Unit 1
Starting points

Welcome to MU123!

This unit is the first step on a mathematical journey which will be full of ideas and exploration. Mathematics has a fundamental role in almost every aspect of modern life, from the media and business to science and medicine, and beyond. As mathematics plays such an important part in the world, using mathematics in practical situations is one of the four main themes of the module. You'll see how you can use mathematics in everyday life – for example, how to use it to make sense of the numbers you see in the media, and how to interpret tables of data. And you'll see some more specialised applications of mathematics, such as how mathematics is used in developing route-planning software to help drivers get from one place to another as quickly as possible.

But there's more to mathematics than solving practical problems: the subject is also fascinating in its own right. It can be appreciated for its beauty, and it can surprise, delight and challenge you just as much as subjects like literature and art. Abstract mathematics is the second theme of the module. You'll be introduced to some surprising facts about numbers and number patterns, and you'll learn how it's possible to prove that certain facts hold not just for a few numbers, but for infinitely many. Remarkably, some areas of abstract mathematics that initially seemed to have no practical applications now have important uses: for example, modern-day data encryption relies heavily on an abstract subject known as number theory, as you will see.

The third theme of the module is developing your mathematical skills. You need to be able to work confidently with numbers and other elements of mathematics if you're to be effective in using mathematics in your everyday life, in your workplace or in further modules. You'll learn new mathematical skills in statistics, geometry and trigonometry, and some fundamental skills in algebra, that will open up new possibilities for your future study and career options.

The final theme of the module is mathematical communication. You'll learn how to interpret mathematics that's presented to you, and how to present your mathematics to others so that it can be easily understood. You'll find that the feedback you get from your tutor is particularly helpful in helping you to develop your skills in mathematical communication.

The four themes of the module are interlinked: most parts of the module involve more than one of them. In summary, in MU123 you will develop your mathematical skills and your abilities in mathematical communication by studying both practical and abstract mathematical ideas.

Figure 1 The two sides of the Ishango bone

Mathematics has been used to describe and explain the world for thousands of years. The regular marks engraved on a piece of bone found in Africa, known as the Ishango bone (Figure 1), appear to indicate that people were counting and thinking mathematically over 20 000 years ago. More sophisticated mathematical activities such as measuring size and using money were also adopted by many civilisations thousands of years ago.

Introduction

This book contains the first four of the fourteen units in the module; this first unit introduces you to the main themes of the module and the different teaching materials that are used. It also helps you to revise some basic mathematical skills and to start studying mathematics more effectively.

Section 1 gives you some brief advice on studying the module.

In Sections 2 and 3 you will revise some basic mathematical skills, such as rounding numbers, using your calculator and working with negative numbers, fractions and percentages. Your study of later units will be much easier if you are fluent in these basic skills, so these sections give you an opportunity to practise them and make sure that there are no gaps in your understanding. In these sections you will also see some everyday applications of mathematics, and how the media (newspapers, television, radio and the internet) use numbers.

You will need to keep the MU123 Guide to hand throughout your study of Sections 2 and 3, as it contains several calculator activities that you will be asked to do. If you are not using the recommended calculator, then you may need your calculator manual as well. If you do not have the MU123 Guide to hand when you reach these activities, then you can return to them later.

Section 4 introduces you to abstract mathematics. You will see how two simple ideas – adding odd numbers and folding paper strips – can be developed mathematically, with intriguing results.

The final two sections in this unit will help you to think constructively about your study methods. Section 5 gives advice on doing your module assignments and introduces you to the topic of how to write mathematics so that it can be easily understood by others. Section 6 looks at how you might review and improve your study methods.

1 Studying MU123

This short first section tells you what to expect in the module, and helps you to start planning a method of study that will be effective for you.

1.1 The module units

Each unit of MU123 is structured in a similar way. As well as explanations of the mathematics, there are worked examples to study and activities to do. The worked examples show you how to do the mathematics and how to set out your working, and in the activities you are asked to do some mathematics yourself. The best way to learn mathematics is to practise it, so it is important to work through the activities carefully, and to do as many as you can. There are solutions at the end of each unit, which you can use to check your answers, or to obtain a hint if you are stuck!

Key facts and strategies are highlighted in pink boxes, so that you can refer to them easily. You will also see blue boxes, like the one below, which tell you some of the rich history of mathematics, or contain other interesting items.

> 'Exercise is the beste instrument in learnyng.'
>
> Robert Recorde (1510–1558), mathematician.

Next to some worked examples and activities you will see a computer icon, like the one alongside Activity 1 below. This icon means that you need to use your computer, for example for software and tutorial clips (clips in which tutors explain worked examples).

As you work through each unit you may find it helpful to keep the Handbook to hand. You can use it to look up ideas covered in other units, and you can annotate it with helpful notes and examples.

The first activity asks you to watch the first module video, which introduces you to some of the ideas in the module and gives you a glimpse of what it is like to be a mathematician. It doesn't contain any details that you have to *learn*, so there is no need to take notes – just sit down and enjoy it! If it is not convenient for you to watch the video now, then you can leave it until later, and continue working through the unit. But try to watch the video as soon as possible, as it will help set the scene for studying the unit.

Activity 1 The first module video

Video

Watch the video 'Welcome to MU123 Discovering mathematics'.

1.2 Studying effectively

Before you start on the mathematics in MU123, it's a good idea to check that you're organised to study effectively, especially if this is your first module with The Open University. This subsection will help you to do that, and there is further advice throughout the module.

Finding time and using it effectively

It is important for you to think about how your study of MU123 will fit in with the rest of your life, and what adjustments you may need to make.

Each unit is designed to take about 13 hours of study for an average student: the time that you will need will depend on your mathematical background, and some units may take you longer than others. You will also need to allow some extra time for the assignment questions and other activities such as tutorials. It is important to keep up with the schedule in the study planner as much as possible, or you could find that you have run out of time to study the units needed to complete an assignment before its cut-off date. So try to plan your study times, fitting them in with your other commitments, to make sure that you do not fall behind.

Some people study late in the evening while others are more alert first thing in the morning. Which are you: a nightingale or a lark – or neither? What is the smallest study period that is likely to be useful to you? Some students find that even a ten-minute burst is enough to try an activity or to keep them going and set them musing, whereas others find that they need at least half an hour for effective study.

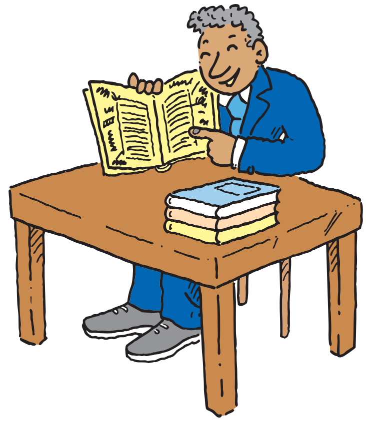
> 'I can work at home and in my office, and those are the two places I work most. Just anywhere where I've got a pad of paper and a biro ... But if you're sitting waiting in an airport lounge, which for many people would be a very boring experience, for a mathematician it isn't. Get out some paper and have a think about things.'
>
> Timothy Gowers, Professor of Mathematics, University of Cambridge, and Fields medallist.

If you do not have continuous access to a computer, then it is worth checking which sections, or parts of sections, can be studied without one, and planning ahead accordingly. However, you are advised to study the module material in the order in which it is presented, as in later parts you will often need some of the skills and knowledge covered earlier.

It can take some time to find out what study pattern works best for you, and it's worth thinking about this as you work through this first unit.

Making notes

Some students find that making their own notes in a notebook or on sheets of paper is an invaluable part of their learning process, as putting things in their own words helps them to make sense of ideas and remember them. These notes may include summaries of the main ideas, new terminology and notation, things learned from doing the activities, useful examples or even diagrams summarising the main ideas and how they fit together. Other students use different methods to help them to understand and remember the material, such as highlighting bits of the text, jotting brief notes and questions in the margins, and making annotations in the Handbook.



Bal was so delighted with his marginal annotations that he had them published in hardback edition!

Whatever you decide about making notes, you are advised to write out your solutions to the activities, and to keep them organised. It can be useful to refer to them when you do assignment questions – particularly if you annotate them with brief notes about anything you first got wrong but then corrected, or found difficult but then resolved. Writing out your solutions in full will also give you useful practice in writing mathematics well. This is a skill that you will need for the assignment questions, and there is advice on it in Section 5.

It is a good idea to think about how to keep track of any non-urgent questions that you want to ask your tutor by phone or at a tutorial. For example, you could make a list in your notes, or stick labels on the edges of relevant pages.

Getting help

Remember that you are not alone with the module materials. Your tutor is there to help you with any mathematical problems that you encounter, and he or she can also provide advice on other matters to do with your progress on the module, such as what you should do if you are worried about completing a part of the module in time. You can also discuss the module with other students, for example at tutorials or on the online module forum. They may be able to help you with certain topics, and you may be able to help them too. Trying to explain an idea to someone is often an excellent way of learning – especially if they ask lots of questions! You should also find it interesting and reassuring just to see how other students are studying the module.

If you are stuck on a particular mathematical point, then it's worth spending a few minutes trying to resolve it yourself – if you can, then you are likely to learn from the process, and you should remember what you have learned more easily in future. It may help to look back at the material that led up to that point, and make sure that you fully understand it. But you should not spend a large amount of time puzzling over a particular point without making progress. Many difficulties can be resolved rapidly if you contact your tutor or post a message on the module forum, leaving you more time to get on with the rest of the unit.

An occasion when you resolve a problem is often called a 'Eureka moment'. The Greek mathematician Archimedes (c. 287–212 BCE) was interested in all the mathematical sciences, from geometry and arithmetic to mechanics and astronomy. There is a story that he was in the bath one day when he suddenly realised that the way objects displace water gave the answer to a problem that had been puzzling him – he leapt out of the bath shouting 'Eureka!' ('I have found it!').

If a problem arises because you are a little rusty on some of the basic mathematical skills that you are expected to have before the start of the module, then you may find it useful to set aside some extra study time and refer to Maths Help. This is an online resource provided by The Open University, which contains help with basic mathematics. There is a link to Maths Help on the module website.

Figure 2 An Italian postage stamp celebrating Archimedes

2 Working with numbers

This section and Section 3 will help you to revise and practise some of the basic mathematical and calculator skills that you will need to be able to use fluently in later units. If you have already made sure that you have the basic mathematical skills needed for the module, as recommended in the module description, then you should find most of these two sections straightforward.

Most students find that they need to improve their fluency in at least some of the skills, and sort out small gaps in their knowledge, so it is important that you work through these sections thoroughly, doing the activities and checking your answers against the solutions at the end of the unit. You may not be aware of some of the gaps in your knowledge or skills, so even if you feel confident about a topic you should still try the activities to make sure.

If you feel that you need more practice on a topic than is provided in the printed unit, then visit the website to see whether there are any suitable practice quiz questions. For some topics you may also be able to find further practice in Maths Help. There are references in the margins to tell you which topics are covered in Maths Help, and where.

2.1 Getting the order right

Suppose that you are buying a box of wallpaper paste costing £8, and 12 rolls of wallpaper at £14 each. The calculation for the total cost in £ can be written as

$$8 + 12 \times 14.$$

To work out the cost, you first do the multiplication and then do the addition, which gives the answer £176. You know that you should do the multiplication before the addition because you know the context of the calculation.

But will your calculator do the same? Or will it work from left to right and do the addition first and then the multiplication, to obtain the answer £280? Try it! Type the whole calculation into your calculator and press the '=' key.

You should find that your calculator gives the correct answer of £176. This is because the following convention is used in mathematics.

Order of operations: BIDMAS

Carry out mathematical operations in the following order.

$$
\begin{array}{ll}
\textbf{B} & \text{brackets} \\
\textbf{I} & \text{indices (powers and roots)} \\
\textbf{D} & \text{divisions} \\
\textbf{M} & \text{multiplications} \\
\textbf{A} & \text{additions} \\
\textbf{S} & \text{subtractions}
\end{array}
\quad
\begin{array}{l}
\\ \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \text{same precedence} \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \text{same precedence}
\end{array}
$$

When operations have the same precedence, work from left to right.

The BIDMAS rules tell you the order in which to deal with the operations $+$, $-$, $\times$ and $\div$, and also *powers* and *roots*. Remember that to raise a number to a power, you multiply it by itself a specified number of times. For example, 2^3 means three 2s multiplied together:

$$2^3 = 2 \times 2 \times 2.$$

The superscript 3 here is called the power, index or exponent. (The plural of 'index' is 'indices'.) Roots are revised in Unit 3.

2^3 is read as 'two cubed', and 5^2 (in Example 1) is read as 'five squared'. For powers other than 2 or 3, you say 'to the power': for example, 7^4 is read as 'seven to the power four' or 'seven to the four' for short.

Example 1 reminds you how to use the BIDMAS rules. It also illustrates another feature that you will see throughout the module. Some of the worked examples include lines of blue text, marked with icons like . This text tells you what someone doing the mathematics might be thinking, but wouldn't write down. It should help you to understand how you might do a similar calculation yourself.

Example 1 Using the BIDMAS rules

Work out the answers to the calculations below without using your calculator. For help with the BIDMAS rules, see Maths Help Module 1, Subsections 3.5 and 3.6.

(a) $8 - 2 + 5 - 1$ (b) $5 + 12 \div 4$ (c) 4×5^2 (d) $(5 - 3) \times 4$

Solution

(a) The addition and subtractions have the same precedence, so do them in order from left to right.

$8 - 2 + 5 - 1 = 6 + 5 - 1 = 11 - 1 = 10$

(b) Do the division first, then the addition.

$5 + 12 \div 4 = 5 + 3 = 8$

(c) Work out the power first, then do the multiplication.

$4 \times 5^2 = 4 \times 25 = 100$

(d) Do the calculation in brackets first, then do the multiplication.

$(5 - 3) \times 4 = 2 \times 4 = 8$

Activity 2 Using the BIDMAS rules

Work out the answers to the calculations below without using your calculator.

(a) $9 + 7 - 2 - 4$ (b) $2 \times (7 - 4)$ (c) $(3 + 5) \times 3$

(d) $(3 + 4) \times (2 + 3)$ (e) $3^2 + 4^3$

Activity 3 More BIDMAS

Check whether each of these calculations is correct. For those that are incorrect, add brackets to make them correct.

(a) $2 \times 5 + 3 = 16$ (b) $3 + 4 \times 7 = 49$ (c) $1 + 2 \times 3 = 7$

(d) $9 - 3 \times 2 = 3$ (e) $2 \times 3 + 3 \times 5 = 60$

Mathematical terms

There are many terms that have specific meanings in mathematics. When the meaning of a term is explained in the module, the term is printed in bold. Important terms and their definitions are also collected together in the Glossary in the Handbook.

Some terms used for calculations are explained below.

Some terms for calculations

The **sum** of two numbers is the result of adding them together.

A **difference** between two numbers is the result of subtracting one from the other. There are two possible answers, depending on which way round you take the numbers, but usually the smaller number is subtracted from the larger.

The **product** of two numbers is the result of multiplying them.

A **quotient** of two numbers is the result of dividing one by the other. There are two possible answers, depending on which way round you take the numbers.

Activity 4 Sums and differences

By trying some positive numbers less than 10, can you find the following?

(a) Two numbers with sum 12 and product 32

(b) Two numbers with difference 2 and quotient 2

2.2 Using your calculator

You are expected to use your calculator for most numerical calculations in MU123. You can refer to either the 'Calculator guide' section in the MU123 Guide (if you are using the recommended calculator) or your calculator manual to remind yourself which keys to press. Sometimes you may find that it is quicker to do a simple calculation in your head or on paper.

Occasionally you are asked not to use your calculator. This is usually so that you can practise a technique that you will need to use later when you learn algebra.

When you type a calculation into your calculator, it is important to think about the BIDMAS rules, to ensure that the operations are carried out in the order you intend.

In the next activity you will practise the following basic calculator skills.

- Enter calculations correctly, using the number keys, the $+$, $-$, $\times$ and $\div$ keys and the bracket keys.

- Display answers as either decimals or fractions.
- Use the power keys.
- Correct mistakes in entering calculations.

Activity 5 Getting to know your calculator

Work through Subsection 3.1 of the MU123 Guide.

