

PRIMAL AND DUAL APPROACHES FOR THE CHAMBER ENUMERATION OF REAL HYPERPLANE ARRANGEMENTS *

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Abstract. Hyperplane arrangements is a problem that appears in various theoretical and applied mathematical contexts. This paper focuses on the enumeration of the chambers of an arrangement, a task that most often requires algebraic or numerical computation. Among the recent numerical methods, Rada and Černý’s recursive algorithm outperforms previous approaches, by relying on a specific tree structure and on linear optimization. This paper presents modifications and improvements to this algorithm. It also introduces a dual approach solely grounded on matroid circuits and its associated concepts of *stem vectors*, thus avoiding the need to solve linear optimization problems. Along the way, theoretical properties of arrangements, such as their cardinality and conditions for their symmetry, completeness and connectivity, as well as properties of their various stem vector sets are presented with an analytic viewpoint. It is shown, in particular, that the set of the chambers of an affine arrangement is located between those of two related linear arrangements. This leads to compact forms of the algorithms, which solve less subproblems. The proposed methods have been implemented in Julia and their efficiency is assessed on various instances of arrangements; for the best of them, this efficiency manifests itself by speedup ratios in the range [1.4, 19.3] with an average value of 3.9.

Key words. Duality, hyperplane arrangement, matroid circuit, Motzkin’s alternative, Schläfli’s bound, stem vector, strict linear inequality system, tree algorithm, Winder’s formula.

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1. Introduction. A *hyperplane* of \mathbb{R}^n is a set of the form $H := \{x \in \mathbb{R}^n : v^\top x = \tau\}$, where $v \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $v^\top x = \sum_{i=1}^n v_i x_i$ denotes the Euclidean scalar product of v and x . Usually, v is asked to be nonzero, but we have allowed v to vanish to make some formulas more compact below. For $v_1, \dots, v_p \in \mathbb{R}^n$ and $\tau_1, \dots, \tau_p \in \mathbb{R}$, consider the collection of hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^\top x = \tau_i\}$ for $i \in [1:p]$ (this is our notation for the set of the first p positive integers). The connected parts of the complement of their union, that is $\mathbb{R}^n \setminus (\cup_{i=1}^p H_i)$, are open polyhedrons, called *chambers* or *cells*. *Hyperplane arrangements* is the name given to the discipline that describes this structure [47, 20]. Its study started at least in the 19th century [43, 40, 49] and has continued until the present with theoretical contributions [54, 1, 51, 32], algorithmic developments [18, 26, 37, 17] as well as applications [6]; see also the references therein. Arrangements can also be stated for complex numbers [31] or over finite fields [10]; arrangements of circles on a sphere is also a subject of interest, with

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This is a refereeing version of the paper. This one differs from the official version by the presence of a few additional footnotes in dark blue, like the present one (hence, this dark blue material will not appear in the final version of the paper). The goal of these additional notes is to make the refereeing as easy as possible by providing more pieces of proof and indications. These are not essential for the understanding of the paper. Additional information can also be found in the full version of the paper [16].

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application in biology [8]. A powerful tool to study arrangements is the characteristic polynomial, which contains much information and provides one way of computing the number of chambers (see for instance [54, 2, 46]; see the proof of formula (3.25a) below, for a different analytic approach).

This paper focuses on the numerical *enumeration* of the chambers of an arrangement. Several approaches have been designed for that purpose. The algorithm of Bieri and Nef [4] recursively sweeps the space with hyperplanes, decreasing the dimension of the current space in order to explore arrangements in affine spaces of smaller dimension. Edelsbrunner, O'Rourke and Seidel [19] have designed an asymptotically optimal algorithm. The approach of Avis, Fukuda and Sleumer [3, 44] starts with an arbitrary chamber and moves from chamber to neighboring chamber, using a “reverse search” paradigm, thanks to the connectivity of the graph structure of the chambers (proposition 3.10 below). Rada and Černý [37] use a more efficient tree, called the \mathcal{S} -tree below, obtained by adding hyperplanes incrementally, whereas in previous approaches all hyperplanes are considered from the start. This tree algorithm possesses various interesting properties such as output-polynomiality, meaning that each individual chamber is obtained in polynomial time, and compactness, meaning that the required memory storage is low.

Several pieces of software in algebra or combinatorics, able to deal with arrangements, have been developed: `polymake` [27], `Sagemath` [50], `Macaulay2` [24], `OSCAR` [33]. Some related works, such as the package `CountingChambers.jl` [7], focus on the use of combinatorial symmetries, eventually alongside the deletion-restriction paradigm (see also [52]), to treat arrangements with underlying symmetries and many more hyperplanes. Similar considerations also appear in `TOPCOM` [38, 39] and yield very good results on particular instances.

Improvements to Rada and Černý's algorithm are proposed and benchmarked in [17]. The authors first present heuristics to bypass some computations. Then, they introduce a dual approach based on Gordan's theorem of the alternative [22], by introducing the notion of *stem vector*, closely related to the *circuits* of a *vector matroid* associated with the arrangement. These modifications allow the authors to significantly reduce the number of linear optimization problems (LOPs) to solve, therefore lowering the computing time, or even to completely remove the need of linear optimization. This paper extends the scope of [17] to arrangements with hyperplanes not necessarily containing the origin. We shall see that the heuristics introduced in [17] have natural extensions in this general case. The same is true for the dual approach, which is here grounded on Motzkin's alternative [29]; this one is indeed naturally associated with affine arrangements. These modifications are compared in the penultimate section of the paper.

This contribution is organized as follows. Section 2 presents some notation used throughout the paper as well as Motzkin's theorem of the alternative [29], crucial in this paper, which contributes to both theoretical and algorithmic aspects. Section 3 starts with the introduction of the concept of *hyperplane arrangement*. Then, it gives conditions ensuring some properties of the associated *sign vector* set, like its symmetry and its connectivity. Next, the section introduces the notion of *stem vector*, describes its set, gives its properties and shows how the stem vectors can be used to detect the infeasibility of sign vectors (covering test of proposition 3.16). Finally, the role of the *augmented matrix* is discussed. It is shown, in particular, that the sign vector set of an affine arrangement is located between the sign vector sets of two linear arrangements. Information on the number of chambers is also given or recalled, in particular when this one is in *affine general position*.

The rest of the paper focuses on algorithmic issues. Section 4 first describes the algorithm of [37], its recursion process and its use of linear optimization. Then, we adapt the heuristic ideas proposed in [17] to affine arrangements, which improves the efficiency of the previous algorithm. Section 5 focuses on dual algorithms, which use the stem vectors and often require less computing time. Section 6 shows how a compact form of the algorithms can be constructed, taking advantage of the fact that only half of the symmetric sign vectors need to be stored. Often, this technique also allows the compact algorithms to save computing time. Finally, section 7 presents the instances used to test the algorithms, their features and some numerical results.

Our presentation is more based on linear algebra and (convex) analysis rather than on discrete geometry or algebra. More specifically, the notion of circuit of a vector matroid and the duality concepts of convex analysis are prominent in sections 3, 5 and 6. In some places, even more in the companion paper [16], new proofs to known results are proposed with these points of view. This allows the readers with an analytic bent to have easier access to these results.

This paper is an abridged version of the more detailed report [16].

2. Background. This section begins with a paragraph on the notation. Next, it recalls a few known results that are useful in the paper.

One denotes by \mathbb{Z} , \mathbb{N} and \mathbb{R} the sets of integers, nonnegative integers and real numbers and one sets $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ ($r \in \mathbb{R}$ is said to be *positive* if $r > 0$ and *nonnegative* if $r \geq 0$). For two integers $n_1 \leq n_2$, $[n_1 : n_2] := \{n_1, \dots, n_2\}$ is the set of the integers between n_1 and n_2 . We denote by $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}_{++}^n := \{x \in \mathbb{R}^n : x > 0\}$ the nonnegative and positive orthants, where the inequalities apply componentwise. For a set S , one denotes by $|S|$ its cardinality, by S^c its complement in a set that will be clear from the context and by S^J , for an index set $J \subseteq \mathbb{N}^*$, the set of vectors, whose elements are in S and are indexed by the indices in J . The vector e denotes the vector of all ones, whose size depends on the context. The Hadamard product of u and $v \in \mathbb{R}^n$ is the vector $u \cdot v \in \mathbb{R}^n$, whose i th component is $u_i v_i$. The sign function $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\text{sgn}(t) = +1$ if $t > 0$, $\text{sgn}(t) = -1$ if $t < 0$ and $\text{sgn}(0) = 0$. The sign of a vector x or a matrix M is defined componentwise: $\text{sgn}(x)_i = \text{sgn}(x_i)$ and $[\text{sgn}(M)]_{i,j} = \text{sgn}(M_{i,j})$ for i and j . For $u \in \mathbb{R}^n$, $|u| \in \mathbb{R}^n$ is the vector defined by $|u|_i = |u_i|$ for all $i \in [1 : n]$. The dimension of a space \mathbb{E} is denoted by $\dim(\mathbb{E})$, the range space of a matrix $A \in \mathbb{R}^{m \times n}$ by $\mathcal{R}(A)$, its null space by $\mathcal{N}(A)$, its rank by $\text{rank}(A) := \dim \mathcal{R}(A)$ and its nullity by $\text{null}(A) := \dim \mathcal{N}(A) = n - \text{rank}(A)$ thanks to the rank-nullity theorem. The i th row (resp. column) of A is denoted by $A_{i,:}$ (resp. $A_{:,i}$). Transposition operates after a row and/or column selection: $A_{i,:}^\top$ is a short notation for $(A_{i,:})^\top$ for instance. The vertical concatenation of matrices $A \in \mathbb{R}^{n_1 \times m}$ and $B \in \mathbb{R}^{n_2 \times m}$ is denoted by $[A; B] \in \mathbb{R}^{(n_1+n_2) \times m}$. For $u \in \mathbb{R}^n$, $\text{Diag}(u) \in \mathbb{R}^{n \times n}$ is the square diagonal matrix with $\text{Diag}(u)_{i,i} = u_i$. The orthogonal of a subspace $Z \subseteq \mathbb{R}^n$ is denoted by $Z^\perp := \{x \in \mathbb{R}^n : x^\top z = 0, \text{ for all } z \in Z\}$.

This article makes extensive use of the so-called (there have been many contributors) Motzkin theorem of the alternative [29] [25, theorem 3.17], abbreviated as *Motzkin's alternative* below, whose following simplified expression is appropriate for our purpose (the general version also includes affine equalities and non strict affine inequalities). Let us write it as an equivalence, rather than an alternative¹: for a

¹Because the equivalence (2.2) intervenes in an essential manner in this paper and that we did not find any reference where it appears as simply as here, we give in this footnote two proofs of (2.2) based on two approaches: Guler's theorem 3.17 [25] and Farkas' lemma.

130 matrix $A \in \mathbb{R}^{m \times n}$ and a vector $a \in \mathbb{R}^m$,

131 (2.2) $\exists x \in \mathbb{R}^n : Ax > a \iff \nexists \alpha \in \mathbb{R}_+^m \setminus \{0\} : A^\top \alpha = 0, a^\top \alpha \geq 0.$

132 Gordan's theorem of the alternative [22, 1873] is recovered when $a = 0$:

133 (2.3) $\exists x \in \mathbb{R}^n : Ax > 0 \iff \nexists \alpha \in \mathbb{R}_+^m \setminus \{0\} : A^\top \alpha = 0.$

134 The latter equivalence satisfies the needs in [17] because the inequality systems en-
135 countered in that paper are homogeneous. It will also be helpful below.

136 The next lemma will be applied several times. It is taken from [17, lemma 2.6]
137 and is a refinement of [53, lemma 2.1]. It is useful to get a discriminating property
138 by a small perturbation of a point.

139 **LEMMA 2.1** (discriminating covectors). *Suppose that $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ is a Euclidean*
140 *vector space, $p \in \mathbb{N}^*$ and v_1, \dots, v_p are p distinct vectors of \mathbb{E} . Then, the set of*
141 *vectors $\xi \in \mathbb{E}$ such that $|\{\langle \xi, v_i \rangle : i \in [1:p]\}| = p$ is dense in \mathbb{E} .*

142 3. Hyperplane arrangements.

143 **3.1. Presentation.** Let $n \in \mathbb{N}^*$. A *hyperplane* of \mathbb{R}^n is a set of the form $H :=$
144 $\{x \in \mathbb{R}^n : v^\top x = \tau\}$, where $v \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. This hyperplane H is said to be *proper*
145 if $v \neq 0$ and *improper* otherwise. A proper hyperplane H partitions \mathbb{R}^n into three
146 subsets: H itself and its negative and positive open halfspaces, respectively defined by

147 $H^- := \{x \in \mathbb{R}^n : v^\top x < \tau\} \quad \text{and} \quad H^+ := \{x \in \mathbb{R}^n : v^\top x > \tau\}.$

148 If H is improper and $\tau = 0$, then $H = \mathbb{R}^n$ and $H^- = H^+ = \emptyset$. If H is improper and
149 $\tau \neq 0$, then $H = \emptyset$ and $H^+ = \mathbb{R}^n$ or \emptyset , while $H^- = (H^+)^c$.

1) *Using Guler's theorem 3.17 [25]. This theorem reads with some notation adaptation:*

$$\exists x : Ax > a, Bx \geq b, Cx = c$$

$$\iff \nexists (\alpha, \alpha_0, \beta, \gamma) : A^\top \alpha + B^\top \beta + C^\top \gamma = 0, a^\top \alpha + b^\top \beta + c^\top \gamma = \alpha_0, (\alpha, \alpha_0, \beta) \geq 0, (\alpha, \alpha_0) \neq 0.$$

Taking $B = 0, b = 0, C = 0$ and $c = 0$, one gets

$$\exists x : Ax > a \iff \nexists (\alpha, \alpha_0) \neq 0 : A^\top \alpha = 0, a^\top \alpha = \alpha_0, (\alpha, \alpha_0) \geq 0.$$

In the right-hand side, one cannot have $\alpha = 0$ since, otherwise, $(\alpha, \alpha_0) = 0$, which is forbidden. Therefore, this right-hand side is identical to the right-hand side in (2.2).

2) *Using Farkas' lemma.* The positive dual cone of a cone K is $K^+ := \{y : y^\top x \geq 0, \text{ for all } x \in K\}$ and for a linear map A and a convex cone K , Farkas' lemma reads $\overline{A(K)} = \{y : A^* y \in K^+\}^+$ (we do not specify the spaces for simplicity, see for example [21, proposition 2.45] for more precision); when K is a *polyhedral* closed cone, like here, $A(K)$ is closed [21, proposition 2.18], so that its closure $\overline{A(K)} = A(K)$. Observe that

$$\begin{aligned} & \exists x : Ax > a \\ & \iff \exists t > 0, \exists x, \exists y \geq 0 : Ax = te + y + a \\ & \iff \exists t > 0, te + a \in (A \quad -I) (\mathbb{R}^n \times \mathbb{R}_+^m) = \{\alpha : A^\top \alpha = 0, \alpha \leq 0\}^+ \quad [\text{Farkas}] \\ (2.1) \quad & \iff \exists t > 0, \forall \alpha \geq 0 \text{ verifying } A^\top \alpha = 0, \text{ one has } te^\top \alpha + a^\top \alpha \leq 0. \end{aligned}$$

One can now prove the equivalence (2.2).

[\Rightarrow] A proof by contradiction is easy. One can also give a direct proof, using (2.1). Assume that there is an $\alpha \geq 0$ such that $A^\top \alpha = 0$ and $a^\top \alpha \geq 0$. Then, the right-hand side of (2.1) implies that $e^\top \alpha = 0$ or $\alpha = 0$. We have shown that the right-hand side of (2.2) holds.

[\Leftarrow] By assumption, for all $\alpha \geq 0$ such that $e^\top \alpha = 1$ and $A^\top \alpha = 0$, one has $a^\top \alpha < 0$. Then, by compactness, $\sup\{a^\top \alpha : \alpha \geq 0, e^\top \alpha = 1, A^\top \alpha = 0\} = -t < 0$. It follows that, for all nonzero $\alpha \geq 0$ such that $A^\top \alpha = 0$, one has $(a^\top \alpha)/(e^\top \alpha) \leq -t$ or $te^\top \alpha + a^\top \alpha \leq 0$. This is the right-hand side of (2.1); hence the existence of x such that $Ax > a$.

A *hyperplane arrangement* is a collection of $p \in \mathbb{N}^*$ hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^\top x = \tau_i\}$, for $i \in [1:p]$, where $v_1, \dots, v_p \in \mathbb{R}^n$ and $\tau_1, \dots, \tau_p \in \mathbb{R}$. It is denoted by $\mathcal{A}(V, \tau)$, where $V := [v_1 \dots v_p] \in \mathbb{R}^{n \times p}$ is the matrix made of the vectors v_i 's and $\tau := [\tau_1; \dots; \tau_p] \in \mathbb{R}^{p \times 1}$. The arrangement $\mathcal{A}(V, \tau)$ is said to be *proper* if V has no zero column (i.e., its hyperplanes are proper) and *improper* otherwise (in proposition 4.4, a construction may yield a harmless improper arrangement, which is the reason why we introduce this concept). The arrangement is said to be *linear* if $\tau = 0$ and *affine* in general (therefore, a linear arrangement is just a particular affine arrangement). The arrangement is said to be *centered* if all the hyperplanes have a point in common [2], which is the case if and only if $\tau \in \mathcal{R}(V^\top)$ (proposition 3.5).

