Trivia

January 2024

About the permutahedron instances

Chambers

Now, let us focus on the `perm` instances. We show analytically that both linear and affine versions have the same $(n+1)!$ sign vectors. First, consider the $n(n-1)/2$ hyperplanes of the form $H_{ij} = (e_i - e_j)^{\perp}$. Let x be a given point,

$$x \in \mathbb{R}^n \setminus (\cup_{i,j} H_{ij}) \Leftrightarrow x_1 \neq x_2 \neq \ldots \neq x_n$$

since if there exists a pair (i,j) such that $x_i = x_j$, then $x \in H_{ij}$. Now, there are $n!$ ways to order the coordinates, the $n!$ permutations. Indeed, let σ be a permutation of $[1:n]$, such that $x_{\sigma(1)} > \cdots > x_{\sigma(n)}$. For a given pair of different indices $(i,j) \in [1:n]^2$, the sign vector of $H_{\sigma(i)\sigma(j)}$ is $+1$ if $\sigma(i) > \sigma(j)$ and $+1$ otherwise. Since the sign of every hyperplane H_{ij} is determined, this amounts to precisely $n!$ chambers.

Then we show that each of these $n!$ regions is exactly split in $n+1$ subregions by the remaining n hyperplanes $H_k = \{x_k = 0\}$. This will lead to $(n+1) \times n! = (n+1)!$ regions.

Let σ be a permutation of size n, such that $x_{\sigma(1)} > \cdots > x_{\sigma(n)}$. Then one can have the following configurations :

$$\begin{array}{ll} x_{\sigma(i)} > 0 \ \forall\ i \in [1:n] & x_{\sigma(1)} < 0, x_{\sigma(i)} > 0 \ \forall\ i > 1 \\ \ldots & \ldots \\ x_{\sigma(i)} < 0, x_{\sigma(n)} > 0 \ \forall\ i < n & x_{\sigma(i)} < 0 \ \forall\ i \in [1:n] \end{array}$$

any other combination is of the form

$$\{x_{\sigma(1)} < 0, \ldots, x_{\sigma(i^*)} > 0, \ldots, x_{\sigma(j^*)} < 0, \ldots, x_{\sigma(n)} > 0\}$$

which does not respect the definition of σ. Note that replacing the 0's by $1's$, i.e., considering the affine version, does not change the reasoning : the affine and linear instances are equivalent. This is a particular case of affine instance that is symmetric, the center of symmetry being $(1, \ldots, 1)$.

Stem vectors

It is also possible to explicitly express the set of circuits and therefore the stem vectors. First, recall that the numbers of stem vectors of the instances `perm-n` for `n = 5,6,7,8` are respectively 197, 1172, 8018, 62814. These numbers correspond to sequence A002807 in the OEIS (hyperlien/référence), up to some index shift (number of circuits $= a(n+1)$). The given formula corresponds precisely to the circuits, as detailed in the upcoming proposition. Recall that the matrices defining the instances are given by

$$V_{lin} = [I_n \ M], \ M = [e_i - e_j]_{1 \leq i < j \leq n}, \quad V_{aff} = \begin{bmatrix} I_n & M \\ 1_{1\times n} & 0_{1\times n(n-1)/2} \end{bmatrix}$$

Proposition 0.1 (circuits of the `perm` instances) *The number of circuits of `perm-n` is given by*

$$|\mathcal{C}(\texttt{perm-}n)| = \sum_{k=3}^{n+1} \frac{(k-1)!}{2} \binom{n+1}{k}$$

In particular, the number of circuits of size $k \in [3 : n+1]$ is precisely the kth term of the sum. Moreover, the circuits are identical in the linear and the affine arrangements.

The proof relies on some artificial but useful notions defined next.

Definition 0.2 *[coordinates covered by J] Let $J \subset [1 : p]$. We denote by $c_J := |\{i \in [1 : p] : \exists\, j \in J, (v_j)_i \neq 0\}|$ the number of nonzero lines of $V_{:,J}$.*

Definition 0.3 *[nonzero components in J] Let $J \subset [1 : p]$. We denote by $K_J := \sum_j ||v_j||_1$ the total number of nonzero components of vectors in J (because $||v_j||_1 = \sum_1^n |(v_j)_i|$ and $(v_j)_i \in \{-1, 0, +1\}$ for every $i \in [1 : n]$ and $j \in J$).*

For the V defining `perm-n`, one clearly has $c_J \leq n$ and $K_J \in [k, 2k]$ for any J since the columns of V belong to $\mathbb{R}^n$ and each of them has one or two nonzero components. In what follows, we call "coordinates" a subset of $[1 : n]$, and "components" the value(s) of one (or multiple) vector(s) eventually at some specific coordinates.