2.3 Units of measurement

Many everyday calculations involve measurements of some kind: for example, lengths, times, amounts of money, and so on. In the UK, both the metric and imperial systems of measurements are used. In the metric system, the different units for the same type of quantity are related to each other via powers of ten: for example, 1 metre is the same as 100 (or 10^2) centimetres. In the US, 'metre' is spelt as 'meter'. In the imperial system, the units are related in different ways: for example, 1 stone is the same as 14 pounds. For more help with units, see Maths Help Module 1, Section 2.

This module mostly uses the standard metric system known as the Système Internationale d'Unités (SI units). This system is used by the scientific community generally and is the main system of measurement in nearly every country in the world.

The metric system was founded in France in the wake of the French Revolution, and work on SI units has been ongoing since the middle of the twentieth century. At the time of writing, the only countries who have not adopted SI units as their sole or primary system of measurement are the United States, Burma (Myanmar) and Liberia.

There are seven base SI units, from which all the other units are derived. The **base units** (and their abbreviations) used most frequently in the module are the metre (m), the kilogram (kg) and the second (s). Prefixes are used to indicate smaller or larger units. For example, millimetres, centimetres, metres and kilometres are all used to measure length. The most common prefixes are shown in Table 1, and there is a more extensive list in the Handbook.

Table 1 Some common prefixes

Prefix	Abbreviation	Meaning	Example
milli	m	a thousandth $\left(\frac{1}{1000}\right)$	1 millimetre (mm) $= \frac{1}{1000}$ metre
centi	c	a hundredth $\left(\frac{1}{100}\right)$	1 centimetre (cm) $= \frac{1}{100}$ metre
kilo	k	a thousand (1000)	1 kilometre (km) $=$ 1000 metres

Converting units

Sometimes you need to convert measurements from one unit to another. For example, suppose that you are thinking of installing new kitchen cabinets. If you measure lengths in your kitchen in metres and then find that the dimensions of new kitchen cabinets are given in millimetres, then you will need to convert both sets of measurements to the same units, say millimetres. Or if your answer to a calculation is 0.006 kg, then it will usually be better to convert it to grams, since 6 g is both simpler and easier to imagine. In much of Europe the decimal point is denoted by a comma rather than a dot, so 0.006 kg would be written as 0,006 kg.

To convert from one unit to another, you should first find how many of the smaller units are equivalent to one of the larger units – if the units are metric, then you can tell this from the prefixes. If you want to convert to the smaller unit, then there will be more of these units so you need to multiply by this number. If you want to convert to the larger unit, then there will be fewer of these units so you need to divide by the number. This is illustrated in the next example.

Example 2 Converting measurements to different metric units

(a) Convert 580 cm to m.

(b) Convert 0.65 g to mg.

Solution

(a) To convert to a larger unit, divide.

There are 100 cm in 1 m, so

$$580\,\text{cm} = (580 \div 100)\,\text{m} = 5.8\,\text{m}.$$

(b) To convert to a smaller unit, multiply.

There are 1000 mg in 1 g, so

$$0.65\,\text{g} = (0.65 \times 1000)\,\text{mg} = 650\,\text{mg}.$$

A few other metric units are commonly used alongside the SI units. The metric tonne (t), which is equivalent to 1000 kg, is often used to measure heavy masses, such as vehicles. (There is an explanation of the difference between ‘mass’ and ‘weight’ on page 186 of this book, in Unit 4.) The litre (l) is often used to measure volumes, particularly of liquids, even though the SI unit for volume is the cubic metre. One litre is equivalent to 1000 cubic centimetres (cm^3 or cc), so 1000 litres is equivalent to $1\,\text{m}^3$.

There is an SI unit for temperature – the kelvin (K) – but it is mainly used by scientists. In the UK most people use the Celsius scale, which is part of the metric system. Some people still use the non-metric Fahrenheit scale.

Time is also often measured in non-metric units – you have probably never heard of a kilosecond! Seconds (s), minutes (min), hours (h) and days are used, even though this makes conversion calculations more complicated.

Example 3 Converting units of time

(a) Convert 2.85 hours into minutes.

(b) Convert 54 hours into days.

Solution

(a) To convert to a smaller unit, multiply.

There are 60 minutes in 1 hour, so

$$\begin{aligned} 2.85 \text{ hours} &= (2.85 \times 60) \text{ minutes} \\ &= 171 \text{ minutes.} \end{aligned}$$

(b) To convert to a larger unit, divide.

There are 24 hours in a day, so

$$\begin{aligned} 54 \text{ hours} &= (54 \div 24) \text{ days} \\ &= 2.25 \text{ days.} \end{aligned}$$

If Example 3(a) had asked for 2.85 hours to be converted into hours *and minutes*, then just the 0.85 hours would need to be converted into minutes:

$$0.85 \text{ hours} = (0.85 \times 60) \text{ minutes} = 51 \text{ minutes.}$$

So 2.85 hours is the same as 2 hours and 51 minutes.

Activity 6 Converting units

Make the following conversions.

(a) 6100 m into km (b) 560 kg into t (c) 3.45 hours into minutes

(d) 0.35 g into mg (e) 450 ml into l (f) 75 cm into m

The UK has embraced metric units rather less enthusiastically than most other countries, and many of its traditional imperial units are still used. If you live in the UK, you may be comfortable with measuring kitchen cabinets in millimetres and buying petrol in litres, for example, but you may think of your height in feet and inches, and your weight in stones and pounds. Most British recipe books give both metric and imperial measurements for ingredients. When following recipes, it is advisable to use one or the other, not mix the two! More seriously, NASA lost the Mars Climate Orbiter spacecraft in 1999 as a result of an error caused because one team working on the project used imperial units of measurement while another used metric units.

Figure 3 shows an early metric road sign near Barnes Pool Bridge, Eton, in the UK, which was put up for the 1908 Olympic Games in London. The sign uses 'kilos' as an abbreviation for kilometres.

In 1969 the UK government set up the Metrication Board, with the aim of ensuring substantial adoption of metric units in the UK by 1975. In particular, it was planned that road sign conversion would take place in 1973. However, in 1970 the conversion programme was put on hold indefinitely, and successive British governments negotiated with the European Union to opt out of using metric units on road signs.

Figure 3 An early metric road sign

If you are more familiar with imperial units than metric units, then the rhymes in Figure 4 might help you to remember the approximate sizes of some of the metric units.

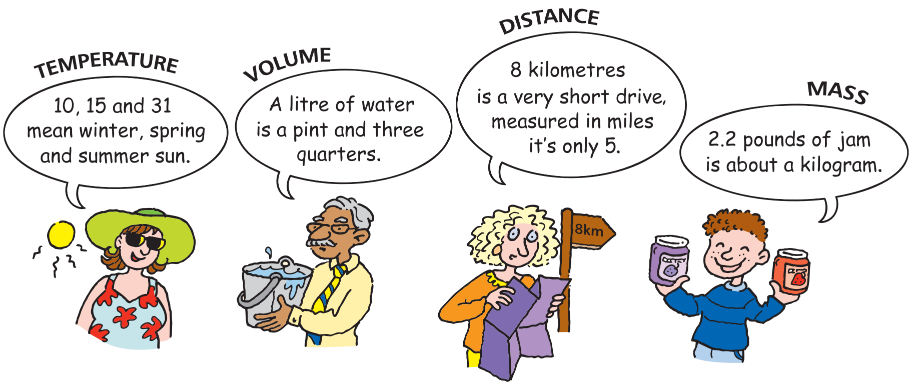


Figure 4 Rhymes for metric units

Alternatively, measuring some items in or around your home in metric units may help you to visualise the sizes of the units – most doors are about 2 metres high, for example.

Using units

When you answer a question that involves units:

- remember to include units in your answers
- check whether you are asked to give your answers in particular units.

2.4 Rounding numbers

When you make a measurement, it is sometimes helpful to *round* your answer. For example, if you were measuring the height of a child, then an answer rounded to the nearest centimetre would probably be adequate. If, say, your measurement was as shown in Figure 5 – between 91 cm and 92 cm but closer to 91 cm – then you would round it down to 91 cm. If your measurement was between 91 cm and 92 cm but closer to 92 cm, then you would round it up. The measurement 91.5 is halfway between the two values, so you could round it either way, but it is usual to round up. For help with rounding numbers, see Maths Help Module 2, Section 1.

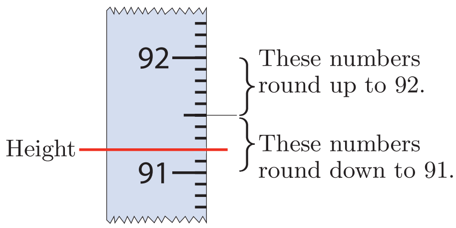


Figure 5 Measuring the height of a child

Another situation where you often need to round numbers is when you are doing calculations, since the answers provided by your calculator can consist of long strings of digits. A (decimal) **digit** is one of the symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

Decimal places

Numbers arising from calculations are sometimes rounded to a particular number of **decimal places**. The decimal places are the positions of the digits to the right of the decimal point, as shown in Figure 6. The abbreviation 'd.p.' is often used for 'decimal place(s)'. For more help with rounding to a number of decimal places, see Maths Help Module 2, Subsection 1.4.

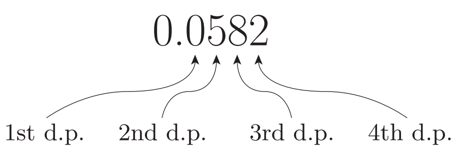


Figure 6 Decimal places

For example, in a calculation involving money, the final answer might be rounded to two decimal places, so that it can be interpreted in pounds and pence.

Once you have decided where to round a number, you should use the following rule to decide whether to round up or down.

Strategy To round a number

Look at the digit immediately after where you want to round.

Round up if this digit is 5 or more, and down otherwise.

When you round a number, you should state how it has been rounded, in brackets after the rounded number. This is shown in the next example.

Example 4 Rounding to a number of decimal places

Round the following numbers as indicated.

(a) 0.0582 to three decimal places

(b) 7.056 83 to one decimal place

(c) 2.3971 to two decimal places

Solution

(a) Look at the digit after the first three decimal places: 0.0582. It is 2, which is less than 5, so round down.

0.0582 = 0.058 (to 3 d.p.)

(b) Look at the digit after the first decimal place: 7.056 83. It is 5, which is 5 or more, so round up.

7.056 83 = 7.1 (to 1 d.p.)

(c) Look at the digit after the first two decimal places: 2.3971. It is 7, which is 5 or more, so round up.

2.3971 = 2.40 (to 2 d.p.)

A long string of digits is easier to read if there are thin spaces between groups of three digits, as in 12 345 and 0.123 45. (Some texts use commas instead of thin spaces.)

Notice that in Example 4(c), a 0 is included after the 4 to make it clear that the number is rounded to *two* decimal places. You should do likewise when you round numbers yourself.

Activity 7 Rounding to a number of decimal places

Round the following numbers as indicated.

(a) 2.2364 to two decimal places

(b) 0.005 47 to three decimal places

(c) 42.598 17 to four decimal places

(d) 7.98 to one decimal place

As an alternative to writing in brackets how you rounded a number, you can replace the equals sign by $\approx$, the 'approximately equals' sign. For example, instead of writing

$$2.3971 = 2.40 \text{ (to 2 d.p.)},$$

you can write

$$2.3971 \approx 2.40,$$

where the symbol $\approx$ is read as 'is approximately equal to'. However, it is usually preferable to state the rounding that you used.

You should not write just

$$2.3971 = 2.40,$$

because this is incorrect: the numbers 2.3971 and 2.40 are not equal.

There have been a few reports of fraudsters in the US using rounding to become rich. They used computer programs to remove tiny amounts of money from lots of bank accounts, by rounding down to the nearest cent and putting the remaining fractions of cents into other accounts. These small amounts were not noticed as missing from any particular account, but they built up into huge sums of money. This is an example of so-called *salami-slicing* or *penny-shaving* fraud, in which thin slivers of money are removed from many accounts. Modern-day banking systems have built-in checks to prevent this type of fraud.

Significant figures

Another way of specifying where a number should be rounded involves looking at its **significant figures**. The first significant figure of a number is its first non-zero digit (from the left). The next significant figure is the next digit, and so on, as illustrated in Figure 7. Common abbreviations for 'significant figure(s)' are 's.f.' and 'sig. fig.'. For more help with rounding to a number of significant figures, see Maths Help Module 2, Subsections 1.5 and 1.6.

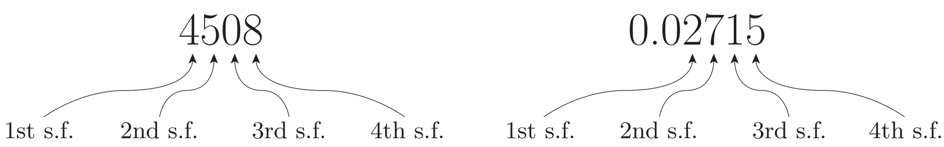


Figure 7 Significant figures

The first significant figure of a number is the most important digit for telling you how big the number is. For example, the digit 4 in the number 4508 in Figure 7 tells you that the number is between four and five thousand, while the digit 2 in the number 0.027 15 tells you that this number is between two hundredths and three hundredths. The second significant figure is the next most important digit for telling you how big the number is, and so on.

The usual '5 or more' rule in the strategy on page 17 is used when rounding to a particular number of significant figures.

Example 5 Rounding to a number of significant figures

Round the following numbers as indicated.

(a) 36.9572 to four significant figures

(b) 0.000 349 to one significant figure

(c) 56.0463 to one significant figure

(d) 0.0198 to two significant figures

Solution

(a) Look at the digit after the first four significant figures: $\underbrace{36.95}\overset{\downarrow}{7}2$. It is 7, which is greater than 5, so round up.

36.9572 = 36.96 (to 4 s.f.)

(b) Look at the digit after the first significant figure: $0.000\,\underbrace{3}\overset{\downarrow}{4}9$. It is 4, which is less than 5, so round down.

0.000 349 = 0.0003 (to 1 s.f.)

(c) Look at the digit after the first significant figure: $\underbrace{5}\overset{\downarrow}{6}.0463$. It is 6, which is greater than 5, so round up.

56.0463 = 60 (to 1 s.f.)

(d) Look at the digit after the first two significant figures: $0.0\underbrace{19}\overset{\downarrow}{8}$. It is 8, which is greater than 5, so round up.

0.0198 = 0.020 (to 2 s.f.)

In Example 5(d) a 0 is included after the 2 to make it clear that the number is rounded to *two* significant figures. You should do likewise when you round numbers yourself.

One or more zeros in a rounded number may be significant figures. For example, the final zero in the number 0.020 in the solution to Example 5(d) is significant: this number is rounded to two significant figures, and the second significant figure just happens to be zero. In contrast, the zero in the number 60 in the solution to Example 5(c) is not significant, as the number is rounded to just one significant figure.

If you are told that a distance is 3700 metres, say, without any information about how the number has been rounded, then you cannot tell whether the zeros are significant. The number 3700 could be the result of rounding 3684 to two significant figures, 3697 to three significant figures, or 3700 to four significant figures, for example. In the first case neither of the zeros is significant, in the second case only the first zero is significant, and in the third case both zeros are significant. This is one reason why it is important to state how a number has been rounded, as shown in Example 5.

When you see a measurement such as 3700 metres with no information about whether or how the number has been rounded, you can usually assume that any zeros at the end are *not* significant.

Activity 8 Rounding to a number of significant figures

Round the following numbers as indicated.

(a) 23 650 to two significant figures

(b) 0.005 47 to one significant figure

(c) 42.598 17 to four significant figures

Other types of rounding

Numbers are also sometimes rounded to the nearest ten, or hundred, or thousand, and so on. For example, if you were writing a report on a concert that had an audience figure of 58 217, then you might choose to round this number to the nearest thousand, to give 58 000, or perhaps to the nearest ten thousand, to give 60 000. The '5 or more' rule is used to decide whether to round up or down. Similarly, you can also round to the nearest metre, or the nearest 10 kilograms, and so on.

Choosing which type of rounding to use

Rounding to a number of significant figures is often the most useful type of rounding to use. For example, the height of a mountain might be usefully quoted as 4559 m to the nearest metre, but if you need to know the height of a woman who is 1.65 m tall then an approximation to the nearest metre is not useful. In each case, however, rounding to three significant figures gives a useful approximation: 4560 m for the height of the mountain and 1.65 m for the height of the woman.