Whilst a proper hyperplane divides \mathbb{R}^n into two nonempty open halfspaces, a proper hyperplane arrangement splits \mathbb{R}^n into nonempty polyhedral convex open sets, called *chambers* (the precise definition of a chamber is given below). This is illustrated in figure 3.1 by two elementary examples that will accompany us throughout the paper. Enumerating the chambers is the problem at hand in this paper and it can be made

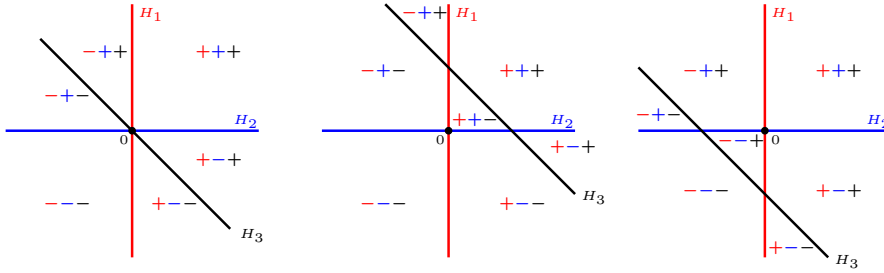


FIG. 3.1. Arrangements in \mathbb{R}^2 specified by the hyperplanes $H_1 := \{x \in \mathbb{R}^2 : x_1 = 0\}$, $H_2 := \{x \in \mathbb{R}^2 : x_2 = 0\}$, $H_3(\text{left}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$, $H_3(\text{middle}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ and $H_3(\text{right}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = -1\}$. The origin is contained in all the hyperplanes but in $H_3(\text{middle})$ and $H_3(\text{right})$, so that the arrangement in the left-hand side is linear with 6 chambers and the other ones are affine with 7 chambers.

precise in the following way. Let

$$\mathfrak{B}([1:p]) := \{(I_+, I_-) \in [1:p]^2 : I_+ \cap I_- = \emptyset, I_+ \cup I_- = [1:p]\}$$

be the collection of *bipartitions* (i.e., partitions into two subsets) of $[1:p]$. With each bipartition $(I_+, I_-) \in \mathfrak{B}([1:p])$, one can associate the set

$$(3.1) \quad C(I_+, I_-) := (\cap_{i \in I_+} H_i^+) \cap (\cap_{i \in I_-} H_i^-).$$

Some of these 2^p sets may be empty, while we are interested in enumerating the nonempty ones, which are called the *chambers* of the arrangement. The collection of these chambers, indexed by the bipartitions of $[1:p]$, is denoted by

$$(3.2) \quad \mathfrak{C}(V, \tau) := \{(I_+, I_-) \in \mathfrak{B}([1:p]) : C(I_+, I_-) \neq \emptyset\}.$$

When $\mathfrak{C}(V, \tau) = \mathfrak{B}([1:p])$, the arrangement $\mathcal{A}(V, \tau)$ is said to be *complete*.

As shown by the following proposition, this problem is equivalent to determining the sign vectors $s \in \{\pm 1\}^p$ that make a set of strict inequalities feasible. The collection of these sign vectors is denoted by

$$(3.3) \quad \mathcal{S}(V, \tau) := \{s \in \{\pm 1\}^p : s \cdot (V^\top x - \tau) > 0 \text{ for some } x \in \mathbb{R}^n\},$$

where “ \cdot ” denotes the Hadamard product. A sign vector $s \in \{\pm 1\}^p$ in $\mathcal{S}(V, \tau)$ is said to be *feasible*, while it is said to be *infeasible* if it is in the complementary set

$$\mathcal{S}(V, \tau)^c := \{\pm 1\}^p \setminus \mathcal{S}(V, \tau).$$

For $s \in \mathcal{S}(V, \tau)$, a point x verifying the system of strict inequalities in (3.3) is called a *witness point* of s [37]. It is often more convenient to work with these sign vectors $s \in \{\pm 1\}^p$ rather than with the bipartitions (I_+, I_-) of $[1 : p]$ and we shall do so in the rest of the paper. To establish the correspondence between the bipartitions (I_+, I_-) of $[1 : p]$ and the sign vectors s of $\{\pm 1\}^p$, one uses the following bijection

$$(3.4) \quad \phi : (I_+, I_-) \in \mathfrak{B}([1 : p]) \mapsto s \in \{\pm 1\}^p, \quad \text{where } s_i = \begin{cases} +1 & \text{if } i \in I_+ \\ -1 & \text{if } i \in I_- \end{cases}$$

whose inverse is given by

$$\phi^{-1} : s \in \{\pm 1\}^p \mapsto (\{i \in [1 : p] : s_i = +1\}, \{i \in [1 : p] : s_i = -1\}) \in \mathfrak{B}([1 : p]).$$

PROPOSITION 3.1 (chambers and sign vectors). *The map ϕ given by (3.4) is a bijection between the chamber set $\mathfrak{C}(V, \tau)$ and the sign vector set $\mathcal{S}(V, \tau)$.*

Proof. Let $(I_+, I_-) \in \mathfrak{B}([1 : p])$ and $s := \phi((I_+, I_-))$. One has

$$\begin{aligned} (I_+, I_-) \in \mathfrak{C}(V, \tau) &\iff \exists x \in \mathbb{R} : v_i^\top x > \tau_i \text{ for } i \in I_+ \text{ and } v_i^\top x < \tau_i \text{ for } i \in I_- \\ &\iff \exists x \in \mathbb{R} : s \cdot (V^\top x - \tau) > 0 \\ &\iff s \in \mathcal{S}(V, \tau). \end{aligned}$$

This proves the bijectivity of $\phi : \mathfrak{C}(V, \tau) \rightarrow \mathcal{S}(V, \tau)$ and concludes the proof. \square

A consequence of this proposition is that it is equivalent to determine the chamber set $\mathfrak{C}(V, \tau)$ (geometric viewpoint) or the sign vector set $\mathcal{S}(V, \tau)$ (analytic viewpoint).

By this proposition, an arrangement $\mathcal{A}(V, \tau)$ is complete if and only if $\mathcal{S}(V, \tau) = \{\pm 1\}^p$. Note that the proposition does not assume that the hyperplanes are different. Observe also that an arrangement with identical hyperplanes is not complete².

When the hyperplanes are linear (i.e., $\tau = 0$), the description of $\mathcal{S}(V, \tau)$ has various reformulations, sometimes in very different areas of mathematics: see [17] for some of them and the references therein for others.

3.2. Properties. Hyperplane arrangements benefit from a myriad of properties. In this section, we mention and prove some of them, those that are relevant for the enumeration of chambers. Most of these properties extend, with adjustments, to affine arrangements those that are valid for linear arrangements, in particular those presented in [17] and in the references therein. For further developments and different viewpoints, see for instance [47, 54, 1, 32].

Let $V = [v_1 \cdots v_p] \in \mathbb{R}^{n \times p}$, $\tau = (\tau_1, \dots, \tau_p) \in \mathbb{R}^p$ and $r := \text{rank}(V)$. In the sequel, we consider the arrangement $\mathcal{A}(V, \tau)$ formed by the p hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^\top x = \tau_i\}$ for $i \in [1 : p]$.

The next proposition gives conditions characterizing the fact that two hyperplanes are parallel or identical (two hyperplanes H and \tilde{H} are said to be *parallel* if they have

²If $H_i = H_j$ for some $i \neq j$ in $[1 : p]$, then either [for all $(I_+, I_-) \in \mathfrak{C}(V, \tau)$: $\{i, j\} \subseteq I_+$ or $\{i, j\} \subseteq I_-$] or [for all $(I_+, I_-) \in \mathfrak{C}(V, \tau)$: $(i, j) \subseteq I_+ \times I_-$ or $(i, j) \subseteq I_- \times I_+$]; equivalently, either [for all $s \in \mathcal{S}(V, \tau)$: $s_i = s_j$] or [for all $s \in \mathcal{S}(V, \tau)$: $s_i = -s_j$].

the same parallel subspace, that is, if $H - H = \tilde{H} - \tilde{H}$). Discarding identical hyperplanes is important to simplify the task of the algorithms enumerating the chambers and the proposition explains how to detect them from the columns of the matrix $[V; \tau^\top]$. Identical hyperplanes prevent an arrangement from being connected (proposition 3.10) and from being in general position (definitions 3.25 and 3.29). Below, we say that two vectors v and $\tilde{v} \in \mathbb{R}^n$ are *colinear* if there is an $\alpha \in \mathbb{R}^*$ such that $\tilde{v} = \alpha v$ (hence v and \tilde{v} vanish simultaneously). A proof of the proposition is given in [16].

PROPOSITION 3.2 (parallel and identical hyperplanes). *Let $H = \{x \in \mathbb{R}^n : v^\top x = \tau\}$ and $\tilde{H} = \{x \in \mathbb{R}^n : \tilde{v}^\top x = \tilde{\tau}\}$ be two nonempty hyperplanes. Then,*

- 1) *H and \tilde{H} are parallel if and only if v and \tilde{v} are colinear in \mathbb{R}^n ,*
- 2) *$H = \tilde{H}$ if and only if (v, τ) and $(\tilde{v}, \tilde{\tau})$ are colinear in $\mathbb{R}^n \times \mathbb{R}$.*

The next proposition identifies some modifications of (V, τ) that have no effect on the sign vector set $\mathcal{S}(V, \tau)$. For a matrix M , we denote by $\text{sgn}(M)$ the matrix defined by $[\text{sgn}(M)]_{i,j} = \text{sgn}(M_{i,j})$ for all i, j . Point 1 of the next proposition is related to proposition 3.2(2). A proof of the proposition is given in [16].

PROPOSITION 3.3 (equivalent arrangements). *Let $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$.*

- 1) *If $D \in \mathbb{R}^{p \times p}$ is a nonsingular diagonal matrix, then $\mathcal{S}(VD, D\tau) = \text{sgn}(D)\mathcal{S}(V, \tau)$.*
- 2) *If $M \in \mathbb{R}^{m \times n}$, then $\mathcal{S}(MV, \tau) \subseteq \mathcal{S}(V, \tau)$ with equality if M is injective.*

A consequence of proposition 3.3(2) is that, as far as the sign vector set $\mathcal{S}(V, \tau)$ is concerned, the rank r of $V \in \mathbb{R}^{n \times p}$ is more relevant than its row dimension n . Indeed, $r \leq n$ and when $r < n$, one can ignore $n - r$ dependent rows of V , without modifying $\mathcal{S}(V, \tau)$. More specifically, assuming that the last $n - r$ rows of V are linearly dependent of its first r rows, one can write

$$V = \begin{bmatrix} I_r \\ A \end{bmatrix} V_{[1:r],:},$$

for some matrix $A \in \mathbb{R}^{(n-r) \times p}$. Since $[I_r; A]$ is injective, one has $\mathcal{S}(V, \tau) = \mathcal{S}(V_{[1:r],:}, \tau)$ by proposition 3.3(2). Now, the dimensions of $V_{[1:r],:}$ do not involve n , so that this presentation indicates that the role of n is not very relevant and explains why many results below show r instead of n .

Now that we have identified the chamber set $\mathfrak{C}(V, \tau)$ with the sign vector set $\mathcal{S}(V, \tau)$ (proposition 3.1), we are led to the introduction of the notion of symmetry, which naturally presents itself in $\{\pm 1\}^p$.

DEFINITIONS 3.4 (symmetric sign vector set). A set of sign vectors $S \subseteq \{\pm 1\}^p$ with $p \in \mathbb{N}^*$ is said to be *symmetric* if $-S = S$; otherwise, it is said *asymmetric*. For a given set $S \subseteq \{\pm 1\}^p$, one says that $s \in \{\pm 1\}^p$ is *symmetric in S* if $\pm s \in S$. \square

This notion of symmetry intervenes in the design of the algorithms computing $\mathcal{S}(V, \tau)$. In particular, when this set is symmetric, only half of it need to be computed. Using the definition (3.3) of $\mathcal{S}(V, \tau)$, it follows immediately that

$$(3.5) \quad \mathcal{S}(V, 0) \text{ is symmetric.}$$

The next proposition shows that this symmetry property occurs for $\mathcal{S}(V, \tau)$ if and only if the arrangement is centered.³

³Proposition 3.5 is not true when $\mathcal{A}(V, \tau)$ is improper. For example, for $V := [0 \ 0]$ and $\tau^\top := [1 \ 0]$, $\mathcal{A}(V, \tau)$ is improper, but $\mathcal{S}(V, \tau) = \emptyset$ is symmetric, while $\tau \notin \mathcal{R}(V^\top)$. Actually, if $\mathcal{A}(V, \tau)$ is improper

PROPOSITION 3.5 (symmetry characterization). *Let $\mathcal{A}(V, \tau)$ be a proper arrangement. Then, the following properties are equivalent:*

- (i) $\mathcal{S}(V, \tau)$ is symmetric,
- (ii) $\tau \in \mathcal{R}(V^\top)$,
- (iii) the arrangement is centered.

Proof. [(i) \Rightarrow (ii)] One can decompose τ as follows:

$$(3.6a) \quad \tau = \tau^0 + V^\top \hat{x},$$

where $\tau^0 \in \mathcal{N}(V)$ and $\hat{x} \in \mathbb{R}^n$. We pursue by contraposition, assuming that $\tau^0 \neq 0$. Hence $I := \{i \in [1:p] : \tau_i^0 \neq 0\}$ is nonempty. Define $s \in \{\pm 1\}^p$ by

$$(3.6b) \quad s_I := \text{sgn}(\tau_I^0),$$

while s_{I^c} is defined below in order to get

$$s \notin \mathcal{S}(V, \tau) \quad \text{and} \quad -s \in \mathcal{S}(V, \tau).$$

These properties suffice to prove the implication “(i) \Rightarrow (ii)”. Set $S_I := \text{Diag}(s_I)$.

To prove that $s \notin \mathcal{S}(V, \tau)$, whatever $s_{I^c} \in \{\pm 1\}^{I^c}$ is, observe that $\alpha_I := |\tau_I^0| \in \mathbb{R}_+^I \setminus \{0\}$ verifies

$$V_{:,I} S_I \alpha_I = 0 \quad \text{and} \quad (\tau_I^0)^\top S_I \alpha_I = \|\tau_I^0\|_2^2 \geq 0.$$

By Motzkin’s alternative (2.2) with $A = S_I V_{:,I}^\top$ and $a = S_I \tau_I^0$, this is equivalent to

$$\nexists x \in \mathbb{R}^n : \quad S_I V_{:,I}^\top x > S_I \tau_I^0.$$

Hence, whatever $s_{I^c} \in \{\pm 1\}^{I^c}$ is, there is no $x \in \mathbb{R}^n$ such that $s \cdot (V^\top x - \tau^0) > 0$.

Now, using (3.6a), we see that there is no $x \in \mathbb{R}^n$ such that $s \cdot (V^\top x - \tau) > 0$, which proves $s \notin \mathcal{S}(V, \tau)$.

Let us now show that $-s \in \mathcal{S}(V, \tau)$, for some s_{I^c} to specify. Observe that there is no $\alpha_I \in \mathbb{R}_+^I \setminus \{0\}$ such that

$$-V_{:,I} S_I \alpha_I = 0 \quad \text{and} \quad -|\tau_I^0|^\top \alpha_I \geq 0$$

because the last inequality, with $\alpha_I \geq 0$ and $|\tau_I^0| > 0$, implies that $\alpha_I = 0$. By Motzkin’s alternative (2.2) with $A = -S_I V_{:,I}^\top$ and $a = -|\tau_I^0| = -S_I \tau_I^0$, this is equivalent to

$$(3.6c) \quad \exists x \in \mathbb{R}^n : \quad -S_I V_{:,I}^\top x > -S_I \tau_I^0.$$

Since the columns of V are nonzero, a small perturbation of x can maintain (3.6c) and ensures that the components of $V_{:,I^c}^\top x$ are nonzero (use, for example, the discriminating lemma 2.1 with the zero vector and the distinct v_i ’s with $i \in I^c$). Next, choosing $s_{I^c} := -\text{sgn}(V_{:,I^c}^\top x)$ and setting $S_{I^c} := \text{Diag}(s_{I^c})$ leads to

$$(3.6d) \quad -S_{I^c} V_{:,I^c}^\top x > 0 = -S_{I^c} \tau_{I^c}^0.$$

and $\mathcal{S}(V, \tau) \neq \emptyset$, $\mathcal{S}(V, \tau) \neq \emptyset$ is not symmetric (if $s \in \mathcal{S}(V, \tau)$ and $v_i = 0$, one must have $s_i \tau_i < 0$; as a result $-s \notin \mathcal{S}(V, \tau)$ since $-s_i \tau_i$ is not negative). For these reasons, we have assumed that $\mathcal{A}(V, \tau)$ is proper in proposition 3.5.

Thanks to (3.6c) and (3.6d), there is an $x \in \mathbb{R}^n$ such that $-s \cdot (V^\top x - \tau^0) > 0$. Now, using (3.6a), we see that there is an $x \in \mathbb{R}^n$ such that $-s \cdot (V^\top x - \tau) > 0$, which proves that $-s \in \mathcal{S}(V, \tau)$.

[(ii) \Leftrightarrow (iii)] Property (ii) is equivalent to the existence of $\hat{x} \in \mathbb{R}^n$ such that $V^\top \hat{x} = \tau$, which is itself equivalent to the fact that the hyperplanes have the point \hat{x} in common, which means that the arrangement is centered.

