hw1

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May 30, 2020

1 Task1

HW 1

2020年5月4日 10:57

(a) associative: $A(BC)=(AB)C$

set $A_{n\times n}=(a_{ij})$ $B_{n\times n}=(b_{ij})$ $C_{n\times n}=(c_{ij})$

$AB=(d_{ij})$ $BC=(e_{ij})$

$ABC=(f_{ij})$ $A(BC)=(g_{ij})$

$\Rightarrow d_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}+b_{nj}$, $j,i=1,2,\cdots n$

$e_{ij}=b_{i1}c_{1j}+b_{i2}c_{2j}+\cdots+b_{in}c_{nj}$, $i,j=1,2,\cdots n$

$f_{ij}=d_{i1}c_{1j}+d_{i2}c_{2j}+\cdots+d_{in}c_{nj}$, $i,j=1,2,\cdots n$

$g_{ij}=a_{i1}e_{1j}+a_{i2}e_{2j}+\cdots+a_{in}e_{nj}$, $i,j=1,2,\cdots n$

$\Rightarrow \forall\, i,j=1,2,\cdots n$

$f_{ij}=d_{i1}c_{1j}+d_{i2}c_{2j}\cdots+d_{in}c_{nj}$

$=(a_{i1}b_{11}+a_{i2}b_{21}\cdots+a_{in}b_{n1})c_{1j}+$

$(a_{i1}b_{12}+a_{i2}b_{22}\cdots+a_{in}b_{n2})c_{2j}+$

$\cdots(a_{i1}b_{1n}+a_{i2}b_{2n}\cdots+a_{in}b_{nn})c_{nj}$

$=a_{i1}(b_{11}c_{1j}+b_{12}c_{2j}+\cdots+b_{1n}c_{nj})+$

$a_{i2}(b_{21}c_{1j}+\cdots+b_{2n}c_{nj})\cdots$

$+a_{in}(b_{n1}c_{1j}+\cdots b_{nn}c_{nj})$

$=a_{i1}e_{1j}+a_{i2}e_{2j}+\cdots a_{in}e_{nj}$

$=g_{ij}$

$\Rightarrow (AB)C=A(BC)$

distributive: $A(B+C) = AC+BC$

$(A+B)C = AC+BC$

$$[(A+B)C]_{ij} = \sum_k (A_{ik}+B_{ik})C_{kj}$$

$$= \sum_k A_{ik}C_{kj} + \sum_k B_{ik}C_{kj}$$

$$= (AC)_{ij} + (BC)_{ij}$$

$\Rightarrow (A+B)C = AC+BC$

works similarly $A(B+C) = AB+AC$

Commtative: $AB = BA$

set $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$AB \neq BA$

commtative is incorrect.

1b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$[A\,E] = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$r_2 - r_1 \rightarrow$ $\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \end{bmatrix}$

$r_2 - r_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$r_2 - 3r_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2} \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 1 & 0 & 0 \end{bmatrix} \quad \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 1\times4\times0 + 2\times6\times1 + 3\times1\times0 - 3\times4\times1 - 2\times1\times0 - 1\times6\times0$$

$$= 12 - 12 = 0$$

$\Rightarrow$ A is a singular matrix $\Rightarrow$ A is not invertible

1c) $L = (A^TA)^{-1}A^T$

$R = A^T(AA^T)^{-1}$

$Ax = I$
$A^TAx = A^TI$
$x = (A^TA)^{-1}A^T$

$xA = I$
$xAA^T = IA^T$
$x = A^T(AA^T)^{-1}$

$A^T \in R^{2\times3}$, $(A^TA)^{-1} \in R^{3\times3}$ hance A^T can mutiply $(A^TA)^{-1}$

$(A^TA)^{-1} \in R^{3\times3}$, $A \in R^{2\times3}$, hance the size of $(A^TA)^{-1}$ can't mutiply A^T

$\therefore$ A has only right-Pseudo Inverse

$\rightarrow$

∴ A has only right-Pseudo Inverse

1d) Set matrix W and vectors $\vec{v}$

if $W * \vec{v} = \lambda \vec{v}$ $(\lambda \neq 0)$

these vectors are called eigenvectors and λ is called eigenvalues

A squared symmetric matrix $A = QDQ^T$, where the columns of Q are the eigenvectors of A and D is a diagonal matrix where the entries are the corresponding eignvalues

These matrices are important in machine learning.

