they are large. For example, the prime factorisation of the number 899 is 29×31, so to find this factorisation you would have to test each prime number from 2 to 29 to see whether it is a factor of 899.

Despite much research, no one has managed to find a quick method for finding large prime factors of a number. So multiplying two large prime numbers together is a process that is quick to carry out but slow to reverse. For example, you could quickly multiply the primes 29 and 31 to obtain the answer 899, but if you were given the number 899 and asked to find the prime factors, then it would take you much longer!

A computer can multiply two 150-digit prime numbers in seconds, but if a suitable computer program were given the result, then it would probably not be able to find the two prime factors within a human lifetime. Mathematicians have found a clever way to exploit this fact to design secure encryption systems. With the advent of internet banking and online purchasing, the security of personal information such as account details and credit card numbers has become an essential field of computer science. It is number theory that underpins these computer security systems.



You can use prime factorisations to help you find the lowest common multiples and highest common factors of sets of numbers.

Example 2 Using prime factorisations to find LCMs and HCFs

Find the lowest common multiple and highest common factor of 84 and 280.

Solution

Write out the prime factorisations, with a column for each different prime.

$$84 = 2^2 \times 3 \qquad \times 7$$
$$280 = 2^3 \qquad \times 5 \times 7$$

To find the LCM, circle the *highest* power of the prime in each column.

$$84 = 2^2 \times 3 \qquad \times 7$$
$$280 = 2^3 \qquad \times 5 \times 7$$

Now multiply together the circled numbers.

The LCM of 84 and 280 is

$$2^3 \times 3 \times 5 \times 7 = 840.$$

To find the HCF, circle the *lowest* power of the prime in each column, considering only the primes that occur in all the rows.

$$84 = 2^2 \times 3 \qquad \times 7$$
$$280 = 2^3 \qquad \times 5 \times 7$$

Now multiply together all the circled numbers.

The HCF of 84 and 280 is

$$2^2 \times 7 = 28.$$

The methods used in Example 2 are summarised below.

Strategy To find the LCM or HCF of two or more numbers

- Find the prime factorisations of the numbers.
- To find the LCM, multiply together the highest power of each prime factor occurring in any of the numbers.
- To find the HCF, multiply together the lowest power of each prime factor common to all the numbers.

Activity 11 Using prime factorisations to find LCMs and HCFs

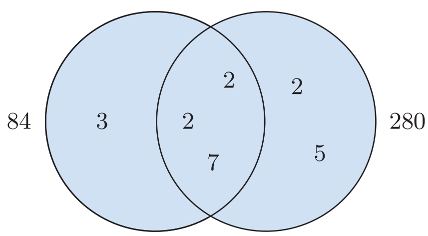
Use prime factorisations to find the lowest common multiple and highest common factor of each of the following sets of numbers.

(a) 18 and 30 (b) 9, 18 and 30

The next activity illustrates a method that can be useful when you want to find the lowest common multiple or highest common factor of just two numbers, which are fairly small.

Activity 12 Finding LCMs and HCFs of two numbers

In the diagram below, one circle contains all the prime factors of 84, and the other circle contains all the prime factors of 280. The common prime factors of 84 and 280 are in the overlap of the two circles.



(a) Can you explain how you could use this diagram to find the lowest common multiple and highest common factor of 84 and 280? (In Example 2 a different method was used to determine that the LCM is 840 and the HCF is 28.)

(b) Check your answer to Activity 11(a) by drawing a similar diagram for 18 and 30 and using it to find the lowest common multiple and highest common factor of these numbers.

1.5 Powers

In the previous subsection you worked with powers of prime numbers. This subsection is all about powers.

As you know, 'raising a number to a power' means multiplying the number by itself a specified number of times. For example, raising 2 to the power 3 gives

$$2^3 = 2 \times 2 \times 2.$$

Here the number 2 is called the **base number** or just **base**, and the superscript 3 is called the **power**, **index** or **exponent**. The word 'power' is also used to refer to the *result* of raising a number to a power – for example, we say that 2^3 is a **power** of 2. When we write expressions like 2^3, we say that we are using **index form** or **index notation**.

As you saw in Unit 1, the plural of 'index' in this context is 'indices'. The word 'index' has several different meanings in English – confusingly, some have plural 'indices', while others have plural 'indexes'! For example, there are *indexes* at the backs of the module books.

The **square** and **cube** of a number are the results of raising it to the powers 2 and 3, respectively. For example, the square of 2 is $2^2 = 4$, and the cube of 2 is $2^3 = 8$. Remember that, for example, 2^2 is read as 'two squared' and 2^3 is read as 'two cubed'. The power 2^5 is read as 'two to the power five' or 'two to the five'. Sometimes you may hear 2^5 read as 'two to the fifth', which is short for 'two to the fifth power'. This is potentially confusing, as it could be interpreted as $2^{1/5}$, but the meaning is normally clear from the context. Other indices are read in a similar way to 2^5.

Standard large numbers like a billion and a trillion can be conveniently described in index form. You can see from the following table that it is easier to look at the index than to count the number of zeros!

Table 1 Standard large numbers

Name	Number	Number in index form
million	1 000 000	10^6
billion (UK and US)	1 000 000 000	10^9
trillion (UK and US)	1 000 000 000 000	10^{12}

The word 'billion' meant 10^{12} rather than 10^9 in the UK until 1974, when the British government decided to switch to the American meaning to avoid confusion in financial markets. Similarly, the word 'trillion' has traditionally meant 10^{18} in the UK, but there has recently been a switch to the American meaning, 10^{12}. Many European countries still use these alternative meanings of 'billion' and 'trillion'.

A *googol* is 10^{100}, but this number is of limited use, as it is greater than the number of atoms in the observable universe! The word was invented by a child, nine-year-old Milton Sirotta, in 1938. He was asked by his uncle, the American mathematician Edward Kasner (1878–1955), what name he would give to a really large number. The word 'googol' gave rise, via a playful misspelling, to the name of the internet search engine Google.

The superscript notation for powers was introduced by the French philosopher and mathematician René Descartes (1596–1650). It was Descartes who wrote the famous philosophical statement *'Cogito ergo sum'*, commonly interpreted in English as 'I think, therefore I am'.

The next few pages describe some basic rules for carrying out calculations with numbers written in index form. It is worth getting to know these rules, as they will be useful later.

Multiplying numbers in index form

Sometimes you need to multiply numbers in index form. For example, suppose that you want to multiply 10^2 by 10^3. You can do this as follows:

$$10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5.$$

You can see that the total number of 10s multiplied together is the sum of the indices, $2 + 3 = 5$. In general we have the following fact.

To multiply numbers in index form that have the same base number, add the indices:

$$a^m \times a^n = a^{m+n}.$$

Example 3 Multiplying powers

Write each of the following products concisely in index form.

(a) $3^4 \times 3^5$ (b) 5×5^9 (c) $2^4 \times 3^7$ (d) $2^3 \times 7 \times 2^2 \times 7^2$

(e) 9×3^5

Solution

(a) $3^4 \times 3^5 = 3^{4+5} = 3^9$

(b) Multiplying 5^9 by 5 increases the index by 1, because the number 5 is the same as 5^1.

$5 \times 5^9 = 5^{1+9} = 5^{10}$

(c) The product $2^4 \times 3^7$ cannot be written any more concisely, as the base numbers are different.

(d) $2^3 \times 7 \times 2^2 \times 7^2 = 2^{3+2} \times 7^{1+2} = 2^5 \times 7^3$

(e) The base numbers are different, but they can be made the same.

$9 \times 3^5 = 3^2 \times 3^5 = 3^{2+5} = 3^7$

Activity 13 Multiplying powers

(a) Write each of the following products concisely in index form.

(i) $3^4 \times 3^3$ (ii) $7^2 \times 7$ (iii) $10^2 \times 10^3 \times 10^4$ (iv) $3^4 \times 5^{12}$

(v) 8×2^5 (vi) 9×3

(b) Use the facts that $294 = 2 \times 3 \times 7^2$ and $441 = 3^2 \times 7^2$ to find the prime factorisation of 294×441.

Activity 14 Making an estimate by multiplying powers

Make an estimate to check the claim made on page 153 that a googol (10^{100}) is greater than the number of atoms in the observable universe. To make your estimate, assume that all the atoms in the universe are hydrogen atoms (it is thought that hydrogen atoms account for about 90% of the mass of the universe), and use the following very rough approximations.

The number of hydrogen atoms in a kilogram is about 10^{27}.
The mass of a star and its planets is about 10^{30} kg.
The number of stars in a galaxy is about 10^{12}.
The number of galaxies in the observable universe is about 10^{11}.

Dividing numbers in index form

Suppose that you want to divide 10^5 by 10^2. You can do this as follows:

$$10^5 \div 10^2 = \frac{10^5}{10^2} = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10}.$$

If you now divide both top and bottom of the fraction by 10, and then by 10 again, you obtain

$$\frac{\cancel{10} \times \cancel{10} \times 10 \times 10 \times 10}{\cancel{10} \times \cancel{10}} = 10 \times 10 \times 10 = 10^3.$$

You can see that the final number of 10s multiplied together is the difference of the indices, $5 - 2 = 3$.

In general we have the following fact.

To divide numbers in index form that have the same base number, subtract the indices:

$$\frac{a^m}{a^n} = a^{m-n}.$$

Activity 15 Dividing powers

(a) Write each of the following quotients concisely in index form.

(i) $7^6 \div 7^2$ (ii) $\dfrac{2^{16}}{2^8}$ (iii) $\dfrac{5^{21}}{3^5}$ (iv) $\dfrac{3^7}{3}$

(b) Use the facts that $3456 = 2^7 \times 3^3$ and $12 = 2^2 \times 3$ to find the prime factorisation of $3456 \div 12$.

Raising a number in index form to a power

Suppose that you want to find $(10^2)^3$, the cube of 10^2. To raise any number to the power 3, you multiply three 'copies' of the number together. So

$$(10^2)^3 = 10^2 \times 10^2 \times 10^2 = (10 \times 10) \times (10 \times 10) \times (10 \times 10) = 10^6.$$

The total number of 10s multiplied together is the product of the indices, $2 \times 3 = 6$. In general we have the following fact.

To raise a number in index form to a power, multiply the indices:

$$(a^m)^n = a^{mn}.$$

Activity 16 Raising numbers in index form to powers

Write each of the following numbers concisely in index form.

(a) $(5^2)^4$ (b) $(7^3)^2$ (c) $(3^5)^3 \times 3^2$ (d) $\dfrac{(2^5)^2}{2^2}$ (e) $\left(\dfrac{2^5}{2^2}\right)^2$

Raising a product or quotient to a power

There are two more facts that are often useful when you are working with powers. Notice that

$$\begin{aligned}(2 \times 10)^3 &= (2 \times 10) \times (2 \times 10) \times (2 \times 10)\\ &= 2 \times 2 \times 2 \times 10 \times 10 \times 10\\ &= 2^3 \times 10^3.\end{aligned}$$

This is an example of the first fact in the box below. The second fact is similar, but it applies to quotients rather than products.

A power of a product is the same as a product of powers;
a power of a quotient is the same as a quotient of powers:

$$(a \times b)^n = a^n \times b^n; \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Activity 17 Finding powers of products and quotients

(a) (i) Use the fact that $21 = 3 \times 7$ to find the prime factorisation of 21^4.

(ii) Use the fact that $24 = 2^3 \times 3$ to find the prime factorisation of 24^3.

(b) Express the following numbers as fractions in their simplest form without using your calculator.

(i) $\left(\frac{2}{7}\right)^2$ (ii) $\left(\frac{3}{4}\right)^3$

Raising negative numbers to powers

So far you have worked with natural numbers raised to powers. Other numbers, such as negative numbers, can also be raised to powers. For example,

$$(-2)^2 = (-2) \times (-2) = 4$$

and

$$(-2)^3 = (-2) \times (-2) \times (-2) = 4 \times (-2) = -8.$$

Remember that a negative number times a negative number is a positive number, and a negative number times a positive number is a negative number.

Activity 18 Calculating powers of negative numbers

Calculate the following powers.

(a) $(-3)^2$

(b) $(-3)^3$

(c) $(-2)^4$

(d) $(-1)^4$

(e) $(-1)^5$

In calculations like those in Activity 18, every pair of negative numbers multiplies together to give a positive number. So you can see that

- a negative number raised to an even power is positive,
- a negative number raised to an odd power is negative.

In this subsection you have seen five rules that you can use when you are working with numbers in index form. These rules apply to powers of any type of number, including negative numbers and numbers that are not whole. They are known as **index laws**, and they are summarised in the box below. You will meet four other index laws later in the unit.

Some index laws

$a^m \times a^n = a^{m+n}$ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

$(a \times b)^n = a^n \times b^n$ $\qquad \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

2 Rational numbers

Many numbers besides the natural numbers are needed for everyday mathematics. This section is about the *rational numbers*, which include the natural numbers and many of the other numbers that you are used to working with.

2.1 What is a rational number?

A **rational number** is a number that can be written in the form

$$\frac{\text{integer}}{\text{integer}},$$

that is, as an integer divided by an integer. Remember that the *integers* are

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots .$$

The following numbers can be written in as an integer divided by an integer.

- *Any fraction*
  Fractions such as $\frac{3}{4}$ and $\frac{11}{2}$ are already written in the form above and so they are rational numbers.
- *Any mixed number*
  Mixed numbers like $5\frac{1}{3}$ can be written as top-heavy fractions, so they are rational numbers. For example, $5\frac{1}{3} = \frac{16}{3}$.
- *Any whole number*
  For example, $7 = \frac{7}{1}$ and $0 = \frac{0}{1}$.
- *Any decimal number with a finite number of digits after the decimal point*
  For example,

  $$0.23 = \frac{23}{100} \quad \text{and} \quad 41.2058 = \frac{412\,058}{10\,000}.$$

- *Some decimal numbers with infinitely many digits after the decimal point*
  For example,

  $$0.333\,333\ldots = \tfrac{1}{3}.$$

- *The negative of any of the numbers above*
  For example,

  $$-4 = \frac{-4}{1} \quad \text{and} \quad -\frac{3}{4} = \frac{-3}{4}.$$

Perhaps you are now wondering whether there are any numbers that are not rational numbers! Well, some decimal numbers with infinitely many digits after the decimal point are not rational numbers. (Remember that *infinite* means 'endless, without limit'; *finite* is the opposite of infinite.) But nearly every number that you use in a real-world application of mathematics is a rational number.

The rest of this subsection is about the decimal forms of rational numbers. In particular, you will find out exactly which numbers with infinitely many digits after the decimal point are rational numbers.