[(ii) \Rightarrow (i)] Let $\hat{x} \in \mathbb{R}^n$ be such that $V^\top \hat{x} = \tau$. If $s \in \mathcal{S}(V, \tau)$, then $s \cdot (V^\top x - \tau) > 0$ for some $x \in \mathbb{R}^n$, hence $-s \cdot (V^\top (2\hat{x} - x) - \tau) = -s \cdot (V^\top (-x) + \tau) > 0$, implying that $-s \in \mathcal{S}(V, \tau)$. \square

It is short to prove (see [16]) that

$$(3.7) \quad \mathcal{S}(V, -\tau) = -\mathcal{S}(V, \tau).$$

Let us define the *symmetric* and *asymmetric parts* of $\mathcal{S}(V, \tau)$ by

$$(3.8) \quad \mathcal{S}_s(V, \tau) := \mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau) \quad \text{and} \quad \mathcal{S}_a(V, \tau) := \mathcal{S}(V, \tau) \setminus \mathcal{S}_s(V, \tau).$$

Clearly, by (3.7), $\pm s \in \mathcal{S}(V, \tau)$ when $s \in \mathcal{S}_s(V, \tau)$, while $-s \notin \mathcal{S}(V, \tau)$ when $s \in \mathcal{S}_a(V, \tau)$ ⁴. This justifies the names given to $\mathcal{S}_s(V, \tau)$ and $\mathcal{S}_a(V, \tau)$. One also observes that (see [16])

$$(3.9) \quad \mathcal{S}_s(V, -\tau) = -\mathcal{S}_s(V, \tau) = \mathcal{S}_s(V, \tau) \quad \text{and} \quad \mathcal{S}_a(V, -\tau) = -\mathcal{S}_a(V, \tau).$$

PROPOSITION 3.6 (symmetry in $\mathcal{S}(V, \tau)$).

- 1) $\mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau)$, with equality if and only if $\mathcal{S}(V, \tau)$ is symmetric,
- 2) $\mathcal{S}_s(V, \tau) = \mathcal{S}(V, 0)$.

Proof. 1a) Let us first show that $\mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau)$. Let $s \in \mathcal{S}(V, 0)$, so that $s \cdot (V^\top x) > 0$ for some $x \in \mathbb{R}^n$. Then, $s \cdot (V^\top (tx) - \tau) > 0$ for t large enough, implying that $s \in \mathcal{S}(V, \tau)$.

2) [\subseteq] If $s \in \mathcal{S}_s(V, \tau)$, one has $\pm s \in \mathcal{S}(V, \tau)$ and there are points x and \tilde{x} such that $s \cdot (V^\top x - \tau) > 0$ and $-s \cdot (V^\top \tilde{x} - \tau) > 0$. After adding these inequalities side by side, one gets $s \cdot (V^\top (x - \tilde{x})) > 0$, i.e., $s \in \mathcal{S}(V, 0)$. [\supseteq] By point 1a, with τ and $-\tau$.

1b) If $\mathcal{S}(V, 0) = \mathcal{S}(V, \tau)$, $\mathcal{S}(V, \tau)$ is symmetric since so is $\mathcal{S}(V, 0)$ by (3.5). Conversely, if $\mathcal{S}(V, \tau)$ is symmetric, then $\mathcal{S}(V, \tau) = -\mathcal{S}(V, \tau) = \mathcal{S}(V, -\tau)$ by (3.7), so that $\mathcal{S}(V, \tau) = \mathcal{S}_s(V, \tau)$; next $\mathcal{S}(V, \tau) = \mathcal{S}(V, 0)$ follows from point 2. \square

As a corollary of the previous proposition, one has for a matrix $V \in \mathbb{R}^{n \times p}$ of rank r and a vector $\tau \in \mathbb{R}^p$ such that $\mathcal{S}(V, \tau) \neq \emptyset$:

$$(3.10) \quad 2^r \leq |\mathcal{S}(V, \tau)| \leq 2^{\tilde{p}},$$

where $\tilde{p} := |\{i \in [1:p] : V_{:,i} \neq 0\}|$.

Proof. Let $I := \{i \in [1:p] : V_{:,i} = 0\}$. By the assumption $\mathcal{S}(V, \tau) \neq \emptyset$, τ_I has no zero component and any $s \in \mathcal{S}(V, \tau)$ verifies $s_I := -\text{sgn}(\tau_I)$. Therefore, one can only consider the components of the sign vectors that are not in I , which amounts to assuming that $I = \emptyset$. For the lower bound, one has $\mathcal{S}(V, \tau) \supseteq \mathcal{S}(V, 0)$ by proposition 3.6(1) and $|\mathcal{S}(V, 0)| \geq 2^r$ by [17, (4.6a)]. The upper bound is clear since, by definition, $\mathcal{S}(V, \tau)$ is included in $\{\pm 1\}^p$, which has cardinality 2^p . \square

⁴If $s \in \mathcal{S}_a(V, \tau)$, then $s \notin \mathcal{S}_s(V, \tau)$ by definition of $\mathcal{S}_a(V, \tau)$, hence $s \notin \mathcal{S}(V, -\tau)$ by (3.8)₁ and $s \in \mathcal{S}(V, \tau)$, so that $-s \notin \mathcal{S}(V, \tau)$ by (3.7).

302 More precise lower and upper bounds on $|\mathcal{S}(V, 0)|$ and $|\mathcal{S}(V, \tau)|$ are given in proposi-
 303 tions 3.26 and 3.31, as well as in (4.8) below.

Proposition 3.6 tells us that $\mathcal{S}(V, \tau)$ is symmetric if and only if it is invariant with respect to $\tau \in \mathbb{R}^p$ (since it is then equal to $\mathcal{S}(V, 0)$ whatever τ is). In the same spirit, one can give yet another characterization of the symmetry of $\mathcal{S}(V, \tau)$, now in terms of the invariance of the chamber existence with respect to $\tau \in \mathbb{R}^p$ and $s \in \mathcal{S}(V, \tau)$, provided that V has no vanishing column. Let us introduce the possibly empty sets, associated with $s \in \{\pm 1\}^p$, denoted by

$$C_\tau(s) := \{x \in \mathbb{R}^n : s \cdot (V^\top x - \tau) > 0\},$$

which are in one to one correspondence with the sets denoted by $C(I_1, I_2)$ in (3.1), thanks to the map ϕ defined in proposition 3.1. Denote by $C_\tau(s)^\infty := \{d \in \mathbb{R}^n : x + \mathbb{R}_+ d \subseteq C_\tau(s)\}$ the asymptotic cone of $C_\tau(s)$ (called recession cone in [41, § 8]). One has⁵

$$C_\tau(s)^\infty = \{d \in \mathbb{R}^n : s \cdot (V^\top d) \geq 0\}.$$

304 Its interior reads

$$305 \quad (3.11) \quad \text{int } C_\tau(s)^\infty = \{d \in \mathbb{R}^n : s \cdot (V^\top d) > 0\} = C_0(s),$$

306 The announced invariance results is the following. For a proof, see [16].

307 **PROPOSITION 3.7** (symmetry and chamber existence invariance). *Let $\mathcal{A}(V, \tau)$ be*
 308 *a proper arrangement. Then, the following properties are equivalent:*

- 309 (i) $\mathcal{S}(V, \tau)$ is symmetric,
- 310 (ii) $\text{int } C_\tau(s)^\infty \neq \emptyset$ for all $s \in \mathcal{S}(V, \tau)$,
- 311 (iii) $C_0(s) \neq \emptyset$ for all $s \in \mathcal{S}(V, \tau)$,
- 312 (iv) $C_{\tau'}(s) \neq \emptyset$ for all $\tau' \in \mathbb{R}^p$ and all $s \in \mathcal{S}(V, \tau)$.

313 The notions of adjacency and connectivity presented below are crucial in some
 314 approaches for computing $\mathcal{S}(V, \tau)$ [3, 44] and are related to *graph theory*.

315 **DEFINITION 3.8** (adjacency in $\{\pm 1\}^p$). Two sign vectors s^1 and $s^2 \in \{\pm 1\}^p$ are
 316 said to be *adjacent* if they differ by a single component. \square

317 **DEFINITION 3.9** (connectivity in $\{\pm 1\}^p$). A *path* of length l in $S \subseteq \{\pm 1\}^p$ is
 318 a finite set of sign vectors $s^0, \dots, s^l \in S$ such that s^k and s^{k+1} are adjacent for all
 319 $k \in [0 : l - 1]$; in which case the path is said to be joining s^0 to s^l in S . One says that
 320 a subset $S \subseteq \{\pm 1\}^p$ is connected if any pair of elements of S can be joined by a path
 321 in S . \square

322 One can transfer the notions of adjacency and connectivity in $\{\pm 1\}^p$ (resp. $\mathcal{S}(V, \tau)$) to
 323 $\mathfrak{B}([1 : p])$ (resp. $\mathfrak{C}(V, \tau)$), thanks to the bijection ϕ defined by (3.4) and proposition 3.1,
 324 thus providing a geometric point of view: two chambers are adjacent if their sets I_+
 325 (or I_-) differ by a single index, which means that they are on either side of (or
 326 separated by) a single hyperplane; while connectivity means that one can join any
 327 two chambers by a continuous path in \mathbb{R}^n that never crosses two or more hyperplanes
 328 simultaneously.

329 The next proposition indicates that, provided the hyperplanes are all different (see
 330 proposition 3.2(2) for an anytical expression of this property), the sign vectors of an

⁵Given $x \in C_\tau(s)$, hence satisfying $s \cdot (V^\top x - \tau) > 0$, and writing $s \cdot (V^\top(x + td) - \tau) = s \cdot (V^\top x - \tau) + ts \cdot V^\top d$ with $t \in \mathbb{R}$, we see that this quantity remains positive for all $t \geq 0$ if and only if $s \cdot V^\top d \geq 0$, independently of $x \in C_\tau(s)$.

arrangement form a connected set. The proof is similar to the one of [15, proposition 4.5], see [16].

PROPOSITION 3.10 (connectivity of \mathcal{S}). *The set $\mathcal{S}(V, \tau)$ of sign vectors of a proper affine arrangement is connected if and only if its hyperplanes are all different. In this case, any elements s and \tilde{s} of $\mathcal{S}(V, \tau)$ can be joined by a path in $\mathcal{S}(V, \tau)$ of length $l := \sum_{i \in [1:p]} |\tilde{s}_i - s_i|/2 \leq p$ and there is no path in $\mathcal{S}(V, \tau)$ joining s and \tilde{s} of smaller length.*

3.3. Stem vectors. The notion of *stem vector* has been rediscovered in [17] for a linear arrangement $\mathcal{A}(V, 0)$ (a similar notion is indeed presented in [55, § 6.2] under the name of *signed circuit*) and it is extended in this section to an affine arrangement $\mathcal{A}(V, \tau)$. It is based on the notion of *circuit* of the vector matroid formed by the columns of V and its subsets of linearly independent columns. It is useful to determine algebraically the complement

$$\mathcal{S}^c := \{\pm 1\}^p \setminus \mathcal{S}$$

of the sign vector set $\mathcal{S} \equiv \mathcal{S}(V, \tau)$ in $\{\pm 1\}^p$. Indeed, as we shall see, a stem vector is a particular sign vector in $\{\pm 1\}^J$ for some $J \subseteq [1:p]$ and proposition 3.16 below will tell us that a sign vector s is in \mathcal{S}^c if and only if s_J is a stem vector for some $J \subseteq [1:p]$. This property is used throughout sections 5 and 6. It also results immediately that, knowing the stem vectors, it is possible to generate completely \mathcal{S}^c (by the crude dual algorithm 5.3). Here are the details.

Recall that a *circuit* of the *vector matroid* defined by the columns of $V \in \mathbb{R}^{n \times p}$ and its subsets of linear independent columns [34, proposition 1.1.1] is formed of the indices of a set of columns of V that are linearly dependent, whose strict subsets are the indices of linearly independent columns of V [34, proposition 1.3.5(iii)]. In compact mathematical terms, the collection $\mathcal{C} \equiv \mathcal{C}(V)$ of the circuits associated with the matrix $V \in \mathbb{R}^{n \times p}$ is defined by

$$(3.12) \quad \mathcal{C}(V) := \{J \subseteq [1:p] : J \neq \emptyset, \text{null}(V_{:,J}) = 1, \text{null}(V_{:,J_0}) = 0 \text{ for all } J_0 \subsetneq J\},$$

where “null” denotes the nullity (i.e., the dimension of the null space) and “ \subsetneq ” denotes strict inclusion. The stem vectors are defined from the circuits of V , with the desire to validate proposition 3.16 below. Recall that, with our notation, a sign vector $\sigma \in \{\pm 1\}^J$ for some $J := \{j_1, \dots, j_{|J|}\} \subseteq [1:p]$ is a vector $(\sigma_{j_1}, \dots, \sigma_{j_{|J|}})$ where the σ_j ’s are in $\{-1, +1\}$.

Note that an index set $J \subseteq [1:p]$ verifying $\text{null}(V_{:,J}) = 1$ is not necessarily a circuit of V , but we have, nevertheless, the following property (see [17, proposition 3.11], for instance), which will be used several times.

LEMMA 3.11 (matroid circuit detection). *Suppose that $I \subseteq [1:p]$ is such that $\text{null}(V_{:,I}) = 1$ and that $\alpha \in \mathcal{N}(V_{:,I}) \setminus \{0\}$. Then, $J := \{i \in I : \alpha_i \neq 0\}$ is a matroid circuit of V and the unique one included in I .*

DEFINITIONS 3.12 (stem vector). A *stem vector* of the arrangement $\mathcal{A}(V, \tau)$ is a sign vector $\sigma \in \{\pm 1\}^J$ for some $J \subseteq [1:p]$ satisfying

$$(3.13) \quad \begin{cases} J \in \mathcal{C}(V) \\ \sigma = \text{sgn}(\eta) \text{ for some } \eta \in \mathbb{R}^J \text{ verifying } \eta \in \mathcal{N}(V_{:,J}) \setminus \{0\} \text{ and } \tau_J^\top \eta \geq 0. \end{cases}$$

A stem vector is said to be *symmetric* if $\tau_J^\top \eta = 0$ and *asymmetric* otherwise (these properties do not depend on the chosen vector η , as shown in remark 3.13(3) below).

We denote respectively by

$$\mathfrak{S}(V, \tau), \quad \mathfrak{S}_s(V, \tau) \quad \text{and} \quad \mathfrak{S}_a(V, \tau) := \mathfrak{S}(V, \tau) \setminus \mathfrak{S}_s(V, \tau)$$

the sets of stem vectors, symmetric stem vectors and asymmetric stem vectors of the arrangement $\mathcal{A}(V, \tau)$. We denote by $\mathfrak{J} : \mathfrak{S}(V, \tau) \rightarrow \mathcal{C}(V)$ the map that associates with a stem vector $\sigma \in \{\pm 1\}^J$ its circuit $J := \mathfrak{J}(\sigma)$. \square

These definitions deserve some explanations and comments.

REMARKS 3.13. 1) When the arrangement is linear ($\tau = 0$), one recovers definition 3.9 in [17].

2) The circuits are defined from V , while the stem vectors are defined from $[V; \tau^\top]$; the latter depend on τ , which is not the case of the former.

3) *A calculation method from $\mathcal{C}(V)$.* One can associate with a circuit $J \in \mathcal{C}(V)$, either one asymmetric stem vector or two symmetric stem vectors (there are no other possibilities). Take indeed a circuit $J \in \mathcal{C}(V)$. Then, by (3.12), $\text{null}(V_{:,J}) = 1$ and any $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ has no zero component (since $\text{null}(V_{:,J_0}) = 0$ for all $J_0 \subsetneq J$). Therefore, $\text{sgn}(\eta) \in \{\pm 1\}^J$ for any $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$. Now, there may be two complementary cases.

(a) Either $\tau_J \in \mathcal{N}(V_{:,J})^\perp$, in which case $\tau_J^\top \eta = 0$ for all $\eta \in \mathcal{N}(V_{:,J})$ and, according to (3.13), there are two symmetric and opposite stem vectors associated with J , namely $\pm \text{sgn}(\eta_0)$ for some arbitrary $\eta_0 \in \mathcal{N}(V_{:,J}) \setminus \{0\}$.

(b) Or $\tau_J \notin \mathcal{N}(V_{:,J})^\perp$, in which case $\tau_J^\top \eta \neq 0$ for some $\eta \in \mathcal{N}(V_{:,J})$ and, actually, for all $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ since $\text{null}(V_{:,J}) = 1$. In this case, there is a single asymmetric stem vector associated with J , namely $\text{sgn}(\eta_+)$, for some $\eta_+ \in \mathcal{N}(V_{:,J})$ such that $\tau_J^\top \eta_+ > 0$.

We have shown, in particular, that the symmetry (resp. asymmetry) property of a stem vector does not depend on the choice of $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ (resp. satisfying $\tau_J^\top \eta > 0$).

4) The stem vectors may have different sizes, because the circuits may have different sizes.

5) The sets $\mathfrak{S}(V, \tau)$, $\mathfrak{S}_s(V, \tau)$ and $\mathfrak{S}_a(V, \tau)$ are neither vector spaces nor groups. However, given a stem vector $\sigma \in \{\pm 1\}^J$, one can consider $-\sigma$ as the opposite of σ in $\{\pm 1\}^J$, so that $-\sigma \in \{\pm 1\}^J$ (with the same J). With this meaning given to $-\sigma$, one defines

$$(3.14) \quad -\mathfrak{S}(V, \tau) := \{-\sigma \in \{\pm 1\}^J : \sigma \in \mathfrak{S}(V, \tau) \text{ and } J := \mathfrak{J}(\sigma)\}.$$

Proposition 3.14(1) below claims that $\sigma \in \mathfrak{S}_s(V, \tau)$ when $\pm\sigma \in \mathfrak{S}(V, \tau)$, which justifies a posteriori the qualifier “symmetric” given to the stem vectors in $\mathfrak{S}_s(V, \tau)$.