Proof. First, the rank of the matrix V defining both arrangement is clearly n. Therefore, the circuits are of size at most $n+1$, and at least 3, because there are no colinear vectors.

Let us show that any circuit J of size k, $c_J \in \{k-1, k\}$. Let $c_J = l$ and denote by $i_1, \dots, i_l$ the associated coordinates.

- suppose that $l \leq k-2$. By definition, the vectors v_j for $j \in J$ have nonzero components only at coordinates $i_1, \dots, i_l$. Therefore, V_J is a submatrix of $[e_{i_1}, e_{i_2}, \dots, e_{i_l}, e_k - e_{k'}]$ (assuming $i_1 \leq k < k' \leq i_l$ for the presentation). However, this matrix is clearly of rank $l \leq k-2$ so its circuits are of size $\leq k-2+1 = k-1 < k$.
- suppose that $l \geq k+1$. By definition of V, $k \leq K_J \leq 2k$ since each v has one or two nonzero components. Since J is a circuit, $\operatorname{null}(V_J) = 1$, meaning $V_J\eta = 0 \in \mathbb{R}^n$ for some $\eta \in \mathbb{R}^J_*$. The indices $k \notin \{i_1, \dots, i_l\}$ clearly verify $(V_J\eta)_k = \sum_{j\in J} (v_j)_k \eta_j = 0$, since $(v_j)_k = 0$ for every $j \in J$. Now, choose an index $i \in i_1, \dots, i_l$. Using the equalities $V_J\eta = 0$ and $(V_J\eta)_i = \left(\sum_{j\in J} v_j\eta_j\right)_i = \sum_{j\in J} (v_j)_i \eta_j = 0$, there must be at least two vectors v_j having a nonzero component i to have $V_J\eta = \sum_j v_j\eta_j = 0 \in \mathbb{R}^n$. Therefore, for all the l coordinates $i_1, \dots, i_l$, there are at least $2l$ nonzero coordinates in the combined vectors $\{v_j : j \in J\}$, which implies $K_J \geq 2l$. This is a contradiction with $K_J \leq 2k = 2|J|$.

Now that the circuits J of size k verify $c_J \in \{k-1, k\}$, one only needs to count these two types of circuits for every $k \in [3 : n+1]$. Let $k \in [3 : n+1]$, and define

$$C_1(n,k) := \frac{(k-1)!}{2} \binom{n}{k-1}, \quad C_2(n,k) := \frac{(k-1)!}{2} \binom{n}{k}.$$

Note that for $k = n+1$, $C_2(n,k) = 0$ since there are only n coordinates. We show there are $C_1(n,k)$ circuits of size k with $c_J = k-1$ and $C_2(n,k)$ circuits of size k with $c_J = k$. Since these are the only possibilities for c_J, the total number of circuits will be the claimed result :

$$\sum_{k=3}^{n+1} \frac{(k-1)!}{2} \left[\binom{n}{k-1} + \binom{n}{k}\right] = \sum_{k=3}^{n+1} \frac{(k-1)!}{2} \binom{n+1}{k}.$$

Let $k \in [3 : n+1]$ and J be a circuit of size k such that $c_J = k$ (if $k = n+1$ there is nothing to do, there are no such circuits). Let $i_1, i_2, \dots, i_k$ be the coordinates associated to J. Since there are k

coordinates, one has $K_J \geq 2k$. However, one also has $K_J \leq 2k$: every vector must have two nonzero components, meaning $J \subset \{e_i - e_j\}_{1\leq i<j\leq n}$.

Moreover, since $c_J = k$, there are exactly two vectors with a nonzero component at coordinate i_1, two (other) vectors with a nonzero component at coordinate i_2, and so on. Consider a sequence of the k indices, now named $j_1 < j_2 < \cdots < j_k$. Clearly $e_{j_1} - e_{j_2}, e_{j_2} - e_{j_3}, \dots, e_{j_{k-1}} - e_{j_k}, e_{j_1} - e_{j_k}$ is a submatrix of nullity one, since $(+1, \dots, +1, -1)$ is in its null space and the first $k-1$ vectors form a family of rank $k-1$: the corresponding indices form a circuit of size k.