2 Task2

2.1 2a

2.1.1 1

Exception

$$E_{\omega \sim p(\omega)}[f] = \sum_{w \in \Omega} P(\omega) f(\omega)$$

Variance

$$var = E[f^2] - E[f]^2$$

they are unlinear operator

2.1.2 2

$$E(A) = 1 \times \frac{4}{18} + 2 \times \frac{1}{18} + 3 \times \frac{6}{18} + 4 \times \frac{2}{18} + 5 \times \frac{1}{18} + 6 \times \frac{4}{18} = 3.39$$

$$var(A) = \frac{(1-3.39)^2 \times 4 + (2-3.39)^2 \times 1 + (3-3.39)^2 \times 6 + (4-3.39)^2 \times 2 + (5-3.39)^2 \times 1 + (6-3.39)^2 \times}{17}$$

$$E(B) = 2.78$$

$$var(B) = 3.01$$

$$E(C) = 3.44$$

$$var(C) = 3.08$$

2.1.3 3

$$A : KL(p||q) = \sum P(x) log \frac{P(x)}{Q(x)} = 0.19$$

$$B : KL(p||q) = 0.25$$

$$C : KL(p||q) = 0.02$$

C is closest to a fair, uniform die.

2.2 2b

suffer from cold is event x, suffer from back-pain is event y

$$p(y|x) = 0.3$$

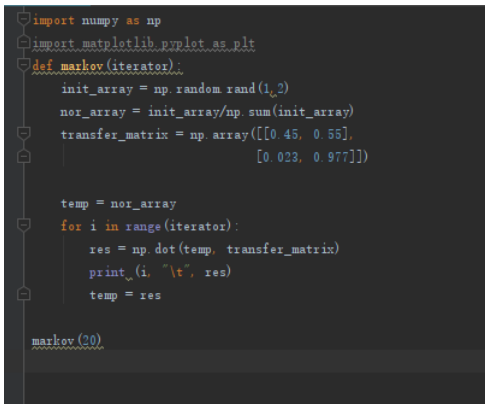
$$p(x) = 0.03$$

$$p(y|!x) = 0.1$$

$$p(y) = p(x,y) + p(!x,y) = p(y|x) \times p(x) + p(y|!x) \times p(!x) = 0.106$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = 0.085$$

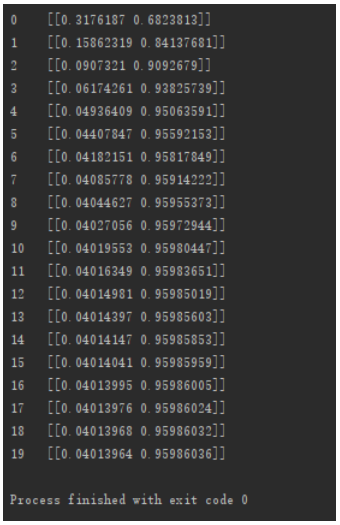
2.3 2c



```
import numpy as np
import matplotlib.pyplot as plt
def markov(iterator):
    init_array = np.random.rand(1,2)
    nor_array = init_array/np.sum(init_array)
    transfer_matrix = np.array([[0.45, 0.55],
                                [0.023, 0.977]])

    temp = nor_array
    for i in range(iterator):
        res = np.dot(temp, transfer_matrix)
        print(i, "\t", res)
        temp = res

markov(20)
```



```
0    [[0.3176187 0.6823813]]
1    [[0.15862319 0.84137681]]
2    [[0.0907321 0.9092679]]
3    [[0.06174261 0.93825739]]
4    [[0.04936409 0.95063591]]
5    [[0.04407847 0.95592153]]
6    [[0.04182151 0.95817849]]
7    [[0.04085778 0.95914222]]
8    [[0.04044627 0.95955373]]
9    [[0.04027056 0.95972944]]
10   [[0.04019553 0.9598044 7]]
11   [[0.04016349 0.95983651]]
12   [[0.04014981 0.95985019]]
13   [[0.04014397 0.95985603]]
14   [[0.04014147 0.95985853]]
15   [[0.04014041 0.95985959]]
16   [[0.04013995 0.95986005]]
17   [[0.04013976 0.95986024]]
18   [[0.04013968 0.95986032]]
19   [[0.04013964 0.95986036]]

Process finished with exit code 0
```

My plan is failed.

3 Task3

3.1 3a

$$H(p) = \sum P(x) log \frac{1}{P(x)} = 0.89$$

$$maxH(p) = \sum 0.25 log \frac{1}{0.25} = 1.39$$

The maximal entropy follows uniform distribution.

3.2 3b

3.2.1 1

$$min \sum_{i=1}^{4} p_i \log(p_i)$$

$$s.t. 6 = \sum_{i=1}^{4} 2p_i i, (p_i - 1) \leq 0$$

3.2.2 2

$$min_{\alpha,\beta} L(p_i, \lambda, \beta) = \sum_{i=1}^{4} p_i \log(p_i) + \alpha \sum_{i=1}^{4} (p_i - 1) + \beta (\sum_{i=1}^{4} 2p_i i - 6)$$

3.2.3 3

$$\frac{\partial L}{\partial p_i} = \log(p_i) + \frac{1}{ln2} + \alpha + 2\beta i,\ i = 1, 2, 3, 4$$

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{4}(p_i - 1)$$

$$\frac{\partial L}{\partial \beta} = (\sum_{i=1}^{4} 2p_i i - 6)$$

we need to solve the minimal value, but the $p_i - 1$is not 0, we can adjust α_i to make L become $-\infty$.