6) A matrix $V \in \mathbb{R}^{n \times p}$ of rank r has at most $\binom{p}{r+1}$ circuits and this bound is reached if and only if the columns of V are in linear general position (definition 3.25 below) [12]; in that case, the circuits are exactly the selections of $r+1$ columns of V^6 . This number can be exponential in p .

7) For $j \in [1 : p]$, one has $\{j\} \in \mathcal{C}(V)$ if and only if $V_{:,j} = 0$. If $J \in \mathcal{C}(V)$ and $|J| \geq 2$, $V_{:,J}$ has no zero column. \square

⁶We use here the further defined notion of *linear general position* (definition 3.25) of the columns of $V \in \mathbb{R}^{n \times p}$, which can be written: $\text{rank}(V_{:,I}) = \min(|I|, r)$, for all $I \subseteq [1 : p]$. When V has that property, any r columns are linearly independent and any set of $r+1$ columns forms a circuit of V . In this case, V has at least $\binom{p}{r+1}$ circuits, which is actually the maximum number of circuits for a matrix V of rank r [12].

It is easy to see that [16]

$$(3.15) \quad -\mathfrak{S}(V, \tau) = \mathfrak{S}(V, -\tau) \quad \text{and} \quad -\mathfrak{S}_a(V, \tau) = \mathfrak{S}_a(V, -\tau).$$

Here are some more properties of the stem vectors, which are direct consequences of their definition. The properties stated in proposition 3.14 can be symbolically represented like in figure 3.2. In the next proposition, we use the symbol “ \cup ” for the disjoint union of sets.

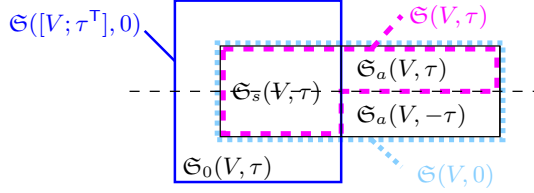


FIG. 3.2. Symbolic representation of the sets $\mathfrak{S}(V, \tau)$, $\mathfrak{S}_s(V, \tau)$, $\mathfrak{S}_a(V, \tau)$, $\mathfrak{S}(V, 0)$, $\mathfrak{S}_0(V, \tau)$ and $\mathfrak{S}([V; \tau^T], 0)$, respecting propositions 3.14, 3.21 and 3.23. The horizontal dashed line aims at representing the reflexion between a stem vector σ and its opposite $-\sigma$: $\mathfrak{S}_s(V, \tau)$, $\mathfrak{S}(V, 0)$, $\mathfrak{S}_0(V, \tau)$ and $\mathfrak{S}([V; \tau^T], 0)$ are symmetric in the sense that $-\mathfrak{S}_s(V, \tau) = \mathfrak{S}_s(V, \tau)$, $-\mathfrak{S}(V, 0) = \mathfrak{S}(V, 0)$, $-\mathfrak{S}_0(V, \tau) = \mathfrak{S}_0(V, \tau)$ and $-\mathfrak{S}([V; \tau^T], 0) = \mathfrak{S}([V; \tau^T], 0)$. By propositions 3.15 and 3.21, the diagram simplifies when $\tau \in \mathcal{R}(V^T)$, since then $\mathfrak{S}_a(V, \tau) = \mathfrak{S}_a(V, -\tau) = \mathfrak{S}_0(V, \tau) = \emptyset$ and there is only one set left.

PROPOSITION 3.14 (stem vector properties). Let $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$. Then,

- 1) $\mathfrak{S}(V, \tau) \cap \mathfrak{S}(V, -\tau) = \mathfrak{S}_s(V, \tau) = \mathfrak{S}_s(V, -\tau)$,
- 2) $\mathfrak{S}(V, \tau) \cup \mathfrak{S}(V, -\tau) = \mathfrak{S}(V, 0)$,
- 3) $\mathfrak{S}_s(V, \tau) \cup \mathfrak{S}_a(V, \tau) \cup \mathfrak{S}_a(V, -\tau) = \mathfrak{S}(V, \tau) \cup \mathfrak{S}_a(V, -\tau) = \mathfrak{S}(V, 0)$.

Proof. 1) The last equality can be deduced from the first one, so that only the latter needs to be proved.

[\subseteq] Let $\sigma \in \mathfrak{S}(V, \tau) \cap \mathfrak{S}(V, -\tau)$. Then, on the one hand, $\sigma = \text{sgn}(\eta) \in \{\pm 1\}^J$ for some $J \in \mathcal{C}(V)$ and some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^T \eta \geq 0$ and, on the other hand, $-\sigma = \text{sgn}(\tilde{\eta}) \in \{\pm 1\}^J$ (the same J , see remark 3.13(5)) for some $\tilde{\eta} \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^T \tilde{\eta} \geq 0$. Since $\text{null}(V_{:,J}) = 1$ by (3.12), $\tilde{\eta} = \alpha \eta$ for some $\alpha \in \mathbb{R}^*$. Then, $-\sigma = \text{sgn}(\tilde{\eta}) = \text{sgn}(\alpha) \text{sgn}(\eta) = \text{sgn}(\alpha) \sigma$ shows that $\text{sgn}(\alpha) = -1$, so that $0 \leq \tau_J^T \tilde{\eta} = \alpha(\tau_J^T \eta)$. Hence $\tau_J^T \eta \leq 0$, so that $\tau_J^T \eta = 0$ and σ is symmetric.

[\supseteq] Since $\mathfrak{S}_s(V, \tau) \subseteq \mathfrak{S}(V, \tau)$, it suffices to show that $\mathfrak{S}_s(V, \tau) \subseteq \mathfrak{S}(V, -\tau)$ or $-\mathfrak{S}_s(V, \tau) \subseteq \mathfrak{S}(V, \tau)$ by (3.15). If $\sigma \in \mathfrak{S}_s(V, \tau)$ and $J := \mathfrak{J}(\sigma)$, one has $J \in \mathcal{C}(V)$, $\sigma = \text{sgn}(\eta)$ for some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^T \eta = 0$. Then, clearly, $-\sigma = \text{sgn}(-\eta)$ with $-\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^T(-\eta) = 0$. Therefore, $-\sigma \in \mathfrak{S}(V, \tau)$.

2) [\subseteq] It suffices to show that $\mathfrak{S}(V, \tau) \subseteq \mathfrak{S}(V, 0)$ for an arbitrary τ , which is quite clear since a stem vector $\sigma := \text{sgn}(\eta) \in \mathfrak{S}(V, \tau)$ with $J := \mathfrak{J}(\sigma)$ must satisfy one more property (namely $\tau_J^T \eta \geq 0$) than those in $\mathfrak{S}(V, 0)$.

[\supseteq] Let $\sigma \in \mathfrak{S}(V, 0)$ and $J = \mathfrak{J}(\sigma)$. Then, $\sigma := \text{sgn}(\eta) \in \{\pm 1\}^J$ for some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$. We see that $\sigma \in \mathfrak{S}(V, \tau)$ if $\tau_J^T \eta > 0$, $\sigma \in \mathfrak{S}(V, -\tau)$ if $\tau_J^T \eta < 0$ and both sets $\mathfrak{S}(V, \tau) \cap \mathfrak{S}(V, -\tau) = \mathfrak{S}_s(V, \tau)$ if $\tau_J^T \eta = 0$.

3) Let us first show that the sets in the left-hand side are disjoint. The sets $\mathfrak{S}_s(V, \tau)$ and $\mathfrak{S}_a(V, \tau)$ are disjoint by their definition. By point 1, $\mathfrak{S}_s(V, -\tau) = \mathfrak{S}_s(V, \tau)$, so that $\mathfrak{S}_s(V, \tau)$ and $\mathfrak{S}_a(V, -\tau)$ are disjoint by their definition. Next, one cannot find an $s \in \mathfrak{S}_a(V, \tau) \cap \mathfrak{S}_a(V, -\tau)$, since s would be in $\mathfrak{S}_s(V, \tau)$ by point 1, which is in contradiction with $s \in \mathfrak{S}_a(V, \tau)$.

Now, the first equality follows from $\mathfrak{S}(V, \tau) = \mathfrak{S}_s(V, \tau) \cup \mathfrak{S}_a(V, \tau)$ and the second equality follows from $\mathfrak{S}_s(V, \tau) \cup \mathfrak{S}_a(V, \tau) = \mathfrak{S}(V, \tau)$, $\mathfrak{S}_s(V, \tau) \cup \mathfrak{S}_a(V, -\tau) = \mathfrak{S}_s(V, -\tau) \cup \mathfrak{S}_a(V, -\tau) = \mathfrak{S}(V, -\tau)$ and point 2. \square

In complement to the characterizations of the centered arrangements in propositions 3.5 (symmetry of \mathcal{S}) and 3.7 (chamber existence invariance), the following characterization is given in terms of the absence of asymmetric stem vector.

PROPOSITION 3.15 (centered arrangement and symmetric stem vector set). *For an affine hyperplane arrangement, the following properties are equivalent:*

- (i) *the arrangement is centered,*
- (ii) *all the stem vectors are symmetric.*

Proof. For $J \subseteq [1:p]$ and $\eta \in \mathbb{R}^J$, we denote by $\bar{\eta} \in \mathbb{R}^p$ the extended vector associated with η that is defined by $\bar{\eta}_j = \eta_j$ for $j \in J$ and $\bar{\eta}_j = 0$ for $j \notin J$.

[(i) \Rightarrow (ii)] If the arrangement is centered, one has $\tau \in \mathcal{R}(V^\top) = \mathcal{N}(V)^\perp$ by proposition 3.5. A stem vector is of the form $\text{sgn}(\eta)$ with $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^\top \eta \geq 0$ for some $J \in \mathcal{C}(V)$. Then, the extended vector $\bar{\eta}$ is in $\mathcal{N}(V)$, so that $\tau_J^\top \eta = \tau^\top \bar{\eta} = 0$ (since $\tau \in \mathcal{N}(V)^\perp$ and $\bar{\eta} \in \mathcal{N}(V)$), showing that the stem vector is symmetric.

[(ii) \Rightarrow (i)] If all the stem vectors are symmetric, then $\tau_J^\top \eta = 0$ for all $J \in \mathcal{C}(V)$ and $\eta \in \mathcal{N}(V_{:,J})$. If $\bar{\eta}$ extends such an η and if we show that these $\bar{\eta}$'s generate $\mathcal{N}(V)$, we will have $\tau \in \mathcal{N}(V)^\perp = \mathcal{R}(V^\top)$, implying that the arrangement is centered (proposition 3.5).

Let $r := \text{rank}(V)$, so that $\dim \mathcal{N}(V) = p - r$. To conclude the proof, it suffices to find $p - r$ vectors $\eta \in \mathcal{N}(V_{:,J})$, with $J \in \mathcal{C}(V)$, so that their extensions $\bar{\eta}$ are linearly independent.

By definition of the rank, one can find a set of r linearly independent columns of V ; let J_0 denote the set of their indices. Denote the other column indices by $\{j_1, \dots, j_{p-r}\} := [1:p] \setminus J_0$ and set $J'_i := J_0 \cup \{j_i\}$ for $i \in [1:p-r]$. From lemma 3.11, J'_i contains a unique circuit, which is denoted by $J_i \in \mathcal{C}(V)$, and $\eta_i \in \mathcal{N}(V_{:,J_i}) \setminus \{0\}$ has no zero component. Necessarily, j_i is in J_i (since, otherwise, $J_i \subseteq J_0$, in which case $V_{:,J_i}$ would be injective and J_i would not be a circuit) and j_i is not in $J_{i'}$ for $i' \in [1:p-r] \setminus \{i\}$ (j_i is not in $J'_{i'}$ by construction, hence not in $J_{i'} \subseteq J'_{i'}$). Hence, the vectors $\bar{\eta}_i$ extending $\eta_i \in \mathcal{N}(V_{:,J_i})$, $i \in [1:p]$, are linearly independent in \mathbb{R}^p . \square

The previous proposition can be rephrased in many ways, in particular as the following equivalence

$$(3.16) \quad \tau \in \mathcal{R}(V^\top) \iff \mathfrak{S}(V, \tau) = \mathfrak{S}(V, 0).$$

The following proposition extends naturally [17, proposition 3.10] to the affine arrangements considered in this paper. The possibility of having the equivalence (3.17) was a certificate for the appropriateness of the proposed definition 3.12 of stem vector. The role of this equivalence is important in the design of algorithms having a dual aspect, like those developed in section 5. The proof of the proposition is grounded on duality, via Motzkin's alternative (2.2).

PROPOSITION 3.16 (generating \mathcal{S}^c from the stem vectors). *For $s \in \{\pm 1\}^p$,*

$$(3.17) \quad s \in \mathcal{S}(V, \tau)^c \iff s_J \in \mathfrak{S}(V, \tau) \text{ for some } J \subseteq [1:p].$$

Proof. [\Rightarrow] Take $s \in \mathcal{S}(V, \tau)^c$. Our goal is to show that the index set $J \subseteq [1:p]$ in the right-hand side of (3.17) can be determined as one satisfying the following two

properties:

$$(3.18a) \quad \{x \in \mathbb{R}^n : s_j(v_j^\top x - \tau_j) > 0 \text{ for all } j \in J\} = \emptyset,$$

$$(3.18b) \quad \forall J_0 \subsetneq J, \{x \in \mathbb{R}^n : s_j(v_j^\top x - \tau_j) > 0 \text{ for all } j \in J_0\} \neq \emptyset.$$

To show that a J satisfying (3.18a) and (3.18b) exists, let us start with $J = [1:p]$, which verifies (3.18a), since $s \in \mathcal{S}(V, \tau)^c$. Next, remove one index j from $[1:p]$ if (3.18a) holds for $J = [1:p] \setminus \{j\}$. Pursuing the elimination of indices j in this way, one finally obtain an index set J satisfying (3.18a) and $\{x \in \mathbb{R}^n : s_j(v_j^\top x - \tau_j) > 0$ for all $j \in J \setminus \{j_0\}\} \neq \emptyset$ for all $j_0 \in J$. Then, (3.18b) clearly holds.

We claim that, for a J satisfying (3.18a) and (3.18b), s_J is a stem vector, which will conclude the proof of the implication.

To show that s_J , with J verifying (3.18a)-(3.18b), is a stem vector, we stick to definition 3.12 and start by showing that J is a matroid circuit. By (3.18a), $J \neq \emptyset$. Next, by Motzkin's alternative (2.2) with $A := \text{Diag}(s_J)V_{:,J}^\top$ and $a := s_J \cdot \tau_J$, (3.18a) and (3.18b) read

$$(3.18c) \quad \exists \alpha \in \mathbb{R}_+^J \setminus \{0\} \text{ such that } V_{:,J}(s_J \cdot \alpha) = 0, \tau_J^\top(s_J \cdot \alpha) \geq 0,$$

$$(3.18d) \quad \forall J_0 \subsetneq J, \nexists \alpha' \in \mathbb{R}_+^{J_0} \setminus \{0\} \text{ such that } V_{:,J_0}(s_{J_0} \cdot \alpha') = 0, \tau_{J_0}^\top(s_{J_0} \cdot \alpha') \geq 0.$$

From these properties, one deduces that $\alpha > 0$ ($\alpha \geq 0$ by (3.18c) and α has no zero component since otherwise (3.18d) would not hold) and that $\text{null}(V_{:,J}) \geq 1$ ($s_J \cdot \alpha \in \mathcal{N}(V_{:,J}) \setminus \{0\}$). To show that $\text{null}(V_{:,J}) = 1$, we proceed by contradiction. Suppose that there is a nonzero $\alpha'' \in \mathbb{R}^J$ that is not colinear with α and that verifies $V_{:,J}(s_J \cdot \alpha'') = 0$. Since α and α'' are nonzero and not colinear, they have at least two components and one can find $r \in \mathbb{R}$ such that $\beta := \alpha'' - r\alpha \in \mathbb{R}^J$ has at least one positive and one negative component (take for instance $r := (r_1 + r_2)/2$, where $r_1 := \max\{r \in \mathbb{R} : r\alpha \leq \alpha''\} < r_2 := \min\{r \in \mathbb{R} : \alpha'' \leq r\alpha\}$). One can also assume that $\tau_J^\top(s_J \cdot \beta) \geq 0$ (otherwise replace β by $-\beta$, which also has at least one positive and one negative component; one can check below that this sign inversion has no unpleasant impact on the reasoning). Now, set $t := 1/\max\{-\beta_j/\alpha_j : j \in J\}$, which is positive, and $J_0 := \{j \in J : \alpha_j + t\beta_j > 0\}$, so that $J \setminus J_0 = \{j \in J : \alpha_j + t\beta_j = 0\}$. Using the fact that β has positive components and the definition of t , we see that $\emptyset \neq J_0 \subsetneq J$. Let us introduce $\alpha' := \alpha + t\beta \geq 0$, which verifies $\alpha'_j > 0$ for $j \in J_0 \neq \emptyset$ and $\alpha'_j = 0$ for $j \in J \setminus J_0 \neq \emptyset$. Therefore,

$$\begin{aligned} & V_{:,J_0}(s_{J_0} \cdot \alpha'_{J_0}) \\ &= V_{:,J}(s_J \cdot \alpha') \quad [\alpha'_{J \setminus J_0} = 0] \\ &= V_{:,J}(s_J \cdot \alpha) + t V_{:,J}(s_J \cdot \beta) \quad [\alpha' := \alpha + t\beta] \\ &= t V_{:,J}(s_J \cdot \alpha'') - r t V_{:,J}(s_J \cdot \alpha) \quad [V_{:,J}(s_J \cdot \alpha) = 0, \beta = \alpha'' - r\alpha] \\ &= 0 \quad [V_{:,J}(s_J \cdot \alpha'') = V_{:,J}(s_J \cdot \alpha) = 0] \end{aligned}$$

and

$$\begin{aligned} \tau_{J_0}^\top(s_{J_0} \cdot \alpha'_{J_0}) &= \tau_J^\top(s_J \cdot \alpha') \quad [\alpha'_{J \setminus J_0} = 0] \\ &= \tau_J^\top(s_J \cdot \alpha) + t \tau_J^\top(s_J \cdot \beta) \quad [\alpha' := \alpha + t\beta] \\ &\geq 0 \quad [\tau_J^\top(s_J \cdot \alpha) \geq 0, \tau_J^\top(s_J \cdot \beta) \geq 0, t > 0]. \end{aligned}$$

⁷One has $r_1 < r_2$, since otherwise $\alpha'' = r\alpha$, which would infringe the noncolinearity of α and α'' . Next, $r\alpha \leq \alpha''$ (resp. $\alpha'' \leq r\alpha$) fails for at least one component, so that β has at least one negative (resp. positive) component.