Since the choice of $i_1, \dots, i_k$ is arbitrary, this gives $\binom{n}{k}$ possibilities. Now, for a fixed choice of $i_1, \dots, i_k$, let us justify there are $(k-1)!/2$ possible circuits of size k for these indices. There are $k!$ possible ways to order the indices, the permutations of $[1:k]$. However, the next paragraph shows the resulting circuits are independent by circular permutation and by symmetry.

Let $\sigma \in \mathfrak{S}([1:k])$, and denote by $i_{\sigma(1)}, \dots, i_{\sigma(k)}$ the indices of the coordinates in order modified by σ. The vectors of V that form a circuit for this order are precisely the vectors $\pm(e_{i_{\sigma(1)}} - e_{i_{\sigma(2)}})$, $\pm(e_{i_{\sigma(2)}} - e_{i_{\sigma(3)}}), \dots, \pm(e_{i_{\sigma(1)}} - e_{i_{\sigma(k)}})$. However, the indices given by a circular permutation of σ, namely, $i_{\sigma(1+j_0)}, i_{\sigma(2+j_0)}, \dots, i_{\sigma(k+j_0)}$, form a circuit with the same vectors. Similarly, the index sequence $i_{\sigma(k)}, i_{\sigma(k-1)}, \dots, i_{\sigma(1)}$ form a circuit with these same vectors. Summarizing these observations, there are $C_2(n,k)$ stem vectors of this form : the invariance by circular permutation and by symmetry divide $k!$ by k and by 2 respectively ($k \geq 3$). Let us explain this for $k = 4$. For simplicity, we assume $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4$. One has $4!$ ways to order the set $\{1,2,3,4\}$, but for instance

$$\begin{array}{llll}
[\{1-2-3-4\}] & \{e_1-e_2, e_2-e_3, e_3-e_4, e_1-e_4\} & = \{e_3-e_4, e_2-e_3, e_1-e_2, e_1-e_4\} & [4-3-2-1] \\
=[2-3-4-1] & \{e_2-e_3, e_3-e_4, e_1-e_4, e_1-e_2\} & = \{e_2-e_3, e_1-e_2, e_1-e_4, e_3-e_4\} & [3-2-1-4] \\
=[3-4-1-2] & \{e_3-e_4, e_1-e_4, e_1-e_2, e_2-e_3\} & = \{e_1-e_2, e_1-e_4, e_3-e_4, e_2-e_3\} & [2-1-4-3] \\
=[4-1-2-3] & \{e_1-e_4, e_1-e_2, e_2-e_3, e_3-e_4\} & = \{e_1-e_4, e_3-e_4, e_2-e_3, e_1-e_2\} & [1-4-3-2]
\end{array}$$

where the orders on the right are obtained by circular permutation and the left by circular permutation after symmetry. A different order, for instance $\{1,3,2,4\}$, would involve new vectors such as $e_1 - e_3$, which means a different circuit is considered. For the affine instance of `perm-n`, since $\zeta_J = 0$, the stem vector is symmetric.

Let us justify these are the only stem vectors of size k with $c_J = k$. Let $v_{i_1} = \pm(e_{j_1} - e_{j_2})$ be fixed. Since there is exactly one other vector with $v_{j_2} \neq 0$, of the form $\pm(e_{j_2} - e_{j_3})$, pursuing the same reasoning on j_3, after some iterations the new vector will be $\pm(e_{j_{k'}} - e_{j_{k'}})$ with $j_{k'} = j_1$ since there is a finite number of indices. If $k' = k$, we have found a circuit of size k as before. If $k' < k$, the sequence of indices $j_1, \dots, j_k'$ and the associated vectors form a circuit of size $k' < k$, which is a contradiction with the definition of a circuit. (The subset is composed of at least two smaller circuits : consider $e_1 - e_2, e_2 - e_3, e_1 - e_3, e_4 - e_5, e_5 - e_6, e_4 - e_6$ for instance : with $k = 6$, it verifies $K_J = 12$, $c_J = 6$ but is composed of two circuits, meaning it is not a circuit.)