Rounding answers appropriately

Often, the measurements that you have used in a calculation give you an indication of the amount of rounding that you should use for your answer.

For example, the road distance from Paris to Lyon is 465 km. Suppose that you want to convert this distance into miles. You can use the fact that

$$1\,\mathrm{km} = 0.621\,371\,192 \text{ miles (to 9 s.f.)}.$$

Multiplying the distance in km by the conversion factor gives the distance in miles as

$$465 \times 0.621\,371\,192 = 288.937\,604\,3.$$

It is inappropriate to leave the answer as 288.937 604 3 miles, as this suggests that the distance has been measured very carefully indeed! The original measurement seems to be given to three significant figures, so your answer should be similarly rounded – to three, or possibly fewer, significant figures. Rounding to three significant figures gives the distance as 289 miles.

To see why this amount of rounding is appropriate, let's consider what the actual distance could be. It was given as 465 km to the nearest kilometre, so it could be anything from 464.5 km up to (but not including) 465.5 km, as shown in Figure 8.

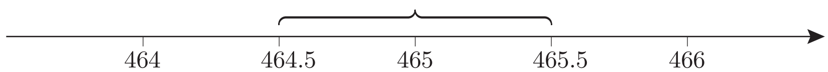


Figure 8 Numbers that round to 465

Now, 464.5 km is equivalent to 288.626 918 7 miles, and 465.5 km is equivalent to 289.248 289 9 miles. Since the actual distance lies between these two values, it is certainly 289 miles when rounded to three significant figures, as you can see from Figure 9. So the amount of rounding was appropriate.

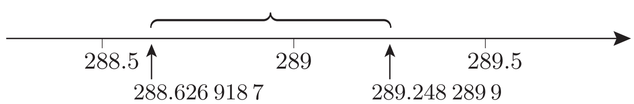


Figure 9 Numbers between 288.626 918 7 and 289.248 289 9

When you are rounding answers, you should round to no more significant figures than the number of significant figures in the least precise number in the calculation. For example, in the calculation above, the measurement and the conversion factor were rounded to three and nine significant figures, respectively, so the answer should be rounded to no more than three significant figures.

Sometimes it is appropriate to round to fewer significant figures than the number of significant figures in the least precise number. A full analysis of rounding is outside the scope of the module, so activities and TMA questions will often state what rounding to use in your answers. Otherwise, rounding to the number of significant figures in the least precise number used should be acceptable. The number of *significant figures* an answer is stated to is known as the **precision** of the answer.

Activity 9 Rounding an answer appropriately

In this activity you are asked to convert 465 km into miles again, but this time using the following less precise conversion factor:

> 1 km is approximately equal to 0.62 miles.

(a) Do the calculation and round your answer to the nearest mile. Compare your answer to the answer found in the calculation on page 21, and comment on why they are not the same.

(b) Round your answer appropriately.

Rounding at the right time

The next activity shows that it is important not to round too early in a calculation.

Activity 10 Rounding at different stages of a calculation

Suppose that you want to calculate the length, in miles, of a return journey to a town 36 km away. Use the conversion factor 1 km = 0.621 371 192 miles (to 9 s.f.) to carry out the following calculations.

(a) Convert 36 km to miles, and round your answer to the nearest mile. Use this answer to find the total length of the journey.

(b) Convert 36 km to miles. With the unrounded answer still in your calculator display, type '× 2' into your calculator, and press the '=' key to obtain the total length of the journey. Round your answer to the nearest mile.

(c) Comment on which of parts (a) and (b) is the better way of carrying out the calculation.

As you saw in Activity 10, if you do a calculation in two or more steps, and round your answer after one of these steps, then your final answer may be inaccurate. This is known as a **rounding error**.

It was reported in the *New York Times* in 1991 that Tina Lubin, a public servant who dealt with the $14.8 billion payroll for the city government of New York, had 'learned to think of $10 million as a rounding error'!

Whenever you use an earlier answer in a later calculation, you should use all the digits that your calculator provided for the earlier answer, to avoid rounding errors. This is known as 'using full calculator precision'. You may not need all the digits, but it is usually not clear how many digits you do need, so it is simplest to use them all.

If you cannot use a full-calculator-precision number displayed on your calculator immediately, then you can note it down for later, or store it in your calculator's memory. You will be reminded about how to use the memory key on your calculator later in the module.

When you use a full-calculator-precision number that has a long string of digits after the decimal point, you do not need to include all the digits in the working that you write down. You can write down just a few of them – usually at least three digits after the decimal point, or at least three significant digits – and use the symbol '...' to show that you have omitted the rest.

The symbol '...' is called an *ellipsis* and is used when something has been left out. It is read as 'dot, dot, dot'.

For example, if your calculator gives you the number 2.478 260 87, and you then use the calculator to multiply this number by 6, then you might write down

$$2.478\ldots \times 6 = 14.869\ldots = 14.87 \text{ (to 4 s.f.)}.$$

When you omit some of the digits of a number just before you round it, you should make sure that you have written down enough digits so that someone reading your working can see that the rounding is correct. This is done for the number '14.869...' above.

Considering the context

When you round an answer, it is also important to consider the context. For example, if you are calculating how many cupboards will fit along your kitchen wall and your answer is 7.9, then you should round *down* and buy only seven cupboards, because eight cupboards wouldn't fit! On the other hand, if you are painting your kitchen and need 1.2 tins of paint, then you should round *up* and buy two tins; otherwise you will run out of paint.

The box below summarises the key points about rounding.

Rounding answers

- Use full calculator precision *throughout* calculations, to avoid rounding errors.
- Round your answer appropriately, taking account of the measurements used and the context.
- Check that you have followed any instructions on rounding given in a question.

2.5 Checking your answers

When you carry out a calculation, it is helpful to have a rough idea of the expected answer. For help with estimating answers, see Maths Help Module 2, Section 2. If your answer is very different from your estimate, then you know to look for a mistake. The mistake could be in your working, or it might have occurred when you used your calculator.

One simple way to estimate an answer is to think about the context. For example, if you were totting up a weekly shopping bill for a family of four, then you might expect an answer between £100 and £200. You would be surprised if your answer turned out to be £1500 or £15!

Another way to estimate an answer is to round all the numbers in the calculation – perhaps to one significant figure, or to nearby numbers that are easy to work with – and carry out the calculation with the rounded numbers. It should be possible to do this fairly quickly, either in your head or on a piece of paper, as the next example shows. This type of estimate may not highlight a mistake in your working, as you could make the same mistake when you do the calculation with the rounded numbers. But it will help you to spot mistakes that happen when you use your calculator.

Example 6 Estimating an answer

The road distances in kilometres between three places in Scotland are shown overleaf. Suppose that you are planning a round trip in which you start at Edinburgh, visit Perth and Glasgow, and return to Edinburgh, in a minibus whose fuel consumption is 12 kilometres per litre of fuel.

(a) Estimate the amount of fuel needed for the trip.

(b) Use your calculator to work out the amount of fuel needed, to the nearest litre.

Solution

(a) Round the numbers to one significant figure.

An estimate for the total distance in km is

$$70 + 100 + 80 = 250.$$

The minibus can travel about 10 km on 1 litre of fuel, so the amount of fuel needed, in litres, is approximately

$$250 \div 10 = 25.$$

So an estimate for the amount of fuel needed is 25 litres.

(b) The amount of fuel needed, in litres, is

$$(69 + 95 + 83) \div 12 = 21 \text{ (to the nearest whole number).}$$

So the amount of fuel needed is 21 litres, to the nearest litre.

The answer is fairly close to the estimate, so there is no evidence of a mistake.

Checking for calculator mistakes

If your answer to a calculation seems to be wrong (you might know this because you estimated it) and you suspect that you made a mistake when you used your calculator, then the first thing to check is whether you have mistyped something. If the calculation is displayed on your calculator screen, then you should check the numbers and operations carefully, and edit the calculation to correct any errors.

The next thing to check is whether the calculation you entered was the correct one. You need to think about the BIDMAS rules. For example, if you had intended to carry out the calculation in Example 6(b), and had typed

$$69 + 95 + 83 \div 12,$$

then you would have obtained the wrong answer, because your calculator would do the division before the additions. You need to include the brackets, as in the solution above.

If you still cannot find a mistake, then you can try breaking the calculation into simpler steps. For example, to do the calculation in Example 6, you could first work out the total distance, which is 247 km, and then divide 247 by 12 to find the amount of fuel in litres. For help with checking answers, see Maths Help Module 2, Section 3.

If nothing seems to be wrong with the way you used your calculator, then you may need to check your estimate, or refine it – *it* could be the problem!

Here is a summary of some key points to check when using your calculator.

Checking your answers when using your calculator

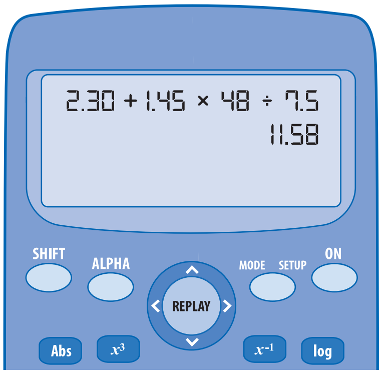
- Have you entered the calculation correctly?
- Have you used brackets where needed?
- Is the answer reasonable? Think about the context or work out an estimate.

Activity 11 Spotting errors in a calculation

A craftswoman makes handmade jewellery boxes. It takes her 2 hours and 30 minutes to make each box, and 1 hour and 45 minutes to apply the decoration. She works for 7.5 hours each day. Suppose that you need to know how many working days it would take the craftswoman to make and decorate 48 jewellery boxes.

(a) Estimate the number of days needed.

(b) A student typed the calculation shown below into a calculator, and concluded that the number of days needed is 12. Try to identify the two mistakes.



(c) Use your calculator to find how many days are needed, and round your answer appropriately.

In this section you have revised some basic skills in working with numbers; you will need to use these skills frequently throughout the rest of the module.

3 Negative numbers and fractions

Negative numbers, fractions and percentages (which can be thought of as a type of fraction) occur frequently in both everyday and abstract mathematics. In this section you will revise some basic skills with these types of numbers and see some everyday applications of these skills.

3.1 Negative numbers

You are probably familiar with negative numbers in the context of temperatures. For example, the minus sign in '$-5°$ C' indicates a temperature five degrees below zero. Negative numbers are also used to represent debt. For help with negative numbers, see Maths Help Module 1, Subsections 1.9 and 1.10.

You can think of all numbers as lying on a line, called the **number line**. Figure 10 shows part of the number line, with the positions of the integers marked. The positive numbers are to the right of zero, and the negative numbers are to the left. Zero itself is neither positive nor negative.

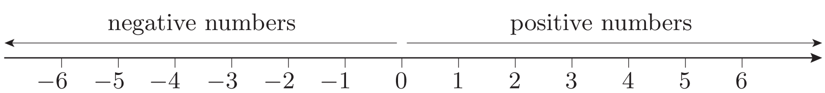


Figure 10 The number line

An **integer** is any one of the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. The numbers on the number line get bigger as you go from left to right. For example, the number -1 is greater than the number -3, since -1 lies further to the right. The number -1 is read as 'minus 1' or 'negative 1'. You may find it helpful to use Figure 10 when you do the next activity.

Activity 12 Comparing temperatures

The noon temperatures on six consecutive winter days are given in the table below.

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
$-3°$ C	$-6°$ C	$-2°$ C	$2°$ C	$0°$ C	$-4°$ C

On which days was the noon temperature lower than on Sunday?

Higher or lower?

In 2007 a lottery scratchcard game was withdrawn, within a week of being launched, because players could not understand the negative numbers involved. To win a prize, players had to scratch away a window to reveal a temperature lower than the temperature shown on the card. As the game had a winter theme, the displayed temperature was usually negative. Many players thought that -3 was lower than -4, for example, and complained to the lottery company when they thought they had won a prize but were told that they had not.

You will often need to use negative numbers in MU123. The rest of this subsection reminds you how to add, subtract, multiply and divide them.

Adding and subtracting negative numbers

No matter what number you start with – whether it is positive, negative or zero – if you want to add a *positive* number to it then you move along the number line to the right. For example,

$$-2 + 5 = 3,$$

as illustrated in Figure 11. See Maths Help Module 1, Subsections 3.16–3.19.

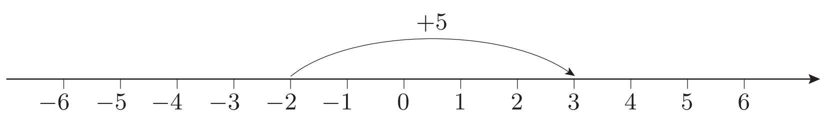


Figure 11 Adding a positive number

Similarly, to subtract a *positive* number you move along the number line to the left. For example,

$$-1 - 4 = -5,$$

as illustrated in Figure 12.

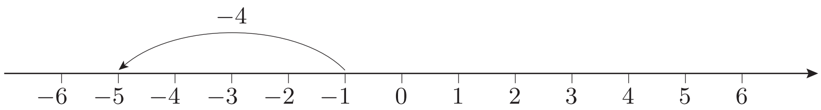


Figure 12 Subtracting a positive number

Activity 13 Adding and subtracting positive numbers

Work out the following calculations without using your calculator.

(a) $-6 + 2$ (b) $-1 + 3$ (c) $2 - 7$ (d) $-3 - 4$ (e) $5 - 7 - 2$

To add and subtract *negative* numbers, use the rules in the box below.

Adding and subtracting negative numbers

Adding a negative number is the same as subtracting the corresponding positive number.

Subtracting a negative number is the same as adding the corresponding positive number.

The next example illustrates these rules. There is a tutorial clip for this example, as indicated by the icon in the margin. If you find the example difficult to follow, try watching the clip. The clip also discusses the reasons behind the rules.

Notice that some of the negative numbers in this example are enclosed in brackets. This is because no two of the mathematical symbols $+$, $-$, $\times$ and $\div$ should be written next to each other, as that would look confusing. So if you want to show that you are adding -2 to 4, for example, then you should put brackets around '-2' and write $4 + (-2)$, not $4 + -2$.

Tutorial clip

Example 7 Adding and subtracting negative numbers

Work out the following calculations without using your calculator.

(a) $-3+(-6)$ (b) $4+(-2)$ (c) $0-(-6)$ (d) $1-(-2)$

(e) $-2-(-3)$

Solution

To *add* a negative number, *subtract* the corresponding positive number.

(a) $-3+(-6) = -3-6 = -9$

(b) $4+(-2) = 4-2 = 2$

To *subtract* a negative number, *add* the corresponding positive number.

(c) $0-(-6) = 0+6 = 6$

(d) $1-(-2) = 1+2 = 3$

(e) $-2-(-3) = -2+3 = 1$

Activity 14 Adding and subtracting negative numbers

Work out the following calculations without using your calculator.

(a) $2+(-7)$ (b) $-8+(-5)$ (c) $1-(-3)$ (d) $-6-(-9)$

(e) $-4-(-4)$ (f) $3-(-2)+(-4)$ (g) $7+(-6)-3$

So far, adding and subtracting zero has not been mentioned. As you would expect, this has no effect on the number you start with: for example

$$-4+0=-4 \quad \text{and} \quad 3-0=3.$$

Multiplying and dividing negative numbers

Now let's look at how to multiply and divide negative numbers. For help with multiplying and dividing negative numbers, see Maths Help Module 1, Subsections 3.20–3.22. First consider the multiplication $3\times(-2)$. This means 'three lots of -2', so

$$\begin{aligned} 3\times(-2) &= (-2)+(-2)+(-2) \\ &= -2-2-2 \\ &= -6. \end{aligned}$$

The order in which you multiply numbers doesn't matter, so the calculation above also tells you that

$$(-2) \times 3 = -6.$$

This example illustrates the first rule in the box below.

Multiplying and dividing negative numbers

When two numbers are multiplied or divided:

- if the signs are *different*, then the answer is *negative*
- if the signs are *the same*, then the answer is *positive*.

The following table might help you to remember these rules:

	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

For example, if you want to multiply a positive number by a negative number, look for the entry corresponding to the row and column headings $+$ and $-$, respectively: the entry is $-$, so the answer is negative.

The parts of the numbers other than the signs are just multiplied or divided in the usual way. The next example illustrates these rules. Its tutorial clip explains the calculations in more detail, and also contains more explanation of the rules.