These last two outcomes are in contradiction with (3.18d), as expected.

To show that $J \in \mathcal{C}$ defined by (3.12), we still have to prove that $V_{:,J_0}$ is injective when $J_0 \subsetneq J$. Equivalently, it suffices to show that any $\beta \in \mathcal{N}(V_{:,J})$ with some zero component vanishes. We proceed by contradiction. If there is a $\beta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ with a zero component, $s_J \cdot \alpha$ and β would be two linearly independent vectors in $\mathcal{N}(V_{:,J})$ (since $s_J \cdot \alpha$ has no zero component), contradicting $\text{null}(V_{:,J}) = 1$.

Now, since $s_J = \text{sgn}(s_J \cdot \alpha)$, since $s_J \cdot \alpha \in \mathcal{N}(V_{:,J})$ and $\tau_J^\top(s_J \cdot \alpha) \geq 0$ by (3.18c) and since J is a matroid circuit of V , s_J is a stem vector (definition 3.12).

[\Leftarrow] Since s_J is a stem vector, it reads $s_J := \text{sgn}(\eta) \in \{\pm 1\}^J$ for some $J \in \mathcal{C}$ and some $\eta \in \mathbb{R}^J$ satisfying $V_{:,J}\eta = 0$ and $\tau_J^\top \eta \geq 0$. Then, $\alpha := s_J \cdot \eta = |\eta|$ is in $\mathbb{R}_+^J \setminus \{0\}$ and verifies $V_{:,J}(s_J \cdot \alpha) = 0$ and $\tau_J^\top(s_J \cdot \alpha) \geq 0$. By Motzkin's alternative (2.2), there is no $x \in \mathbb{R}^n$ such that $s_J \cdot (V_{:,J}^\top x - \tau_J) > 0$. Hence, there is certainly no $x \in \mathbb{R}^n$ such that $s \cdot (V^\top x - \tau) > 0$. This means that $s \in \mathcal{S}(V, \tau)^c$. \square

We say that $s \in \mathcal{S}(V, \tau)^c$ covers a sign vector $\sigma \in \{\pm 1\}^J$ for some $J \subseteq [1:p]$ if $s_J = \sigma$. Given a set of stem vectors \mathfrak{S} , checking whether a sign vector s covers some $\sigma \in \mathfrak{S}$ is called below a *covering test*. This operation is an essential step of the dual algorithms of section 5.

REMARK 3.17. One might wonder whether having a sign vector $s \in \{\pm 1\}^p$ such that $\pm s \in \mathcal{S}(V, \tau)^c$ would imply that one has $\pm s_J \in \mathfrak{S}(V, \tau)$ for some $J \in [1:p]$. This implication does not hold. Equivalently, for a given $s \in \{\pm 1\}^p$, the two nonempty sets $\{J \subseteq [1:p] : s_J \in \mathfrak{S}(V, \tau)\}$ and $\{J \subseteq [1:p] : s_J \in -\mathfrak{S}(V, \tau)\}$ may have an empty intersection (note that its union may also differ from \mathcal{C}). This is the case, for example, when $V := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $\tau^\top := [0 \ 0 \ 1 \ 2]$ (add one hyperplane perpendicular to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in figure 3.1(middle)). See [16] for a detailed analysis of this example. More is said on this situation in the equivalence (3.20) below. \square

3.4. Augmented matrix. According to definition 3.3, verifying whether $s \in \mathcal{S}(V, \tau)$ amounts to checking whether there is an $x \in \mathbb{R}^n$ such that $s \cdot (V^\top x - \tau) > 0$ or, equivalently, whether there is a pair $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$s \cdot ([V; \tau^\top]^\top [x; \xi]) > 0 \quad \text{and} \quad \xi = -1.$$

The first condition above reads $s \in \mathcal{S}([V; \tau^\top], 0)$ and refers to the linear arrangement in \mathbb{R}^{n+1} governed by the *augmented matrix* $[V; \tau^\top]$. This presentation of the problem shows that there must be links between the following sign vector sets and between the following stem vector sets

$$(3.19a) \quad \mathcal{S}(V, 0), \quad \mathcal{S}(V, \tau) \quad \text{and} \quad \mathcal{S}([V; \tau^\top], 0),$$

$$(3.19b) \quad \mathfrak{S}(V, 0), \quad \mathfrak{S}(V, \tau) \quad \text{and} \quad \mathfrak{S}([V; \tau^\top], 0).$$

For example, we already know the inclusions $\mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau)$ and $\mathfrak{S}(V, \tau) \subseteq \mathfrak{S}(V, 0)$ from propositions 3.6 and 3.14(2).

This section aims at identifying a few properties where the augmented matrix $[V; \tau^\top]$ intervenes. In section 3.4.1, some links between the sets in (3.19a) are highlighted. Section 3.4.2 establishes some connections between the circuits of V and $[V; \tau^\top]$, as well as between the stem vector sets in (3.19b). In section 3.4.3, one observes that the identity obtained in proposition 3.18(4) makes it easy to deduce a formula for $|\mathcal{S}(V, \tau)|$ (proposition 3.24) and a known bound on $|\mathcal{S}(V, \tau)|$ (proposition 3.31), which is reached if and only if the arrangement is in *affine general position* (proposition 3.28 and definition 3.29). The role of the augmented matrix in the specification of the notion of *affine general position* is also pointed out.

Viewing an affine arrangement in $x \in \mathbb{R}^n$ as the intersection of a linear arrangement in $(x, \xi) \in \mathbb{R}^{n+1}$ with the affine space $\{(x, \xi) \in \mathbb{R}^{n+1} : \xi = -1\}$ is called the *method of coning* in [32, definition 1.15].

3.4.1. Sign vectors of the augmented matrix. Recall the definition of $\mathcal{S}_s(V, \tau)$ and $\mathcal{S}_a(V, \tau)$ in (3.8) and the properties (3.9). Some of the properties stated in proposition 3.18(1) can be symbolically represented like in figure 3.3. In the next proposition, we use the symbol “ \cup ” for the disjoint union of sets.

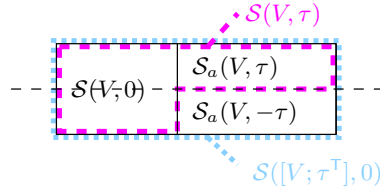


FIG. 3.3. Symbolic representation of the sets $\mathcal{S}(V, 0)$, $\mathcal{S}(V, \tau)$, $\mathcal{S}_a(V, \tau)$ and $\mathcal{S}([V; \tau^T], 0)$, respectively (3.7), (3.8), (3.9) and propositions 3.6 and 3.18. The horizontal dashed line aims at representing the reflection between a sign vector s and its opposite $-s$: $\mathcal{S}(V, 0)$, $\mathcal{S}([V; \tau^T], 0)$ and $\mathcal{S}([V; \tau^T], 0)^c$ are symmetric in the sense of definition 3.4.

PROPOSITION 3.18 (properties with $\mathcal{S}([V; \tau^T], 0)$). *Let $\mathcal{A}(V, \tau)$ be an arrangement with $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$. Then, the following properties hold.*

- 1) $\mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau) = \mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau) \subseteq \mathcal{S}([V; \tau^T], 0) = \mathcal{S}(V, \tau) \cup \mathcal{S}(V, -\tau)$.
- 2) $\mathcal{S}(V, \tau)^c \cup \mathcal{S}(V, -\tau)^c = \mathcal{S}(V, 0)^c \supseteq \mathcal{S}(V, \tau)^c \supseteq \mathcal{S}([V; \tau^T], 0)^c = \mathcal{S}(V, \tau)^c \cap \mathcal{S}(V, -\tau)^c$.
- 3) $\mathcal{S}(V, 0) \cup \mathcal{S}_a(V, \tau) \cup \mathcal{S}_a(V, -\tau) = \mathcal{S}([V; \tau^T], 0)$.
- 4) $2|\mathcal{S}(V, \tau)| = |\mathcal{S}(V, 0)| + |\mathcal{S}([V; \tau^T], 0)|$.
- 5) $2|\mathcal{S}(V, \tau)^c| = |\mathcal{S}(V, 0)^c| + |\mathcal{S}([V; \tau^T], 0)^c|$.

Proof. 1) The first equality repeats proposition 3.6(2), using (3.8), the first inclusion repeats proposition 3.6(1) and the second inclusion is straightforward: if $s \in \mathcal{S}(V, \tau)$, one has $s \cdot (V^T x - \tau) > 0$ for some $x \in \mathbb{R}^n$ or $s \cdot ([V; \tau^T]^T [x; -1]) > 0$, implying that $s \in \mathcal{S}([V; \tau^T], 0)$.

Consider now the last equality. [\subseteq] Let $s \in \mathcal{S}([V; \tau^T], 0)$, so that $s \cdot (V^T x + \tau \xi) > 0$ for some $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}$. By homogeneity, it follows that $s \in \mathcal{S}(V, \tau)$ if $\xi < 0$, that $s \in \mathcal{S}(V, -\tau)$ if $\xi > 0$ and that $s \in \mathcal{S}(V, 0) = \mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau)$ if $\xi = 0$. [\supseteq] By the second inclusion, $\mathcal{S}(V, \tau) \subseteq \mathcal{S}([V; \tau^T], 0)$ and $\mathcal{S}(V, -\tau) = -\mathcal{S}(V, \tau) \subseteq -\mathcal{S}([V; \tau^T], 0) = \mathcal{S}([V; \tau^T], 0)$.

2) Take the complement of the sets in point 1.

3) Let us first show that the sets are disjoint. By (3.8) and proposition 3.6(2), one has $\mathcal{S}_a(V, \pm\tau) = \mathcal{S}(V, \pm\tau) \setminus \mathcal{S}(V, 0)$, so that $\mathcal{S}(V, 0) \cap \mathcal{S}_a(V, \pm\tau) = \emptyset$. Use the same arguments to get that

$$\begin{aligned} \mathcal{S}_a(V, \tau) \cap \mathcal{S}_a(V, -\tau) &= [\mathcal{S}(V, \tau) \cap \mathcal{S}(V, 0)^c] \cap [\mathcal{S}(V, -\tau) \cap \mathcal{S}(V, 0)^c] \\ &= [\mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau)] \cap \mathcal{S}(V, 0)^c \\ &= \mathcal{S}(V, 0) \cap \mathcal{S}(V, 0)^c \\ &= \emptyset. \end{aligned}$$

Consider now the identity:

$$\begin{aligned} \mathcal{S}(V, 0) \cup \mathcal{S}_a(V, \tau) \cup \mathcal{S}_a(V, -\tau) &= \mathcal{S}(V, 0) \cup [\mathcal{S}(V, \tau) \cap \mathcal{S}(V, 0)^c] \cup [\mathcal{S}(V, -\tau) \cap \mathcal{S}(V, 0)^c] \\ &= \mathcal{S}(V, 0) \cup \mathcal{S}(V, \tau) \cup \mathcal{S}(V, -\tau) \\ &= \mathcal{S}([V; \tau^T], 0) \quad [\text{point 1}]. \end{aligned}$$

4) The identity results from

$$\begin{aligned} |\mathcal{S}([V; \tau^T], 0)| &= |\mathcal{S}(V, \tau)| + |\mathcal{S}(V, -\tau)| - |\mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau)| \quad [\text{point 1}] \\ &= 2|\mathcal{S}(V, \tau)| - |\mathcal{S}(V, 0)| \quad [(3.7), (3.8), \text{proposition 3.6(2)}]. \end{aligned}$$

5) Each set in point 4 is a part of $\{\pm 1\}^p$ of cardinality 2^p . Hence, the identity in point 4 gives

$$2(2^p - |\mathcal{S}(V, \tau)^c|) = (2^p - |\mathcal{S}(V, 0)^c|) + (2^p - |\mathcal{S}([V; \tau^T], 0)^c|).$$

Point 5 follows after subtracting 2^{p+1} from both sides. \square

REMARKS 3.19. 1) Let us emphasize the meaning of the inclusions in proposition 3.18(1): the *affine* arrangement $\mathcal{A}(V, \tau)$ has its sign vectors (in bijection with its chambers, by proposition 3.1) containing those of the *linear* arrangement $\mathcal{A}(V, 0)$ and contained in those of the *linear* arrangement $\mathcal{A}([V; \tau^T], 0)$.

2) As a corollary of proposition 3.18(2), one has (see remark 3.17):

$$(3.20) \quad \pm s \in \mathcal{S}(V, \tau)^c \iff s_J \in \mathfrak{S}([V; \tau^T], 0) \text{ for some } J \subseteq [1 : p].$$

Indeed $\pm s \in \mathcal{S}(V, \tau)^c$ if and only if $s \in \mathcal{S}(V, \tau)^c \cap \mathcal{S}(V, -\tau)^c = \mathcal{S}([V; \tau^T], 0)^c$ (proposition 3.18(2)), which is equivalent to $s_J \in \mathfrak{S}([V; \tau^T], 0)$ for some $J \in [1 : p]$ (proposition 3.16). Note that $\mathfrak{S}([V; \tau^T], 0)$ is symmetric, so that the properties in (3.20) are also equivalent to the fact that $s_J \in -\mathfrak{S}([V; \tau^T], 0)$ for some $J \subseteq [1 : p]$. \square

3.4.2. Circuits and stem vectors of the augmented matrix. The next propositions highlight connections between the circuits and the stem vectors of V and those of the augmented matrix $[V; \tau^T]$. Recall from proposition 3.5 that an arrangement is centered if and only if $\tau \in \mathcal{R}(V^T)$. Note also that

$$(3.21a) \quad \text{rank}([V; \tau^T]) = \begin{cases} \text{rank}(V) & \text{if } \tau \in \mathcal{R}(V^T) \\ \text{rank}(V) + 1 & \text{otherwise,} \end{cases}$$

$$(3.21b) \quad \text{null}([V; \tau^T]) = \begin{cases} \text{null}(V) & \text{if } \tau \in \mathcal{R}(V^T) \\ \text{null}(V) - 1 & \text{otherwise.} \end{cases}$$

The formula of $\text{rank}([V; \tau^T])$ is clear and the one of $\text{null}([V; \tau^T])$ can be deduced from (3.21a) by the rank-nullity theorem.

PROPOSITION 3.20 (circuits of V and $[V; \tau^T]$). *Let $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$. Then, the following properties are equivalent:*

- (i) $\mathcal{C}(V) = \mathcal{C}([V; \tau^T])$,
- (ii) $\mathcal{C}(V) \subseteq \mathcal{C}([V; \tau^T])$,
- (iii) $\tau \in \mathcal{R}(V^T)$, meaning that the arrangement $\mathcal{A}(V, \tau)$ is centered.

Proof. [(i) \Rightarrow (ii)] Clear.

[(ii) \Rightarrow (iii)] Let $J \in \mathcal{C}(V)$. Then, $\text{null}(V_{:,J}) = 1$ by (3.12). By assumption, $J \in \mathcal{C}([V; \tau^T])$, so that $\text{null}([V; \tau^T]_{:,J}) = 1$, as well. By (3.21b), $\tau_J \in \mathcal{R}(V_{:,J}^T) =$

635 $\mathcal{N}(V; \cdot, J)^\perp$. According to remark 3.13(3.a), the stem vectors associated with J are
 636 symmetric. Since J is arbitrary in $\mathcal{C}(V)$, one has $\mathfrak{S}(V, \tau) = \mathfrak{S}_s(V, \tau)$, implying that
 637 the arrangement is centered (proposition 3.15).