Now, we consider the circuits J such that $c_J = k-1$. With a similar argument, let $k' \in [1:k]$ be the number of vectors v_j for $j \in J$ such that $v_j \in \{e_i\}_{i\in[1:n]}$. Clearly $K_J = k' \times 1 + (k-k') \times 2 = 2k - k'$. Since $c_J = k-1$ coordinates are involved in the circuits, $K_J \geq 2(k-1) = 2k-2$ total coordinates, meaning $k' \leq 2$. If $k' = 0$, this reduces to the previous subcase : the subset is composed of two or more circuits (but two have a common index, for instance $e_1 - e_2, e_2 - e_3, e_1 - e_3, e_1 - e_4, e_1 - e_5, e_4 - e_5$). If $k' = 1$, denoting by e_{i^*} the associated vector, since $K_J = 2k-1$, the $k-1$ vectors of the form $e_i - e_j$ without the vector e_{i*} already form a circuit since they cover $k-1$ coordinate and are $k-1$. In other words, the component of η corresponding to e_{i^*} equals zero : this is a contradiction. The only possibility is $k' = 2$, meaning there exists a pair of indices (i^*, j^*) such that e_{i^*}, e_{j^*} belong to the columns of V_J.

Now, since $c_J = k-1$ and $K_J = 2k-2$, for every coordinate $i_1, \dots i_{k-1}$ must have two vectors having a nonzero component in this coordinate. Since e_{i^*} and e_{j^*} have only one nonzero component in positions i^* and j^*, one vector must be of the form $\pm(e_{i^*} - e_{i_k})$, one of the form $\pm(e_{j^*} - e_{i'_k})$, and the reasoning is pursued as before in the case $c_J = k-1$. Essentially, one of the vectors $e_i - e_j$ is split into the pair (e_i, e_j).

When counting the circuits, since the choice of the $k-1$ coordinates is arbitrary, there are $\binom{n}{k-1}$ possibilities. Then, there are $(k-1)!$ ways to order the coordinates and $(k-1)!/2$ taking into account the symmetry. However, there is no invariance by circular permutation of the involved $k-1$ indices. Indeed, permuting the order would (for instance) change the pair $\{e_{i_j}, e_{i_{j+1}}\}$ into $\{e_{i_{j+1}}, e_{i_{j+2}}\}$, which is a different circuit. This justifies the assumption about $C_1(n,k)$. Moreover, since $\{e_i, e_j\}$ replaces $e_i - e_j$, in the associated stem vector $\eta_i + \eta_j = 0$, meaning $\zeta_J^\top \eta = 0$: the circuits are symmetric even for the affine version of the instances.

To conclude the proof, let us consider the stem vectors of size $n+1$. Clearly they cannot have $c_J = n+1$ since $c_J \leq n$. Necessarily these circuits are of the form described by the point $c_J = k-1$, otherwise the amount of nonzero coordinate is incorrect. Since all n coordinates must be chosen, and $\binom{n}{n} = 1$, only the order of the coordinates matter : as seen previously, this number is $n!/2$. □

Alternate proof of the number of chambers of the cross-polytope separability arrangement

In [BEK21], section 6.4 page 14, the cross-polytopes are defined as n-dimensional polytope with the $2n$ vertices $\mathcal{V} = \{\pm e_i\}_{i \in [1:n]}$. From there, the associated separability arrangement has the $2n$ hyperplanes defined by $\{(1; v)^\perp : v \in \mathcal{V}\}$. The chamber number is indicated as $2 \times 3^n - 2^n$, verified numerically for reasonable n's and corresponding to sequence A027649 of the OEIS. The methods presented in [Ath96], focusing on hyperplanes with two coordinates, can justify this value, according to [BEK21]. In what follows, a proof by recurrence of the announced cardinal is shown, then an explicit enumeration of them is proposed. Note that the enumeration also justifies the number of chambers. The proof adopts the formalism of the tree algorithm. We denote $\mathcal{S}_n$ the chambers formed by the $2n$ hyperplanes.