Rational numbers as decimals

Every rational number can be written as a decimal. To do this, you divide the integer on the top of the fraction by the integer on the bottom. For example,

$$\tfrac{5}{8} = 5 \div 8 = 0.625.$$

When you find a decimal in this way, there are two possibilities for the outcome. You might get a decimal number that has only a finite number of digits after the decimal point. This is called a **terminating decimal**. For example, the decimal form 0.625 of $\frac{5}{8}$ is terminating – it has only three digits after the decimal point. Alternatively, you might get a decimal number with a block of one or more digits after the decimal point that repeats indefinitely. For example,

$$\tfrac{2}{3} = 0.666\,666\ldots$$

and

$$\tfrac{7}{54} = 0.1\underbrace{296}\,\underbrace{296}\,\underbrace{296}\,\underbrace{296}\,\underbrace{296}\ldots.$$

A decimal like this is called a **recurring decimal**.

Of course, if you use your calculator to divide the integer on the top of a fraction by the integer on the bottom, then you will only be able to see the first few digits of the answer. You can obtain more digits by carrying out long division by hand or by using mathematical software. However, the number of digits on your calculator is adequate for most practical purposes.

There are two alternative notations for indicating a recurring decimal. You can either put a dot above the first and last digit of the repeating block, or you can put a line above the whole repeating block. For example,

$$\tfrac{2}{3} = 0.\dot{6} = 0.\overline{6}$$

and

$$\tfrac{7}{54} = 0.1\dot{2}9\dot{6} = 0.1\overline{296}\,.$$

If you would like to know why you always get either a terminating or recurring decimal when you write a rational number as a decimal, then take a look at the document explaining this on the module website. You need to think about long division!

So every rational number, when written in decimal form, is a terminating or recurring decimal. But is the reverse true? That is, is every terminating or recurring decimal a rational number? We know that the first sentence here is true, but that doesn't mean that we know that the reverse is true. Every poodle is a dog, but not every dog is a poodle!

Certainly, every *terminating* decimal is a rational number, since it can be written in the form of an integer divided by an integer, as you saw at the beginning of this subsection. It is less obvious that every *recurring* decimal is a rational number, but in fact this is true as well. If you go on to study further mathematics modules, then you may learn how to convert recurring decimals to fractions. It is done using basic algebra.

So another way to think of the rational numbers is as follows.

> The rational numbers are the decimal numbers that are terminating or recurring.

Now consider the number below – it has an infinite number of digits after the decimal point, and they follow a pattern of larger and larger blocks of 0s separated by individual 1s:

$$0.010\,010\,001\,000\,010\,000\,01\ldots.$$

This number is not a terminating decimal and it is not a recurring decimal, as it does not have a *fixed number* of digits that keep repeating. So it is not a rational number! You will learn more about numbers that are not rational in the next section. First, however, it is important to make sure that you are proficient with arithmetical operations on fractions. In the next few subsections you can revise and practise these operations, and learn more about powers.

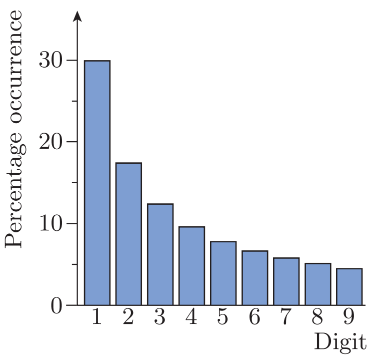


Figure 8 The percentage occurrences of $1, 2, \ldots, 9$ as the first digits of numbers in a table of data

One situation where numbers occur in the real world is in tables of data, such as the prices of stocks and shares, sports statistics, the populations of towns, utility bills, and so on. You might expect that if you were to investigate the first digits of the numbers in such a table then you would find that each of the possible digits $1, 2, \ldots, 9$ occurs equally often. Surprisingly, this is usually not the case. In fact, the digit 1 tends to occur about 30.1% of the time, the digit 2 about 17.6% of the time, and the larger digits less and less frequently, up to the digit 9, which tends to occur about 4.6% of the time. The chart in Figure 8 illustrates how often each digit tends to occur.

This phenomenon was investigated in the 1930s by the American physicist Frank Benford (1883–1948). He analysed thousands of tables of data, and found a mathematical formula that predicts the percentage occurrences of each digit. This formula is known as *Benford's law*. The first rigorous mathematical explanation of it was provided in 1995. Benford's law has been successfully applied to fraud detection, since tables of invented data usually do not have this property.

2.2 Adding and subtracting fractions

You saw in the previous subsection that every rational number can be expressed as a fraction. It is important that you are able to add, subtract, multiply and divide fractions, not only by using your calculator, but also on paper.

One reason for this is that it is useful to be able to do arithmetic with simple fractions without having to resort to your calculator. Another reason is that you will need to use these techniques when you learn algebra later in the module, and you cannot use your calculator for fractions that contain letters! This subsection, and the next, give you a chance to practise these techniques.

First let's look at adding and subtracting fractions. Remember that you can add or subtract fractions only if they are of the same type – that is, if they have the same denominator.

For example, if one-fifth of a pizza is eaten and then another two-fifths of the pizza is eaten, then altogether three-fifths of the pizza has been eaten (Figure 9):

$$\tfrac{1}{5} + \tfrac{2}{5} = \tfrac{3}{5}.$$

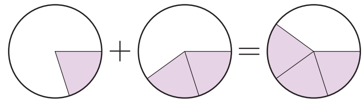


Figure 9 $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

If fractions are not of the same type, then you cannot add or subtract them directly. For example, if two-thirds of a pizza is eaten and then another quarter of the pizza is eaten, then it is not so clear what fraction of the pizza has been eaten. You cannot carry out the following addition directly (Figure 10):

$$\tfrac{2}{3} + \tfrac{1}{4}.$$

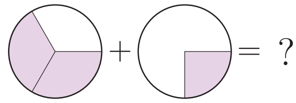


Figure 10 $\frac{2}{3} + \frac{1}{4} = ?$

In this situation, you need to change the fractions into equivalent fractions of the same type – that is, you need to write them with a **common denominator**. Here you can change the fractions into twelfths (Figure 11):

$$\tfrac{2}{3} + \tfrac{1}{4} = \tfrac{8}{12} + \tfrac{3}{12} = \tfrac{11}{12}.$$

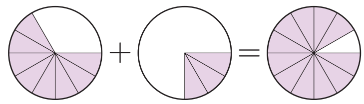


Figure 11 $\frac{8}{12} + \frac{3}{12} = \frac{11}{12}$

So eleven-twelfths of the pizza has been eaten.

Remember that to find a fraction equivalent to a particular fraction, you multiply (or divide) top and bottom of the fraction by the same number. For example,

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \quad \text{and} \quad \frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12}.$$

The next example illustrates some more things to remember when you add and subtract fractions. You should find it helpful to watch the tutorial clip, and there is more detail on fraction calculations in Maths Help. For advice on adding and subtracting fractions, see Maths Help Module 1, Subsections 3.10–3.11.

Tutorial clip

Example 4 Adding and subtracting fractions

Carry out the following fraction additions and subtractions.

(a) $\dfrac{3}{8} + \dfrac{1}{8}$ (b) $\dfrac{4}{9} + \dfrac{5}{6}$ (c) $\dfrac{6}{7} - \dfrac{2}{3}$ (d) $1\tfrac{2}{3} + 4\tfrac{1}{2}$

Solution

(a) The denominators are the same, so add the numerators. Write the answer in its simplest form, by cancelling.

$$\frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{\overset{1}{\cancel{4}}}{\underset{2}{\cancel{8}}} = \frac{1}{2}$$

(b) Make the denominators the same. Write each fraction with denominator 18. Some advice on how to choose the common denominator is given after the strategy box below.

$$\frac{4}{9}+\frac{5}{6}=\frac{8}{18}+\frac{15}{18}=\frac{23}{18}=1\tfrac{5}{18}$$

The answer $\frac{23}{18}$ cannot be simplified by cancelling. You can write it as the mixed number $1\frac{5}{18}$ or just leave it as $\frac{23}{18}$.

(c) Write each fraction with denominator 21.

$$\frac{6}{7}-\frac{2}{3}=\frac{18}{21}-\frac{14}{21}=\frac{4}{21}$$

(d) Add the integer parts of the mixed numbers together, and add the fractional parts together, separately.

$$1\tfrac{2}{3}+4\tfrac{1}{2}=1+4+\tfrac{2}{3}+\tfrac{1}{2}=5+\tfrac{4}{6}+\tfrac{3}{6}=5+\tfrac{7}{6}=5+1\tfrac{1}{6}=6\tfrac{1}{6}$$

Here is the strategy that was used in Example 4.

Strategy To add or subtract fractions

- Make sure that the denominators are the same. (You may need to write each fraction as an appropriate equivalent fraction.)
- Add or subtract the numerators.
- Write the answer in its simplest form.

The trickiest part of adding and subtracting fractions is making the denominators the same. How do you decide what the common denominator should be?

For example, the calculation $\frac{4}{9}+\frac{5}{6}$ in Example 4(b) was carried out by writing each fraction with denominator 18, but how was the number 18 chosen?

The number chosen has to be a common multiple of 9 and 6, because you must be able to get it by multiplying the denominator 9 by something, and also by multiplying the denominator 6 by something.

Any common multiple of 9 and 6 will do. One way to obtain a common multiple of 6 and 9 is just to multiply them together. However, if you want to keep the numbers in the calculation as small as possible, and keep the cancelling of the answer to a minimum, then the best number to choose is the *lowest* common multiple of 9 and 6, which is 18.

Once you have chosen 18, then since $18 = 2 \times 9$ and $18 = 3 \times 6$, you have to multiply the numerator and denominator of the first and second fractions by 2 and 3, respectively, as done in Example 4(b).

Here are some fraction additions and subtractions for you to try.

Activity 19 Adding and subtracting fractions

(a) Carry out the following fraction additions and subtractions without using your calculator.

(i) $\frac{2}{9}+\frac{4}{9}$ (ii) $\frac{7}{8}+\frac{9}{24}$ (iii) $\frac{11}{14}-\frac{3}{14}$ (iv) $\frac{5}{6}-\frac{1}{4}$

(v) $2\frac{1}{7}+4\frac{2}{7}$ (vi) $3\frac{3}{4}-1\frac{1}{5}$ (vii) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

(b) In 2010, half of all UK car drivers were under the age of 35, and a seventh were over the age of 65. What fraction were between 35 and 65 years of age?

2.3 Multiplying and dividing fractions

Suppose that three-quarters of the three-year-olds in a village attend a nursery, and half of them attend full-time. What fraction of the toddlers attend nursery full-time? The answer is a half of three-quarters, and you can see from Figure 12 that this is three-eighths. That is,

$$\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}.$$

This is an example of the following general rule.

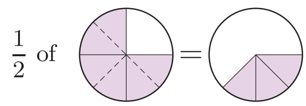


Figure 12 $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$

Strategy To multiply fractions

Multiply the numerators together and multiply the denominators together.

For advice on multiplying fractions, see Maths Help Module 1, Subsection 3.12. The next example illustrates some more things to remember when you multiply fractions.

Tutorial clip

Example 5 Multiplying fractions

Carry out the following fraction multiplications.

(a) $\frac{2}{5}\times\frac{4}{7}$ (b) $2\times\frac{3}{7}$ (c) $\frac{2}{3}\times\frac{5}{6}$ (d) $3\frac{2}{3}\times\frac{5}{6}$

Solution

(a) $\frac{2}{5}\times\frac{4}{7}=\frac{8}{35}$

(b) Here you can either use the strategy, as is done below, or use the fact that 2 lots of 3 sevenths is 6 sevenths.

$2\times\frac{3}{7}=\frac{2}{1}\times\frac{3}{7}=\frac{6}{7}$

(c) Here the number 2 is a factor of the numerator of the first fraction and also a factor of the denominator of the second fraction, so it is a factor of both the numerator and denominator of the product. It is easier to cancel factors like this *before* multiplying.

$$\frac{2}{3} \times \frac{5}{6} = \frac{\overset{1}{\cancel{2}}}{3} \times \frac{5}{\underset{3}{\cancel{6}}} = \frac{5}{9}$$

(d) To multiply by a mixed number, first convert it to a top-heavy fraction. $3\frac{2}{3} \times \frac{5}{6} = \frac{11}{3} \times \frac{5}{6} = \frac{55}{18} = 3\frac{1}{18}$

The answer can be left as $\frac{55}{18}$ if you wish.

Here are some fraction multiplications for you to try.

Activity 20 Multiplying fractions

(a) Carry out the following fraction multiplications.

(i) $\frac{5}{8} \times \frac{3}{10}$ (ii) $\frac{4}{5} \times 3$ (iii) $1\frac{1}{3} \times 2\frac{5}{6}$

(b) At a particular college, two-fifths of the students have jobs, and of these a quarter work for more than 35 hours per week. What fraction of the students work for more than 35 hours per week?

In the next activity you are asked to use your knowledge of fractions to spot a mistake in a newspaper article.

Activity 21 Fractions in the media

The newspaper clipping below is fictitious, but it contains a mistake similar to one that appeared in a real newspaper article. Can you find and correct the mistake? (Assume that half of teenagers are girls and half are boys.)

Teenage smoking rate still too high

A quarter of teenage girls and a sixth of teenage boys smoke regularly, research has shown. That's five-twelfths of the teenage population - nearly half of all teenagers.

Now let's look at how to divide fractions. The rule for doing this can be conveniently described using the idea of the *reciprocal* of a number.

A number and its **reciprocal** multiply together to give 1. So, for example,

0.25 is the reciprocal of 4, since $0.25 \times 4 = 1$;

$\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$, since $\frac{3}{2} \times \frac{2}{3} = 1$. (1)

Another way to think of the reciprocal of a number is that it is 1 divided by the number. For example,

the reciprocal of 5 is $\frac{1}{5} = 0.2$.

As you can see from the line marked (1) above, to find the reciprocal of a fraction, you just 'turn it upside down'. For example,

the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$;

the reciprocal of $\frac{1}{4}$ is $\frac{4}{1} = 4$;

the reciprocal of 2 is $\frac{1}{2}$, since $2 = \frac{2}{1}$.

Now suppose that you have a length of string, and you plan to cut it into two-metre pieces. How many pieces will you get? The answer is the length of the string in metres, divided by 2.

Next suppose that you want to cut the string into pieces one-third of a metre long. How many pieces will you get this time? The answer is the length of the string in metres, divided by $\frac{1}{3}$. But how do you divide by $\frac{1}{3}$?

Well, you get three pieces for every metre of string (Figure 13), so you need to multiply the length of the string by 3. So dividing by $\frac{1}{3}$ is the same as multiplying by 3, the reciprocal of $\frac{1}{3}$.

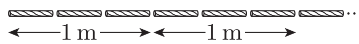


Figure 13 A length of string cut into one-third metre pieces

What if you want to cut the string into pieces two-thirds of a metre long – how many pieces will you get this time (Figure 14)? That is, how do you divide by $\frac{2}{3}$? Well, the number of pieces two-thirds of a metre long is half of the number of pieces one-third of a metre long. So to divide by $\frac{2}{3}$ you multiply by 3 and then by $\frac{1}{2}$, which is the same as multiplying by $\frac{3}{2}$. So dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$.