[(iii) \Rightarrow (i)] Let $J \subseteq [1 : p]$ and $J_0 \subsetneq J$. When $\tau \in \mathcal{R}(V^\top)$, one has $\tau_J \in \mathcal{R}(V_{:,J}^\top)$
 and $\tau_{J_0} \in \mathcal{R}(V_{:,J_0}^\top)$, so that (3.21b) yields

$$\text{null}([V; \tau^\top]_{:,J}) = \text{null}(V_{:,J}) \quad \text{and} \quad \text{null}([V; \tau^\top]_{:,J_0}) = \text{null}(V_{:,J_0}).$$

638 It follows that $\text{null}([V; \tau^\top]_{:,J}) = 1$ and $\text{null}([V; \tau^\top]_{:,J_0}) = 0$ for all $J_0 \subsetneq J$ if and only
 639 if $\text{null}(V_{:,J}) = 1$ and $\text{null}(V_{:,J_0}) = 0$ for all $J_0 \subsetneq J$. In other words, $J \in \mathcal{C}([V; \tau^\top])$ if
 640 and only if $J \in \mathcal{C}(V)$. We have shown that $\mathcal{C}([V; \tau^\top]) = \mathcal{C}(V)$. \square

641 The implication (ii) \Rightarrow (i) of lemma 3.20 is not based on the fact that one
 642 would always have $\mathcal{C}(V) \supseteq \mathcal{C}([V; \tau^\top])$, which is not true. As a counter-example,
 643 take $V = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $\tau^\top = [0 \ 0 \ 1 \ 2]$, in which case one has $\mathcal{C}(V) = \{\{3, 4\}, \{1, 2, 3\},$
 644 $\{1, 2, 4\}\}$, while $\mathcal{C}([V; \tau^\top]) = \{\{1, 2, 3, 4\}\}$. Actually, the property $\mathcal{C}(V) \supseteq \mathcal{C}([V; \tau^\top])$,
 645 which is therefore weaker than those in lemma 3.20, has various other equivalent in-
 646 teresting formulations, including $\mathfrak{S}_s(V, \tau) = \mathfrak{S}([V; \tau^\top], 0)$, as shown by the following
 647 proposition. Recall figure 3.2 for a symbolic representation of the stem vector sets.

648 PROPOSITION 3.21 (stem vectors of $\mathcal{A}(V, \tau)$ and $\mathcal{A}([V; \tau^\top], 0)$). *Consider an*
 649 *arrangement $\mathcal{A}(V, \tau)$. Then,*

$$650 \quad (3.22) \quad \mathfrak{S}_a(V, \tau) \cap \mathfrak{S}([V; \tau^\top], 0) = \emptyset \quad \text{and} \quad \mathfrak{S}_s(V, \tau) \subseteq \mathfrak{S}([V; \tau^\top], 0),$$

651 *with equality in the last inclusion if the arrangement is centered. More precisely, the*
 652 *following properties are equivalent:*

- 653 (i) $\mathfrak{S}_s(V, \tau) = \mathfrak{S}([V; \tau^\top], 0)$,
- 654 (ii) $\mathfrak{S}_s(V, \tau) \supseteq \mathfrak{S}([V; \tau^\top], 0)$,
- 655 (iii) $\mathcal{C}(V) \supseteq \mathcal{C}([V; \tau^\top])$,
- 656 (iv) $\tau_J \in \mathcal{R}(V_{:,J}^\top)$, for all $J \in \mathcal{C}([V; \tau^\top])$.

657 *Proof.* 1) [(3.22)₁] Let $\sigma \in \mathfrak{S}([V; \tau^\top], 0)$ and $J := \mathfrak{J}(\sigma)$. Then, $\sigma = \text{sgn}(\eta)$ for
 658 some $\eta \in \mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\}$. This η verifies $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^\top \eta = 0$. It
 659 follows that $\sigma \notin \mathfrak{S}_a(V, \tau)$, either because $J \notin \mathcal{C}(V)$ or because $J \in \mathcal{C}(V)$, in which
 660 case $\sigma \in \mathfrak{S}_s(V, \tau)$ by the properties of η .

661 [(3.22)₂] Let $\sigma \in \mathfrak{S}_s(V, \tau)$ and $J := \mathfrak{J}(\sigma)$. Then, $J \in \mathcal{C}(V)$ and $\sigma = \text{sgn}(\eta)$ for
 662 some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^\top \eta = 0$. It follows that $\eta \in \mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\}$. To
 663 show that $\sigma \in \mathfrak{S}([V; \tau^\top], 0)$, one still has to verify that $J \in \mathcal{C}([V; \tau^\top])$ (definition 3.12).
 664 First, $J \neq \emptyset$, since $J \in \mathcal{C}(V)$. Next, $\text{null}([V; \tau^\top]_{:,J}) = 1$, since $\eta \in \mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\}$
 665 and $\text{null}([V; \tau^\top]_{:,J}) \leq \text{null}(V_{:,J}) = 1$ (because $J \in \mathcal{C}(V)$). Finally, for all $J_0 \subsetneq J$, one
 666 has $\text{null}([V; \tau^\top]_{:,J_0}) = 0$, since $\text{null}([V; \tau^\top]_{:,J_0}) \leq \text{null}(V_{:,J_0}) = 0$ (because $J \in \mathcal{C}(V)$).

667 2) Suppose now that the arrangement $\mathcal{A}(V, \tau)$ is centered (or $\tau \in \mathcal{R}(V^\top)$) and
 668 let us show that $\mathfrak{S}_s(V, \tau) \supseteq \mathfrak{S}([V; \tau^\top], 0)$ (this could also be viewed as a consequence
 669 of the implication (iv) \Rightarrow (i) proved below). Let $\sigma \in \mathfrak{S}([V; \tau^\top], 0)$ and $J := \mathfrak{J}(\sigma)$.
 670 Then, $J \in \mathcal{C}([V; \tau^\top])$ and $\sigma = \text{sgn}(\eta)$ for some $\eta \in \mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\}$. Then,
 671 $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^\top \eta = 0$. We see that it suffices now to observe that $J \in \mathcal{C}(V)$,
 672 which results from the implication (iii) \Rightarrow (i) of proposition 3.20.

673 3) Consider now the equivalences (i)-(iv).

674 [(i) \Leftrightarrow (ii)] By (3.22).

675 [(ii) \Rightarrow (iii)] Let $J \in \mathcal{C}([V; \tau^\top])$. By remark 3.13(3.a), there is a stem vector
 676 $\sigma \in \{\pm 1\}^J$ that is in $\mathfrak{S}([V; \tau^\top], 0)$, hence in $\mathfrak{S}_s(V, \tau)$ by (ii). This latter fact implies
 677 that $J \in \mathcal{C}(V)$.

678 [(iii) \Rightarrow (iv)] Let $J \in \mathcal{C}([V; \tau^\top])$. By (iii), $J \in \mathcal{C}(V)$, so that $\text{null}([V; \tau^\top]_{:,J}) =$
 679 $\text{null}(V_{:,J})$ (these nullities = 1). Then, (3.21b) implies that $\tau_J \in \mathcal{R}(V_{:,J}^\top)$.
 680 [(iv) \Rightarrow (ii)] Let $\sigma = \text{sgn}(\eta) \in \mathfrak{S}([V; \tau^\top], 0)$ and $J := \mathfrak{J}(\sigma)$. Since $\eta \in$
 681 $\mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\} \subseteq \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau^\top \eta = 0$, it suffices to show that $J \in$
 682 $\mathcal{C}(V)$. This property follows from $J \in \mathcal{C}([V; \tau^\top]_{:,J})$, since $J \in \mathcal{C}([V; \tau^\top])$, from
 683 $\mathcal{C}([V; \tau^\top]_{:,J}) = \mathcal{C}(V_{:,J})$, by $\tau_J \in \mathcal{R}(V_{:,J}^\top)$ and the implication (iii) \Rightarrow (i) of proposi-
 684 tion 3.20, and from $\mathcal{C}(V_{:,J}) \subseteq \mathcal{C}(V)$. \square

EXAMPLES 3.22. 1) An example in which the equivalent properties of proposition 3.21 hold, but not those of proposition 3.20, is given by $V = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $\tau^\top = [0 \ 0 \ 0 \ 1]$. One has

$$\mathcal{C}(V) = \{\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\} \quad \text{and} \quad \mathcal{C}([V; \tau^\top]) = \{\{1, 2, 3\}\}.$$

685 We see that $\mathcal{C}([V; \tau^\top])$ is included in $\mathcal{C}(V)$ but is not equal to it. Note also that
 686 $\tau \notin \mathcal{R}(V^\top)$,

$$\begin{aligned} 687 \quad \mathfrak{S}_s(V, \tau) &= \mathfrak{S}([V; \tau^\top], 0) = \left\{ \pm \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in \{\pm 1\}^{\{1,2,3\}} \right\} \quad \text{and} \\ 688 \quad \mathfrak{S}_a(V, \tau) &= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \{\pm 1\}^{\{3,4\}}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in \{\pm 1\}^{\{1,2,4\}} \right\}. \\ 689 \end{aligned}$$

690 All this is in agreement with propositions 3.20 and 3.21.

691 2) Property (iv) of proposition 3.21 does not imply that $\tau_K \in \mathcal{R}(V_{:,K}^\top)$, for $K :=$
 692 $\cup\{J \in \mathcal{C}([V; \tau^\top])\}$. To see this, take $V = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$ and $\tau^\top = [0 \ 0 \ 1 \ 0 \ 0 \ 1]$. One has
 693 $\mathcal{C}([V; \tau^\top]) = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$, $\tau_J \in \mathcal{R}(V_{:,J}^\top)$ for all $J \in \mathcal{C}([V; \tau^\top])$ and $K = [1 : 6]$,
 694 but $\tau \notin \mathcal{R}(V^\top)$. \square

695 Recall (3.21)₂. In the algorithm of section 6.3, it will be interesting to partition
 696 $\mathfrak{S}([V; \tau^\top], 0)$ in $\mathfrak{S}_s(V, \tau)$ and $\mathfrak{S}_0(V, \tau)$, where

$$697 \quad \mathfrak{S}_0(V, \tau) := \mathfrak{S}([V; \tau^\top], 0) \setminus \mathfrak{S}_s(V, \tau).$$

699 The stem vectors of $\mathfrak{S}_0(V, \tau)$ can be recognized thanks to the following proposition,
 700 whose short proof is given in [16].

701 PROPOSITION 3.23 ($\mathfrak{S}_0(V, \tau)$ and $\mathfrak{S}_s(V, \tau)$ characterizations). *Let $\mathcal{A}(V, \tau)$ be an*
 702 *arrangement with $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$. Let $\sigma \in \mathfrak{S}([V; \tau^\top], 0)$ and $J = \mathfrak{J}(\sigma)$. Then,*

$$703 \quad (3.23a) \quad \sigma \in \mathfrak{S}_0(V, \tau) \iff \begin{cases} J \in \mathcal{C}([V; \tau^\top]) \setminus \mathcal{C}(V) \\ \sigma = \text{sgn}(\eta) \text{ for some } \eta \in \mathcal{N}([V; \tau^\top]_{:,J}) \setminus \{0\}. \end{cases}$$

$$704 \quad (3.23b) \quad \sigma \in \mathfrak{S}_s(V, \tau) \iff J \in \mathcal{C}(V) \iff \text{null}(V_{:,J}) = 1 \iff \tau_J \in \mathcal{R}(V_{:,J}^\top).$$

706 **3.4.3. Sign vector set cardinality.** Proposition 3.18(4) and Winder's formula
 707 of $|\mathcal{S}(V, 0)|$ (linear arrangement) make it possible to give expressions of $|\mathcal{S}(V, \tau)|$
 708 and $|\mathcal{S}_a(V, \tau)|$ having the flavor of Winder's. Formula (3.25a) below is given by Za-
 709 slavsky [54, corollary 5.9, page 68], who makes its connection with a cardinality for-
 710 mula using a characteristic polynomial of the arrangement [54, theorem A, page 18];
 711 an approach that looks rather different from ours.

Recall that, for a matrix $V \in \mathbb{R}^{n \times p}$ without zero column, *Winder's formula* of the cardinality of $\mathcal{S}(V, 0)$ reads [52, 1966] (see also [15, § 4.2.1])

$$(3.24) \quad |\mathcal{S}(V, 0)| = \sum_{J \subseteq [1:p]} (-1)^{\text{null}(V_{:,J})},$$

where the term in the right-hand side corresponding to $J = \emptyset$ is 1 (one takes the convention that $\text{null}(V_{:,\emptyset}) = 0$).

PROPOSITION 3.24 (cardinality of $\mathcal{S}(V, \tau)$). *Consider a proper affine arrangement $\mathcal{A}(V, \tau)$, with $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$. Then,*

$$(3.25a) \quad |\mathcal{S}(V, \tau)| = \sum_{\substack{J \subseteq [1:p] \\ \tau_J \in \mathcal{R}(V_{:,J}^T)}} (-1)^{\text{null}(V_{:,J})},$$

$$(3.25b) \quad |\mathcal{S}_a(V, \tau)| = \sum_{\substack{J \subseteq [1:p] \\ \tau_J \notin \mathcal{R}(V_{:,J}^T)}} (-1)^{\text{null}(V_{:,J})-1},$$

where the term in the right-hand side of (3.25a) corresponding to $J = \emptyset$ is 1 (it is considered that $\tau_J \in \mathcal{R}(V_{:,J}^T)$ for $J = \emptyset$).

Proof. Using proposition 3.18(4), Winder's formula (3.24) and (3.21b), one gets

$$\begin{aligned} 2|\mathcal{S}(V, \tau)| &= |\mathcal{S}(V, 0)| + |\mathcal{S}([V; \tau^T], 0)| \\ &= \sum_{J \subseteq [1:p]} (-1)^{\text{null}(V_{:,J})} + \sum_{J \subseteq [1:p]} (-1)^{\text{null}([V; \tau^T]_{:,J})} \\ &= 2 \sum_{\substack{J \subseteq [1:p] \\ \tau_J \in \mathcal{R}(V_{:,J}^T)}} (-1)^{\text{null}(V_{:,J})} + \sum_{\substack{J \subseteq [1:p] \\ \tau_J \notin \mathcal{R}(V_{:,J}^T)}} \left[(-1)^{\text{null}(V_{:,J})} + (-1)^{\text{null}(V_{:,J})-1} \right] \\ &= 2 \sum_{\substack{J \subseteq [1:p] \\ \tau_J \in \mathcal{R}(V_{:,J}^T)}} (-1)^{\text{null}(V_{:,J})}, \end{aligned}$$

since $(-1)^{\text{null}(V_{:,J})} + (-1)^{\text{null}(V_{:,J})-1} = 0$ (note that $\text{null}(V_{:,J}) > 0$ if $\tau_J \notin \mathcal{R}(V_{:,J}^T)$ ⁸). Formula (3.25a) follows. Formula (3.25b) of $|\mathcal{S}_a(V, \tau)|$ comes from (3.8)₂, $\mathcal{S}_s(V, \tau) = \mathcal{S}(V, 0)$ and proposition 3.18(1), which implies that $|\mathcal{S}_a(V, \tau)| = |\mathcal{S}(V, \tau)| - |\mathcal{S}(V, 0)|$. \square

Note that one recovers (3.24) from (3.25a) when the arrangement is centered (i.e., $\tau \in \mathcal{R}(V^T)$). By proposition 3.18(1), $\mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau)$, so that $|\mathcal{S}(V, 0)| \leq |\mathcal{S}(V, \tau)|$, but this inequality is not easy to deduce from (3.24) and (3.25a), since the terms of the sums in the right-hand sides of these formulas may be negative and positive.

Formula (3.25a) is usually not easy to evaluate because the number of terms in the sum can be large. It is therefore sometimes useful to have a bound on $|\mathcal{S}(V, \tau)|$ that is easier to compute than the exact formula. Proposition 3.18(4), joined to Schläfli's bound on $|\mathcal{S}(V, 0)|$ (linear arrangement), makes it possible to recover a known bound on $|\mathcal{S}(V, \tau)|$ and to clarify the conditions under which this bound is reached. Recall that, for a matrix $V \in \mathbb{R}^{n \times p}$ of rank r , *Schläfli's bound* on the cardinality of $\mathcal{S}(V, 0)$

⁸By contraposition, if $\text{null}(V_{:,J}) = 0$, $V_{:,J}^T$ is surjective, hence $\tau_J \in \mathcal{R}(V_{:,J}^T)$.

reads [43, p. 211] (see also [15, proposition 4.15])

$$(3.26) \quad |\mathcal{S}(V, 0)| \leq 2 \sum_{i \in [0 : r-1]} \binom{p-1}{i}.$$

Winder [52, 1966, corollary] showed that the upper bound in (3.26) is reached if the arrangement $\mathcal{A}(V, 0)$ is in *linear general position*, a concept defined as follows.

DEFINITION 3.25 (linear general position). Let be given $V \in \mathbb{R}^{n \times p}$ of rank r , without zero column. The linear arrangement $\mathcal{A}(V, 0)$ is (or the columns of V are) said to be in *linear general position* if the following equivalent properties hold

$$\begin{aligned} \forall I \subseteq [1 : p] : \quad \dim(\cap_{i \in I} H_i^0) &= n - \min(|I|, r), \\ \forall I \subseteq [1 : p] : \quad \text{rank}(V_{:,I}) &= \min(|I|, r), \end{aligned}$$

where $H_i^0 := \{x \in \mathbb{R}^n : V_{:,i}^\top x = 0\}$ for $i \in [1 : p]$. \square

The first condition has a geometric nature, while the second one has an algebraic flavor. Their equivalence comes from the fact that $\dim(\cap_{i \in I} H_i^0) = n - \text{rank}(V_{:,I})$ ⁹. Observe that the inequality $\text{rank}(V_{:,I}) \leq \min(|I|, r)$ always holds. This linear general position property is clearly less restrictive than the injectivity of V , since it holds for an injective V but does not impose $p \leq n$. In [17, proposition 4.10], it is shown analytically that the linear general position is also necessary to have equality in (3.26). Let us summarize these facts in a proposition.