Example 8 Multiplying and dividing negative numbers

Tutorial clip

Work out the following calculations without using your calculator.

(a) $(-5) \times 6$ (b) $9 \div (-3)$ (c) $(-3) \times (-7)$

(d) $(-70) \div (-10)$ (e) $(-2) \times 3 \times (-4)$

Solution

(a) A negative times a positive (different signs) gives a negative.

$(-5) \times 6 = -30$

(b) A positive divided by a negative (different signs) gives a negative.

$9 \div (-3) = -3$

(c) A negative times a negative (same signs) gives a positive.

$(-3) \times (-7) = 21$

(d) A negative divided by a negative (same signs) gives a positive.

$(-70) \div (-10) = 7$

(e) Do the multiplications one at a time. In the first multiplication, a negative times a positive gives a negative. Then this negative, times a negative, gives a positive.

$(-2) \times 3 \times (-4) = (-6) \times (-4) = 24$

Activity 15 Multiplying and dividing negative numbers

Work out the following calculations without using your calculator.

(a) $5 \times (-3)$ (b) $(-2) \times (-4)$ (c) $6 \times (-10)$ (d) $25 \div (-5)$

(e) $(-49) \div (-7)$ (f) $(-36) \div 12$ (g) $(-2) \times (-5) \times (-4)$

A minus indicating a negative in a calculation has the same precedence as subtraction in the BIDMAS rules. For example, in the calculation -3^2, the power is dealt with first, so the answer is -9. The calculation does not mean the square of -3, so the answer is not $(-3) \times (-3) = 9$.

Despite their importance in modern-day mathematics, negative numbers were rejected by some British mathematicians as late as the eighteenth century. Francis Maseres (1731–1824) wrote that they 'darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple'.

However, over a thousand years earlier, the Indian mathematician Brahmagupta (598–670) wrote down the rules for adding, subtracting, multiplying and dividing negative numbers, in terms of debt (negative numbers) and fortune (positive numbers).

The result of multiplying any number (whether positive, negative or zero) by zero is zero. For example,

$$4 \times 0 = 0, \quad 0 \times (-3) = 0, \quad 0 \times 0 = 0.$$

However, no number can be divided by zero. For example,

$$-3 \div 0$$

has no meaning. (This is because the answer to this calculation would have to be a number such that if you multiplied it by zero you would get -3, and there is no such number.)

On the other hand, the result of dividing zero by any non-zero number is zero. For example,

$$0 \div (-3) = 0.$$

Brahmagupta was also the first known person to write down rules for doing arithmetic with the number zero, though later medieval mathematicians remained confused about how a symbol used to represent the concept of nothing could itself be a number.

The best way to become confident with operations on negative numbers – as with all topics in mathematics – is to practise.

Practise, practise, practise!

Throughout your studies in mathematics, try as many activities as you can. You may *think* that you understand the mathematics you are reading, but you need to try it yourself to be sure. The more you practise, the more confident and fluent you will become, and the better you will remember how to do the mathematics when it comes up again. Each unit of the module contains plenty of practice activities, both within the text and in the practice quizzes.

Negative numbers on your calculator

Another way to practise calculations involving negative numbers is to make up some examples of your own and check your answers on your calculator. The next activity reminds you how to use your calculator for negative number calculations.

Activity 16 Using your calculator for negative numbers

Work through Subsection 3.2 of the MU123 Guide.

3.2 Fractions

Although decimal numbers are used in many everyday situations, there are occasions when fractions are appropriate. For example, in 2002 it was reported that the populations of four-fifths of the bird species, half of the plant species and a third of the insect species on arable farmland in Great Britain had declined. (Source: Robinson, R.A. and Sutherland, W.J. (2002) 'Post-war changes in arable farming and biodiversity in Great Britain', *Journal of Applied Ecology*, vol. 39, pp. 157–76.) The fractions here make it clear that the populations of most of the bird species had declined, but that the insect species had fared better.

Fractions are also important in mathematics, particularly in algebra. In this subsection you will revise how to work with fractions, and you will see some everyday uses of them. For help with fractions, see Maths Help Module 1, Subsection 1.7.

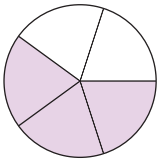


Figure 13 $\frac{3}{5}$ of a disc

A **fraction** is a number that describes the relationship between part of something and the whole. For example, the disc in Figure 13 is divided into five equal parts, of which three are shaded. To express this, we write that $\frac{3}{5}$ (three-fifths) of the disc is shaded. The fraction $\frac{3}{5}$ can also be written as 3/5.

The top number in a fraction is called the **numerator** and the bottom number is called the **denominator**.

So the fraction $\frac{3}{5}$ has numerator 3 and denominator 5.

Fractions can be converted to decimal form by dividing the numerator by the denominator. For example,

$$\tfrac{3}{5} = 3 \div 5 = 0.6 \quad \text{and} \quad \tfrac{2}{11} = 2 \div 11 = 0.181\,818\,18\ldots.$$

However, it is usually best not to convert fractions to decimals, but to leave them as they are. This is especially true for a fraction that does not have a short exact decimal form, such as $\frac{2}{11}$.

Equivalent fractions

Each fraction can be written in many different, but **equivalent**, forms. For example, $\frac{1}{2}$ is the same as $\frac{2}{4}$, $\frac{3}{6}$ and $\frac{4}{8}$, as you can see from Figure 14. You can use the following method to convert between different forms of a fraction.

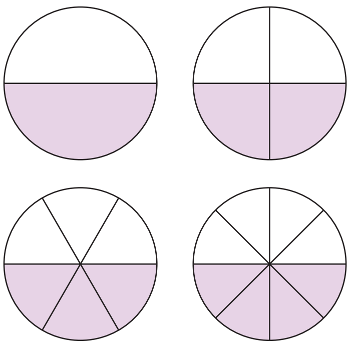


Figure 14 $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$

To find a fraction equivalent to a given fraction

Multiply or divide the numerator and denominator by the same non-zero number.

For example,

$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24} \quad \text{and} \quad \frac{25}{50} = \frac{25 \div 5}{50 \div 5} = \frac{5}{10}.$$

Simplifying fractions

When you divide the top and bottom of a fraction by a whole number larger than 1, you get an equivalent fraction with a smaller numerator and denominator. This is known as **cancelling** the fraction. When a fraction has been cancelled to give the smallest possible numerator and denominator (but still whole numbers), it is said to be in its **simplest form** or **lowest terms**. For help with simplifying fractions, see Maths Help Module 1, Subsection 1.8.

For example, the fraction $\frac{24}{30}$ can be cancelled by dividing top and bottom by 6. This calculation would normally be set out as follows:

$$\frac{\overset{4}{\cancel{24}}}{\underset{5}{\cancel{30}}} = \frac{4}{5}.$$

The fraction $\frac{4}{5}$ is in its simplest form because there is no whole number that divides exactly into both 4 and 5.

Cancelling can be carried out in stages. For example, to cancel $\frac{24}{30}$ you could divide top and bottom first by 2, and then by 3:

$$\frac{\overset{\overset{4}{\cancel{12}}}{\cancel{24}}}{\underset{\underset{5}{\cancel{15}}}{\cancel{30}}} = \frac{4}{5}.$$

Activity 17 Writing fractions in their simplest forms

(a) Express each of the following fractions in its simplest form, without using your calculator.

(i) $\frac{7}{21}$

(ii) $\frac{48}{72}$

(iii) $\frac{35}{105}$

(b) In a survey of 1200 students, 720 said that they have a part-time job. What fraction of the students is this? Give your answer in its simplest form.

Fractions (and percentages, which are discussed in Section 3.3) are often used in media headlines because of their impact. For example, if a media article were to be written about the survey in Activity 17(b), then it might have the headline ‘Three-fifths of students have part-time jobs’. This would have more impact than ‘720 out of 1200 students questioned have part-time jobs’.

However, the fractions and percentages in headlines can sometimes be misleading. When you read a headline that involves a fraction or a percentage of a group of people, it is worth reading the article to see whether it includes the answers to the following questions.

- How many people were included in the survey?
- Are the people in the survey representative of the overall population?

For example, you might not be impressed by the headline ‘Three-fifths of students have part-time jobs’ if you found that it was based on interviewing five students in a supermarket, and three of them were working on the tills!

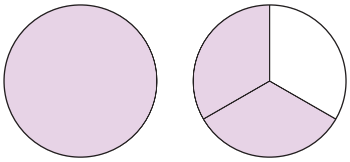


Figure 15 The mixed number $1\frac{2}{3}$

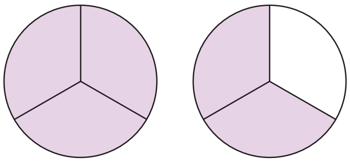


Figure 16 The improper fraction $\frac{5}{3} = 1\frac{2}{3}$

Mixed numbers and improper fractions

A **proper fraction** is a fraction in which the numerator is smaller than the denominator, such as $\frac{2}{3}$.

A number that consists of a whole number plus a proper fraction is called a **mixed number**. For example, the mixed number $1\frac{2}{3}$ is illustrated in Figure 15. Each mixed number can also be written as an **improper fraction** – a fraction in which the numerator is larger than the denominator. For example, you can see from Figure 16 that $1\frac{2}{3}$ contains five thirds altogether, so it is the same as $\frac{5}{3}$. An improper fraction is also known as a **top-heavy fraction**.

Example 9 Converting between mixed numbers and top-heavy fractions

(a) Write $2\frac{5}{8}$ as a top-heavy fraction.

(b) Write $\frac{13}{4}$ as a mixed number.

Solution

(a) There are eight eighths in one whole, so $2\frac{5}{8}$ can be written as two lots of eight eighths plus five more eighths.

$$2\frac{5}{8} = \frac{2 \times 8 + 5}{8} = \frac{21}{8}$$

(b) Divide 4 into 13. The answer is 3, remainder 1.

$$\frac{13}{4} = 3\frac{1}{4}$$

Activity 18 Converting between mixed numbers and top-heavy fractions

(a) Write $5\frac{2}{3}$ as a top-heavy fraction.

(b) Write $\frac{18}{5}$ as a mixed number.

Fractions of quantities

Sometimes you need to calculate fractions of quantities. For example, if you have a recipe that serves eight people, and you want to make it for three people or thirty people, say, then you have to scale the quantities of the ingredients. The next example shows you two ways of doing this. The second method involves fractions and is slightly quicker.

Example 10 Scaling a recipe

A recipe for eight people specifies 750 g of strawberries. What quantity of strawberries would be needed for three people?

Solution

First method

Work out the quantity of strawberries needed for one person. Use this to find the quantity of strawberries needed for three people.

The quantity of strawberries needed for one person is

$$750\,\mathrm{g} \div 8 = 93.75\,\mathrm{g}.$$

So the quantity of strawberries needed for three people is

$$3 \times 93.75\,\mathrm{g} = 281.25\,\mathrm{g} = 280\,\mathrm{g} \text{ (to 2 s.f.)}.$$

Second method

Find the fraction of the original quantity that is needed, and use this to calculate the quantity of strawberries needed.

Three-eighths of the original quantity is required. So the quantity of strawberries needed for three people is

$$\tfrac{3}{8} \times 750\,\mathrm{g} = 3 \div 8 \times 750\,\mathrm{g} = 281.25\,\mathrm{g} = 280\,\mathrm{g} \text{ (to 2 s.f.)}.$$

Of course, instead of working out the correct quantity of strawberries for three people, you could just halve the quantity for eight people and serve bigger portions!

In Example 10, the answer to a question was calculated in two different ways. There are often many different ways to solve a problem, so you may sometimes find that you have used a method different from one used in the unit or suggested by someone else. As long as your reasoning and answer are correct, this doesn't usually matter. However, it is a good idea to look at any model solution provided, as it may suggest an alternative and possibly quicker method that you could use to solve a similar problem in the future.

Activity 19 Calculating fractions of quantities

(a) Work out the following fractions of quantities.

(i) $\frac{4}{5}$ of 60 ml

(ii) $\frac{5}{8}$ of 20 kg

(b) A recipe for potato curry for 6 people uses 900 g of potatoes. If you are making the curry for 20 people, what quantity of potatoes do you need?

In theory you can scale any recipe to cater for a group of any size, but in practice you may wish to adjust your answers a little. It is important that you have enough food for everyone, but you may also wish to minimise waste and cost. The larger the group of diners, the more likely it is that a few people will eat only small portions or none at all, so caterers often use guidelines such as the following:

> Allow 150 g of potatoes per person for up to 10 people; for more than 10 people, allow 125 g per person.

In general, when you are using mathematics to make practical decisions, it is important to think about whether your calculations are appropriate for the situation.

Fractions on your calculator

In the next activity you will use your calculator to carry out calculations involving fractions.

Activity 20 Using your calculator for fractions

Work through Subsection 3.3 of the MU123 Guide.

3.3 Percentages

You may have noticed that relatively few types of fraction are used in the media – the only types commonly used are halves, thirds, quarters, and perhaps fifths or eighths. You don't usually come across fractions like $\frac{9}{20}$ or $\frac{37}{54}$, because fractions like these are difficult to visualise and compare. Instead, **percentages** are often used. A percentage is a fraction with a denominator of 100. It is often written using the symbol %. For example, $\frac{25}{100} = 25\%$.

This subsection reminds you of some basic techniques for calculating with percentages, and shows you how they can be used to make comparisons and describe changes. It also shows you how some of the numbers and percentages quoted in the media might have been manipulated to portray an author's point of view, or to make a news story seem more dramatic than it really is. For help with percentages, see Maths Help Module 3, Subsection 3.1.

The term 'per cent' means 'per 100'. So, for example, if the packaging of a cake tells you that 20% of the cake is fat, then it means that $\frac{20}{100}$ (or $\frac{1}{5}$) of the cake is fat. The symbol % is read as 'per cent'.

A percentage can be converted to a fraction or a decimal. To do this, you first write the percentage in the form of a fraction with denominator 100, then you simplify this to get a fraction, or divide out to get a decimal. For example,

$$45\% = \frac{45}{100} = \frac{9}{20}$$

and

$$45\% = \frac{45}{100} = 45 \div 100 = 0.45.$$

Similarly,

$$100\% = \frac{100}{100} = 1.$$

To convert the other way – from a fraction or decimal to a percentage – you just need to multiply by 100, and your answer is a percentage. Because 100% = 1, this does not change the value of the fraction or decimal; it just allows it to be written as a percentage. For example,

$$\tfrac{2}{5} = \left(\tfrac{2}{5} \times 100\right)\% = 40\%$$

and

$$0.015 = (0.015 \times 100)\,\% = 1.5\%.$$

For help with these conversions, see Maths Help Module 3, Subsection 3.2.

Activity 21 Converting between percentages, fractions and decimals

(a) Complete the following table.

Percentage	Decimal	Fraction
60%		
		$\frac{7}{8}$
	1.35	

(b) Convert 3.8% to a decimal.

It is often helpful to express one number as a percentage of another. This is done by expressing the first number as a fraction of the second number, and then converting the fraction to a percentage, by multiplying by 100, and the answer is a percentage. This is summarised in the box below.

Strategy To express a number as a percentage of another number

Calculate

$$\left(\frac{\text{first number}}{\text{second number}} \times 100\right)\%.$$

Example 11 Expressing a number as a percentage of another number

In a survey of 1500 mature students, 465 agreed with the statement that higher education is vital for getting a new career. What percentage of the group is this?

Solution

Write down the fraction and convert it to a percentage.

The fraction of students who agreed with the statement is

$$\frac{465}{1500}.$$

So the percentage of students who agreed with the statement is

$$\left(\frac{465}{1500} \times 100\right)\% = 31\%.$$

Activity 22 Expressing a number as a percentage of another number

In the survey in Example 11, 420 of the 1500 mature students said that their main reason for going to university was the potential to earn more money. What percentage of the group is this?

Using percentages to make comparisons

One advantage of using percentages rather than fractions is that they are easy to compare. For example, if you were informed that $\frac{7}{9}$ of the pupils in a year group at a school attained the nationally expected standard in English, while $\frac{3}{4}$ of the year group attained the nationally expected standard in mathematics, then you might not be able to tell immediately which subject had the better performance. On the other hand, if you were informed that 78% and 75% of the year group attained the expected levels in English and mathematics, respectively, then you would know immediately that the performance in English is slightly better.