Proposition 0.4 *The separability arrangement of the crosspolytope of dimension n, composed of the hyperplanes $(1; v)^\perp$ for $v \in \{\pm e_i\}_{i \in [1:n]}$, has $2 \times 3^n - 2^n$ chambers.*

Proof. We proceed by recurrence on n. When $n = 1$, the two hyperplanes are $(1,1)^\perp$, $(1,-1)^\perp$: there are four regions, which is $2 \times 3^1 - 2^1 = 4$. Now, suppose the result is justified for n. In what follows, the added dimension (with the 1 coordinate) is indexed with 0, meaning indices 1 to $n+1$ correspond to the positions of the ± 1 defining the normal vectors. First, remark that the matrices for dimension n and $n+1$ can be written as

$$V^n = \begin{bmatrix} 1_n & & 1_n \\ I_n & -I_n & \end{bmatrix} \quad V^{n+1} = \begin{bmatrix} V^n_{1,:} & 1 & 1 \\ V^n_{2:n+1,:} & 0_n & 0_n \\ 0_{2n} & 1 & -1 \end{bmatrix}$$

The main idea of the proof is as follows : as only the two new vectors have a nonzero coordinate in the $n+1$-th dimension, adding $(1, e_{n+1})^\perp$ would duplicate the set of chambers ; but adding afterwards $(1, -e_{n+1})^\perp$ does not duplicate again (it is spanned by the other new vector). With the formula $2 \times 3^n - 2^n$, we will show two things. For every $s \in S^1 \subset \mathcal{S}_n$, with $|S^1| = 2^n$, s has 4 descendants ; for every $s \in S^2 \subset \mathcal{S}_n$, with $|S^2| = 2 \times (3^n - 2^n)$, s has 3 descendants. This means the total number of descendants is $4 \times 2^n + 3 \times 2(3^n - 2^n) = 2 \times 3^{n+1} - (6-4)2^n = 2 \times 3^{n+1} - 2^{n+1}$.

For that purpose, recall that when adding one hyperplane, a sign vector s has two descendants if and only if the associated region is crossed by the hyperplane, i.e., there exists d^s such that $s_i v_i^\top d^s > 0, v_{new}^\top d^s = 0$. Following the same reasoning, when adding a second hyperplane, s can have 4 descendants if and only if there exists a d^s inside the region on the intersection of both added hyperplanes.

Now, let us focus briefly on the two new hyperplanes, $(1; 0_n; 1)^\perp, (1; 0_n; -1)^\perp$, i.e., $\{d \in \mathbb{R}^{n+2} : d_0 + d_{n+1} = 0\}, \{d \in \mathbb{R}^{n+2} : d_0 - d_{n+1} = 0\}$. Their intersection is $\{d \in \mathbb{R}^{n+2} : d_0 = 0 = d_{n+1}\}$. Let us justify there are precisely 2^n chambers of $\mathcal{S}_n$ around this intersection. Let $s \in \mathcal{S}_n$, its system of inequations is

$$\begin{cases} \forall\ i \in [1:n], & s_i(d_0 + d_i) > 0 \\ \forall\ i \in [1:n], & s_{i+n}(d_0 - d_i) > 0 \end{cases}$$

Now, the chambers that are around the intersection of the two new hyperplanes need to have directions d with $d_0 = 0$ and $d_{n+1} = 0$. The coordinate d_{n+1} does not intevene in the above system - this is because of the independence of the two added vectors. However, when adding the constraint $d_0 = 0$, the above system becomes

$$\begin{cases} \forall\ i \in [1:n], & s_i d_i > 0 \\ \forall\ i \in [1:n], & -s_{i+n} d_i > 0 \end{cases}$$

which means s_i and s_{i+n} must be opposite. Therefore, $s \in \mathcal{S}_n \subset \{\pm 1\}^{2n}$ has four descendants if and only if $s_{n+1:2n} = -s_{1:n}$: there are 2^n possibilities. To summarize, we have shown that "s has 4 descendants $\Rightarrow s_{n+1:2n} = -s_{1:n}$; the converse is straightforward by reversing the computations. To finish this part of the proof, we need to justify the 2^n possible chambers described are indeed in $\mathcal{S}^n$: their corresponding systems are verified in $\mathbb{R}^{n+1}$ by the vectors $(0; w)$ for $w \in \{\pm 1\}^n$, and by the vectors $(0; w; 0)$ in $\mathbb{R}^{n+2}$.

Now, one must justify the remaining $2(\times 3^n - 2^n)$ chambers are split in 3. Remark that these chambers, by the above reasoning, do not verify $s_{1:n} = -s_{n+1:2n}$. Therefore, there exists i such that $s_i = s_{i+n}$, so the corresponding equations $s_i(d_0 + d_i) > 0, s_i(d_0 - d_i) > 0$ mean d_0 cannot be 0.