PROPOSITION 3.26 (bound on $|\mathcal{S}(V, 0)|$). Consider a proper linear arrangement $\mathcal{A}(V, 0)$, with $V \in \mathbb{R}^{n \times p}$ of rank r . Then, (3.26) holds. Furthermore, equality holds in (3.26) if and only if the arrangement is in linear general position.

For instance, the arrangement on the left-hand side pane in figure 3.1 is in linear general position and verifies (3.26) with equality. Also, linear general position generally occurs for a matrix V of rank r that is randomly generated, since then one usually has $\text{rank}(V_{:,I}) = \min(|I|, r)$ for all $I \subseteq [1 : p]$.

The general position for an affine arrangement, like the one in the middle and right-hand side panes of figure 3.1, is different from that specified by definition 3.25. It is usually given a definition in geometric terms. We are also going to give it an algebraic expression (definition 3.29), since this one easily provides necessary and sufficient conditions to have equality in a bound on $|\mathcal{S}(V, \tau)|$ (proposition 3.31). The definition is grounded on proposition 3.28 below; lemma 3.27 will also be useful. The proofs and more discussions can be found in [16].

LEMMA 3.27 (contribution to the affine general position). Consider a proper affine arrangement $\mathcal{A}(V, \tau)$, with $V \in \mathbb{R}^{n \times p}$ of rank r and $\tau \in \mathbb{R}^p$. Then, the following two conditions are equivalent:

- (i) $|\mathcal{S}([V; \tau^\top], 0)| = 2 \sum_{i \in [0 : r]} \binom{p-1}{i},$
- (ii) $\forall I \subseteq [1 : p] : \text{rank}([V; \tau^\top]_{:,I}) = \min(|I|, r + 1).$

PROPOSITION 3.28 (affine general position). Let be given $V \in \mathbb{R}^{n \times p}$ of rank r , without zero column, and $\tau \in \mathbb{R}^p$. Set $H_i := \{x \in \mathbb{R}^n : V_{:,i}^\top x = \tau_i\}$ for $i \in [1 : p]$.

⁹Use $\dim(\cap_{i \in I} H_i^0) = \text{null}(V_{:,I}^\top)$, the rank-nullity theorem and $\text{rank}(V_{:,I}^\top) = \text{rank}(V_{:,I})$.

Then, the following properties are equivalent:

$$(3.27a) \quad \forall I \subseteq [1:p] : \begin{cases} \cap_{i \in I} H_i \neq \emptyset \text{ and } \dim(\cap_{i \in I} H_i) = n - |I| & \text{if } |I| \leq r \\ \cap_{i \in I} H_i = \emptyset & \text{if } |I| \geq r + 1, \end{cases}$$

$$(3.27b) \quad \forall I \subseteq [1:p] : \begin{cases} \text{rank}(V_{:,I}) = |I| & \text{if } |I| \leq r \\ \text{rank}([V; \tau^T]_{:,I}) = r + 1 & \text{if } |I| \geq r + 1, \end{cases}$$

$$(3.27c) \quad \forall I \subseteq [1:p] : \begin{cases} \text{rank}(V_{:,I}) = \min(|I|, r) \\ \text{rank}([V; \tau^T]_{:,I}) = \min(|I|, r + 1). \end{cases}$$

DEFINITION 3.29 (affine general position). A proper affine arrangement $\mathcal{A}(V, \tau)$ is said to be in *affine general position* if the equivalent properties of proposition 3.28 hold. \square

REMARKS 3.30. 1) The two conditions in (3.27c) are independent of each other. For the arrangement in the left-hand side pane of figure 3.1, the first condition in (3.27c) holds (linear general position) but not the second one. For the arrangement defined by $V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\tau^T = [0 \ 0 \ 1]$, the second condition in (3.27c) holds but not the first one.

2) Even for a linear proper arrangement, we make a difference between *linear general position* (definition 3.25) and *affine general position* (definition 3.29), since an arrangement $\mathcal{A}(V, 0)$ can be considered as a linear arrangement or an affine arrangement $\mathcal{A}(V, \tau)$ with $\tau = 0$. We see on (3.27c) and from the first remark that the notion of affine general position is more restrictive than the notion of linear general position since it requires one more independent condition.

3) A centered proper arrangement $\mathcal{A}(V, \tau)$, with $V \in \mathbb{R}^{n \times p}$ of rank r , can be in affine general position, only if $p = r$ (take $I = [1:p]$ in (3.27c)₂).

4) Affine general position usually holds when V of rank r and τ are randomly generated. \square

Condition (3.27a) is the one that is usually given to define the affine general position of an affine arrangement $\mathcal{A}(V, \tau)$ [48, p. 287]; it has a geometric nature. Condition (3.27c) is the form that suits the needs of the proof of the following proposition. We have not found elsewhere the fact that the affine general position is necessary to have equality in (3.28) (for equality in (3.28) when the arrangement is in general position, see [54, (5.7)₁]). The right-hand side of (3.28) is sequence A008949 in [30] (see also [45]).

PROPOSITION 3.31 (bound on $|\mathcal{S}(V, \tau)|$). Consider a proper arrangement $\mathcal{A}(V, \tau)$, with $V \in \mathbb{R}^{n \times p}$ of rank r and $\tau \in \mathbb{R}^p$. Then,

$$(3.28) \quad |\mathcal{S}(V, \tau)| \leq \sum_{i \in [0:r]} \binom{p}{i},$$

with equality if and only if the arrangement is in affine general position, in the sense of definition 3.29.

Proof. Observe first that $\text{rank}([V; \tau^T]) \leq r + 1$. Then, by (3.26), one has

$$(3.29a) \quad |\mathcal{S}(V, 0)| \leq 2 \sum_{i \in [0:r-1]} \binom{p-1}{i},$$

$$(3.29b) \quad |\mathcal{S}([V; \tau^T], 0)| \leq 2 \sum_{i \in [0:r]} \binom{p-1}{i}.$$

Using these estimates in proposition 3.18(4) provides

$$\begin{aligned}
 |\mathcal{S}(V, \tau)| &= \frac{1}{2} |\mathcal{S}(V, 0)| + \frac{1}{2} |\mathcal{S}([V; \tau^\top], 0)| \\
 &\leq \binom{p-1}{0} + \binom{p-1}{1} + \cdots + \binom{p-1}{r-1} \\
 &\quad + \underbrace{\binom{p-1}{0}}_{\binom{p}{0}} + \underbrace{\binom{p-1}{1}}_{\binom{p}{1}} + \cdots + \underbrace{\binom{p-1}{r-2}}_{\binom{p}{r-1}} + \underbrace{\binom{p-1}{r-1} + \binom{p-1}{r}}_{\binom{p}{r}} \\
 &= \sum_{i \in [0:r]} \binom{p}{i},
 \end{aligned}
 \tag{3.29c}$$

which is the bound (3.28).

By the previous reasoning, equality holds in (3.28) if and only if equalities hold in (3.29a) and (3.29b). By proposition 3.26 and lemma 3.27, these last equalities are equivalent to the affine general position condition (3.27c). \square

For instance, the arrangements in the middle and right-hand side panes in figure 3.1 are in affine general position and verifies (3.28) with equality ($p = 3$ and $r = 2$).

Observe from (3.29c) that $2 \sum_{i \in [0:r-1]} \binom{p-1}{i} = \sum_{i \in [0:r]} \binom{p}{i} - \binom{p-1}{r}$, so that the bound (3.26) on $|\mathcal{S}(V, 0)|$ is lower than the bound (3.28) on $|\mathcal{S}(V, \tau)|$, unless $r = p$, in which case the affine arrangement is centered (necessarily $\tau \in \mathcal{R}(V^\top)$) and can be viewed as a translated linear arrangement.

4. Chamber computation - Primal approaches. This section starts the algorithmic part of the paper, which focuses on the computation of the sign vector set $\mathcal{S} \equiv \mathcal{S}(V, \tau)$, defined by (3.3), of the considered arrangement $\mathcal{A}(V, \tau)$. By proposition 3.1, the bijection ϕ , defined by (3.4), establishes a one-to-one correspondence between these sign vectors and the chambers of the arrangement. In this section, we assume that the arrangement is proper, which means that V has only nonzero columns:

$$(4.1) \quad \forall j \in [1:p] : \quad V_{:,j} \neq 0.$$

Section 6 describes compact versions of some algorithms. Finally, section 7 compares these different algorithms on various instances of arrangements.

Many algorithms have been designed to list the chambers of an arrangement (see the introduction). Most of them adopt a *primal* strategy, in the sense that they focus on the realization of the inequality system $s \cdot (V^\top x - \tau) > 0$ in (3.3), by trying to compute *witness points* $x \in \mathbb{R}^n$. Section 4.1 describes this \mathcal{S} -tree mechanism of [37], while section 4.2 adapts to affine arrangements some of the enhancements brought to this algorithm in [17] for linear arrangements.

4.1. Primal \mathcal{S} -tree algorithm. For $k \in [1:p]$, define the partial sign vector set $\mathcal{S}_k \subseteq \{\pm 1\}^k$ and its complement \mathcal{S}_k^c in $\{\pm 1\}^k$ by

$$(4.2) \quad \mathcal{S}_k \equiv \mathcal{S}_k(V, \tau) := \mathcal{S}(V_{:, [1:k]}, \tau_{[1:k]}) \quad \text{and} \quad \mathcal{S}_k^c := \{\pm 1\}^k \setminus \mathcal{S}_k.$$

Hence, \mathcal{S}_k is the sign vector set of the arrangement associated with the matrix $V_{:, [1:k]} \in \mathbb{R}^{n \times k}$ and the vector $\tau_{[1:k]} \in \mathbb{R}^k$. Let us denote by v_i the i th column of V , by τ_i the i th component of τ and by $H_i := \{x \in \mathbb{R}^n : v_i^\top x = \tau_i\}$ the i th hyperplane. The \mathcal{S} -tree is a tree structure, whose k level contains the sign vectors in \mathcal{S}_k . Therefore, in addition to its empty root, the complete \mathcal{S} -tree has p levels and the bottom one is $\mathcal{S}_p = \mathcal{S}(V, \tau)$. The first level is $\mathcal{S}_1 = \{+1, -1\}$, because the inequalities $(+1)(v_1^\top x_+ - \tau_1) > 0$ and $(-1)(v_1^\top x_- - \tau_1) > 0$ are satisfied by the following two witness points, located on either side of the hyperplane H_1 :

$$(4.3) \quad x_+ := (\tau_1 + 1)v_1/\|v_1\|^2 \quad \text{and} \quad x_- := (\tau_1 - 1)v_1/\|v_1\|^2.$$

The level $k+1$ is obtained by considering the additional pair $(v_{k+1}, \tau_{k+1}) \in \mathbb{R}^n \times \mathbb{R}$, which defines the hyperplane H_{k+1} . It can be constructed from the level k as follows. By the general assumption (4.1), every node $s \in \mathcal{S}_k$ may have one or two children, namely $(s, +1)$ and/or $(s, -1)$. Geometrically, there are two children if and only if the chamber associated with s is divided in two parts by the hyperplane H_{k+1} , but this geometric view is not easy to detect algebraically in terms of sign vectors (see below). Figure 4.1 shows the three levels of the \mathcal{S} -tree corresponding to the arrangement in the middle pane of figure 3.1. Now, instead of searching the children of every $s \in \mathcal{S}_k$

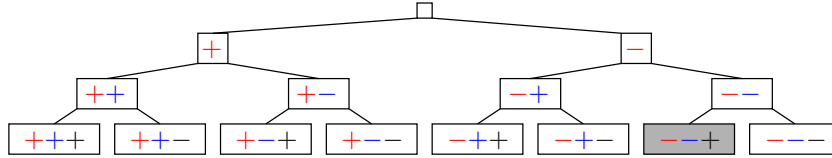


FIG. 4.1. \mathcal{S} -tree of the arrangement in the middle pane of figure 3.1. The gray node is actually absent from the tree, since there is no chamber associated with $s = (-1, -1, +1)$ (no x such that $s \cdot (V^\top x - \tau) > 0$).

in order to obtain \mathcal{S}_{k+1} , the \mathcal{S} -tree will be constructed by a depth-first search [37], in order to avoid having to keep \mathcal{S}_k in memory, which can be large. In this approach, at most p nodes along a path from the root node to a leaf node must be stored at a time. Note that, in the case of a linear arrangement (i.e., $\tau = 0$) or, more generally, a centered arrangement (i.e., $\tau \in \mathcal{R}(V^\top)$), $\mathcal{S}(V, \tau)$ is symmetric (proposition 3.5) and only half of the sign vectors must be computed.

In the algorithm descriptions, it is assumed that the problem data (V, τ) is known and we do not repeat this data on entry of the functions. A function can modify its arguments. Let us now outline the algorithm exploring the \mathcal{S} -tree, called **p.stree** (algorithm 4.1, “p” for “primal”), which uses for this purpose a recursive procedure called **p.stree.rec** (algorithm 4.2).

ALGORITHM 4.1 (**p.stree**). // primal \mathcal{S} -tree algorithm

1. **p.stree.rec**($+1, x_+$) // x_+ given by (4.3)₁
2. **p.stree.rec**($-1, x_-$) // x_- given by (4.3)₂

ALGORITHM 4.2 (**p.stree.rec**($s \in \{\pm 1\}^k, x \in \mathbb{R}^n$)).

1. **if** ($k = p$)
2. Output s and **return** // s is a leaf of the \mathcal{S} -tree; end the recursion
3. **endif**
4. **if** ($v_{k+1}^\top x = \tau_{k+1}$)

```

894      5.  p_stree_rec((s, +1), x + εvk+1)      // (s, +1) ∈ Sk+1
895      6.  p_stree_rec((s, -1), x - εvk+1)      // (s, -1) ∈ Sk+1
896      7.  return
897      8.  endif
898      9.  sk+1 := sgn(vk+1Tx - τk+1)      // (s, sk+1) ∈ Sk+1
899      10. p_stree_rec((s, sk+1), x)
900      11. if ((s, -sk+1) is feasible with witness point  $\tilde{x}$ )
901      12.  p_stree_rec((s, -sk+1),  $\tilde{x}$ )      // (s, -sk+1) ∈ Sk+1
902      13. endif
903

```

904 The algorithm `p_stree` executes the recursive algorithm `p_stree_rec` for con-
905 structing the descendants of the nodes “+1” and “-1” of the first level of the \mathcal{S} -tree.
906 For its part, the algorithm `p_stree_rec` constructs the descendants of a node $s \in \mathcal{S}_k$,
907 knowing a witness point, that is a point $x \in \mathbb{R}^n$ in the chamber associated with s ,
908 hence $s_i(v_i^T x - \tau_i) > 0$ for $i \in [1 : k]$. Let us examine its instructions.

- 909 • If $k = p$ (instructions 1..3), the node s is a leaf of the \mathcal{S} -tree and has no child.
910 Then, `p_stree_rec` just outputs s (it prints it or stores it, depending on the user’s
911 wish) and returns to the calling procedure.
- 912 • Instructions 4..8 consider the case when x is exactly in the hyperplane H_{k+1} , that
913 is when $v_{k+1}^T x = \tau_{k+1}$ (in section 4.2.2, the mechanism used in that case will also
914 be applied when x is sufficiently closed to H_{k+1}): then s has two children $(s, \pm 1)$,
915 since, for an easily computable sufficiently small $\varepsilon > 0$, $x_\pm^\varepsilon := x \pm \varepsilon v_{k+1}$ satisfies
916 $s_i(v_i^T x_\pm^\varepsilon - \tau_i) > 0$, for all $i \in [1 : k]$ and $\pm(v_{k+1}^T x_\pm^\varepsilon - \tau_{k+1}) > 0$. Note that if
917 $x \in H_{k+1}$, then H_{k+1} is not identical to a previous hyperplane H_i , for $i \in [1 : k]$,
918 since x does not belong to any of these H_i ’s.
- 919 • In the sequel $v_{k+1}^T x \neq \tau_{k+1}$, so that $s_{k+1} := \text{sgn}(v_{k+1}^T x - \tau_{k+1}) \in \{\pm 1\}$ and
920 $(s, s_{k+1}) \in \mathcal{S}_{k+1}$ with witness point x . The instructions 9..10 deal with that situa-
921 tion, asking to compute the descendants of (s, s_{k+1}) .
- 922 • Instructions 11..13 examine whether $(s, -s_{k+1})$ is also a child of s , which amounts
923 to determining whether the following system has a solution $\tilde{x} \in \mathbb{R}^n$ (see below how
924 this can be done):

$$925 \quad (4.4) \quad \begin{cases} s_i(v_i^T \tilde{x} - \tau_i) > 0, & \text{for } i \in [1 : k] \\ -s_{k+1}(v_{k+1}^T \tilde{x} - \tau_{k+1}) > 0. \end{cases}$$

926 If this is the case, the descendants of $(s, -s_{k+1})$ are searched using `p_stree_rec`.

927 To determine whether the strict inequalities (4.4) are compatible, one can, like
928 in [37], recast the problem as a linear optimization problem (LOP) and check whether
929 its optimal value is negative. The linear optimization problem reads

$$930 \quad (4.5) \quad \begin{aligned} & \min_{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}} \quad \alpha \\ & \text{s.t.} \quad s_i(v_i^T x - \tau_i) + \alpha \geq 0, \quad \text{for } i \in [1 : k] \\ & \quad \quad -s_{k+1}(v_{k+1}^T x - \tau_{k+1}) + \alpha \geq 0 \\ & \quad \quad \alpha \geq -1. \end{aligned}$$

931 This optimization problem is feasible (by taking an arbitrary $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$
932 sufficiently large) and bounded (i.e., its optimal value is bounded below, here by -1),
933 so that it has a solution [5, theorem 19.1]. Denote it by $(\bar{x}, \bar{\alpha})$. It is clear that (4.4)
934 is feasible if and only if $\bar{\alpha} < 0$. This equivalence can then be used as a feasibility
935 criterion for (4.4). One can find in [16] a discussion on the possibility to detect

whether $(s, -s_{k+1})$ is a symmetric sign vector in \mathcal{S}_{k+1} (in the sense of definition 3.4) by optimization technique.