In the example above there is only one group – a year group at a school – but percentages also make it easy to compare figures for different groups. For example, if you were told that 121 pupils from a year group at one school and 86 pupils from the corresponding year group at another school had achieved a certain standard, then you could not tell which school had performed better because the year group at one school might have many more pupils than the year group at the other school. To compare the performance of the two schools, you need to calculate the percentage of pupils at each school who achieved the standard.

A comparison which takes account of underlying numbers in this way is called a **relative comparison**. Here the comparison is relative to the numbers of pupils in the year groups at the two schools. If no account is taken of the underlying numbers – for example, if you just compare the numbers of pupils achieving the standard – then the comparison is known as an **absolute comparison**.

Activity 23 Making a relative comparison

Some fictional results from two English schools in 2009 are given in the table below.

	Number of pupils tested	Number of pupils achieving five or more GCSEs at grades C and above
School A	194	121
School B	130	86

Calculate, for each school, the percentage of pupils who achieved five or more GCSEs at grades C and above. Which school had the better performance, on this measure?

Even when it is clear that a relative comparison is fairer than an absolute one, it is not always clear what the comparison should be *relative to*. For example, the relative comparison in Activity 23 shows which of the two schools had the better performance, but it certainly does not show which school had the better teaching. It might be that the pupils at one of the schools tended to have poorer skills when they started at the school than the pupils at the other school. A fair relative comparison would need to take figures for this issue, and probably others, into account. This is why school league tables do not always provide a good way to compare schools. 'Value-added' measures of performance, which take into account the attainment of the pupils at the time when they start at the school, are better, but it is difficult to devise a truly fair method of comparison.

You might like to watch out for examples of absolute and relative comparisons in the media, as different viewpoints can be put forward depending on the comparison used. For example, an article reporting that a police force had solved fewer crimes one year than it had solved in the previous year might lead you to think that the crime-solving performance of the force had deteriorated.

However, if it turned out that fewer crimes were reported in the second year than in the first, then a relative comparison might show that the crime-solving performance of the force had improved. When you read a media report it is worth thinking about what the viewpoint of the author might be, and whether the figures could be analysed in a different way. We'll return to this topic at the end of this section.

Percentages of quantities

Sometimes you need to work out a percentage of a quantity. For example, Mike works in sales. His yearly bonus is 3% of his total sales for that year. What will his bonus be if his total sales for a year are £300 000? Calculations like this can be worked out using the strategy below.

Strategy To calculate a percentage of a quantity

Change the percentage to a fraction or a decimal, and multiply by the quantity.

Example 12 Calculating a percentage of a quantity

What is 3% of £300 000?

Solution

$$3\% \text{ of } £300\,000 = \tfrac{3}{100} \times £300\,000 = £9000.$$

Activity 24 Calculating percentages of quantities

(a) Work out the following.

- (i) 30% of 150 g
- (ii) 110% of 70 ml
- (iii) 0.5% of £220

(b) To sell an item, an internet auction site charges a fee of £1.50 for the insertion of an advertisement, together with fees of 9% of the first £30 of the selling price and 5% of the remainder of the selling price. If you use the site to sell an item for £75, what is the total fee that you pay?

Percentage increases and decreases

Another common use of percentages is in indicating how quantities have changed. For example, the depreciation in the value of a car during a year, and the change in house prices from one month to the next, can both be conveniently described by percentages.

A percentage increase or decrease is calculated by expressing the increase or decrease as a fraction of the original value, and then converting the fraction to a percentage, by multiplying by 100, and the answer is a percentage. This is summarised in the box below.

Strategy To calculate a percentage increase or decrease

Calculate

$$\left(\frac{\text{actual increase or decrease}}{\text{original value}} \times 100\right)\%.$$

For help with percentage increases and decreases, see Maths Help Module 3, Subsections 3.3–3.5.

Example 13 Calculating a percentage increase

Last year 1450 students enrolled on a mathematics course. This year 1870 students have enrolled. What is the percentage increase in the number of students?

Solution

The actual increase is $1870 - 1450 = 420$.

So the increase as a percentage of the original number is

$$\left(\frac{420}{1450} \times 100\right)\% = 29\% \text{ (to 2 s.f.)}.$$

Hence there is a 29% increase in the number of students.

Activity 25 Calculating a percentage decrease

The number of complaints received by a customer services department has fallen from 145 to 125 over the last month. What is the percentage decrease?

Often you know about a percentage increase or decrease in the value of something, and you want to work out the new value. For example, if an item you want to buy is priced at £599 and the shop is advertising a '15% off day', then you might want to calculate the reduced price. There are two main ways of working out the new value that results from a percentage increase or decrease. The next example shows you both methods.

Example 14 Calculating a value resulting from a percentage decrease

A computer originally priced at £599 is reduced by 15% in a sale. What is the new price?

Solution

First method

Calculate the decrease in price and subtract it from the original price.

The decrease in price is

$$15\% \text{ of } £599 = 0.15 \times £599 = £89.85.$$

So the reduced price is

$$£599 - £89.85 = £509.15.$$

Second method

Use the fact that you have to pay $(100 - 15)\,\%$ of the original price.

The reduced price is $(100 - 15)\,\% = 85\%$ of the original price. So the reduced price is

$$85\% \text{ of } £599 = 0.85 \times £599 = £509.15.$$

When you use the second method to calculate the result of a percentage *increase*, you have to multiply the original value by a percentage greater than 100%, as illustrated in the next example.

Example 15 Calculating a value resulting from a percentage increase

The rent on a flat is £800 per month and is to be raised by 5%. What is the new rent?

Solution

The new rent is $(100 + 5)\,\% = 105\%$ of the original rent. So the new rent is

$$105\% \text{ of } £800 = 1.05 \times £800 = £840.$$

Activity 26 Calculating values resulting from percentage changes

(a) Work out the new price of a car if the original price was £15 400 and the price has been reduced by 20%.

(b) If a weekly wage of £360 is increased by 2.5%, what is the new weekly wage?

(c) If a barrel of oil costs \$90 and the price rises by 100%, what is the new price?

You may have found part (c) of Activity 26 quite surprising: if something increases by 100%, then it doubles. You can also work out that if something increases by 200% then it triples, and if something increases by 300% then it quadruples, and so on.

Here is another example that you might find surprising. From the third quarter of 2005 to the third quarter of 2007, the average price of a house in the UK rose from £165 000 to £200 000. The actual increase was $£200\,000 - £165\,000 = £35\,000$, so the percentage increase was

$$\frac{35\,000}{165\,000} \times 100\% = 21\% \text{ (to 2 s.f.)}.$$

By the fourth quarter of 2008, the average house price had fallen back to about £165 000 again. This is a percentage decrease of

$$\frac{35\,000}{200\,000} \times 100\% = 18\% \text{ (to 2 s.f.)}.$$

So the average house price rose by 21%, but had to fall by only 18% to get back to the original value! This is because the rise started from a smaller value than the fall did.

In 2008, the UK government reduced the rate of value added tax (VAT) from 17.5% to 15%, in response to the economic situation at the time. Many people thought that this meant prices should drop by 2.5%, but in fact the reduction was smaller: only about 2.1%. To see why, consider an item that cost £100 exclusive of VAT. When the VAT rate was 17.5%, the item cost £117.50, and when the VAT rate was 15%, it cost £115. So the VAT cut caused the price to decrease by £2.50 from an original price of £117.50, and hence the percentage decrease was

$$\frac{2.5}{117.5} \times 100\% = 2.1\% \text{ (to 2 s.f.)}.$$

Many shops had to put up notices explaining the perceived discrepancy. Some of the notices included worked examples like the one above!

Making sense of numbers in the media

The final activity in this section asks you to use some of the techniques that you've met to make sense of two fictitious newspaper cuttings. Although both cuttings use the same data, their conclusions appear contradictory.

Table 2 shows the total UK government spending in England and the amount spent on public order, in 2002–3 and 2006–7. The public order category includes spending on the police, fire services, law courts, prisons and associated research.

Table 2 UK government spending in England (£ billion)

	2002–3	2006–7
Total expenditure	274.2	359.2
Public order and safety	18.7	23.7

Source: `www.hm-treasury.gov.uk`

The two cuttings below illustrate how the figures in Table 2 can provide evidence for either praise or criticism of the government's spending on public order in England, depending on how the numbers are manipulated. To understand the numbers in the cuttings, you also need to know that the population of England was about 50 million in 2007.

> **Spending on public order rises by 27% in 4 years**
>
> Government spending on public order rose from £18.7 billion in 2002 to £23.7 billion in 2006. This increase of £5 billion represents an extra £100 spent on public order for every man, woman and child throughout England.

> **PUBLIC ORDER SERVICES LOSE OUT BY £800 MILLION**
>
> In 2002, out of every £1 it spent, the government spent a miserly 6.8p on public order. By 2006, it had fallen to 6.6p – a drop of 3%. Based on 2006 government spending figures, this represents a loss to law and order of almost £800 million.

Figure 17 Two fictional newspaper articles based on the data in Table 2

Activity 27 Checking the figures

(a) Explain how the figures of 27% and £100 in the cutting on the left of Figure 17 were derived.

(b) For each of 2002–3 and 2006–7, calculate the percentage of total expenditure that was spent on public order. Check that these percentages correspond to the amounts of 6.8p and 6.6p in the cutting on the right of Figure 17.

(c) How much would the government have spent on public order in 2006–7 if it had spent the same percentage of total expenditure as in 2002–3? Give your answer to three significant figures.

(d) Use your rounded answer to part (c) to explain how the figure of £800 million in the second cutting has been derived.

(e) What criticisms could you make of each article?

In this subsection you have revised some of the ways that percentages are used, and met the idea of absolute and relative comparisons. You have also seen how percentages and other numbers can be used in the media to promote particular points of view. You may read media articles more critically in future!

4 Thinking mathematically

In the previous two sections, you concentrated on practical mathematics and the sorts of calculations that you can do to describe or understand everyday situations. In this section you will explore some mathematical ideas that are interesting in their own right. These ideas come from a branch of mathematics known as pure mathematics. This is mathematics that does not necessarily have practical applications, but is studied because it is interesting and intriguing, and often beautiful.

> ‘I really think that solving a mathematical puzzle is a little bit like trying to find *who done it* in a murder mystery.’
>
> Marcus du Sautoy, Professor of Mathematics, University of Oxford.

4.1 An odd pattern

This first subsection invites you to think about a mathematical problem that arises from curiosity about numbers. First, here are the names given to various special types of numbers.

The usual counting numbers

$$1, 2, 3, 4, \ldots$$

are called the **natural numbers** (or positive integers).

Each natural number is either even or odd. The **even** natural numbers,

$$2, 4, 6, 8, \ldots,$$

are those that can be divided by 2 exactly; that is, an even number can be divided ‘evenly’ into two parts. For example,

$$8 \div 2 = 4.$$

The **odd** natural numbers,

$$1, 3, 5, 7, \ldots,$$

are those that cannot be divided by 2 exactly; that is, when an odd number is divided by 2, there is 1 left over. For example,

$$7 \div 2 = 3 \text{ remainder } 1.$$

There might be an odd number of socks in your sock drawer!

Figure 18 Odd socks

It is not just the *positive* integers that are either even or odd. For example, 0, -2 and -4 are even numbers, and -1 and -3 are odd numbers. However, this subsection is about positive integers, so, for example, ‘the first four odd numbers’ means the first four positive ones: 1, 3, 5 and 7.

The **square numbers**

$$1, 4, 9, 16, \ldots$$

are obtained by multiplying each natural number by itself:

$$\begin{aligned} 1 &= 1 \times 1 = 1^2, \\ 4 &= 2 \times 2 = 2^2, \\ 9 &= 3 \times 3 = 3^2, \\ 16 &= 4 \times 4 = 4^2, \end{aligned}$$

and so on. The square numbers can be represented as patterns of dots arranged as squares, as shown in Figure 19. These patterns explain why multiplying a number by itself is called *squaring*.

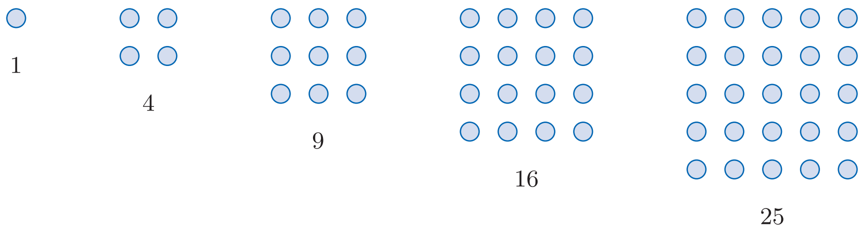


Figure 19 The first five square numbers as patterns of dots

Perhaps you hadn't thought of representing numbers as patterns of dots before, but you will see shortly that this can be helpful in discovering properties of numbers.

Activity 28 Types of numbers

Write down the following numbers.

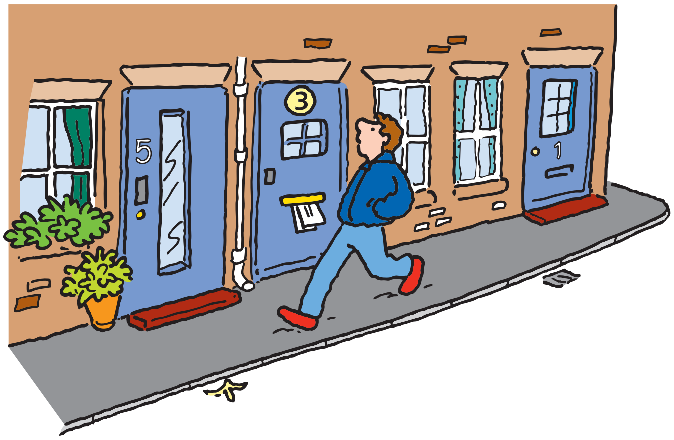
(a) The sixth natural number

(b) The sixth even number

(c) The sixth odd number

(d) The sixth square number

Now imagine that you are walking down one side of a street, looking at the house numbers $1, 3, 5, \ldots$. What happens if you add these numbers up?



Houses are not always numbered with odd numbers on one side of the street and even numbers on the other. In remote parts of Australia they are sometimes numbered according to their distance from a junction, so house 265 is 2650 metres from the junction, for example. Houses may also be numbered according to when they were built: 1 for the first house, 2 for the second, and so on. The numbering system used may depend on what information it is important to convey!

Activity 29 Adding odd numbers

(a) Complete the following table of sums of odd numbers.

How many odd numbers	Sum
1	$1 = 1$
2	$1 + 3 = 4$
3	$1 + 3 + 5 =$
4	$1 + 3 + 5 + 7 =$
5	$1 + 3 + 5 + 7 + 9 =$
6	$1 + 3 + 5 + 7 + 9 + 11 =$

(b) What do you notice about these sums?

In Activity 29 you might have spotted a rather surprising result. The sums look familiar – they are all square numbers. Moreover, each sum is the square of the number of odd numbers that are added. It looks as if adding consecutive odd numbers starting from 1 *always* results in the square of the number of odd numbers that are added. At this stage, this statement is a **conjecture** – an informed guess about what might be true, from considering a few cases. So far, there is not enough information to conclude that what we have observed will always happen, no matter how many odd numbers are added.

If we use the letter n to represent any natural number, then the conjecture can be expressed in the following neat way.

Conjecture

If you add up the first n odd numbers, then the sum is always n^2.

To develop more confidence in this conjecture, you can check that it works for cases you haven't tried. According to the conjecture, when you add the first seven odd numbers the answer should be 7^2, which is 49. Checking this sum gives $1 + 3 + 5 + 7 + 9 + 11 + 13$, which is indeed equal to 49.

At this point, it seems increasingly likely that the conjecture is true, and you can check it for many natural numbers n. But no amount of checking of individual cases can prove that it is true for *all* natural numbers n. However, it turns out that we can prove this by considering patterns of dots.

You have seen that the square numbers can be represented as square patterns of dots. What about the odd numbers? One way to represent them is as L-shaped patterns of dots. Figure 20 shows the first four odd numbers represented in this way.