Consider such a s for which a feasible $d \in \mathbb{R}^{n+1}$ has $d_0 > 0$ (by symmetry, the same is true if $d_0 < 0$). Now, in $\mathbb{R}^{n+2}$, the line $\{(d; t) : t \in \mathbb{R}\}$ verifies the $2n$ equations of s, which are independent of coordinate $n+1$. But for the two added hyperplanes, the system with $(-,-)$ cannot be verified :

$$-d_0 - d_{n+1} > 0, -d_0 + d_{n+1} > 0$$

is impossible to verify if $d_0 > 0$. The system $(+,+)$ is verified for $t = 0$, the system $(+,-)$ for t positive enough and $(-,+)$ for t negative enough.

To conclude, the proof has shown that among the $2 \times 3^n - 2^n$ chambers, a specific subset of 2^n chambers is precisely around the intersection of the two added hyperplanes, meaning they are split in 4 chambers of the arrangement in dimension $n+1$. Meanwhile, the remaining $2 \times (3^n - 2^n)$ are only split in 3 chambers in $\mathbb{R}^{n+2}$. This amounts to the announced $2 \times 3^{n+1} - 2^{n+1}$. □

Proposition 0.5 *The chambers of the n-crosspolytope arrangement, can be explicitly obtained. They are the sign vectors that do not have pairs $(i, i+n)$ and $(j, j+n)$ such that $s_i = s_{i+n} = -s_j = -s_{j+n}$.*

Proof. Using proposition 0.4, it is known the sign vectors of the form $(s, -s)$ for $s \in \{\pm 1\}^n$ are all feasible : indeed, their corresponding systems are verified with $d = (0; s_i)$, and they amount to 2^n. Now, the conditions making a sign vector infeasible will be clarified. Let $i \neq j$ be two integers of $[1:n]$ such that $s_i = s_{i+n} = -s_j = -s_{j+n}$. The corresponding four equations are

$$\begin{cases} s_i(d_0 + d_i) > 0 & -s_i(d_0 + d_j) > 0 \\ s_i(d_0 - d_i) > 0 & -s_i(d_0 - d_j) > 0 \end{cases}$$

which implies, summing the left ones, that $s_i d_0 > 0$ and $-s_i d_0 > 0$ summing the right ones. So clearly the system is infeasible. In terms of stem vectors and duality [dgp1], $e_0 + e_i, e_0 - e_i, e_0 + e_j, e_0 - e_j$ are linearly dependent (the stem vector is $\pm(1, 1, -1, -1)$).

Therefore, what remains is to count the number of sign vectors verifying this property, or equivalently, those not verifying it, and that are not of the form $(s, -s)$. By symmetry, we will consider the sign vectors with more $+1$'s than -1's. First, note that if there are n $+1$'s and n -1's, either the sign vector is of the form $(s, -s)$ and therefore already counted, or there are pairs $(i, i+n)$ and

$(j, j+n)$ verifying the above condition (if not of the form $(s, -s)$, then there exists an index i with $s_i = s_{i+n}$, and because there are n $+1$'s and n -1's there is a j with $s_j = s_{j+n}$ with $s_i = -s_j$). Thus one can count the sign vectors having $k < n$ -1's, and by symmetry multiplying by 2 at the end will be sufficient.

First, if there is $k = 0$ -1's, the only possibility is 1_{2n}. For $k = 1$, there are $2n$ possibilities - choosing any index to put the -1. For $k = 2$, any possibility except having s_i and s_{i+n} equal to -1, i.e., $2 \times (d-1) \times (d)$. Continuing this reasoning, for k values at -1 one needs to dispatch the k indices at positions such that there is no pair of indices $(i, i+n)$ both with a -1. This means one chooses k of the pairs $(i, i+n)$, labelled $(i_1, i_1+n), (i_2, i_2+n) \ldots, (i_k, i_k+n)$, and among them changes one of the signs s_{i_j} or s_{i_j+n}. This amounts to, for k changes, $\binom{n}{k}2^k$.

From there, generating the chambers is straightforward. Let us verify we recover the total number of chambers :

$$\sum_{k=0}^{k=n-1} \binom{n}{k} 2^k = \sum_{k=0}^{k=n} \binom{n}{k} 2^k - 2^n = 3^n - 2^n$$

Now, by symmetry, one gets $2 \times (3^n - 2^n)$; adding the 2^n sign vectors of the $(s, -s)$ form, one gets $2 \times 3^n - 2^n$, meaning every chamber has been identified. $\square$