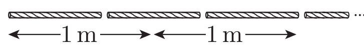


Figure 14 A length of string cut into two-third metre pieces

These are examples of the following general rule.

Strategy To divide by a fraction

Multiply by its reciprocal.

Here are some more examples.

Tutorial clip

Example 6 Dividing fractions

Carry out the following fraction divisions.

(a) $\frac{4}{7} \div \frac{5}{6}$ (b) $\frac{5}{6} \div \frac{1}{4}$ (c) $\frac{3}{5} \div 2$

Solution

(a) $\frac{4}{7} \div \frac{5}{6} = \frac{4}{7} \times \frac{6}{5} = \frac{24}{35}$

(b) Here, once you have turned the second fraction upside down, there are factors that you can cancel before multiplying.

$$\frac{5}{6} \div \frac{1}{4} = \frac{5}{6} \times \frac{4}{1} = \frac{5}{\underset{3}{\cancel{6}}} \times \frac{\overset{2}{\cancel{4}}}{1} = \frac{10}{3} = 3\tfrac{1}{3}$$

(c) $\frac{3}{5} \div 2 = \frac{3}{5} \div \frac{2}{1} = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$

Here are some fraction divisions for you to try. For advice on dividing fractions, see Maths Help Module 1, Subsections 3.13–3.14.

Activity 22 Dividing fractions

(a) Carry out the following fraction divisions without using your calculator.

(i) $6 \div \frac{4}{3}$ (ii) $\frac{3}{8} \div \frac{11}{24}$ (iii) $1\frac{1}{3} \div 1\frac{7}{9}$

Hint: In part (iii) you must turn the mixed numbers into top-heavy fractions before dividing.

(b) A factory worker makes a particular type of metal component. It takes him $1\frac{1}{4}$ hours to make each component, and he works a $37\frac{1}{2}$-hour week. How many components can he make in a week?

2.4 Negative indices

Consider the following powers of 2:

$$2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ \ldots .$$

You could extend this list as far to the right as you like. But does it also make sense to extend it to the left? That is, do all the powers in the following list mean something?

$$\ldots,\ 2^{-3},\ 2^{-2},\ 2^{-1},\ 2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ \ldots$$

The powers with positive indices have the pattern shown below.

$$\dots \quad 2^{-3} \quad 2^{-2} \quad 2^{-1} \quad 2^0 \quad 2^1 \xleftarrow{\div 2} 2^2 \xleftarrow{\div 2} 2^3 \xleftarrow{\div 2} 2^4 \xleftarrow{\div 2} 2^5 \quad \dots$$

If you assume that this pattern continues leftwards, then 2^0 must be the number that you get by dividing 2^1 by 2. That is,

$$2^0 = 2^1 \div 2 = 2 \div 2 = 1.$$

Similarly, 2^{-1} must be the number that you get by dividing 2^0 by 2. That is,

$$2^{-1} = 2^0 \div 2 = 1 \div 2 = \tfrac{1}{2}.$$

If the pattern continues in this way, then the zero and negative indices must have the meanings suggested in Table 2.

Table 2 Powers of 2

Power	$\dots$	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3	2^4	2^5	$\dots$
Meaning	$\dots$	$\frac{1}{2^3}$	$\frac{1}{2^2}$	$\frac{1}{2}$	1	2	2^2	2^3	2^4	2^5	$\dots$

But do these meanings make sense? Do they work with the index laws that you met in Subsection 1.5? For example, one of the index laws is

$$\frac{a^m}{a^n} = a^{m-n}.$$

If you use this rule and the meanings in Table 2 to work out 2^4 divided by 2^5, then you obtain

$$\frac{2^4}{2^5} = 2^{4-5} = 2^{-1} = \frac{1}{2}.$$

This makes sense, because

$$\frac{2^4}{2^5} = \frac{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}}}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times \underset{1}{\cancel{2}} \times 2} = \frac{1}{2}.$$

Similarly, if you use the same rule and the meanings in Table 2 to work out 2^3 divided by 2^3, then you obtain

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0 = 1,$$

which also makes sense, since 2^3 divided by 2^3 is 1.

It turns out that the meanings of negative and zero indices suggested in Table 2 do work with all the index laws you saw in Section 1. So, these are the meanings that are used. They are summarised below, and you can think of them as two further index laws.

Negative and zero indices

A non-zero number raised to the power zero is 1:

$$a^0 = 1.$$

A non-zero number raised to a negative power is the reciprocal of the number raised to the corresponding positive power:

$$a^{-n} = \frac{1}{a^n}.$$

The rules above hold for all appropriate numbers. So, for example, in the second rule a can be any number except 0; it cannot be 0 because you cannot divide by 0. The first rule also holds for all values of a except 0 (the power 0^0 has no meaning).

The second index law above tells you that, in particular,

$$a^{-1} = \frac{1}{a}.$$

So raising a number to the power -1 is the same as finding its reciprocal. For example,

$$\left(\tfrac{2}{3}\right)^{-1} = \tfrac{3}{2}.$$

Example 7 Working with negative and zero indices

Find the values of the following numbers.

(a) 17^0 (b) 3^{-2} (c) 0.4^{-3} (d) $\left(\tfrac{3}{4}\right)^{-1}$ (e) $\left(\tfrac{3}{4}\right)^{-2}$

Solution

(a) $17^0 = 1$

(b) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(c) $0.4^{-3} = \dfrac{1}{0.4^3} = \dfrac{1}{0.064} = 15.625$

(d) $\left(\tfrac{3}{4}\right)^{-1} = \tfrac{4}{3}$

(e) $\left(\tfrac{3}{4}\right)^{-2} = \dfrac{1}{\left(\tfrac{3}{4}\right)^2} = \dfrac{1}{\left(\tfrac{9}{16}\right)} = 1 \times \tfrac{16}{9} = \tfrac{16}{9}$

Alternatively, you can use the index law $(a^m)^n = a^{mn}$.

$\left(\tfrac{3}{4}\right)^{-2} = \left(\left(\tfrac{3}{4}\right)^{-1}\right)^2 = \left(\tfrac{4}{3}\right)^2 = \tfrac{4^2}{3^2} = \tfrac{16}{9}$

Activity 23 Working with negative and zero indices

Find the values of the following numbers, without using your calculator.

(a) $\left(\frac{1}{2}\right)^0$ (b) 7^{-1} (c) 7^{-2} (d) $\left(\frac{1}{3}\right)^{-1}$ (e) $\left(\frac{2}{5}\right)^{-1}$
(f) $\left(\frac{1}{3}\right)^{-2}$ (g) $\left(\frac{2}{5}\right)^{-2}$ (h) $\left(\frac{1}{3}\right)^{-3}$ (i) $(-2)^{-3}$

2.5 Scientific notation

In this subsection you will see one reason why negative and zero indices are useful. Some of the numbers used in mathematics, science, medicine and economics are very big or very small. For example, the population of the world at the time of writing is estimated to be about 7 801 000 000 people, and the mass of the Sun is about 1 990 000 000 000 000 000 000 000 000 000 kilograms. In contrast, the mass of an atom of hydrogen is about 0.000 000 000 000 000 000 000 000 001 674 kilograms. It's hard to tell how many zeros there are in these numbers without laboriously counting them.

Numbers like these are more usefully expressed in **scientific notation** (which is also called **standard form**). To write a number in scientific notation, you express it as a number between 1 and 10, multiplied by a power of 10. The number between 1 and 10 can be equal to 1 but not 10. Any number can be expressed in this form – here are some examples:

$$\begin{aligned} 250 &= 2.5 \times 10^2, \\ 25 &= 2.5 \times 10^1, \\ 2.5 &= 2.5 \times 10^0, \\ 0.25 &= 2.5 \times 10^{-1}. \end{aligned}$$

Here is a strategy that you can use to express a number in scientific notation.

Strategy To express a number in scientific notation

(a) Place a decimal point between the first and second significant digits to give a number between 1 and 10.

(b) Count to find the power of 10 by which this number should be multiplied (or divided) to restore it to the original number.

Tutorial clip

Example 8 Converting numbers to scientific notation

Express the following numbers in scientific notation.

(a) 523.4 (b) 0.006 71

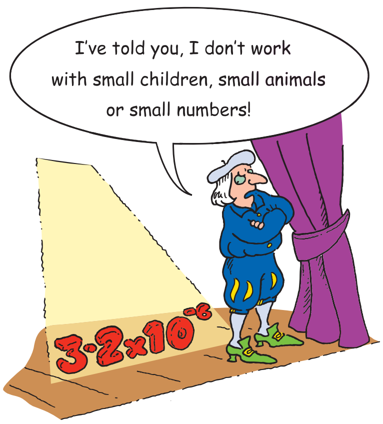
Solution

(a) Place a decimal point between the first and second significant digits. This gives 5.234. To restore this number to the original number, you need to move the decimal point 2 places to the right. That is, you need to multiply by 10^2.

$523.4 = 5.234 \times 10^2$

(b) Place a decimal point between the first and second significant digits. This gives 6.71. To restore this number to the original number, you need to move the decimal point 3 places to the left. That is, you need to divide by 10^3, which is the same as multiplying by $\frac{1}{10^3}$, or 10^{-3}.

$0.00671 = 6.71 \times 10^{-3}$



You might be wondering why scientific notation involves numbers between 1 and 10, rather than a different size of number. For example, why is it preferable to write 75 000 as 7.5×10^4 rather than, say, 75×10^3 or 0.75×10^5? The reason is simply that this notation has been agreed as the one that everyone will use. Using consistent notation makes it easy to compare numbers and carry out calculations. For example, you can tell immediately that 2.1×10^5 is greater than 7.5×10^4 by comparing the powers of ten. It is less easy to tell whether 2.1×10^5 is greater than 75×10^3.

To convert a number from scientific notation back to ordinary notation, you just need to carry out the multiplication or division. Since the multiplication or division is by a power of ten, this involves moving the decimal point. For example,

$$8.22 \times 10^4 = 82\,200,$$
$$1.7 \times 10^{-2} = 1.7 \div 10^2 = 0.017.$$

Activity 24 Converting to and from scientific notation

(a) Write the following numbers in scientific notation.

(i) 7723 (ii) 50 007 000 (iii) 0.100 34 (iv) 0.000 208

(b) Write the following quantities in scientific notation, using the approximate values given at the start of this subsection:

(i) the population of the world

(ii) the mass of the Sun

(iii) the mass of a hydrogen atom.

(c) Convert the following numbers from scientific notation to ordinary notation.

(i) 7.04×10^3 (ii) 4.52×10^4 (iii) 7.3×10^{-2}

(iv) 2.045×10^{-5}

Calculations using numbers in scientific notation

You will use your calculator to carry out most calculations involving numbers in scientific notation, and you will be asked to practise this in an activity at the end of this subsection. However, there may be occasions, perhaps when you are making a quick estimate, when it is more convenient to work out the answer by hand, using the index laws that you met earlier. The next example illustrates how to do this.

Example 9 Calculating with numbers in scientific notation

Carry out the following calculations without using your calculator. The brackets in parts (a) and (c) are included to help make the calculations clear. They are not essential because the calculations would mean the same without the brackets, by the BIDMAS rules.

(a) $(4 \times 10^9) \times (6 \times 10^{-7})$

(b) $\dfrac{4 \times 10^2}{8 \times 10^4}$

(c) $\left(8.2 \times 10^{-2}\right) - \left(5.4 \times 10^{-3}\right)$ Give the answer to part (c) to two significant figures.

Solution

(a) Use the index law $a^m \times a^n = a^{m+n}$ to multiply the powers of ten.

$$(4\times 10^9)\times(6\times 10^{-7}) = (4\times 6)\times(10^9\times 10^{-7}) = 24\times 10^{9-7} = 24\times 10^2$$

To write the answer in scientific notation, write the first number, 24, in scientific notation and use the same index law again.

$24 \times 10^2 = 2.4 \times 10^1 \times 10^2 = 2.4 \times 10^3$

(b) Use the index law $\dfrac{a^m}{a^n} = a^{m-n}$ to divide the powers of ten.

$$\frac{4 \times 10^2}{8 \times 10^4} = \frac{4}{8} \times \frac{10^2}{10^4} = 0.5 \times 10^{2-4} = 0.5 \times 10^{-2}$$

Write the answer in scientific notation.

$0.5 \times 10^{-2} = 5 \times 10^{-1} \times 10^{-2} = 5 \times 10^{-3}$

(c) To add or subtract numbers in scientific notation, first write the numbers so that the powers of 10 are the same.

$8.2 \times 10^{-2} = 8.2 \times 10 \times 10^{-3} = 82 \times 10^{-3}$, so

$$\left(8.2 \times 10^{-2}\right) - \left(5.4 \times 10^{-3}\right) = \left(82 \times 10^{-3}\right) - \left(5.4 \times 10^{-3}\right).$$

Now 82 lots of 10^{-3} subtract 5.4 lots of 10^{-3} is the same as $(82 - 5.4)$ lots of 10^{-3}.

$$\begin{aligned}\left(82 \times 10^{-3}\right) - \left(5.4 \times 10^{-3}\right) &= (82 - 5.4) \times 10^{-3} \\ &= 76.6 \times 10^{-3} \\ &= 77 \times 10^{-3} \text{ (to 2 s.f.).}\end{aligned}$$

Write the answer in scientific notation.

$77 \times 10^{-3} = 7.7 \times 10 \times 10^{-3} = 7.7 \times 10^{-2}$.

So $\left(8.2 \times 10^{-2}\right) - \left(5.4 \times 10^{-3}\right) = 7.7 \times 10^{-2}$ (to 2 s.f.).

You can use some of the methods illustrated in Example 9 to do the following two activities.

Activity 25 Financial bailouts

The following headline appeared in a British newspaper in February 2009, following the collapse of several banks.

Bailouts add £1.5 trillion to Britain's public debt.

A trillion is 10^{12}, and a million is 10^6. The British population in 2009 was approximately 61 million.

(a) Write both quantities quoted above in scientific notation.

(b) Work out the figure that should go in the gap in the extended headline below. (Give your answer to two significant figures.)

Bailouts add £1.5 trillion to Britain's public debt – that's about $\boxed{}$ for each person!

Activity 26 The world's smallest guitar

Figure 15 shows a scanning electron microscope image of a tiny 'nano guitar', made out of silicon at Cornell University in 1997. According to the press release, it is 10 micrometres long, and each of the six strings is about 50 nanometres wide. A micrometre is 10^{-6} metres and a nanometre is 10^{-9} metres. Use these conversion factors to answer the following questions.

(a) An ordinary guitar is about 1 metre long. How many times smaller is the nano guitar?

(b) A human hair is about 100 micrometres wide. How many times smaller is the width of a string of the nano guitar?