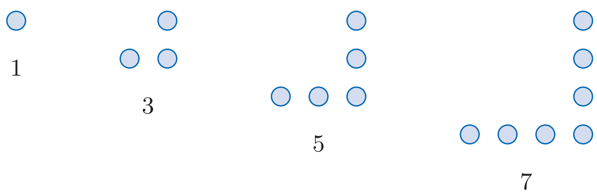


Figure 20 L-shaped patterns for 1, 3, 5 and 7

Let's consider the sum of these four odd numbers. It is $1 + 3 + 5 + 7 = 16$, and you know that this number of dots can be arranged in the square pattern shown in Figure 21.

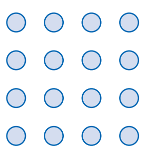


Figure 21 The 4×4 square

Now, a good question is: can you combine the four L-shaped patterns of dots to make the square pattern? Figure 22 shows how this can be done – the lines show the four separate L-shaped patterns.

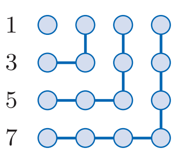


Figure 22 Four odd numbers make a 4×4 square

This picture is very suggestive and you are surely itching to add to it the next L-shaped pattern of 9 dots and so make a 5×5 square! This is shown in Figure 23.

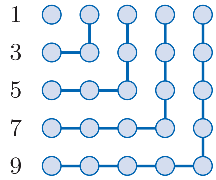


Figure 23 Five odd numbers make a 5×5 square

You can make larger and larger squares of dots by adding larger and larger L-shaped patterns of dots. At each stage you add on the next odd number of dots, and the result is the next square number. So, if you put together the first n of the L-shaped patterns of dots, where n is a natural number, then the result is a square of n^2 dots. This is shown in Figure 24.

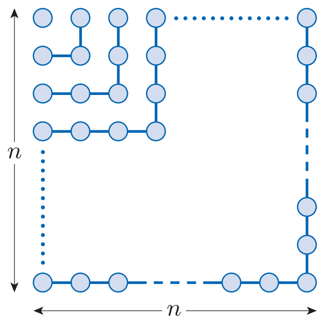


Figure 24 n odd numbers make an $n \times n$ square

Because you can do this for *any* natural number n, you can see that the conjecture is true. That is, the sum of the first n odd numbers is always n^2. So the conjecture has now been proved.

A mathematical statement that has been proved is called a **theorem** or a **result**. So we now have the following result.

Result

If you add up the first n odd numbers, then the sum is always n^2.

Activity 30 Using the result

Use the result above to find the sum of the first 100 odd numbers.

Such is the power of mathematics – it is certainly easier to use the result than to add 100 numbers!

> ‘The mathematician’s patterns, like the painter’s or the poet’s, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.’
>
> G.H. Hardy, in *A Mathematician’s Apology*, Cambridge University Press, 1940.

Figure 25 G.H. Hardy (1877–1947)

In this subsection a result about adding odd numbers was proved using geometric reasoning. However, for many results there is no geometric proof. Instead, mathematicians often use *algebra*. You will learn about algebra later in the module, and in Unit 9 you will see an algebraic proof of a formula for adding up sequences of numbers.

4.2 From folding to fractals

In this subsection, you will see how recognising a pattern that arises from simply folding a strip of paper leads to some surprising and far-reaching ideas. Origami, the ancient art of paper folding, has many practical applications, from the design of solar panels on satellites and space telescopes to modelling the performance of airbags.

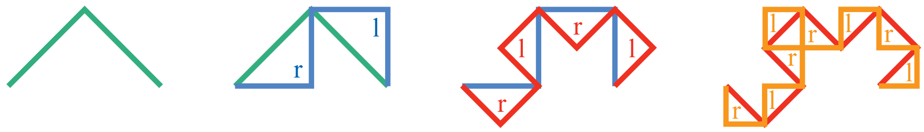
In the video that you watched in Activity 1, you saw a strip of paper being folded in half repeatedly. The strip was partially unfolded after each fold, and the patterns that were seen by looking at the strip edge-on are shown in Figure 26.

(a) First fold (b) Second fold (c) Third fold (d) Fourth fold

Figure 26 Folds in a strip of paper

It is difficult to fold a piece of paper in half more than about six or seven times, even if you start with a long strip. For a long time it was thought that the limit was eight folds, but twelve folds have now been achieved. So only a few patterns can be generated by physically folding a paper strip. If you want to try folding a strip of paper yourself to produce the patterns in Figure 26, then you need to be careful to always fold the strip the ‘same way’, or you will obtain different patterns.

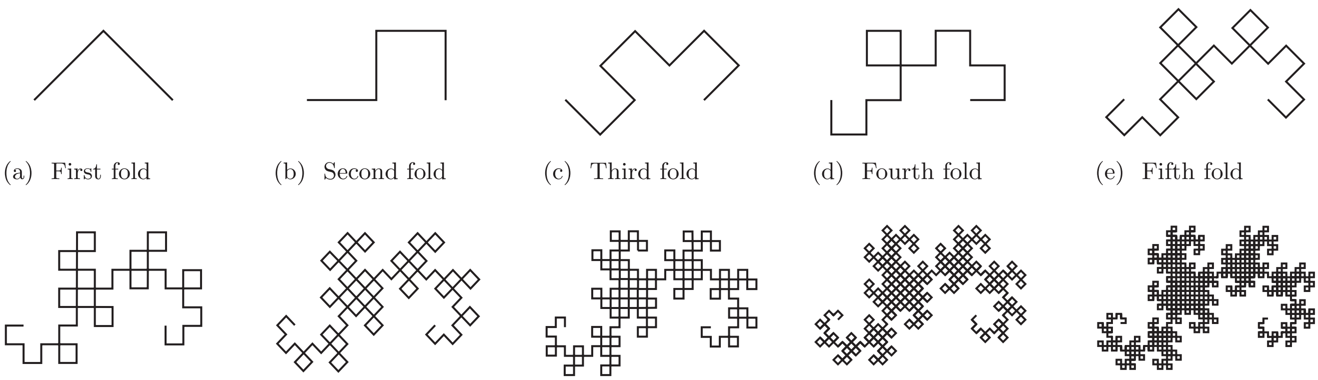
However, by thinking mathematically you can generate the pattern that corresponds to as many folds as you like! If you think carefully about the patterns in Figure 26 (and quite a bit of thought is needed), you can see that to get from one pattern to the next you replace each line in the old pattern by two new lines at right angles to each other – this is what happens when you make a new fold. The two new lines are either to the right or to the left of the old line: the first pair of new lines is to the right, the next pair to the left, and so on. This is illustrated in Figure 27.



(a) (b) (c) (d)

Figure 27 How to get each edge pattern from the one before

You can use this process to find the next few patterns. The first ten patterns are shown in Figure 28.



(a) First fold (b) Second fold (c) Third fold (d) Fourth fold (e) Fifth fold

(f) Sixth fold (g) Seventh fold (h) Eighth fold (i) Ninth fold (j) Tenth fold

Figure 28 The first ten patterns

From a mathematical point of view, the pattern can be developed to correspond to *infinitely* many folds, when, amazingly, it is no longer a collection of lines, but forms a filled-in shape, as shown in Figure 29. **Infinite** means endless and without limit. This shape is called the *Heighway dragon*, and it is an example of a **fractal**.

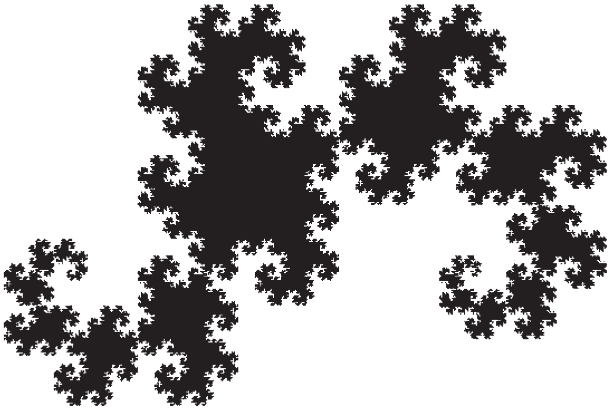


Figure 29 The pattern after infinitely many folds

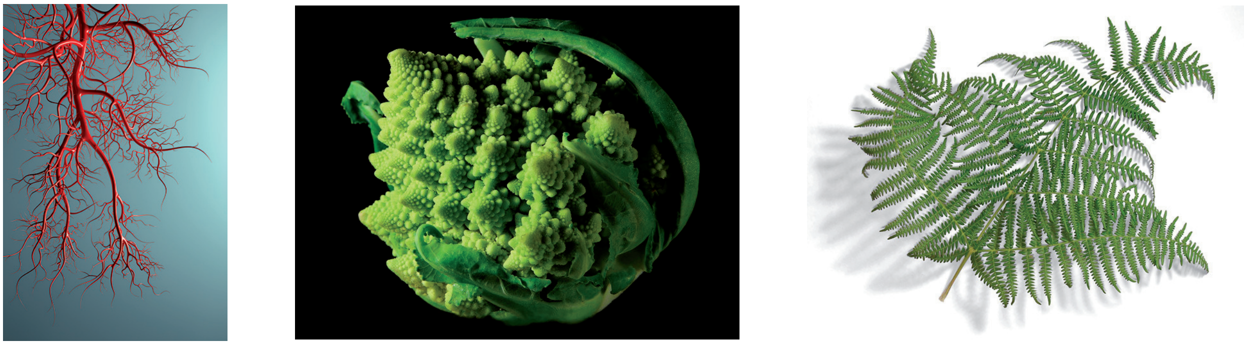
The Heighway dragon is sometimes called the *Jurassic Park dragon*, as it was printed in copies of Michael Crichton's novel *Jurassic Park*. It is also sometimes known as the *Harter–Heighway dragon*, or even just the *dragon curve*. It was first investigated by NASA physicists John Heighway, William Harter and Bruce Banks and was described by Martin Gardner in *Scientific American* in 1967.

The word 'fractal' was coined by the French mathematician Benoit Mandelbrot to describe a shape that is irregular at all scales, no matter how closely you look. Many fractals can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole shape. A shape that has this property is said to be *self-similar*. In Figure 29, you can see that many parts of the Heighway dragon have a shape that is similar to the whole shape.

'As to the word *fractal*, I coined it on some precisely datable evening in the winter of 1975, from a very concrete Latin adjective, fractus, which denoted a stone's shape after it was hit very hard.'

Benoit Mandelbrot, quoted in Hargittai, I. and Laurent, T.C. (eds) (2002) *Symmetry 2000*, pp. 133–41, Portland Press.

Mathematicians have carried out a great deal of research into the properties of fractals in recent years. Fractals are also used in many practical situations, from modelling internet traffic and fluctuations in world stock markets, to medical research. They also abound in nature, for example in the structure of clouds and snowflakes, and in the patterns of lightning. Our own bodies contain a myriad of self-similar fractal systems: for instance, our circulatory systems have this structure – the branching of tiny capillaries is similar to the branching of major arteries and veins. Some more examples of self-similar shapes are shown in Figure 30.



(a) (b) (c)

Figure 30 Self-similarity in (a) a human lung, (b) Romanesco broccoli and (c) a fern

Isn't it surprising that a shape as curly as the Heighway dragon arises naturally from straight lines and right angles? Another surprising fact is that, despite the rough shape of the Heighway dragon, you can cover a flat surface with Heighway-dragon-shaped 'tiles', with no gaps or overlaps! In fact this can be done in several different ways, two of which are shown in Figure 31.

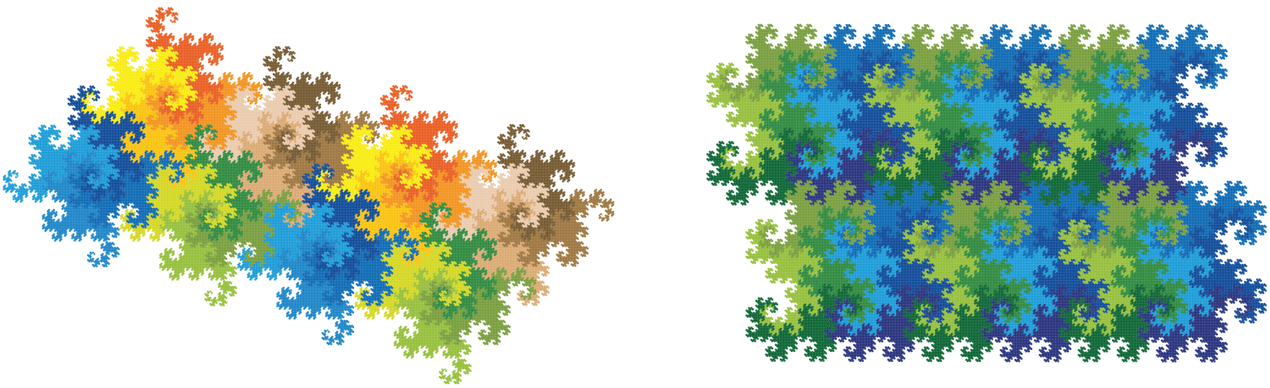


Figure 31 Heighway dragon tilings

In this section you have seen some surprising patterns emerge from simple beginnings – adding odd numbers and folding paper. In the first example you saw how by thinking mathematically – representing numbers as patterns of dots – you could prove that a property was true for any number of odd numbers, not just for those you had checked. In the second example you saw how by thinking about a pattern mathematically you could extend it beyond what is possible physically, with intriguing results. These examples illustrate just a little of the power and beauty of the kind of mathematics that fascinates many mathematicians. You can look for mathematics in all kinds of situations: asking yourself the tantalising question 'What would happen if . . . ?' is often a good way to start!

'Now let me explain a wonderful thing: the more mathematics you learn, the more opportunities you will find for asking new questions.'

Ian Stewart, Professor of Mathematics, University of Warwick (2006) in *Letters to a young mathematician*, New York, Basic Books.

5 Preparing your assignments

This section helps you to prepare for the MU123 assignments and gives you some advice on how to tackle them. The instructions for submitting your assignments are on the module website.

The first subsection helps you to review your progress on the module so far, and the second subsection provides some suggestions for how to approach the iCMAs. The final subsection is based around a sample TMA question. It provides some guidelines to help you to write the kind of clearly explained answers that your tutor will be looking for when reviewing your TMAs and EMA. Learning how to express your mathematics well, so that a reader can easily understand it, is an important part of learning mathematics.

Évariste Galois (1811–1832) was a brilliant young mathematician who died in a duel at the age of 20. He spent some of his final hours editing his mathematical manuscripts and writing a summary of his discoveries. It took mathematicians many years to understand his notes, but they now form the basis of a branch of advanced mathematics called Galois theory. A moral of his story is that you should write down your ideas in good time, and explain them clearly, so they can be easily understood by others. Another moral is to avoid duelling!

Figure 32 A French postage stamp celebrating Galois

When should you start your assignment questions?

Most students find that it is best to start each assignment question fairly soon after studying the module material on which the question is based. For example, you might find it helpful to do each assignment question immediately after studying the relevant section or sections in the unit. This allows you to focus closely on the topics and may alert you to the need to re-read some sections. Alternatively, you might prefer to work through the whole of each unit before tackling the assignment questions on the unit. This has the advantage that you might find some questions on the earlier parts of a unit easier when you have studied the whole unit. Also, if you leave a short gap between studying the material relevant to an assignment question and attempting the question, then you should find that you are better able to remember key ideas, since the more times you use a concept or technique, the better you remember it. This should make your study of future units and modules easier.

It is usually not a good idea to defer starting an assignment until close to the cut-off date. This is because you may need time to revise some topics or contact your tutor with questions, and you are unlikely to be able to produce your best work if you are under time pressure. Also, something unexpected may happen near the cut-off date, so you should allow some contingency time.

5.1 Reviewing your progress

Before you start on the assignment questions for a unit, it is a good idea to spend a few moments thinking about the progress that you have made while studying the unit. You should check whether there are any topics that you need help with, or any topics on which you need more practice. If you sort out any problems, then you should find it easier to do the assignment questions, and you will also find it easier to understand the material in the later units. If you are an experienced student, then you have probably already paused to take stock of your progress, whether formally or not. It is a good idea to do this every so often.

In the next activity you are asked to look back over your work on this unit and use the practice quiz to help you to assess your progress. The practice quizzes for each unit can be accessed from the module website, along with other resources for each unit – you may have tried some of the questions earlier in the unit. The quizzes are similar in style to the iCMA questions, so you can also use them to familiarise yourself with the process of answering iCMA questions before you attempt the first iCMA.