3.2.4 4

$$\theta_D = min_{p_i} L$$

3.2.5 5

steepest descent

3.3 3c

```python
import numpy as np

def cal_rosenbrock(x):

    #compute rosenbrock
    return sum(100.0*(x[1:]-x[:-1]**2.0)**2.0 + (1-x[:-1])**2.0)



def rosen_der(x):
    xm = x[1:-1]
    xm_m1 = x[:-2]
    xm_p1 = x[2:]
    der = np.zeros_like(x)
    der[1:-1] = 200*(xm-xm_m1**2) - 400*(xm_p1 - xm**2)*xm - 2*(1-xm)
    der[0] = -400*x[0]*(x[1]-x[0]**2) - 2*(1-x[0])
    der[-1] = 200*(x[-1]-x[-2]**2)
    return der


def armijo(x,error,alpha = 1):

    # armijo search alpha
    def loss(alpha,x,loss = 0):
        temp = []
        for i in range(20):
            temp.append(x[i])
        for i in range(20):
            temp[i] -= alpha * error[i]
        for i in range(19):
            loss += 100 * (temp[i + 1] - temp[i] ** 2) ** 2 + (temp[i] - 1) ** 2

        return loss

    def check(alpha,x):
        return loss(alpha,x) > loss_0 - sigma * alpha * np.dot(error,error)

    sigma = 0.02
    rho = 0.4
    loss_0 = loss(0,x)

    if check(alpha,x) == False:
        alpha = 1
    elif check(alpha,x):
        alpha = 1
        while check(alpha,x):
            alpha *= rho

    return alpha

def for_rosenbrock_func(max_iter_count = 10000):
    pre_x = np.zeros((20,),dtype=np.float32)
    loss = 19
    cnt = 0
    while loss > 0.001 and cnt < max_iter_count:
        error = np.zeros((20,), dtype=np.float32)

        #compute grade
        error = rosen_der(pre_x)

        #find best solution

        alpha = armijo(pre_x,error)

        for j in range(20):
            pre_x[j] -= alpha * error[j]

        loss = cal_rosenbrock(pre_x) # 最小值为0

        print("count: ", cnt, "the loss:", loss,"step:",alpha)
        cnt += 1
    return pre_x

if __name__ == '__main__':
    w = for_rosenbrock_func()
    print(w)
```

```
count:  4633 the loss:  0.0010194874476283644 step:  0.00128
count:  4634 the loss:  0.0010188045446355432 step:  0.00128
count:  4635 the loss:  0.0010183825612841702 step:  0.00128
count:  4636 the loss:  0.0010144268380365418 step:  0.0005120000000000001
count:  4637 the loss:  0.001012136528391494 step:  0.0032
count:  4638 the loss:  0.0010112044967840461 step:  0.00128
count:  4639 the loss:  0.0010104321054811294 step:  0.00128
count:  4640 the loss:  0.0010098866035725962 step:  0.00128
count:  4641 the loss:  0.0010064342646813884 step:  0.0005120000000000001
count:  4642 the loss:  0.001006120468804994 step:  0.008
count:  4643 the loss:  0.0009975901953431787 step:  0.0005120000000000001
[0.99999785 1.0000044  0.999994   1.0000042  0.9999955  0.99999684
 0.99999607 0.9999826  0.9999781  0.9999437  0.9998957  0.99978805
 0.9995723  0.9991493  0.9982871  0.9965756  0.9931411  0.9862981
 0.9727148  0.94603527]
```

in step 4642 we achieve x=[0.99999785 1.0000044 0.999994 1.0000042 0.9999955

0.99999684 0.99999607 0.9999826 0.9999781 0.9999437 0.9998957 0.99978805
0.9995723 0.9991493 0.9982871 0.9965756 0.9931411 0.9862981 0.9727148 0.94603527]

3.4 3d

3.4.1 1

Batch:

$$\theta_{j+1} = \theta - \alpha\delta_\theta J(\theta)$$

all the datas are used in the computing, so the result is exact but the calculation time is large and occupy much resource. stochastic:

$$\theta_{j+1} = \theta - \alpha\delta_\theta J(\theta, x^i, y^i)$$

every iteration only uses oone point in SGD, so the need of calculation time is less . But not all iterations can reach to the global optimum. Mini-Batch:

$$\theta_{j+1} = \theta - \alpha\delta_\theta J(\theta, x^{i:i+n}, j^{i:i+n})$$

This allows a balance between calculated amount and computing time,but the convergence is not very ideal.

3.4.2 2

When we use new data to iterate an inertia, we gives the old direction aweight too. The computing is stabler but not always useful when we need to run faster.