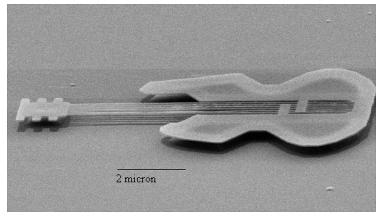


Figure 15 The nano guitar

In the next activity you will learn how to use your calculator to carry out calculations involving scientific notation.

Activity 27 Scientific notation on your calculator

Work through Subsection 3.5 of the MU123 Guide. If you do not have the MU123 Guide to hand, then you can come back to this activity later.

3 Irrational numbers and real numbers

3.1 What is an irrational number?

In the previous section you saw that many of the numbers that you use in everyday mathematics are *rational* numbers – they can be expressed in the form of an integer divided by an integer.

You saw that all rational numbers have decimal forms that are either terminating or recurring, and so the following number is not a rational number:

$$0.010\,010\,001\,000\,010\,000\,01\ldots.$$

But perhaps numbers like this are not 'proper' numbers? Perhaps the rational numbers form a sensible system of numbers that we can use for all the usual purposes, and we can ignore decimals like the one above? Let's consider whether this suggestion is workable.

One reason why we need numbers is so we can measure things, such as length. To measure length, you first need to decide on a unit of measure. The unit could be a centimetre, a metre, an inch or any other convenient length – it doesn't matter what it is, as long as it is used consistently.

Suppose that we decide to measure lengths in cm. Here are the lengths of some lines measured using this unit.

$1\,\text{cm}$

$2\,\text{cm}$

$3\,\text{cm}$

$\frac{1}{2}\,\text{cm}$

$1\frac{19}{27}\,\text{cm}$

The numbers here, 1, 2, 3, $\frac{1}{2}$ and $1\frac{19}{27}$, are all rational numbers. But is the length of *every* line, measured in centimetres, a rational number?

Consider the diagonal lines in the tiling pattern in Figure 16. The pattern is made of four square tiles, each with sides 1 cm long, and each tile is half green and half yellow. The diagonal lines form the sides of a green square. Suppose that the length of these diagonal lines is d cm.

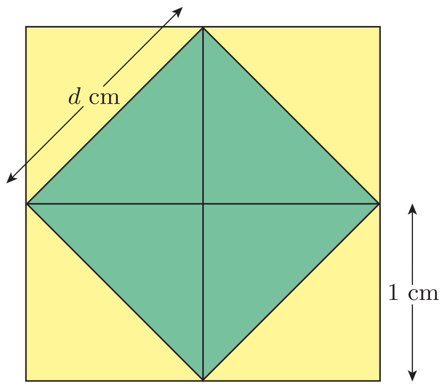


Figure 16 A tiling pattern

The whole pattern is a square with sides 2 cm long, so its area is $2 \times 2 = 4\,\text{cm}^2$. The green square covers half the total area, so its area is $\frac{1}{2} \times 4 = 2\,\text{cm}^2$. Therefore

$$d \times d = 2, \quad \text{that is,} \quad d^2 = 2.$$

So the length of the sides of the green square, measured in cm, is a number whose square is 2.

For help with calculating areas, see Maths Help Module 7, Subsection 3.1.

Now it turns out that there is no rational number whose square is 2. This is not obvious, but it can be proved in an argument that takes about half a page. So the length of the sides of the green square, measured in centimetres, is not a rational number. A proof that there is no rational number whose square is 2 is available on the module website – take a look if you are interested.

Many other lines can be drawn, using similar patterns, that have lengths that are not rational numbers. This is true no matter what unit of measurement you choose. Of course, in practice you can approximate these lengths by rational numbers, but a sensible system of numbers should include the numbers that are the exact lengths of these lines.

So the rational numbers by themselves do not form a workable system of numbers. We must include many more numbers to obtain such a system, and the new numbers that we must include are the ones with decimal forms that are not terminating or recurring – that is, we must include the decimals with an infinite number of digits after the decimal point but no repeating block of digits. These numbers are called the **irrational numbers** – they are the numbers that are not rational.

One of these numbers is the number whose square is 2. This number is called the *square root* of 2, and is denoted by $\sqrt{2}$. Here it is to the first 120 decimal places:

$$\begin{aligned}&1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,948\,073\\&\quad 176\,679\,737\,990\,732\,478\,462\,107\,038\,850\,387\,534\,327\,641\,572\,735\,013\,846\\&\quad 230\,912\,297\,024\ldots.\end{aligned}$$

The irrational numbers also include the positive number whose square is 3, which is denoted by $\sqrt{3}$, and many other *roots* of rational numbers. You will learn more about these later in this section. Another irrational number is the number

$$\pi = 3.141\,592\,653\,589\,793\,238\,46\ldots,$$

which you encountered in Unit 2. This is an important number in mathematics, and you will see it used frequently in some of the later units of the module. You have also seen that

$$0.010\,010\,001\,000\,010\,000\,01\ldots$$

is an irrational number, but there is nothing special about this one, except that its digits have a pattern – one that is different from the type of pattern found in the decimal forms of rational numbers.

The irrational numbers together with the rational numbers form the **real numbers**. These numbers are sufficient to represent the length of any line or curve. Each point on the number line represents a real number, so the number line is often called the **real line**. The positions of some real numbers on the number line are shown in Figure 17.

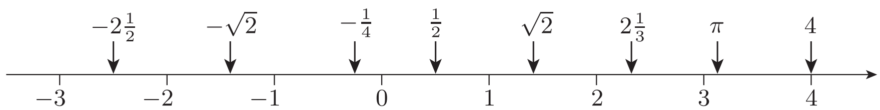


Figure 17 The real line

There are infinitely many rational numbers and infinitely many irrational numbers.

The Pythagoreans, who were followers of the philosophy of the Greek mathematician Pythagoras (c. 569 BCE – c. 494 BCE), one of the most elusive figures of antiquity, developed a theory of proportions between natural numbers. They believed in a harmony between the world and natural numbers and their theory can be considered to be an early version of our theory of rational numbers. However, they soon discovered that not all relations between quantities (such as lengths) can be expressed by such proportions. So the idea of 'irrational numbers' dawned early in the history of mathematics, about two and a half thousand years ago.

Figure 18 shows a useful way to think about the types of numbers that you have learned about in this unit. It illustrates the facts that all of the natural numbers are also integers, all of the integers are also rational numbers and all of the rational numbers are also real numbers.

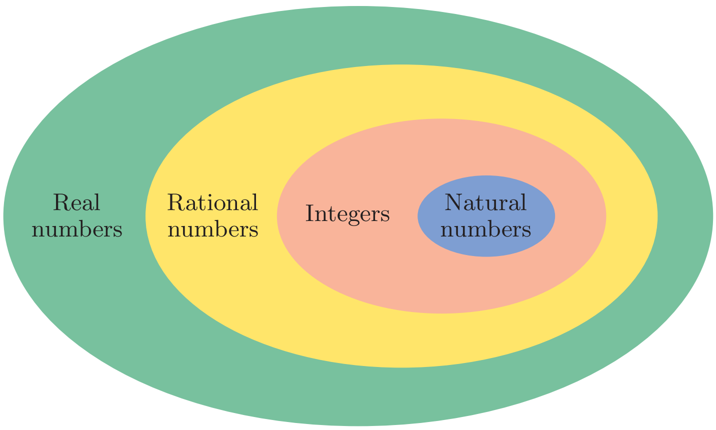


Figure 18 Types of numbers

If you go on to take further mathematics modules, then you can learn about yet another type of number. The *complex numbers* include all the numbers above, and also many 'imaginary' numbers, such as the square root of -1. The idea of imaginary numbers might seem strange, but the complex numbers have a huge number of useful practical applications.

3.2 Roots of numbers

As you have seen, the number whose square is 2 is called the *square root* of 2.

In general, a **square root** of a number is a number that when multiplied by itself gives the original number. For example, 6 is a square root of 36, since $6 \times 6 = 36$. The number -6 is also a square root of 36, since $(-6) \times (-6) = 36$.

Every positive number has two square roots – a positive one and a negative one. However, when we say *the* square root of a number, we mean the positive one. The symbol '$\pm$', which means 'plus or minus', can be useful when you are working with square roots; for example, you can write that the two square roots of 36 are ± 6.

The *positive* square root of a positive number is denoted by the symbol $\sqrt{}$. For example,

$$\sqrt{36} = 6.$$

> The symbol $\sqrt{}$ for roots was introduced by René Descartes (see page 153).

Because $\sqrt{}$ means the positive square root, it is incorrect to write, for example,

'if $x^2 = 4$, then
$x = \sqrt{4} = \pm 2$'.

What you should write is

'if $x^2 = 4$, then
$x = \pm\sqrt{4} = \pm 2$'.

There are other types of roots apart from square roots. A **cube root** of a number is a number such that if you multiply three 'copies' of it together, you get the original number. For example, 4 is a cube root of 64, because

$$4 \times 4 \times 4 = 64.$$

And -4 is a cube root of -64, because

$$(-4) \times (-4) \times (-4) = -64.$$

Similarly, a fourth root of a number is a number such that if you raise it to the power 4 you get the original number. For example, 5 and -5 are both fourth roots of 625, because $5^4 = 625$ and $(-5)^4 = 625$. Similarly, numbers can have fifth, sixth and seventh roots, and so on.

The (positive) cube root of a positive number is denoted by $\sqrt[3]{}$, the positive fourth root of a positive number is denoted by $\sqrt[4]{}$, and so on. So, for example, $\sqrt[3]{64} = 4$ and $\sqrt[4]{625} = 5$.

The symbols $\sqrt{\ }$, $\sqrt[3]{\ }$ and so on can also be used with zero under the root sign. Zero has just one square root, one cube root and so on, namely zero.

Activity 28 Finding roots of numbers

Find the following roots of numbers, without using your calculator.

(a) $\sqrt{9}$ (b) $\sqrt[3]{8}$ (c) $\sqrt[3]{27}$ (d) $\sqrt[4]{16}$

(e) Two square roots of 9 (f) Two fourth roots of 16

If you know the square roots of two numbers, then you can use this information to find the square root of the product or a quotient of the numbers. For example, consider the numbers 9 and 25, with square roots 3 and 5, respectively. By the index law $(a \times b)^n = a^n \times b^n$, we know that

$$(3 \times 5)^2 = 3^2 \times 5^2,$$

that is

$$(3 \times 5)^2 = 9 \times 25.$$

So the positive square root of 9×25 is 3×5. That is,

$$\sqrt{9 \times 25} = \sqrt{9} \times \sqrt{25}.$$

This is an example of the first rule in the box below. The second rule is similar, but it applies to quotients rather than products.

A square root of a product is the same as a product of square roots; a square root of a quotient is the same as a quotient of square roots:

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Analogous rules apply to cube roots, fourth roots, and so on.

Activity 29 Finding more roots of numbers

(a) Use the fact that $1764 = 36 \times 49$ to find $\sqrt{1764}$.

(b) Find the following square roots of fractions, without using your calculator.

(i) $\sqrt{\frac{4}{9}}$ (ii) $\sqrt{\frac{36}{49}}$ (iii) $\sqrt{\frac{1}{4}}$

You can use your calculator to find square roots of numbers, and you will get a chance to practise this later in this section.

3.3 Surds

All the roots of numbers that you were asked to find in the last subsection were rational, but most numbers have irrational roots.

In particular, the square root of any natural number that is not a perfect square is irrational. (A *perfect square* is another name for a square number.) So, for example, the following roots are irrational:

$$\sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{6}.$$

Because numbers like these cannot be written down exactly as terminating decimals or fractions, we often leave them just as they are in calculations and in the answers to calculations. For example, we might say that the two square roots of 2 are

$$\sqrt{2} \text{ and } -\sqrt{2}.$$

Or we might say that the answer to a calculation is

$$1+\sqrt{5}.$$

The advantage of this approach is that it allows us to work with exact numbers, rather than approximations. This can help to simplify calculations.

Expressions like

$$\sqrt{2},\quad -\sqrt{2},\quad \sqrt[3]{2},\quad 1+\sqrt{5},\quad \frac{\sqrt{7}}{3},\quad 1-2\sqrt{5}\quad \text{and}\quad \sqrt{2}+\sqrt{3}$$

are called *surds*. That is, a **surd** is a numerical expression containing one or more irrational roots of numbers.

> The word 'surd' is derived from the same Latin word as 'absurd'! The original Latin word is 'surdus', which means deaf or silent.

Surds are usually written concisely, in a similar way to formulas. Multiplication signs are usually omitted, though sometimes it is necessary or helpful to include them. Also, where a number and a root are multiplied together, it is conventional to write the number first. So the penultimate surd in the list above is written as

$$1-2\sqrt{5},\ \text{rather than}\ 1-\sqrt{5}\times 2,\ \text{say}.$$

It is also usually helpful to write surds in the simplest form possible. In the rest of this subsection you will learn some ways to simplify surds.

First, you can sometimes use the rule

$$\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$$

to make the number under a square root sign smaller. For example,

$$\sqrt{12}=\sqrt{4\times 3}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}.$$

You can simplify a square root in this way whenever the number under the square root sign has a factor that is a perfect square greater than 1. Remember that the first ten perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Here are some more examples of simplifying surds.

Tutorial clip

Example 10 Simplifying square roots

Simplify the following surds, where possible.

(a) $\sqrt{18}$ (b) $\sqrt{10}$ (c) $\sqrt{60}$ (d) $\sqrt{80}$

Solution

(a) Write 18 as the product of a perfect square and another number, then use the rule $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

(b) The factors of 10 are 1, 2, 5 and 10. None of the factors greater than 1 is a perfect square, so the surd $\sqrt{10}$ cannot be simplified. It is already in its simplest form.

(c) Write 60 as the product of a perfect square and another number.

$\sqrt{60} = \sqrt{4 \times 15} = \sqrt{4} \times \sqrt{15} = 2\sqrt{15}$

Now check whether 15 has any square factors. It doesn't, so the surd can't be simplified any further.

(d) Write 80 as the product of a perfect square and another number.

$\sqrt{80} = \sqrt{4 \times 20} = \sqrt{4} \times \sqrt{20} = 2\sqrt{20}$

The number 20 has a square factor, so the surd can be simplified further.

$\sqrt{80} = 2\sqrt{20} = 2\sqrt{4 \times 5} = 2\sqrt{4} \times \sqrt{5} = 2 \times 2\sqrt{5} = 4\sqrt{5}$

In Example 10(d) the square root of 80 was simplified by first using the fact that the perfect square 4 is a factor of 80. The working can be shortened by instead using the fact that the larger perfect square 16 is a factor of 80. This gives

$$\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}.$$

So it's most efficient to begin with the largest square factor that you can spot, but if it turns out that there is a larger one, then you can simplify the root in stages, as in Example 10(d).

Activity 30 Simplifying square roots

Simplify the following surds, where possible, without using your calculator.