Practice Quiz

Activity 31 Checking your progress

(a) Are you confident about the mathematical skills covered in this unit? Look back at your answers to the activities, and any notes you made, to identify where you might need more practice. You might like to complete the table below, to help you to organise your thoughts.

Topic	Confident	Need more practice	Unhappy with this topic
BIDMAS			
Using your calculator			
Units of measurement			
Rounding			
Checking answers			
Negative numbers			
Fractions			
Percentages			

(b) Try the practice quiz questions on this unit if you have not already done so, to check your understanding of the topics above.

(c) If you are still not confident in some areas, then plan what you will do to improve your understanding and skills. You may need to allow some extra time to work through some topics in Maths Help, or to try some more practice quiz questions. If you are not sure of the best way forward, then contact your tutor for advice.

5.2 iCMA questions

When you do the iCMA questions on a unit, you should have the unit and any notes that you made to hand, as you will probably find it helpful to consult them. You will also need a pen or pencil, paper and your calculator.

Make sure that you read each question carefully, so that you understand what is required before you start to work out your answer. You do not have to complete all the questions in an iCMA in one session: you can answer a few questions at a time, in any order, and save your answers. You can change your answers in later sessions if you wish, before submitting the iCMA.

Once you have completed the questions in an iCMA, it is a good idea to read through the questions again, to check that you are happy with your answers and that you have answered as many questions as you can.

Activity 32 Answering iCMA questions

iCMA

Find the first iCMA on the module website. Follow the instructions given there and try some of the questions. Complete as many of the questions as you can before the cut-off date.

If you find a lot of questions in the first iCMA difficult, then you may need to seek advice from your tutor.

5.3 TMA questions

TMAs are more substantial pieces of work than iCMAs. They allow your tutor to assess how you present and explain your mathematical ideas, as well as the accuracy of your mathematics. An example of part of a TMA question is shown below.

> Last year, a town recycled 8750 tonnes of waste, of which 20% was paper. Five years ealier, the town recycled 1130 tonnes of waste paper.
>
> (a) Find the amount of paper that was recycled last year, in tonnes to three significant figures. [4]
>
> (b) Show that the amount of paper that was recycled has risen by approximately 52% over the five year period. [3]

Figure 33 Part of a typical TMA question

TMA questions are usually similar to activities in the units, but they include marks for the question parts, in square brackets at the right-hand side. Usually, the more marks a question part is worth, the more substantial your solution should be.

If you are not sure how to do part of a TMA question, then look back through the unit and any notes that you made, to remind yourself of the methods that you could use. For example, for part (a) of the question above you could look back at Subsection 3.3, which is about percentages, and in particular at Example 12 on page 42, which illustrates how to calculate a percentage of a quantity. If looking back through the unit does not help, and you are stuck, then contact your tutor.

Once you have decided what method seems appropriate for a question part, you need to write out a full and clear solution for your tutor. You may find it helpful to write a rough version first.

Here is an example of a solution to part (a) of the question above that would be awarded full marks by a tutor. The solution has been annotated to show the key steps.

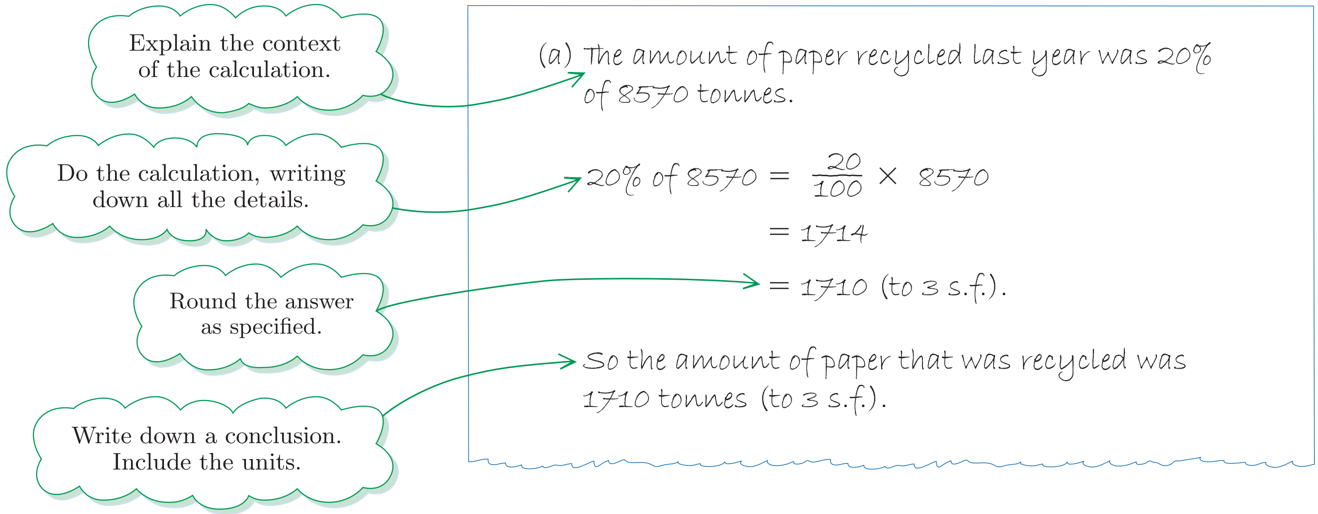


Figure 34 An annotated solution to a typical TMA question

Notice that the solution has been written in sentences, and the final sentence gives a clear conclusion in the context of the question, and includes the appropriate units. This means that anyone reading the solution can easily understand it.

There is always more than one way to write a solution. Here is an alternative solution to the same question part, which would be equally acceptable.

(a) The amount of paper recycled last year was

$$\begin{aligned} 20 \text{ percent of } 8570\text{ t} &= 0.2 \times 8570\text{ t} \\ &= 1714\text{ t} \\ &= 1710\text{ t (to 3 s.f.).} \end{aligned}$$

Figure 35 An alternative solution to the TMA question

In this second solution, the explanation and details of the calculation, and the conclusion, are all given in one sentence. You may be able to do this for a short calculation, but you must make sure that the sentence makes sense and that the answer at the end of the sentence includes the appropriate units.

If you are not sure how much explanation or detail you need to include in your solution to a TMA question, then use the worked examples in the units as a guide. Remember that the blue thinks text is not part of the solutions, but all the other words *are* part of them, and your solutions should include similar amounts of explanation and working.

In the next activity you are asked to write out a solution to part (b) of the TMA question. This question part actually gives you the answer, 52%, and asks you to *show* how it can be worked out. Because of this, none of the marks for this question part will be for the final answer – all three marks will be for working and explanation.

When you are doing a question that gives you the answer, you should not use the given answer in your working. Just work it out as usual, and then check that your answer matches the one given in the question.

Activity 33 Writing a good solution

Try to write out a good solution to part (b) of the TMA question on page 60. Then look at the comments on this activity in the back of the unit.

Activity 34 Improving a solution

Here is an incorrect and poorly explained solution to parts (a) and (b) of the TMA question.

(a) $8570 = 0.2 \times 8570 = 1714$

(b) $1714 - 1130 = 584 = \frac{584}{1714} = .34$

Write down some suggestions for how the solution could be improved. Then look at the comments on this activity in the back of the unit.

Although the solution in Activity 34 could be significantly improved, it does show some understanding of how the question could be answered and so it would be awarded some marks.

There are several other things to notice about this solution. First, equals signs are used incorrectly, which is another reason why the solution is difficult to follow. Whenever you use an equals sign, whatever is on the left of the equals sign must be equal to whatever is on the right. So it is wrong to write

$$8570 = 0.2 \times 8570,$$

as in the solution, because the number on the left, 8570, is not equal to the calculation on the right, 0.2×8570.

Activity 35 Using equals signs correctly

Can you spot two other places in the solution in Activity 34 where equals signs have been used incorrectly?

Another thing to notice about the solution in Activity 34 is that its author should have spotted that the answer to part (b) is wrong, because the correct answer, 52%, is given in the question. It is always worth thinking about whether your answer is likely to be correct. You can often do this even when the answer is not given in the question. For example, if you were doing part (a) of the TMA question and you obtained the answer 17 100 tonnes, say, for the amount of paper recycled last year, then you might have spotted that this answer is too large. You could spot this by considering the context: 17 100 tonnes is more than the total amount of waste recycled. Alternatively you could have estimated that the correct answer is approximately 20% of 10 000 tonnes, which is 2000 tonnes. For help with reading and writing mathematics, and approaches to problem solving, see Maths Help Module 6.

Notice that although the author of the solution wrote down the wrong fraction in part (b), he or she went on to evaluate it correctly as 0.34. If you make a mistake but your calculations following the mistake are correct, then you may be awarded marks for these calculations even though your final answer is incorrect. These marks are known as 'follow-through marks'. Your tutor may use the abbreviation 'F.T.' to indicate follow-through marks or reasoning.

The incorrect solution in Activity 34 illustrates why it is so important to explain your solutions clearly. It is important that your tutor can follow your working, and also that you yourself can follow it if you look over it some time later. If you use mathematics in your workplace, then it is also important that you write down clearly explained solutions for your colleagues.

When you have written out your solution to a TMA question, you should read through the question again, to make sure that you have answered all the parts and given all your answers in the required form. Then read over your solution again, to make sure that your explanations make sense. It can be helpful to do this after a break – you might be surprised to find that what you have written down does not quite say what you meant it to.

Some guidelines for writing good mathematics are summarised below.

Things to remember when writing your own mathematics

- Write in sentences, explaining your reasoning step by step.
- Use link words like 'so' to make your solution easier to read.
- Start each new idea on a new line.
- Use notation correctly, especially equals signs.
- Include units where appropriate.
- Give a conclusion, in the appropriate context.
- Read through what you have written to check that it makes sense.

You have seen that if a question part starts with the word ‘show’, then the answer is given in the question, and all the marks are for explanation and working. Table 3 gives some similar words that you might see in TMA questions, and explains what sorts of answers are expected.

Table 3 Instructions in mathematics questions

Instruction	Explanation
find, determine, calculate, work out	You need to give both an answer and the details of how the answer is worked out. Some of the marks are for the answer, and some are for explanation and working.
show, verify, check	The answer is given to you, and you need to give the details of how it is worked out. All of the marks are for explanation and working.
write down, state, list	You need to give only an answer; no explanation or working is required. All of the marks are for the answer. This is usually because the answer can be found without doing any working.

Explanation and working are nearly always required in your solution to a TMA question, so if in doubt, include them!

TMA

Activity 36 Answering the TMA questions for Unit 1

If you have not already done so, find and read the document on the module website that gives you information about submitting TMAs – it is in the same place as the TMAs. Then open the first TMA and prepare your solutions to the questions on Unit 1. Keep your solutions safe until you have completed the other questions in the TMA.

The box below summarises the main steps that you should follow when doing TMA questions.

Preparing solutions to TMA questions

- Start each question well before the cut-off date.
- Read the questions carefully, noting all the instructions – for example, you may be asked to use a particular method, or round an answer in a particular way.
- If you are stuck, look back at similar examples and activities in the unit.
- Contact your tutor if you need help.
- Write your solutions clearly, giving all the details of your working and explaining it carefully.
- Check that your answers seem reasonable.
- Read the questions and your solutions again, to make sure that you have answered all the parts and followed all the instructions.

6 Reviewing your study methods

This unit is a little different from the other units in the module, as it aims to introduce the module and check that you are well prepared to study it. Now that you have worked through the unit, you should know how the different module components (such as the tutorial clips, practice quizzes, videos and assignments) can help your learning, and you should have improved your fluency in some essential mathematical skills, including using your calculator.

You have been encouraged to think about your studying, and to make changes to improve it, if necessary. Before you go on to the next unit, it's worth spending a few minutes reviewing how your studying of this unit has gone, to see whether there are any changes that it would be helpful to make to your study methods in future.

- Did you manage to find enough time to complete the unit in the period allocated on the study planner? If not, do you need to rearrange or give up some other activities to make more time, or do you need to try to make better use of the time you have allocated for studying? For example, a greater number of shorter study sessions or studying at a different time of the day may be more effective.
- Can you remember the ideas that you have studied? You do not have to remember everything in the unit, but you will need to use some of the techniques and ideas in later parts of the module. The most important ideas are the ones covered in the practice quizzes and assignment questions. Learning actively, by doing the activities and perhaps making some form of notes, will help your understanding and your ability to apply and remember these ideas.
- Can you quickly find any information that you need? For example, if you want to check what 'improper fraction' means, or revise how to round appropriately, then where would you look first – the unit, the book's index, your notes, the glossary in the Handbook, or the summary pages in the Handbook? Try to get to know what information is contained in different sources, and try to keep your notes organised, so that you can easily find information that you need.

At the end of each unit there is a learning checklist, which you can use to make sure that you have acquired the main skills and knowledge taught in the unit. If there are some skills that you do not feel confident about, then you may need to spend more time on them. However, if doing this will use up some of the time that you have allocated for the next unit, then you should contact your tutor to discuss how to proceed, as it is also important to keep up with the schedule in the study planner.

As well as helping you prepare to study the rest of the module, this unit has also introduced some practical applications of mathematics – for example, it has shown you how to critically assess some types of numerical information in the media. It has also offered you a taste of the power and beauty of abstract mathematics. Mathematics really is everywhere, and in MU123 you will have plenty of opportunities to discover this for yourself!

Learning outcomes

After studying this unit, you should be able to:

- find and use the main components of the module, including tutorial clips, practice quizzes, videos and assignments
- plan how you will use your study time effectively
- carry out mathematical operations such as $+$, $-$, $\times$ and $\div$ in the correct order, using the BIDMAS rules
- use your calculator effectively
- understand and use some SI units
- round numbers appropriately to a number of decimal places or significant figures
- check calculations by estimating answers
- understand and use negative numbers, fractions and percentages
- start to critically analyse numerical information in the media
- understand the difference between relative and absolute comparisons
- start to investigate mathematical patterns, make conjectures and appreciate the idea of proof
- start to write mathematics well, using appropriate notation
- review your learning progress and make changes to improve your study methods
- prepare your answers to iCMA and TMA questions.

Solutions and comments on Activities

Solution to Activity 2

(a) $9 + 7 - 2 - 4 = 16 - 2 - 4 = 14 - 4 = 10$

(b) $2 \times (7 - 4) = 2 \times 3 = 6$

(c) $(3 + 5) \times 3 = 8 \times 3 = 24$

(d) $(3 + 4) \times (2 + 3) = 7 \times 5 = 35$

(e) $3^2 + 4^3 = 3 \times 3 + 4 \times 4 \times 4 = 9 + 64 = 73$

Solution to Activity 3

(a) The calculation is incorrect. A correct calculation is $2 \times (5 + 3) = 16$.

(b) The calculation is incorrect. A correct calculation is $(3 + 4) \times 7 = 49$.

(c) The calculation is correct.

(d) The calculation is correct.

(e) The calculation is incorrect. A correct calculation is $2 \times (3 + 3) \times 5 = 60$.

Solution to Activity 4

(a) The numbers are 4 and 8. Their sum is $4 + 8 = 12$ and their product is $4 \times 8 = 32$. (If you didn't spot the answer quickly, then you could have found it by systematically listing the pairs of whole numbers with sum 12 until you found the pair with product 32.)

(b) The numbers are 2 and 4. Their difference is $4 - 2 = 2$ and their quotient is $4 \div 2 = 2$.

Solution to Activity 6

(a) There are 1000 m in 1 km, so

$$6100\,\mathrm{m} = (6100 \div 1000)\,\mathrm{km} = 6.1\,\mathrm{km}.$$

(b) There are 1000 kg in 1 t, so

$$560\,\mathrm{kg} = (560 \div 1000)\,\mathrm{t} = 0.56\,\mathrm{t}.$$

(c) There are 60 minutes in 1 hour, so

$$\begin{aligned} 3.45 \text{ hours} &= (3.45 \times 60) \text{ minutes} \\ &= 207 \text{ minutes.} \end{aligned}$$

(d) There are 1000 mg in 1 g, so

$$0.35\,\mathrm{g} = (0.35 \times 1000)\,\mathrm{mg} = 350\,\mathrm{mg}.$$

(e) There are 1000 ml in 1 l, so

$$450\,\mathrm{ml} = (450 \div 1000)\,\mathrm{l} = 0.45\,\mathrm{l}.$$

(f) There are 100 cm in 1 m, so

$$75\,\mathrm{cm} = (75 \div 100)\,\mathrm{m} = 0.75\,\mathrm{m}.$$

Solution to Activity 7

(a) $2.2364 = 2.24$ (to 2 d.p.)

