Positive-definite and positive semi-definite matrix

1 Notations

- the conjugate transpose of matrix A is denoted as A^H
- the transpose of matrix A is denoted as A^T
- the conjugate of matrix A is denoted as A^*

2 Definition

2.1 Hermitian matrix

In mathematics, a Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own conjugate transpose—that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j:
$A = A^H$
Hermitian matrices can be understood as the complex extension of real symmetric matrices.

3 properties

- a Hermitian matrix's diagonal elements must be real, as they must be their own complex conjugate
- if a square matrix A equals the multiplication of a matrix and its conjugate transpose, that is, $A = BB^H$, then A is a Hermitian positive semi-definite matrix.Furthermore, if B is row full-rank, then A is positive definite.

- for an arbitrary complex valued vector v the product $v^H Av$ is real because of $v^H Av = (v^H Av)^H$

4 How to judge a positive-definite matrix

- a matrix is positive definite if it's symmetric and all its pivots are positive. Just perform elimination and examine the diagonal terms.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

after performing the elimination:

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

the pivots are 1 and -3. -3 is not positive, so the matrix is not positive definite matrix.

- the k-th pivot of a matrix is $d_k = \frac{det(A_k)}{det(A_{k-1})}$ where A_k is the upper left kxk submatrix. All the pivots will be positive if and only if $det(A_k) \geq 0$ for all $1 \leq k \leq n$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4$$

$2, 3, 4 \geq 0$ so the matrix is the positive definite matrix