(a) $\sqrt{8}$ (b) $\sqrt{75}$ (c) $\sqrt{15}$ (d) $\sqrt{56}$ (e) $\sqrt{48}$

Another way in which you can sometimes simplify surds is to simplify products of two or more square roots. Where two identical square roots are multiplied together, this is easily done: for example, $\sqrt{2} \times \sqrt{2} = 2$. Where different square roots are multiplied together, you can use the rule

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}.$$

For example, $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$. Here are some more examples of multiplying square roots in surds.

Example 11 Multiplying roots

Tutorial clip

Simplify the following surds, where possible.

(a) $\left(\sqrt{3}\,\right)^2$ (b) $2\sqrt{5} \times 4\sqrt{5}$ (c) $\sqrt{6} \times \sqrt{3}$ (d) $5\sqrt{2} \times 3\sqrt{10}$

Solution

(a) $\left(\sqrt{3}\,\right)^2 = \sqrt{3} \times \sqrt{3} = 3$

(b) Multiply the numbers together, and multiply the roots together.

$2\sqrt{5} \times 4\sqrt{5} = 8\sqrt{5}\sqrt{5} = 8 \times 5 = 40$

(c) Use the rule $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ to multiply the roots. Simplify the result.

$\sqrt{6} \times \sqrt{3} = \sqrt{6 \times 3} = \sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

(d) Multiply the numbers together, and multiply the roots together using the rule $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$. Simplify the result.

$$\begin{aligned} 5\sqrt{2} \times 3\sqrt{10} &= 15\sqrt{2 \times 10} \\ &= 15\sqrt{20} \\ &= 15\sqrt{4 \times 5} \\ &= 15\sqrt{4}\sqrt{5} \\ &= 15 \times 2\sqrt{5} \\ &= 30\sqrt{5} \end{aligned}$$

Activity 31 Multiplying roots

Simplify the following surds, where possible, without using your calculator.

(a) $(\sqrt{7})^2$ (b) $\sqrt{7} \times 3\sqrt{7}$ (c) $\sqrt{7} \times \sqrt{14}$ (d) $\sqrt{2} \times \sqrt{8}$

(e) $2\sqrt{3} \times 3\sqrt{2}$ (f) $2\sqrt{3} \times 2\sqrt{15}$

You can also sometimes simplify quotients of roots in surds. Both of the rules

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

can be useful.

Tutorial clip

Example 12 Dividing roots

Simplify the following surds.

(a) $\dfrac{\sqrt{15}}{\sqrt{3}}$ (b) $\dfrac{2}{\sqrt{2}}$

Solution

(a) Use the rule $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

$$\frac{\sqrt{15}}{\sqrt{3}} = \frac{\sqrt{3} \times \sqrt{5}}{\sqrt{3}} = \frac{\overset{1}{\cancel{\sqrt{3}}} \times \sqrt{5}}{\underset{1}{\cancel{\sqrt{3}}}} = \sqrt{5}$$

Alternatively, use the rule $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

$$\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}} = \sqrt{5}$$

(b) Use the fact that $2 = \sqrt{2} \times \sqrt{2}$.

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{2}} = \frac{\overset{1}{\cancel{\sqrt{2}}} \times \sqrt{2}}{\underset{1}{\cancel{\sqrt{2}}}} = \sqrt{2}$$

Alternatively, multiply top and bottom by $\sqrt{2}$.

$$\frac{2}{\sqrt{2}} = \frac{2 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Activity 32 Dividing roots

Simplify the following surds without using your calculator.

(a) $\dfrac{\sqrt{10}}{\sqrt{2}}$ (b) $\dfrac{5}{\sqrt{5}}$ (c) $\dfrac{\sqrt{8}}{\sqrt{2}}$ (d) $\dfrac{8}{\sqrt{2}}$

You cannot usually simplify a sum of two different roots, such as $\sqrt{3}+\sqrt{5}$, in a surd. In general,

$$\sqrt{a}+\sqrt{b} \quad \text{is } \textit{not} \text{ equal to} \quad \sqrt{a+b}.$$

For example,

$$\sqrt{3}+\sqrt{5} = 3.96\ldots,$$

whereas

$$\sqrt{3+5} = \sqrt{8} = 2.82\ldots.$$

However, you can add, or subtract, roots that are the same. This is illustrated in the next example.

Example 13 Adding and subtracting roots

Tutorial clip

Simplify the following surds, where possible.

(a) $2\sqrt{3}+4\sqrt{3}$ (b) $5\sqrt{2}-\sqrt{2}$ (c) $\sqrt{3}+2\sqrt{5}$

(d) $\sqrt{12}-\sqrt{3}$

Solution

(a) Two lots of $\sqrt{3}$ plus four lots of $\sqrt{3}$ is six lots of $\sqrt{3}$.

$2\sqrt{3}+4\sqrt{3} = 6\sqrt{3}$

(b) Five lots of $\sqrt{2}$ subtract one lot of $\sqrt{2}$ is four lots of $\sqrt{2}$.

$5\sqrt{2}-\sqrt{2} = 4\sqrt{2}$

(c) The roots in the surd $\sqrt{3}+2\sqrt{5}$ are different (and are in their simplest forms), so the surd cannot be simplified.

(d) First write $\sqrt{12}$ in its simplest form. Then proceed as before.

$\sqrt{12}-\sqrt{3} = \sqrt{4\times 3}-\sqrt{3} = \sqrt{4}\sqrt{3}-\sqrt{3} = 2\sqrt{3}-\sqrt{3} = \sqrt{3}$

Activity 33 Adding and subtracting roots

Simplify the following surds, where possible, without using your calculator.

(a) $\sqrt{3}+\sqrt{3}$ (b) $\sqrt{2}+\sqrt{5}$ (c) $7\sqrt{3}-2\sqrt{3}$ (d) $5\sqrt{8}-2\sqrt{2}$

Here is a summary of some of the ways in which surds can be simplified.

To simplify surds

- Simplify roots of integers with square factors.
- Simplify products and quotients of roots.
- Add or subtract roots that are the same.

3.4 Fractional indices

Earlier in this unit you worked with powers that have negative and zero indices, such as 3^{-2} and 2^0. Meanings were given for these indices, and you saw that with these meanings the index laws work for negative and zero indices.

Meanings can also be given to *fractional* indices in such a way that the index laws work for these indices. For example, consider the power $5^{\frac{1}{2}}$. If this power has a meaning, then by the index law

$$a^m \times a^n = a^{m+n},$$

we have

$$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5,$$

so you can see that a sensible meaning for $5^{\frac{1}{2}}$ is

$$5^{\frac{1}{2}} = \sqrt{5}.$$

(The meaning $5^{\frac{1}{2}} = \sqrt{5}$ is preferable to $5^{\frac{1}{2}} = -\sqrt{5}$ because you would expect $5^{\frac{1}{2}}$ to be positive. For example, you have seen that $5^0 = 1$ and $5^1 = 5$, so you would expect $5^{\frac{1}{2}}$ to be between 1 and 5.)

Similarly, by the index law $a^m \times a^n = a^{m+n}$, we have

$$\begin{aligned} &5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \\ &= 5^{\frac{1}{3}+\frac{1}{3}} \times 5^{\frac{1}{3}} \\ &= 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}. \end{aligned}$$

So we have

$$5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 5^1 = 5,$$

so a sensible meaning for $5^{\frac{1}{3}}$ is

$$5^{\frac{1}{3}} = \sqrt[3]{5}.$$

We make the following definition, which you can think of as another index law.

> Raising a number to the power $\frac{1}{2}$ is the same as taking its square root, raising a number to the power $\frac{1}{3}$ is the same as taking its cube root, and so on:
>
> $$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

This rule, together with the index laws that you have already met, can be used to give a meaning to any fractional index. For example, using the index law

$$(a^m)^n = a^{mn},$$

we obtain

$$5^{\frac{4}{3}} = 5^{\frac{1}{3}\times 4} = \left(5^{\frac{1}{3}}\right)^4 = \left(\sqrt[3]{5}\right)^4.$$

It is worth stating the general rule illustrated here as another index law.

> Raising a number to the power $\dfrac{m}{n}$ is the same as raising the nth root of the number to the power m:
>
> $$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m.$$

The two rules in the boxes above hold for all appropriate numbers. For example, a must be positive, since the notation $\sqrt{}$ applies only to positive numbers.

Here are some more examples of fractional indices.

Example 14 Raising numbers to fractional indices

Find the values of the following powers.

(a) $9^{\frac{1}{2}}$ (b) $4^{\frac{3}{2}}$ (c) $4^{-\frac{3}{2}}$

Solution

(a) $9^{\frac{1}{2}} = \sqrt{9} = 3$

(b) $4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

(c) Use the index law $a^{-n} = \dfrac{1}{a^n}$, then the result of part (b).

$$4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{8} \quad \text{(by part (b))}$$

Activity 34 Raising numbers to fractional indices

Find the values of the following powers, without using your calculator.

(a) $16^{\frac{1}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $4^{-\frac{1}{2}}$ (d) $4^{\frac{5}{2}}$ (e) $27^{\frac{2}{3}}$

Now that you have met fractional indices, you can see that the two rules for square roots that you met earlier,

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b} \quad \text{and} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

are really just index laws in disguise. They are obtained by taking $n = \frac{1}{2}$ in the index laws

$$(a \times b)^n = a^n \times b^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$

which you met in Section 1.

You have seen that the index in a power can be any rational number. Perhaps you are now wondering whether an index can be irrational? For example, does $2^{\sqrt{2}}$ have a meaning? Powers like this do have precise meanings, which you can learn about in detail in more advanced mathematics modules. The basic idea is that since

$$\sqrt{2} = 1.414\,213\,562\,373\,09\ldots,$$

you can work out the value of $2^{\sqrt{2}}$ as accurately as you like by using as many decimal places of the decimal form of $\sqrt{2}$ as you like. For example, one approximation to $2^{\sqrt{2}}$ is

$$2^{1.414} = 2.664\,749\,650\,184\,04\ldots,$$

and a more accurate one is

$$2^{1.414\,213} = 2.665\,143\,103\,797\,72\ldots,$$

and so on. The indices here, 1.414 and 1.414 213, and so on, are rational, as they are terminating decimals.

So the index in a power can be any real number. All the index laws that you have seen in this unit hold for indices and base numbers that are any real numbers (except that the numbers must be appropriate for the operations – for example, you cannot divide by zero, or take a square root of a negative number). Here is a summary of the index laws.

Index laws

$a^m \times a^n = a^{m+n}$ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

$(a \times b)^n = a^n \times b^n$ $\qquad \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

$a^0 = 1$ $\qquad a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

Powers and surds on your calculator

In the final activity of this section you can practise using your calculator for calculations involving powers, scientific notation and surds.

Activity 35 Powers and surds on your calculator

Work through Subsection 3.6 of the MU123 Guide. If you do not have the MU123 Guide to hand, then you can come back to this activity later.

4 Ratios

4.1 What is a ratio?

If you have made vinaigrette salad dressing, then you may remember that the recipe is 3 parts oil to 1 part vinegar. So, for example, you could mix 30 ml oil and 10 ml vinegar, or 120 ml oil and 40 ml vinegar, or perhaps, if you need a lot of salad dressing, 1.5 l oil and 0.5 l vinegar.

We say that the **ratio** of oil to vinegar is

$$3 : 1.$$

The symbol ':' is a colon; it is read as 'to' when it is in a ratio. This ratio is equivalent to

$$30 : 10, \quad \text{and} \quad 120 : 40, \quad \text{and} \quad 1.5 : 0.5.$$

Notice that ratios do not have units. Ratios can contain more than two numbers. For example, to make a particular type of concrete, you need 1 part cement to 2 parts sand to 4 parts gravel. That is, the ratio of cement to sand to gravel is

$$1 : 2 : 4.$$

So, for example, you could mix 1 shovelful of cement with 2 shovelfuls of sand and 4 shovelfuls of gravel, or 5 shovelfuls of cement with 10 shovelfuls of sand and 20 shovelfuls of gravel, and so on, depending on how much concrete you need.

You met a particular type of ratio in Unit 2, when you looked at map scales. You saw that if the scale factor of a map is 500 000, say, then the map scale is often given in the form

$$1 : 500\,000.$$

This is the ratio between a distance on the map and the corresponding distance on the ground.

A ratio is changed to an **equivalent ratio** in the same way that a fraction is changed to an equivalent fraction.

To find a ratio equivalent to a given ratio

Multiply or divide each number in the ratio by the same non-zero number.

If the numbers in a ratio are rational, then the ratio has a simplest form in the same way that fractions do. A ratio is in its **simplest form** when each number in the ratio is a whole number, and these numbers are cancelled down as much as possible – that is, they have no common factors.

Example 15 Simplifying ratios

Express the following ratios in their simplest forms.

(a) $9 : 12 : 6$ (b) $0.5 : 1.25$

Solution

(a) Divide each number by 3.

$$9 : 12 : 6 = \frac{9}{3} : \frac{12}{3} : \frac{6}{3} = 3 : 4 : 2$$

The numbers 3, 4 and 2 have no common factors, so this ratio can't be cancelled down any further.

(b) Multiply by 100 to give integers, then cancel down.

$$0.5 : 1.25 = (100 \times 0.5) : (100 \times 1.25) = 50 : 125 = \frac{50}{25} : \frac{125}{25} = 2 : 5$$

Alternatively, you might spot that you can just multiply by 4.

$$0.5 : 1.25 = (4 \times 0.5) : (4 \times 1.25) = 2 : 5$$

Activity 36 Simplifying ratios

Express the following ratios in their simplest forms.

(a) $18 : 3$ (b) $12 : 60 : 18$ (c) $2 : 0.5 : 1.5$ (d) $6 : 12 : 7$

When you are working with ratios that contain just two numbers, it is sometimes helpful to convert them to the form 'number : 1'. For example, this can help you to compare different ratios. You can convert a ratio to this form by dividing both numbers by the second number. For example,

$$5 : 6 = \frac{5}{6} : \frac{6}{6} = 0.83 : 1 \quad \text{(to 2 d.p.)}.$$

Activity 37 Comparing ratios

A mother has the choice of two different after-school clubs for her child. Club A takes 46 children and has 6 staff, and club B takes 25 children and has 4 staff.

(a) Find the ratio of children to staff for each club in the form 'number : 1', rounding your answer to one decimal place.

(b) Which club has fewer children per member of staff?

Writing ratios in the form 'number : 1' can also help you to find approximate ratios, which can be useful when you want to compare quantities.

Example 16 Finding an approximate ratio

A secondary school has 823 boys and 534 girls on its roll. What is the approximate ratio of boys to girls at the school?

Solution

Find the ratio in the form 'number : 1', approximate the number by a whole number or simple fraction, then simplify the ratio.