(b) $0.005\,47 = 0.005$ (to 3 d.p.)

(c) $42.598\,17 = 42.5982$ (to 4 d.p.)

(d) $7.98 = 8.0$ (to 1 d.p.)

(In part (d), the 0 after the decimal point should be included to show that the number has been rounded to one decimal place.)

Solution to Activity 8

(a) $23\,650 = 24\,000$ (to 2 s.f.)

(b) $0.005\,47 = 0.005$ (to 1 s.f.)

(c) $42.598\,17 = 42.60$ (to 4 s.f.)

(In part (c), the 0 after the 6 should be included to show that the number has been rounded to four significant figures.)

Solution to Activity 9

(a) Using the given conversion factor gives the distance in miles as

$$465 \times 0.62 = 288.3.$$

Rounding to the nearest mile gives the answer 288 miles. This is inaccurate, since we know from the calculation on page 21 that the correct answer to the nearest mile is 289 miles. The conversion factor used in this activity is not precise enough to give an answer correct to the nearest mile.

(b) Rounding to two significant figures gives the answer 290 miles.

(The answer is rounded to two significant figures because two is the number of significant figures in the least precise number used in the calculation. The answer found here agrees with the answer found in the calculation before the activity, 289 miles, because $289 = 290$ (to 2 s.f.).)

Solution to Activity 10

(a) The distance, in miles, to the town is $36 \times 0.621\,371\,192$. The answer displayed on the calculator is $22.369\,362\,91$, which rounds to 22 (to the nearest mile). Multiplying by 2 gives

$$2 \times 22 = 44.$$

So the total distance is calculated as 44 miles.

(b) As in part (a), the distance, in miles, to the town is displayed on the calculator as $22.369\,362\,91$. Multiplying by 2 gives

$$\begin{aligned} 2 \times 22.369\,362\,91 &= 44.738\,725\,82 \\ &= 45 \text{ (to the nearest mile).} \end{aligned}$$

So the total distance is 45 miles.

(c) The answer in part (b) is more accurate. In part (a), rounding too early led to an inaccurate final answer.

Solution to Activity 11

(a) Each jewellery box takes about 4 hours to make and decorate. A working day is about 8 hours, so about two jewellery boxes can be completed in a working day. So it would take about 24 days to complete 48 jewellery boxes.

(b) The student's first mistake was to forget to include brackets around '$2.30 + 1.45$'. So the calculator will first multiply 1.45 by 48, then divide by 7.5, and then add 2.30, which is not what the student intended.

The student's other mistake was to assume that if you add 2 hours and 30 minutes to 1 hour and 45 minutes then the total number of hours is $2.30 + 1.45$. This is not correct, since 2 hours and 30 minutes is 2.5 hours, not 2.30 hours, and 1 hour and 45 minutes is 1.75 hours, not 1.45 hours.

(c) The time needed to make and decorate a jewellery box is

$$\begin{aligned} &2 \text{ hours and } 30 \text{ minutes} + 1 \text{ hour and } 45 \text{ minutes} \\ &= 4 \text{ hours and } 15 \text{ minutes} \\ &= 4.25 \text{ hours.} \end{aligned}$$

Thus the number of days needed to make and decorate 48 jewellery boxes is

$$4.25 \times 48 \div 7.5 = 27.2.$$

This number has to be rounded up, because all 48 boxes must be finished. So 28 days are needed.

(This answer, unlike the student's, is fairly close to the estimate found in part (a).)

Solution to Activity 12

The numbers -6 and -4 lie to the left of -3 on the number line, and the numbers -2, 2 and 0 lie to the right of -3. So the noon temperature was lower on Monday and Friday.

Solution to Activity 13

(a) $-6 + 2 = -4$

(b) $-1 + 3 = 2$

(c) $2 - 7 = -5$

(d) $-3 - 4 = -7$

(e) $5 - 7 - 2 = -2 - 2 = -4$

Solution to Activity 14

(a) $2 + (-7) = 2 - 7 = -5$

(b) $-8 + (-5) = -8 - 5 = -13$

(c) $1 - (-3) = 1 + 3 = 4$

(d) $-6 - (-9) = -6 + 9 = 3$

(e) $-4 - (-4) = -4 + 4 = 0$

(f) $3 - (-2) + (-4) = 3 + 2 - 4 = 5 - 4 = 1$

(g) $7 + (-6) - 3 = 7 - 6 - 3 = 1 - 3 = -2$

Solution to Activity 15

(a) $5 \times (-3) = -15$

(b) $(-2) \times (-4) = 8$

(c) $6 \times (-10) = -60$

(d) $25 \div (-5) = -5$

(e) $(-49) \div (-7) = 7$

(f) $(-36) \div 12 = -3$

(g) $(-2) \times (-5) \times (-4) = 10 \times (-4) = -40$

Solution to Activity 17

(a) **(i)** $\frac{7}{21} = \frac{1}{3}$

(ii) $\frac{48}{72} = \frac{2}{3}$

(iii) $\frac{35}{105} = \frac{1}{3}$

(b) The fraction of the group is

$$\frac{720}{1200} = \frac{3}{5}.$$

(You might have cancelled the fraction like this:

$$\frac{\overset{\overset{\overset{3}{\cancel{9}}}{\cancel{72}}}{\cancel{720}}}{\underset{\underset{\underset{5}{\cancel{15}}}{\cancel{120}}}{\cancel{1200}}} = \frac{3}{5}\,.$$

But there are many different ways to cancel it.)

Solution to Activity 18

(a) $5\frac{2}{3} = \dfrac{5 \times 3 + 2}{3} = \frac{17}{3}$

(b) $\frac{18}{5} = 3\frac{3}{5}$

Solution to Activity 19

(a) **(i)**
$$\begin{aligned} \tfrac{4}{5} \text{ of } 60\,\text{ml} &= \tfrac{4}{5} \times 60\,\text{ml} \\ &= 4 \div 5 \times 60\,\text{ml} \\ &= 48\,\text{ml} \end{aligned}$$

(ii)
$$\begin{aligned} \tfrac{5}{8} \text{ of } 20\,\text{kg} &= \tfrac{5}{8} \times 20\,\text{kg} \\ &= 5 \div 8 \times 20\,\text{kg} \\ &= 12.5\,\text{kg} \end{aligned}$$

(b) The quantity of potatoes needed is

$$\begin{aligned} \tfrac{20}{6} \times 900\,\text{g} &= 20 \div 6 \times 900\,\text{g} \\ &= 3000\,\text{g} \\ &= 3\,\text{kg}. \end{aligned}$$

Solution to Activity 21

(a) The conversions are

$$\begin{aligned} &60\% = \tfrac{60}{100} = 0.6, \\ &60\% = \tfrac{60}{100} = \tfrac{3}{5}, \\ &\tfrac{7}{8} = \left(\tfrac{7}{8} \times 100\right)\% = (7 \div 8 \times 100)\,\% = 87.5\%, \\ &\tfrac{7}{8} = 7 \div 8 = 0.875, \\ &1.35 = (1.35 \times 100)\,\% = 135\%, \\ &1.35 = 135\% = \tfrac{135}{100} = \tfrac{27}{20} = 1\tfrac{7}{20}. \end{aligned}$$

So the completed table is as follows.

Percentage	Decimal	Fraction
60%	0.6	$\frac{3}{5}$
87.5%	0.875	$\frac{7}{8}$
135%	1.35	$\frac{27}{20}$

(b) $3.8\% = \dfrac{3.8}{100} = 0.038$

Solution to Activity 22

The fraction of students is

$$\frac{420}{1500}.$$

So the percentage of students is

$$\left(\frac{420}{1500} \times 100\right)\% = 28\%.$$

Solution to Activity 23

The percentage of pupils at School A who achieved the standard is

$$\left(\frac{121}{194} \times 100\right)\% = 62.4\% \text{ (to 1 d.p.)}.$$

The percentage of pupils at School B who achieved the standard is

$$\left(\frac{86}{130} \times 100\right)\% = 66.2\% \text{ (to 1 d.p.)}.$$

So School B had the better performance.

Solution to Activity 24

(a) **(i)** 30% of $150\,\text{g} = \frac{30}{100} \times 150\,\text{g}$
$= 0.3 \times 150\,\text{g}$
$= 45\,\text{g}$

(ii) 110% of $70\,\text{ml} = \frac{110}{100} \times 70\,\text{ml}$
$= 1.1 \times 70\,\text{ml}$
$= 77\,\text{ml}$

(iii) 0.5% of $£220 = \frac{0.5}{100} \times £220$
$= 0.005 \times £220$
$= £1.10$

(b) The fee paid on the first £30 of the selling price is

$$9\% \text{ of } £30 = \tfrac{9}{100} \times £30 = £2.70.$$

The remainder of the selling price is $£75 - £30 = £45$, and the fee paid on this amount is

$$5\% \text{ of } £45 = \tfrac{5}{100} \times £45 = £2.25.$$

The total fee is the insertion fee plus the two fees above, which is

$$£1.50 + £2.70 + £2.25 = £6.45.$$

Solution to Activity 25

The actual decrease is $145 - 125 = 20$.

So the decrease as a percentage of the original number is

$$\left(\frac{20}{145} \times 100\right)\% = 14\% \text{ (to 2 s.f.)}.$$

Hence there is a 14% decrease in the number of complaints.

Solution to Activity 26

(a) The new price is $(100 - 20)\,\% = 80\%$ of the original price. So the new price is

$$80\% \text{ of } £15\,400 = 0.8 \times £15\,400 = £12\,320.$$

(b) The new wage is $(100 + 2.5)\,\% = 102.5\%$ of the original wage. So the new price is

$$102.5\% \text{ of } £360 = 1.025 \times £360 = £369.$$

(c) The new price is $(100 + 100)\,\% = 200\%$ of the original price. So the new price is

$$200\% \text{ of } \$90 = 2 \times \$90 = \$180.$$

Solution to Activity 27

(a) Government spending rose from £18.7 billion (or bn) to £23.7 bn, which is an increase of £5 bn, as stated in the article. So the percentage increase is

$$\left(\frac{5}{18.7} \times 100\right)\% = 27\% \text{ (to 2 s.f.)}.$$

This explains the figure of 27% in the headline of the cutting on the left.

Since the actual increase in spending was £5 bn, and there were about 50 million people in England in 2007, the increase in spending per person was approximately

$$\begin{aligned} \frac{£5 \text{ bn}}{50 \text{ million}} &= \frac{£5000 \text{ million}}{50 \text{ million}} \\ &= \frac{£5000}{50} \\ &= £100. \end{aligned}$$

This explains the figure of £100 in the headline of the cutting on the left.

(b) The percentage of total expenditure that was spent on public order in 2002–3 was

$$\left(\frac{18.7}{274.2} \times 100\right)\% = 6.8\% \text{ (to 2 s.f.)}.$$

The percentage of total expenditure that was spent on public order in 2006–7 was

$$\left(\frac{23.7}{359.2} \times 100\right)\% = 6.6\% \text{ (to 2 s.f.)}.$$

These percentages correspond to the amounts of 6.8p and 6.6p in the second cutting, because 6.8% of £1 is 6.8p and 6.6% of £1 is 6.6p.

(c) The percentage of total expenditure spent on public order in 2002–3 was approximately 6.8%, or more precisely, 6.819 83. . . %. If this percentage of total expenditure had been spent on public order in 2006–7, then the spending on public order would have been

$$\begin{aligned} &6.819\,83\ldots\% \text{ of } £359.2 \text{ bn} \\ &= \frac{6.819\,83\ldots}{100} \times £359.2 \text{ bn} \\ &= 0.068\,198\,3\ldots \times £359.2 \text{ bn} \\ &= £24.5 \text{ bn (to 3 s.f.)}. \end{aligned}$$

(d) The difference between the amount in part (c) and the amount that the government actually spent on public order in 2006–7 is

$$£24.5 \text{ bn} - £23.7 \text{ bn} = £0.8 \text{ bn} = £800 \text{ million}.$$

This explains how the figure of £800 million was worked out.

(e) The first article, on the left-hand side, emphasises the *absolute* increase in the amount spent, but ignores the fact that prices will have risen over the four-year period as well. So some of the extra £5 bn would be spent just maintaining the level of support that the public received in 2002. The key question here is what new support is being provided for the public – and neither that, nor the amount spent on new support, is stated in the article.

By using a relative comparison, the second article ignores the fact that there was a significant absolute increase in spending on public order, and in total spending. It is not helpful to be told that the percentage of total expenditure that is spent on public order has dropped. That might have been caused by, for example, large increases in spending on health and education, without any loss to spending on public order. A smaller percentage of a larger amount may still be larger than a larger percentage of a smaller amount! The percentage spent has dropped, but again the key question is what effect has that had on the services provided – has there been an overall increase or decrease in those?

Solution to Activity 28

(a) The first six natural numbers are 1, 2, 3, 4, 5, 6, so the sixth natural number is 6.

(b) The first six even numbers are 2, 4, 6, 8, 10, 12, so the sixth even number is 12.

(c) The first six odd numbers are 1, 3, 5, 7, 9, 11, so the sixth odd number is 11.

(d) The first six square numbers are 1, 4, 9, 16, 25, 36, so the sixth square number is 36.

Solution to Activity 29

(a) The completed table is as follows.

How many odd numbers	Sum
1	$1 = 1$
2	$1 + 3 = 4$
3	$1 + 3 + 5 = 9$
4	$1 + 3 + 5 + 7 = 16$
5	$1 + 3 + 5 + 7 + 9 = 25$
6	$1 + 3 + 5 + 7 + 9 + 11 = 36$

(b) All the sums are square numbers – in fact, each sum is the square of the number of odd numbers that are added.

Solution to Activity 30

By the result stated before the activity, the sum of the first 100 odd numbers is $100^2 = 10\,000$.

Solution to Activity 31

If you have any concerns about your responses to this activity, then contact your tutor for advice.

Solution to Activity 33

Here is an example of a solution to part (b) that would be awarded full marks.

(b) The percentage rise is

$$\begin{aligned} &\frac{\text{actual rise}}{\text{original amount}} \times 100\% \\ &= \frac{1714 - 1130}{1130} \times 100\% \\ &= 52\% \text{ (to 2 s.f.)}. \end{aligned}$$

Hence the amount of paper recycled has risen by about 52%, as required.

Did you remember to explain your calculation in words, and to write in sentences? And did you remember to use the *unrounded* value 1714 from part (a), not the rounded value 1710? If you used the rounded value, then you will have obtained the answer 51%, which is not the answer given in the question.

Solution to Activity 34

See how many of the following possible improvements you spotted – but don't worry if you didn't spot them all! The author of the solution could:

- explain the calculations in words and write in sentences
- include a conclusion for each question part, stating the answer clearly in the context of the question
- include the units in part (a)
- round the answer to part (a) to three significant figures, as requested in the question
- check the answer to part (b) against the given answer of 52%, and try to find the mistake
- correct the mistake – the amount of paper recycled five years ago was 1130 tonnes, not 1714 tonnes, so the increase as a fraction of the amount five years ago is $\frac{584}{1130}$, not $\frac{584}{1714}$
- write .34 as 0.34 – a decimal point should have a digit on each side to make it easy to read – it's easy to mistake .34 for 34
- write 0.34 as the percentage 34%, since the question referred to the percentage rise
- use equals signs correctly – see the text following the activity.

Solution to Activity 35

The equals sign in

$$584 = \frac{584}{1714}$$

is used incorrectly. The fraction on the right-hand side is not equal to 584.

The equals sign in

$$\frac{584}{1714} = .34$$

is also used incorrectly. The fraction on the left is not *exactly* equal to 0.34, so the author of the solution should have either used the approximately equals sign or included the rounding precision. That is, he or she should have written either

$$\frac{584}{1714} \approx 0.34$$

or

$$\frac{584}{1714} = 0.34 \text{ (to 2 s.f.).}$$

(The equals sign in

$$1714 - 1130 = 584$$

is correctly used, but it would be better to explain this calculation in a sentence, such as: 'The increase in the amount of paper recycled is $1714\,\mathrm{t} - 1130\,\mathrm{t} = 584\,\mathrm{t}$.' Then a linking word such as 'So' could be used to introduce the next line of the calculation.)