The ratio of boys to girls is

$$\begin{aligned} 823 : 534 &= \frac{823}{534} : \frac{534}{534} \\ &= 1.54\ldots : 1. \end{aligned}$$

You can approximate $1.54\ldots$ by $1.5 = \frac{3}{2}$.

$$\begin{aligned} 1.54\ldots : 1 &\approx \tfrac{3}{2} : 1 \\ &= 3 : 2. \end{aligned}$$

So there are about three boys for every two girls.

The next activity asks you to find two approximate ratios.

Activity 38 Finding approximate ratios

In 2007 the population of the UK was about 61.0 million. About 31.9 million of these people were taxpayers (paying income tax). About 3.9 million of the taxpayers paid tax at the higher rate, and the remainder paid tax at the basic rate or less. Calculate approximate values for the following ratios.

(a) The ratio of taxpayers to non-taxpayers

(b) The ratio of ordinary taxpayers to higher-rate taxpayers (where the ordinary taxpayers are those paying tax at the basic rate or less)

Sometimes you need to divide a quantity in a particular ratio. The next example illustrates how to do this.

Example 17 Dividing a quantity in a ratio

Three flatmates, Amy, Becky and Carol, have agreed to contribute to their joint budget in the ratio $5:2:3$. (For every £5 Amy contributes, Becky contributes £2 and Carol contributes £3.) The flatmates' expenses amount to £1250 per month. How much does each flatmate contribute to this?

Solution

We have to divide £1250 in the ratio $5:2:3$.

The total number of parts in the ratio is $5+2+3=10$.

Amy contributes 5 of the 10 parts, so the amount that she contributes is

$$\frac{5}{10} \times £1250 = £625.$$

Similarly, Becky contributes

$$\frac{2}{10} \times £1250 = £250,$$

and Carol contributes

$$\frac{3}{10} \times £1250 = £375.$$

(Check: $£625 + £250 + £375 = £1250$.)

Activity 39 Dividing a quantity in a ratio

A bottle of screenwash for cars recommends the following ratios of screenwash to water.

Conditions	Ratio of screenwash to water
Summer	1 : 20
Winter	1 : 4
Severe winter	2 : 1

Calculate, to the nearest 100 ml, the volume of screenwash and the volume of water you would need to make 2 litres of diluted screenwash for your car in the following conditions.

(a) Winter

(b) Severe winter

Other forms of ratio

Ratios that contain just two numbers are sometimes written as fractions. For example, the ratio 3 : 2 can be written as $\frac{3}{2}$. The first and second numbers in the ratio become the numerator and denominator of the fraction, respectively. The fraction representing a ratio can also be written as a decimal: for example,

$$3 : 2 = \tfrac{3}{2} = 1.5.$$

This is why you sometimes see ratios given as single numbers. The mathematical term 'rational' arises from the fact that a rational number is the *ratio* of two integers.

The single number that represents a ratio is just the number that is obtained when the ratio is written in the form 'number : 1'. For example,

$$3 : 2 = \tfrac{3}{2} : \tfrac{2}{2} = 1.5 : 1.$$

You can use this fact to convert a ratio given as a single number into the usual colon form. For example, the ratio 1.4 is the same as

$$1.4 : 1 = 14 : 10 = 7 : 5.$$

Activity 40 Converting forms of ratio

Write the following ratios in colon form, and simplify them as much as possible.

(a) $\frac{3}{4}$ (b) 1.75 (c) 0.2

The fact that ratios can be written as single numbers also explains why you often see phrases such as 'the larger ratio'.

Activity 41 Comparing more ratios

In 2003 the United Kingdom had about 59.6 million people and 27.0 million cars, and Germany had 82.5 million people and 45.0 million cars.

(a) Calculate the ratio of cars to people for each of the two countries, to two significant figures.

(b) Which of the two countries had the larger ratio of cars to people?

4.2 Aspect ratios

Many forms of media involve rectangular shapes. For example, photographs, video images and sheets of paper are all usually rectangular. Some rectangles are long and thin, while others are closer to the shape of a square. The shape of a rectangle can be conveniently described using the idea of *aspect ratio*.

The **aspect ratio** of a rectangle is the ratio of its longer side to its shorter side. For example, the aspect ratio of the left-hand rectangle in Figure 19 is 25 : 15, which simplifies to 5 : 3. The aspect ratio of the right-hand rectangle is 10 : 6, which also simplifies to 5 : 3, so these two rectangles have the same aspect ratio. So the two rectangles have the same shape, though the second is smaller.

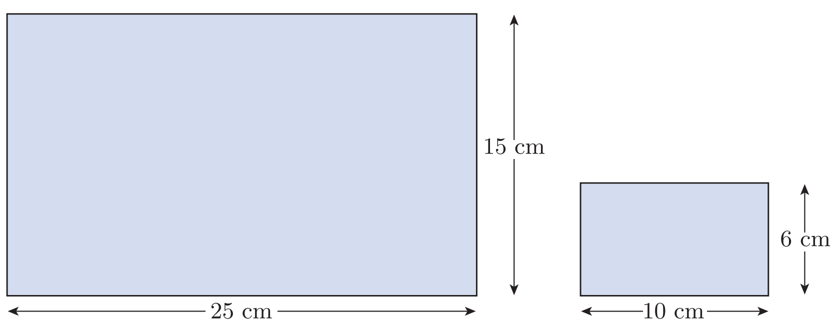
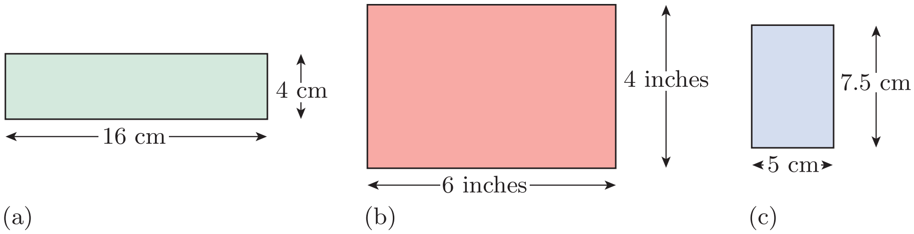


Figure 19 Two rectangles with aspect ratio 5 : 3

Activity 42 Finding aspect ratios

Find the aspect ratios of the following rectangles, in their simplest forms.



A rectangular image can be enlarged or reduced to any rectangle that has the same aspect ratio as the original image. If a different aspect ratio is required, then the image has to be cropped.

Photographs

Most digital cameras used by professional photographers and hobbyists produce images with aspect ratio 3 : 2 or 4 : 3, depending on the camera. (By contrast, most smartphone cameras typically take images with an aspect ratio of 4 : 3 or 16 : 9. This is more in line with the standards for video, which we look at later in this subsection.)

If your camera produces images with aspect ratio 4 : 3, and you want a print of size $20\,\text{cm} \times 15\,\text{cm}$, then this can be made without losing any part of the picture, because $20 : 15 = 4 : 3$. However, if you want a $15\,\text{cm} \times 10\,\text{cm}$ print, which has an aspect ratio of 3 : 2, then your photograph has to be cropped as illustrated in Figure 20(a). Similarly, if your camera produces images with aspect ratio 3 : 2, and you want a print with aspect ratio 4 : 3, then your photograph has to be cropped as illustrated in Figure 20(b).

(a) (b)

Figure 20 (a) A crop of a 4 : 3 image to give a 3 : 2 image. (b) A crop of a 3 : 2 image to give a 4 : 3 image.

Activity 43 Choosing photographic print sizes

The first column of the table below contains some standard photographic print sizes, which are available from many photograph-processing websites. The dimensions are in inches.

Print size	Aspect ratio in simplest form	Aspect ratio in form 'number : 1'
6×4		
7×5		
8×6		
9×6		
10×8		
12×8		

(a) Complete the second and third columns of the table, rounding the numbers in the third column to two decimal places.

(b) Which three print sizes are most appropriate for photographs taken with a camera that produces images with an aspect ratio of 3 : 2? Which print size is the next most appropriate?

Scale factors

If an image that measures $3\,\text{cm} \times 2\,\text{cm}$ is enlarged to $9\,\text{cm} \times 6\,\text{cm}$, then the width and the height both triple. We say that the **scale factor** is 3. Similarly, if the same image is instead reduced to $1.5\,\text{cm} \times 1\,\text{cm}$, then the width and height both halve, and the scale factor is $\frac{1}{2}$. In general,

$$\text{scale factor} = \frac{\text{new length}}{\text{old length}},$$

where the length is the width or height of the image, or the length of anything that appears in the image.

The scale factors displayed on photocopiers are usually expressed as percentages. For example, if you want a photocopier to produce an image that is double the height of the original image, then you need a scale factor of 2, so you would set the copier to enlarge by 200%.

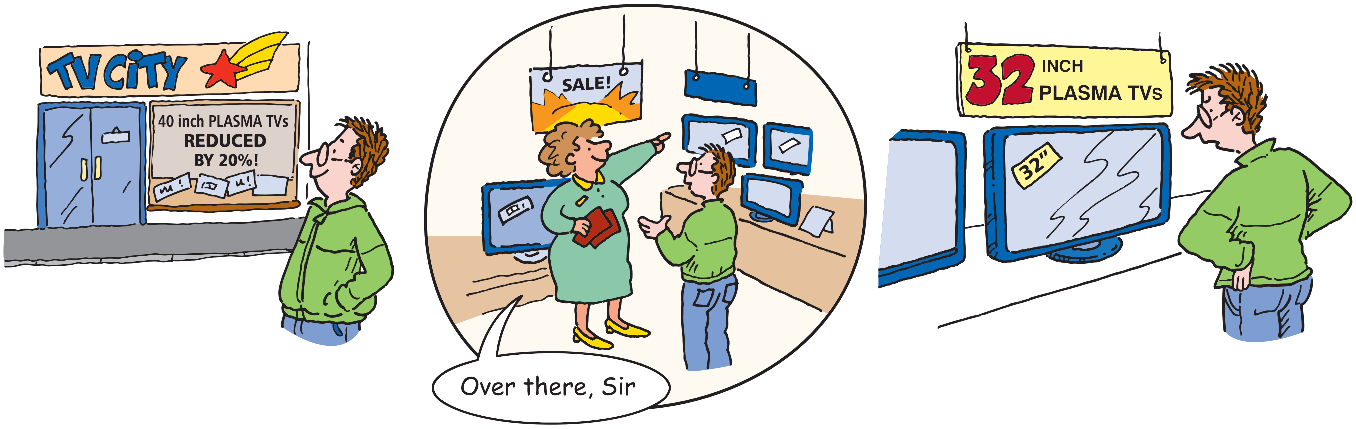
Activity 44 Finding scale factors

Find the scale factors of the following enlargements and reductions. Express each answer both as a number and as a percentage.

(a) An image measuring $4\,\text{cm} \times 3\,\text{cm}$ enlarged to $16\,\text{cm} \times 12\,\text{cm}$

(b) An image measuring $3\,\text{cm} \times 2\,\text{cm}$ enlarged to $7.5\,\text{cm} \times 5\,\text{cm}$

(c) An image measuring $20\,\text{cm} \times 10\,\text{cm}$ reduced to $4\,\text{cm} \times 2\,\text{cm}$



Videos

Aspect ratio is also an important issue for videos. Many older video programmes were made with an aspect ratio of 4 : 3, but in recent years 16 : 9 has become the most common video standard throughout the world. When a 4 : 3 image is displayed on a 16 : 9 screen, the image has to be *pillarboxed* (displayed with black bars on each side), stretched or cropped. Often a combination of these methods is used.

(a) (b) (c)

Figure 21 A 4 : 3 image of Harold Wilson (a) pillarboxed, (b) stretched and (c) cropped to appear on a 16 : 9 screen. The Open University was established by Harold Wilson's Government, and received its Charter on 23 April 1969.

Paper sizes

Finally in this subsection, we consider the aspect ratios of sheets of paper. You are probably familiar with the paper sizes A4, A3, and so on. The largest paper size in this series is A0, the next-largest is A1, and so on. This series of paper sizes is known as the ISO 216 standard. The ISO (International Organization for Standardization) sets standards for a wide range of products, and 216 is the number assigned by this organisation to this particular standard.

The paper sizes in the series were designed so that they all have the same aspect ratio. This means that an A4 image, for example, can be scaled up to an A3 one with no need for cropping. They were also designed to have the additional property that each size of paper is exactly the same size and shape as two of the next-smaller sizes placed side by side. This is illustrated in Figure 22.

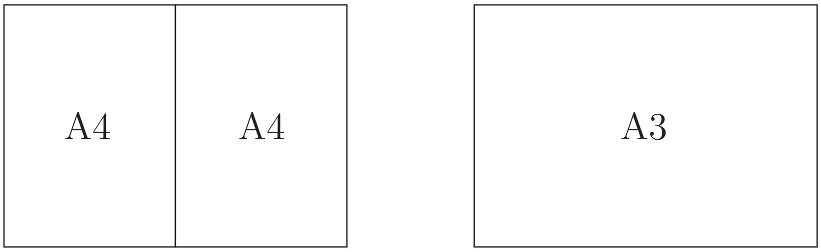


Figure 22 Two sheets of A4 make one sheet of A3

So, for example, if you fold an A3 sheet of paper in half, then it becomes the same size as a sheet of A4.There are various advantages of this property. For example, an envelope sized to fit an A5 sheet of paper will fit an A4 sheet folded in half, or an A3 sheet folded in quarters, and so on. You can see this property of the ISO paper sizes in the next activity.

Screencast

Activity 45 ISO paper sizes

View the screencast on the module website which demonstrates the mathematics behind the paper sizes of the ISO 216 Standard.

The aspect ratio that is needed if the paper sizes are to have the properties described above can be worked out as follows. Suppose that the aspect ratio needed is $a : 1$, where a represents some number.

Consider a sheet of paper with this aspect ratio. If its shorter side has length w cm, say, then its longer side has length aw cm, since $aw : w = a : 1$. This is shown on the left of Figure 23.

You can see from the right of Figure 23 that since two smaller sheets of paper must make one larger sheet, the next-larger size of paper measures $2w$ cm by aw cm.

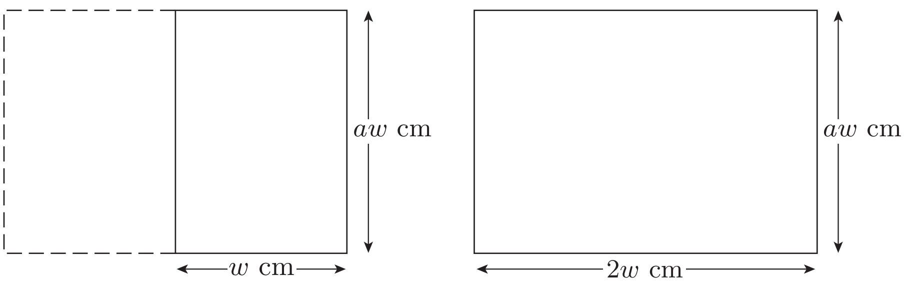


Figure 23 The dimensions of smaller and larger sheets of paper

So the aspect ratios of the two sizes of paper are

$$aw : w \quad \text{and} \quad 2w : aw.$$

These ratios can be simplified to

$$a : 1 \quad \text{and} \quad 2 : a,$$

by dividing each number by w.

Remember that the aspect ratio of a rectangle is the ratio of the length of its *longer* side to the length of its *shorter* side.

Since the two paper sizes have the same aspect ratio, these ratios must be equal. So that we can compare them, let's make the first ratio have second number a, the same as the second ratio. To do this, we multiply both numbers in the first ratio by a. So the two ratios are now

$$a^2 : a \quad \text{and} \quad 2 : a.$$

Since these ratios are equal, you can see that

$$a^2 = 2.$$

The number a must be positive because $a : 1$ is an aspect ratio. Therefore a must be $\sqrt{2}$. So the aspect ratio that is needed is $\sqrt{2} : 1$ – it involves an irrational number! Each size of paper in the ISO 216 standard has an aspect ratio of approximately $\sqrt{2} : 1$.

The ISO 216 paper sizes were developed in Germany in the early 1900s. They were adopted as a standard there in 1922, and soon spread to other European countries. The UK adopted them in 1959, and they were adopted by the International Organization for Standardization in 1975. They are now used throughout the world: at the time of writing the only exceptions are the USA, Canada, Peru, Colombia and the Dominican Republic. The largest ISO 216 paper size, A0, has an area of $1\,\mathrm{m}^2$.

Activity 46 Scaling ISO paper sizes

(a) Use the shorter sides of the rectangles in Figure 23 to work out the scale factor needed to enlarge from one ISO 216 paper size to the next-larger size.

(b) What scale factor is needed to reduce from one ISO 216 paper size to the next-smaller size?

(c) Explain why most photocopying machines offer the scale factors 141% and 71% as standard options for enlarging and reducing.

If you have not already done so, try the iCMA and TMA questions for this unit now.

Learning outcomes

After studying this unit, you should be able to:

- understand multiples and factors of natural numbers
- find lowest common multiples and highest common factors
- begin to investigate some simple properties of numbers
- find prime factorisations of natural numbers
- carry out calculations with numbers in index form, including those with negative and fractional indices
- carry out calculations with fractions
- understand and use scientific notation
- understand the difference between rational and irrational numbers
- simplify surds
- understand and use the concepts of ratio and aspect ratio.

Solutions and comments on Activities

Solution to Activity 1

(a) The first five multiples of 7 are 7, 14, 21, 28 and 35, because

$$1 \times 7 = 7,$$
$$2 \times 7 = 14,$$
$$3 \times 7 = 21,$$
$$4 \times 7 = 28,$$
$$5 \times 7 = 35.$$

(b) Since $4183 \div 11 = 380.272\ldots$, the number 4183 is not a multiple of 11. So the amount of money in the cash box is not correct.

Solution to Activity 2

(a) The multiples of 2 are 2, 4, 6, 8, ...
The multiples of 3 are 3, 6, 9, 12, ...
The multiples of 4 are 4, 8, 12, 16, ...
The multiples of 6 are 6, 12, 18, 24, ...
The multiples of 8 are 8, 16, 24, 32, ...

(i) The LCM of 2 and 3 is 6.

(ii) The LCM of 4 and 6 is 12.

(iii) The LCM of 4 and 8 is 8.

(b) The LCM of 2, 3 and 8 is 24. (This is the smallest number that is a multiple of all of 2, 3 and 8.)

Solution to Activity 3

(a) The smallest number of chocolates is the LCM of 2, 3 and 4, which is 12.

(b) The next suitable number of chocolates is $2 \times 12 = 24$.

Solution to Activity 4

(a) The first factor pair is 1, 20.
The next is 2, 10.
The next is 4, 5.
The next is 5, 4, which is a repeat, so the factors of 20 are 1, 2, 4, 5, 10 and 20.

(b) The first factor pair is 1, 24.
The next is 2, 12.
The next is 3, 8.
The next is 4, 6.
The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

(c) The first factor pair is 1, 45.
The next is 3, 15.
The next is 5, 9.
The factors of 45 are 1, 3, 5, 9, 15 and 45.

Solution to Activity 5

(a) The number 621 is divisible by 3, because the sum of its digits is $6 + 2 + 1 = 9$, which is divisible by 3.

(b) The number 273 is not divisible by 9, because the sum of its digits is $2 + 7 + 3 = 12$, which is not divisible by 9.

Solution to Activity 6

(a) The common factors of 20 and 24 are 1, 2 and 4. So their highest common factor is 4.

(b) The common factors of 12 and 24 are 1, 2, 3, 4, 6 and 12. So their highest common factor is 12.

(c) The common factors of 18, 20 and 24 are 1 and 2. So their highest common factor is 2.

Solution to Activity 7

A number ending in 0, 2, 4, 6 or 8 is even and so is divisible by 2. A number ending in 0 or 5 is divisible by 5. So if a number ends in 0, 2, 4, 5, 6 or 8, then it is divisible by either 2 or 5, and hence it is not prime, unless it is 2 or 5 itself. So every prime number except 2 and 5 ends with one of the other possible digits, 1, 3, 7 or 9. You can see from the list of prime numbers on page 143 that each of these possible digits occurs.

Solution to Activity 8

(a)

Prime number	3	5	7	11	13	17	19	23	29
Remainder	3	1	3	3	1	1	3	3	1

(b)

Prime number	3	5	7	11	13	17	19	23	29
Sum of two squares?	×	✓	×	×	✓	✓	×	×	✓

($5 = 4 + 1$, $13 = 9 + 4$, $17 = 16 + 1$ and $29 = 25 + 4$.)

(c) It seems that the odd prime numbers that can be written as the sum of two square numbers are those that have remainder 1 when they are divided by 4.

To test this conjecture, you could consider the prime numbers 31 and 37, for example.

The first of these numbers, 31, has remainder 3 when it is divided by 4, and it cannot be written as the sum of two square numbers. To see this, notice that if 31 were the sum of two square numbers, then one of the square numbers would be greater than $31/2 = 15.5$, and the other square number would be less than this. So one of the square numbers must be either 16 or 25, and the other must be 1, 4 or 9. Also, one of the square numbers must be odd and the other must be even. So the only possibilities are $16 + 1$, $16 + 9$ and $25 + 4$, and none of these sums is equal to 31.

The second number, 37, has remainder 1 when it is divided by 4, and it can be written as the sum of two square numbers: $37 = 36 + 1$. So the two prime numbers 31 and 37 provide further evidence for the conjecture.

It has been proved that the conjecture above is true. That is, the odd prime numbers that can be written as a sum of two square numbers are those that have remainder 1 when they are divided by 4. This theorem is known as **Fermat's Christmas Theorem**, because the French mathematician Pierre de Fermat (1601–1665) announced it in a letter to Marin Mersenne dated 25 December 1640. Fermat's proof of the theorem was incomplete, however, and the missing steps were provided by the Swiss mathematician Leonhard Euler (1707–1783) about a hundred years later.

Solution to Activity 9

(a) $300 = 2^2 \times 3 \times 5^2$

(b) Any factor tree gives the same answer.

Solution to Activity 10

(a) $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

(b) $855 = 3 \times 3 \times 5 \times 19 = 3^2 \times 5 \times 19$

(c) $1000 = 2^3 \times 5^3$

(d) $847 = 7 \times 11^2$

Solution to Activity 11

$$\begin{aligned} 9 &= \phantom{2 \times {}} 3^2 \\ 18 &= 2 \times 3^2 \\ 30 &= 2 \times 3 \times 5 \end{aligned}$$

(a) The LCM of 18 and 30 is $2 \times 3^2 \times 5 = 90$.
The HCF of 18 and 30 is $2 \times 3 = 6$.

(b) The LCM of 9, 18 and 30 is $2 \times 3^2 \times 5 = 90$.
The HCF of 9, 18 and 30 is 3.

Solution to Activity 12

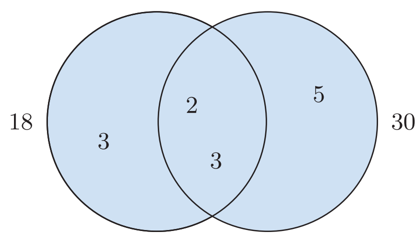
(a) The LCM is the product of all the numbers inside the circles, which is

$$3 \times 2 \times 2 \times 7 \times 2 \times 5 = 840,$$

and the HCF is the product of the numbers in the overlap, which is

$$2 \times 2 \times 7 = 28.$$

(b) The diagram is below.



The LCM of 18 and 30 is the product of all the numbers inside the circles, which is

$$3 \times 2 \times 3 \times 5 = 90.$$

The HCF of 18 and 30 is the product of the numbers in the overlap, which is

$$2 \times 3 = 6.$$

Solution to Activity 13

(a) (i) $3^4 \times 3^3 = 3^{4+3} = 3^7$

(ii) $7^2 \times 7 = 7^{2+1} = 7^3$

(iii) $10^2 \times 10^3 \times 10^4 = 10^{2+3+4} = 10^9$

(iv) $3^4 \times 5^{12}$ cannot be written any more concisely in index form, because the base numbers are different.

(v) $8 \times 2^5 = 2^3 \times 2^5 = 2^{3+5} = 2^8$

(vi) $9 \times 3 = 3^2 \times 3 = 3^{2+1} = 3^3$

(b) $$\begin{aligned} 294 \times 441 &= (2 \times 3 \times 7^2) \times (3^2 \times 7^2) \\ &= 2 \times 3^{1+2} \times 7^{2+2} \\ &= 2 \times 3^3 \times 7^4 \end{aligned}$$

Solution to Activity 14

An estimate for the number of atoms in the observable universe, obtained using the figures in the question, is

$$\begin{aligned} 10^{27} \times 10^{30} \times 10^{12} \times 10^{11} &= 10^{27+30+12+11} \\ &= 10^{80}. \end{aligned}$$

This is much less than a googol, which is 10^{100}.

Solution to Activity 15

(a) (i) $7^6 \div 7^2 = \dfrac{7^6}{7^2} = 7^{6-2} = 7^4$

(ii) $\dfrac{2^{16}}{2^8} = 2^{16-8} = 2^8$

(iii) The quotient $\dfrac{5^{21}}{3^5}$ cannot be written any more concisely in index form, because the base numbers are different.

(iv) $\dfrac{3^7}{3} = 3^{7-1} = 3^6$

(b) $\dfrac{3456}{12} = \dfrac{2^7 \times 3^3}{2^2 \times 3} = 2^{7-2} \times 3^{3-1} = 2^5 \times 3^2$

Solution to Activity 16

(a) $(5^2)^4 = 5^8$

(b) $(7^3)^2 = 7^6$

(c) $(3^5)^3 \times 3^2 = 3^{15} \times 3^2 = 3^{17}$

(d) $\dfrac{(2^5)^2}{2^2} = \dfrac{2^{10}}{2^2} = 2^8$

(e) $\left(\dfrac{2^5}{2^2}\right)^2 = (2^3)^2 = 2^6$

Solution to Activity 17

(a) (i) $21^4 = (3 \times 7)^4 = 3^4 \times 7^4$

(ii) $24^3 = (2^3 \times 3)^3 = \left(2^3\right)^3 \times 3^3 = 2^9 \times 3^3$

(b) (i) $\left(\dfrac{2}{7}\right)^2 = \dfrac{2^2}{7^2} = \dfrac{4}{49}$

(ii) $\left(\dfrac{3}{4}\right)^3 = \dfrac{3^3}{4^3} = \dfrac{27}{64}$

Solution to Activity 18

(a) $(-3)^2 = (-3) \times (-3) = 9$

(b) $$\begin{aligned} (-3)^3 &= (-3) \times (-3) \times (-3) \\ &= 9 \times (-3) = -27 \end{aligned}$$

(c) $$\begin{aligned} (-2)^4 &= (-2) \times (-2) \times (-2) \times (-2) \\ &= 4 \times 4 = 16 \end{aligned}$$

(d) $$\begin{aligned} (-1)^4 &= (-1) \times (-1) \times (-1) \times (-1) \\ &= 1 \times 1 = 1 \end{aligned}$$

(e) $(-1)^5 = (-1)^4 \times (-1) = 1 \times (-1) = -1$

Solution to Activity 19

(a) (i) $\frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}$

(ii) $\frac{7}{8} + \frac{9}{24} = \frac{21}{24} + \frac{9}{24} = \frac{30}{24} = \frac{5}{4} = 1\frac{1}{4}$

(iii) $\frac{11}{14} - \frac{3}{14} = \frac{8}{14} = \frac{4}{7}$

(iv) $\frac{5}{6} - \frac{1}{4} = \frac{10}{12} - \frac{3}{12} = \frac{7}{12}$

(v) $2\frac{1}{7} + 4\frac{2}{7} = 2 + 4 + \frac{1}{7} + \frac{2}{7} = 6\frac{3}{7}$

(vi) $$\begin{aligned} 3\tfrac{3}{4} - 1\tfrac{1}{5} &= 3 - 1 + \tfrac{3}{4} - \tfrac{1}{5} \\ &= 2 + \tfrac{15}{20} - \tfrac{4}{20} = 2\tfrac{11}{20} \end{aligned}$$

(vii) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12} = 1\frac{1}{12}$

(b) Half of all drivers were under the age of 35, so half were aged 35 or over. One-seventh were over the age of 65, so the fraction of drivers who were between 35 and 65 years of age was

$$\tfrac{1}{2} - \tfrac{1}{7} = \tfrac{7}{14} - \tfrac{2}{14} = \tfrac{5}{14}.$$

So in 2010, five-fourteenths of UK drivers were aged between 35 and 65. (This is slightly more than a third of UK drivers, since $\frac{5}{15} = \frac{1}{3}$.)

Solution to Activity 20

(a) (i) $\dfrac{5}{8} \times \dfrac{3}{10} = \dfrac{\overset{1}{\cancel{5}}}{8} \times \dfrac{3}{\underset{2}{\cancel{10}}} = \dfrac{3}{16}$

(ii) $\frac{4}{5} \times 3 = \frac{4}{5} \times \frac{3}{1} = \frac{12}{5} = 2\frac{2}{5}$

(The answer can be left as a top-heavy fraction or written as a mixed number.)

(iii) $1\frac{1}{3} \times 2\frac{5}{6} = \frac{4}{3} \times \frac{17}{6}$

$$= \dfrac{\overset{2}{\cancel{4}}}{3} \times \dfrac{17}{\underset{3}{\cancel{6}}} = \tfrac{34}{9} = 3\tfrac{7}{9}$$

(b) The fraction of students who work for more than 35 hours per week is

$$\dfrac{2}{5} \times \dfrac{1}{4} = \dfrac{\overset{1}{\cancel{2}}}{5} \times \dfrac{1}{\underset{2}{\cancel{4}}} = \dfrac{1}{10}.$$

Solution to Activity 21

If a quarter of teenage girls and less than a quarter of teenage boys smoke, then less than a quarter of all teenagers smoke. The fraction five-twelfths is greater than a quarter, so it cannot be correct.

The fraction $\frac{5}{12}$ is obtained by adding $\frac{1}{4}$ and $\frac{1}{6}$, but this is not the correct calculation. The correct calculation is as follows.

About half of teenagers are girls, so the fraction of teenagers who are girls and smokers is

$$\tfrac{1}{2} \times \tfrac{1}{4} = \tfrac{1}{8}.$$

Similarly, the fraction of teenagers who are boys and smokers is

$$\tfrac{1}{2} \times \tfrac{1}{6} = \tfrac{1}{12}.$$

Therefore the fraction of all teenagers who smoke is

$$\tfrac{1}{8} + \tfrac{1}{12} = \tfrac{3}{24} + \tfrac{2}{24} = \tfrac{5}{24}.$$

(The fraction $\frac{5}{24}$ is halfway between $\frac{1}{4}$ and $\frac{1}{6}$.)

Solution to Activity 22

(a) (i) $6 \div \dfrac{4}{3} = \dfrac{6}{1} \times \dfrac{3}{4} = \dfrac{\overset{3}{\cancel{6}}}{1} \times \dfrac{3}{\underset{2}{\cancel{4}}} = \dfrac{9}{2} = 4\dfrac{1}{2}$

(ii) $\dfrac{3}{8} \div \dfrac{11}{24} = \dfrac{3}{8} \times \dfrac{24}{11} = \dfrac{3}{\underset{1}{\cancel{8}}} \times \dfrac{\overset{3}{\cancel{24}}}{11} = \dfrac{9}{11}$

(iii) $1\frac{1}{3} \div 1\frac{7}{9} = \dfrac{4}{3} \div \dfrac{16}{9} = \dfrac{4}{3} \times \dfrac{9}{16} = \dfrac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{3}}} \times \dfrac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{16}}} = \dfrac{3}{4}$

(b) The number of components that the factory worker can make in a week is

$$37\tfrac{1}{2} \div 1\tfrac{1}{4} = \dfrac{75}{2} \div \dfrac{5}{4} = \dfrac{75}{2} \times \dfrac{4}{5} = \dfrac{\overset{15}{\cancel{75}}}{\underset{1}{\cancel{2}}} \times \dfrac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{5}}} = 30.$$

Solution to Activity 23

(a) $\left(\frac{1}{2}\right)^0 = 1$

(b) $7^{-1} = \frac{1}{7}$

(c) $7^{-2} = \dfrac{1}{7^2} = \dfrac{1}{49}$

(d) $\left(\frac{1}{3}\right)^{-1} = 3$

(e) $\left(\frac{2}{5}\right)^{-1} = \frac{5}{2}$

(f) $\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$

(g) $\left(\frac{2}{5}\right)^{-2} = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

(h) $\left(\frac{1}{3}\right)^{-3} = 3^3 = 27$

(i) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

Solution to Activity 24

(a) (i) $7723 = 7.723 \times 10^3$

(ii) $50\,007\,000 = 5.0007 \times 10^7$

(iii) $0.100\,34 = 1.0034 \times 10^{-1}$

(iv) $0.000\,208 = 2.08 \times 10^{-4}$

(b) (i) The population of the world at the time of writing is about 7.801×10^9 people.

(ii) The mass of the Sun is about 1.99×10^{30} kg.

(iii) The mass of a hydrogen atom is about 1.674×10^{-27} kg.

(c) (i) $7.04 \times 10^3 = 7040$

(ii) $4.52 \times 10^4 = 45\,200$

(iii) $7.3 \times 10^{-2} = 0.073$

(iv) $2.045 \times 10^{-5} = 0.000\,020\,45$

Solution to Activity 25

(a) 1.5 trillion $= 1.5 \times 10^{12}$, which is in scientific notation, and 61 million $= 61 \times 10^6 = 6.1 \times 10^7$.

(b) The amount of public debt per person, in pounds, is

$$\begin{aligned}\frac{1.5 \times 10^{12}}{6.1 \times 10^7} &= \frac{1.5}{6.1} \times \frac{10^{12}}{10^7}\\ &= 0.25 \times 10^{12-7} \quad \text{(to 2 s.f.)}\\ &= 0.25 \times 10^5\\ &= 25\,000\end{aligned}$$

So the headline would be as follows.

> Bailouts add £1.5 trillion to Britain's public debt – that's about £25 000 for each person!

Solution to Activity 26

(a) The approximate length in metres of an ordinary guitar, divided by the length in metres of the nano guitar, is

$$\frac{1}{10 \times 10^{-6}} = \frac{1}{10^{1+(-6)}} = \frac{1}{10^{-5}} = 10^5 = 100\,000.$$

So the nano guitar is 100 000 times smaller than an ordinary guitar.

(You can work out that $\dfrac{1}{10^{-5}} = 10^5$ in either of the following ways. You can use the index law $a^{-n} = \dfrac{1}{a^n}$:

$$\frac{1}{10^{-5}} = 10^{-(-5)} = 10^5.$$

Or you can use the index law $\dfrac{a^m}{a^n} = a^{m-n}$:

$$\frac{1}{10^{-5}} = \frac{10^0}{10^{-5}} = 10^{0-(-5)} = 10^5.)$$

(b) The approximate width in metres of a human hair, divided by the width in metres of a string of the nano guitar, is

$$\begin{aligned}\frac{100 \times 10^{-6}}{50 \times 10^{-9}} &= \frac{100}{50} \times \frac{10^{-6}}{10^{-9}}\\ &= 2 \times 10^{-6-(-9)}\\ &= 2 \times 10^3\\ &= 2000.\end{aligned}$$

So a string of the nano guitar is 2000 times less wide than a human hair.

Solution to Activity 28

(a) $\sqrt{9} = 3$

(b) $\sqrt[3]{8} = 2$

(c) $\sqrt[3]{27} = 3$

(d) $\sqrt[4]{16} = 2$

(e) Two square roots of 9 are ± 3.

(f) Two fourth roots of 16 are ± 2.

Solution to Activity 29

(a) $\sqrt{1764} = \sqrt{36} \times \sqrt{49} = 6 \times 7 = 42$

(b) **(i)** $\sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$

(ii) $\sqrt{\dfrac{36}{49}} = \dfrac{\sqrt{36}}{\sqrt{49}} = \dfrac{6}{7}$

(iii) $\sqrt{\dfrac{1}{4}} = \dfrac{\sqrt{1}}{\sqrt{4}} = \dfrac{1}{2}$

Solution to Activity 30

(a) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

(b) $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

(c) The surd $\sqrt{15}$ is already in its simplest form, since the factors of 15 greater than 1 are 3, 5 and 15, and none of these factors is a square number.

(d) $\sqrt{56} = \sqrt{4 \times 14} = \sqrt{4}\sqrt{14} = 2\sqrt{14}$

(The root $\sqrt{14}$ cannot be simplified.)

(e) $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$

Solution to Activity 31

(a) $(\sqrt{7})^2 = \sqrt{7} \times \sqrt{7} = 7$

(b) $\sqrt{7} \times 3\sqrt{7} = 3 \times 7 = 21$

(c)

$$\begin{aligned}\sqrt{7} \times \sqrt{14} &= \sqrt{7 \times 14}\\ &= \sqrt{7 \times 7 \times 2}\\ &= \sqrt{7 \times 7} \times \sqrt{2}\\ &= 7\sqrt{2}\end{aligned}$$

(d) $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$

(e) $2\sqrt{3} \times 3\sqrt{2} = 6\sqrt{6}$

(f) $$\begin{aligned} 2\sqrt{3}\times 2\sqrt{15} &= 4\sqrt{3\times 15}\\ &= 4\sqrt{3\times 3\times 5}\\ &= 4\sqrt{3\times 3}\times\sqrt{5}\\ &= 4\times 3\sqrt{5}\\ &= 12\sqrt{5}\end{aligned}$$

Solution to Activity 32

(a) $\dfrac{\sqrt{10}}{\sqrt{2}} = \dfrac{\sqrt{2}\times\sqrt{5}}{\sqrt{2}} = \sqrt{5}$

(b) $\dfrac{5}{\sqrt{5}} = \dfrac{\sqrt{5}\times\sqrt{5}}{\sqrt{5}} = \sqrt{5}$

(c) $\dfrac{\sqrt{8}}{\sqrt{2}} = \dfrac{\sqrt{4\times 2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}} = 2$

(d) $\dfrac{8}{\sqrt{2}} = \dfrac{4\times 2}{\sqrt{2}} = \dfrac{4\times\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = 4\sqrt{2}$

Solution to Activity 33

(a) $\sqrt{3}+\sqrt{3} = 2\sqrt{3}$

(b) The roots in the surd $\sqrt{2}+\sqrt{5}$ are different (and are in their simplest forms), so the surd cannot be simplified.

(c) $7\sqrt{3} - 2\sqrt{3} = 5\sqrt{3}$

(d) $$\begin{aligned} 5\sqrt{8} - 2\sqrt{2} &= 5\sqrt{4\times 2} - 2\sqrt{2}\\ &= 5\times 2\sqrt{2} - 2\sqrt{2}\\ &= 10\sqrt{2} - 2\sqrt{2}\\ &= 8\sqrt{2}\end{aligned}$$

Solution to Activity 34

(a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

(b) $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$

(c) $4^{-\frac{1}{2}} = \dfrac{1}{4^{\frac{1}{2}}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

(d) $4^{\frac{5}{2}} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

(e) $27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

Solution to Activity 36

(a) $18 : 3 = 6 : 1$

(b) $12 : 60 : 18 = 2 : 10 : 3$

(c) $2 : 0.5 : 1.5 = 4 : 1 : 3$

(d) The ratio $6 : 12 : 7$ is already in its simplest form.

Solution to Activity 37

(a) The ratio of children to staff for club A is

$$46 : 6 = \frac{46}{6} : \frac{6}{6} \approx 7.7 : 1.$$

The ratio of children to staff for club B is

$$25 : 4 = \frac{25}{4} : \frac{4}{4} \approx 6.3 : 1.$$

(b) Club B has fewer children per member of staff.

Solution to Activity 38

(a) The number of non-taxpayers, in millions, was approximately

$$61.0 - 31.9 = 29.1.$$

So the ratio of taxpayers to non-taxpayers was approximately

$$31.9 : 29.1 = \frac{31.9}{29.1} : \frac{29.1}{29.1} = 1.096\ldots : 1 \approx 1 : 1.$$

So there was about one non-taxpayer for every taxpayer.

(b) The number of ordinary taxpayers, in millions, was approximately

$$31.9 - 3.9 = 28.0.$$

So the ratio of ordinary taxpayers to higher-rate taxpayers was approximately

$$\begin{aligned} 28.0 : 3.9 &= \frac{28.0}{3.9} : \frac{3.9}{3.9}\\ &= 7.179\ldots : 1\\ &\approx 7 : 1.\end{aligned}$$

So there were about seven ordinary taxpayers for every higher-rate taxpayer.

Solution to Activity 39

(a) In winter conditions the recommended ratio of screenwash to water is 1 : 4, so there are $1 + 4 = 5$ parts.

The volume of screenwash needed is

$$\tfrac{1}{5} \times 2000\,\text{ml} = 400\,\text{ml},$$

and the volume of water needed is

$$\tfrac{4}{5} \times 2000\,\text{ml} = 1600\,\text{ml}.$$

(Check: $400 + 1600 = 2000$.)

(b) In severe winter conditions the recommended ratio of screenwash to water is 2 : 1, so there are $2 + 1 = 3$ parts.

The volume of screenwash needed is

$$\tfrac{2}{3} \times 2000\,\text{ml} \approx 1300\,\text{ml},$$

and the volume of water needed is

$$\tfrac{1}{3} \times 2000\,\text{ml} \approx 700\,\text{ml}.$$

(Check: $1300 + 700 = 2000$.)

Solution to Activity 40

(a) $\frac{3}{4} = 3 : 4$

(b) $1.75 = 1.75 : 1 = (1.75 \times 4) : 4 = 7 : 4$

(c) $0.2 = 0.2 : 1 = (0.2 \times 5) : 5 = 1 : 5$

Solution to Activity 41

(a) The ratio of cars to people in the United Kingdom was

$$\frac{27.0}{59.6} \approx 0.45,$$

and the ratio of cars to people in Germany was

$$\frac{45.0}{82.5} \approx 0.55.$$

(b) Germany had the larger ratio of cars to people. (It had about 55 cars for every 100 people, whereas the UK had about 45 cars for every 100 people.)

Solution to Activity 42

(a) The aspect ratio is $16 : 4 = 4 : 1$.

(b) The aspect ratio is $6 : 4 = 3 : 2$.

(c) The aspect ratio is $7.5 : 5 = 15 : 10 = 3 : 2$.

Remember that the aspect ratio of a rectangle is the ratio of its *longer* side to its *shorter* side.

Solution to Activity 43

(a)

Print size	Aspect ratio in simplest form	Aspect ratio in form 'number : 1'
6×4	3 : 2	1.50 : 1
7×5	7 : 5	1.40 : 1
8×6	4 : 3	1.33 : 1
9×6	3 : 2	1.50 : 1
10×8	5 : 4	1.25 : 1
12×8	3 : 2	1.50 : 1

(b) The three most appropriate print sizes for photographs taken with a camera that produces 3 : 2 images are 6×4, 9×6 and 12×8, since all of these have aspect ratio 3 : 2.

The next most appropriate print size is 7×5. This is because $3 : 2 = 1.50 : 1$, and the aspect ratio in the table closest to $1.50 : 1$ is $1.40 : 1 = 7 : 5$. So less of the picture will be lost in a print with aspect ratio 7 : 5 than in a print with aspect ratio 4 : 3, for example.

Solution to Activity 44

(a) The scale factor is $\dfrac{16}{4} = 4 = 400\%$.

(b) The scale factor is $\dfrac{7.5}{3} = 2.5 = 250\%$.

(c) The scale factor is $\dfrac{4}{20} = 0.2 = 20\%$.

Solution to Activity 46

(a) By looking at the shorter sides of the rectangles in Figure 23, you can see that a length of w cm must be scaled up to a length of aw cm. So the scale factor needed is $a = \sqrt{2}$.

(b) Similarly, from Figure 23 you can see that a length of aw cm must be scaled down to a length of w cm, so the scale factor needed is $\dfrac{1}{a} = \dfrac{1}{\sqrt{2}}$.

(c) $\sqrt{2} \approx 1.41 = 141\%$ and $\dfrac{1}{\sqrt{2}} \approx 0.71 = 71\%$, so the given values are the scale factors needed for enlarging or reducing from one ISO 216 paper size to the next size.