For future reference, we quote in a proposition an observation, which is deduced from the algorithm.

PROPOSITION 4.3 (binary \mathcal{S} -tree). *Let $\mathcal{A}(V, \tau)$ be a proper arrangement. Then, the \mathcal{S} -tree is a binary tree.*

Proof. This fact is obtained by construction of the \mathcal{S} -tree in algorithm 4.1–4.2. Note that to have two children in lines 5..6 of algorithm 4.2, one must have $v_{k+1} \neq 0$, hence the assumption of a *proper* arrangement. If $v_{k+1} = 0$, either s has no child (if $\tau_{k+1} = 0$) or the single child $(s, -\text{sgn}(\tau_{k+1}))$. \square

By proposition 4.3, any sign vector in \mathcal{S}_k , with $k \in [1:p-1]$, has either one or two children in \mathcal{S}_{k+1} . The next proposition characterizes the sign vectors of \mathcal{S}_k that have two children in \mathcal{S}_{k+1} . It extends to affine arrangements proposition 4.9 in [15], which is there used to give an analytic version of Winder’s proof of (3.24), giving the cardinality of $\mathcal{S}(V, 0)$. Below, proposition 4.4 will be useful to get the lower bound (4.8) on $|\mathcal{S}(V, \tau)|$, improving (3.10). In the statement of proposition 4.4, $P_{k+1} : \mathbb{R}^n \rightarrow H_{k+1} - H_{k+1}$ denotes the orthogonal projector on the subspace v_{k+1}^\perp that is parallel to the affine space $H_{k+1} := \{x \in \mathbb{R}^n : v_{k+1}^\top x = \tau_{k+1}\}$; while $\hat{x}_{k+1} := \tau_{k+1} v_{k+1} / \|v_{k+1}\|^2$ is the unique point in $\mathcal{N}(P_{k+1}) \cap H_{k+1}$ ¹⁰. We also denote by P_{k+1} the transformation matrix of the projector, so that $P_{k+1} V_{:, [1:k]}$ can be viewed as the product of two matrices (its j th column is $P_{k+1} v_j$, for $j \in [1:k]$). Note that in (4.6b), $\tilde{V} := P_{k+1} V_{:, [1:k]}$ may have zero columns, in which case, by its definition (3.3), the set $\mathcal{S}(\tilde{V}, \tilde{\tau})$ will be nonempty if the corresponding components of $\tilde{\tau}$ do not vanish.

PROPOSITION 4.4 (two child criterion). *Let $V \in \mathbb{R}^{n \times p}$, $s \in \{\pm 1\}^k$ for some $k \in [1:p-1]$ and \hat{x}_{k+1} be the unique point in $\mathcal{N}(P_{k+1}) \cap H_{k+1}$. Then,*

$$\begin{aligned} (s, +1) \text{ and } (s, -1) &\in \mathcal{S}_{k+1} \\ (4.6a) \quad &\iff \exists x \in \mathbb{R}^n : s_i(v_i^\top x - \tau_i) > 0, \text{ for } i \in [1:k], \text{ and } v_{k+1}^\top x - \tau_{k+1} = 0 \\ (4.6b) \quad &\iff s \in \mathcal{S}(P_{k+1} V_{:, [1:k]}, \tau_{[1:k]} - V_{:, [1:k]}^\top \hat{x}_{k+1}). \end{aligned}$$

Proof. To simplify the notation, set $V_k := V_{:, [1:k]}$. The following equivalences prove the result ((4.7a) and (4.7b) are justified afterwards):

$$\begin{aligned} (s, +1) \text{ and } (s, -1) &\in \mathcal{S}(V_{k+1}, \tau_{[1:k+1]}) \\ &\iff \begin{cases} \exists x_+ \in \mathbb{R}^n : s_i(v_i^\top x_+ - \tau_i) > 0, \text{ for } i \in [1:k], \\ \quad \text{and } +(v_{k+1}^\top x_+ - \tau_{k+1}) > 0 \\ \exists x_- \in \mathbb{R}^n : s_i(v_i^\top x_- - \tau_i) > 0, \text{ for } i \in [1:k], \\ \quad \text{and } -(v_{k+1}^\top x_- - \tau_{k+1}) > 0 \end{cases} \\ (4.7a) \quad &\iff \exists x \in \mathbb{R}^n : s_i(v_i^\top x - \tau_i) > 0, \text{ for } i \in [1:k], \text{ and } v_{k+1}^\top x - \tau_{k+1} = 0 \\ (4.7b) \quad &\iff \exists x \in \mathbb{R}^n : s_i([P_{k+1} v_i]^\top x - [\tau_i - v_i^\top \hat{x}_{k+1}]) > 0, \text{ for } i \in [1:k] \end{aligned}$$

¹⁰Indeed,

$$\begin{aligned} \hat{x}_{k+1} \in \mathcal{N}(P_{k+1}) \cap H_{k+1} &\iff P_{k+1} \hat{x}_{k+1} = 0 \text{ and } v_{k+1}^\top \hat{x}_{k+1} = \tau_{k+1} \\ &\iff \hat{x}_{k+1} = \alpha v_{k+1} \text{ for some } \alpha \in \mathbb{R} \text{ and } v_{k+1}^\top \hat{x}_{k+1} = \tau_{k+1} \\ &\iff \hat{x}_{k+1} = \tau_{k+1} v_{k+1} / \|v_{k+1}\|^2. \end{aligned}$$

$$\iff s \in \mathcal{S}(P_{k+1} V_k, \tau_{[1:k]} - V_k^\top \hat{x}_{k+1}).$$

The equivalence in (4.7a) is shown as follows.

[\Rightarrow] Define $t_- := +(v_{k+1}^\top x_+ - \tau_{k+1}) > 0$, $t_+ := -(v_{k+1}^\top x_- - \tau_{k+1}) > 0$, $\alpha_- := t_-/(t_- + t_+) \in (0, 1)$ and $\alpha_+ := t_+/(t_- + t_+) \in (0, 1)$. Then, $x = \alpha_+ x_+ + \alpha_- x_-$ is appropriate since

$$\text{for } i \in [1:k]: \quad s_i(v_i^\top x - \tau_i) = \alpha_+ s_i[v_i^\top x_+ - \tau_i] + \alpha_- s_i[v_i^\top x_- - \tau_i] > 0,$$

$$v_{k+1}^\top x - \tau_{k+1} = \alpha_+[v_{k+1}^\top x_+ - \tau_{k+1}] + \alpha_-[v_{k+1}^\top x_- - \tau_{k+1}] = \alpha_+ t_- - \alpha_- t_+ = 0 \square$$

[\Leftarrow] Take $x_\pm = x \pm \varepsilon v_{k+1}$ for a sufficiently small $\varepsilon > 0$.¹¹

The equivalence in (4.7b) is shown as follows.

[\Rightarrow] The point x in (4.7a) satisfies $x \in H_{k+1}$, so that $x = P_{k+1} x + \hat{x}_{k+1}$, since $x - P_{k+1} x \in \mathcal{N}(P_{k+1}) \cap H_{k+1} = \{\hat{x}_{k+1}\}$ ¹². Furthermore, by (4.7a), one has for $i \in [1:k]$:

$$0 < s_i(v_i^\top x - \tau_i) = s_i(v_i^\top [P_{k+1} x + \hat{x}_{k+1}] - \tau_i) = s_i([P_{k+1} v_i]^\top x - [\tau_i - v_i^\top \hat{x}_{k+1}]),$$

where we have used $P_{k+1}^\top = P_{k+1}$ (P_{k+1} is an orthogonal projector).

[\Leftarrow] Take $x := P_{k+1} x_0 + \hat{x}_{k+1} \in H_{k+1}$ in (4.7a), where x_0 is the x given by (4.7b). \square

Note that if the equivalences in proposition 4.4 hold, then $s \in \mathcal{S}_k$ ¹³.

The next proposition improves and extends to affine arrangement proposition 4.6 in [17]. Its proof and other properties of algorithm 4.1 are given in [16]. We denote by $\text{vect}\{v_1, \dots, v_k\}$ the vector space spanned by the vectors v_1, \dots, v_k .

PROPOSITION 4.5 (incrementation). *Let $V = [v_1 \dots v_p] \in \mathbb{R}^{n \times p}$.*

- 1) *If $s \in \mathcal{S}_k^c$, then $(s, \pm 1) \in \mathcal{S}_{k+1}^c$. Consequently, $|\mathcal{S}_{k+1}^c| \geq 2|\mathcal{S}_k^c|$.*
- 2) *If $v_{k+1} \notin \text{vect}\{v_1, \dots, v_k\}$, then, $(s, \pm 1) \in \mathcal{S}_{k+1}$ for all $s \in \mathcal{S}_k$, $|\mathcal{S}_{k+1}| = 2|\mathcal{S}_k|$ and $|\mathcal{S}_{k+1}^c| = 2|\mathcal{S}_k^c|$.*
- 3) *If $v_{k+1} \in \text{vect}\{v_1, \dots, v_k\}$, $V_{:, [1:k+1]}$ has no zero column, $H_{k+1} \neq H_i$ for $i \in [1:k]$, and $r_k := \dim \text{vect}\{v_1, \dots, v_k\}$, then $|\mathcal{S}_{k+1}| \geq |\mathcal{S}_k| + 2^{r_k-1}$.*

Proof. 1) If $s \in \mathcal{S}_k^c$, there is no $x \in \mathbb{R}^n$ such that $s_i(v_i^\top x - \tau_i) > 0$ for $i \in [1:k]$. Therefore, there is certainly no $x \in \mathbb{R}^n$ satisfying $s_i(v_i^\top x - \tau_i) > 0$ for $i \in [1:k+1]$, with $s_{k+1} \in \{\pm 1\}$. Therefore, $(s, \pm 1) \in \mathcal{S}_{k+1}^c$. This observation implies that $|\mathcal{S}_{k+1}^c| \geq 2|\mathcal{S}_k^c|$.

2) Let Q be the orthogonal projector on $\text{vect}\{v_1, \dots, v_k\}^\perp$ for the Euclidean scalar product. By assumption, $Q v_{k+1} \neq 0$. Let $s \in \mathcal{S}_k$, so that there is an $x \in \mathbb{R}^n$ such that $s_i(v_i^\top x - \tau_i) > 0$ for $i \in [1:k]$. For any $t \in \mathbb{R}$ and $i \in [1:k]$, the points $x_\pm := x \pm t Q v_{k+1}$

¹¹There is an end-of-proof mark above that is a bug of the `siamart220329` Latex style. Ignore it. It is not the end of the proof.

¹²One has $P_{k+1}(x - P_{k+1} x) = P_{k+1} x - P_{k+1}^2 x = 0$, since $P_{k+1}^2 = P_{k+1}$. Furthermore, $x - P_{k+1} x \in H_{k+1}$ since $x \in H_{k+1}$ and $P_{k+1} x \in \text{vect}\{v_1, \dots, v_k\}$.

¹³This is clear if $(s, +1)$ and $(s, -1) \in \mathcal{S}_{k+1}$. This is also the case if $s \in \mathcal{S}(P_{k+1} V_{:, [1:k]}, \tau_{[1:k]} - V_{:, [1:k]}^\top \hat{x}_{k+1})$, since then there is an $x \in \mathbb{R}^n$ such that

$$s \cdot (V_{:, [1:k]}^\top P_{k+1} x - [\tau_{[1:k]} - V_{:, [1:k]}^\top \hat{x}_{k+1}]) > 0$$

or

$$s \cdot (V_{:, [1:k]}^\top [P_{k+1} x + \hat{x}_{k+1}] - \tau_{[1:k]}) > 0,$$

which implies that $s \in \mathcal{S}_k$.

verify $s_i(v_i^\top x_\pm - \tau_i) = s_i(v_i^\top x - \tau_i) > 0$ (because $v_i^\top Q v_{k+1} = 0$). In addition, for $t > 0$ sufficiently large, one has $\pm(v_{k+1}^\top x_\pm - \tau_{k+1}) = \pm(v_{k+1}^\top x - \tau_{k+1}) + t\|Q v_{k+1}\|^2 > 0$ (because $Q^2 = Q$ and $Q^\top = Q$). We have shown that both $(s, +1)$ and $(s, -1)$ are in \mathcal{S}_{k+1} . Therefore, $|\mathcal{S}_{k+1}| \geq 2|\mathcal{S}_k|$.

Now, $|\mathcal{S}_k| + |\mathcal{S}_k^c| = 2^k$, $|\mathcal{S}_{k+1}| + |\mathcal{S}_{k+1}^c| = 2^{k+1}$ and $|\mathcal{S}_{k+1}^c| \geq 2|\mathcal{S}_k^c|$ by point 1. Therefore, one must have $|\mathcal{S}_{k+1}| = 2|\mathcal{S}_k|$ and $|\mathcal{S}_{k+1}^c| = 2|\mathcal{S}_k^c|$ ¹⁴.

3) One has $\text{null}(P_{k+1}) = 1$ (since $\mathcal{N}(P_{k+1}) = \mathbb{R}v_{k+1}$) and $\text{rank}(V_{:, [1:k]}) = r_k$ (by definition). Then, $\text{rank}(P_{k+1} V_{:, [1:k]}) \geq r_k - 1$ ¹⁵.

To apply (3.10) to the arrangement $\mathcal{A}(P_{k+1} V_{:, [1:k]}, \tau_{[1:k]} - V_{:, [1:k]}^\top \hat{x}_{k+1})$, one must show that $v_i^\top \hat{x}_{k+1} \neq \tau_i$ when $i \in [1:k]$ and $P_{k+1} v_i = 0$ (i.e., v_{k+1} and v_i are colinear or H_{k+1} and H_i are parallel by proposition 3.2(1)). This is indeed the case, since $v_i^\top \hat{x}_{k+1} = \tau_i$ would imply that $H_{k+1} = H_i$ (because then \hat{x}_{k+1} would belong to both H_i and H_{k+1} , which are parallel), in contradiction with the assumption. By (3.10), $|\mathcal{S}(P_{k+1} V_{:, [1:k]}, \tau_{[1:k]} - V_{:, [1:k]}^\top \hat{x}_{k+1})| \geq 2^{r_k-1}$.

By proposition 4.4, there are at least 2^{r_k-1} sign vectors in \mathcal{S}_k with two children. Since any $s \in \mathcal{S}_k$ has at least one child when $v_{k+1} \neq 0$, one gets $|\mathcal{S}_{k+1}| \geq |\mathcal{S}_k| + 2^{r_k-1}$. \square

As a corollary of the previous proposition, one gets the following improvement of the lower bound of $|\mathcal{S}(V, \tau)|$ given by (3.10).

COROLLARY 4.6 (lower bound of $|\mathcal{S}(V, \tau)|$). *For a matrix V of rank r without zero column and a vector τ such that all the hyperplanes H_i are different, one has*

$$(4.8) \quad 2^r + 2^{r-1}(p - r) \leq |\mathcal{S}(V, \tau)|.$$

Proof. By a possible change of column order, which only affects the chamber numbering, not the cardinality of $\mathcal{S}(V, \tau)$, one can assume that the first r columns of V are linearly independent. Then, by proposition 4.5(2), $|\mathcal{S}_r| = 2^r$. Next, for $k \in [r : p-1]$, $\dim \text{vect}\{v_1, \dots, v_k\} = r$, so that proposition 4.5(3) implies that $|\mathcal{S}_{k+1}| \geq |\mathcal{S}_k| + 2^{r-1}$. By induction, one gets (4.8). \square

4.2. Preventing some computations. The main computation cost of algorithm 4.1 comes from solving the LOPs (4.5) at some inner nodes. This section describes three ways of bypassing LOPs. They are adapted from [17], where only linear arrangements are considered, and are identified by the letters A, B and C (the letters appearing in the section titles), which will also be used to label more efficient variants of algorithm 4.1. These variants significantly speed up the algorithm by reducing the numbers of LOPs to solve (see section 7).

4.2.1. A - Rank of the arrangement. Instead of starting the \mathcal{S} -tree with the two nodes of $\mathcal{S}_1 = \{+1, -1\}$, like in algorithm 4.1, one can start it with the 2^r nodes of $\mathcal{S}_r = \{\pm 1\}^r$, by considering first a selection of $r := \text{rank}(V)$ linearly independent vectors whose \mathcal{S} -tree is easy to construct without having to solve any LOP. Here are the details.

Numerically, r linearly independent vectors can be found by a QR factorization of V :

$$VP = QR,$$

¹⁴One has $2^{k+1} = |\mathcal{S}_{k+1}| + |\mathcal{S}_{k+1}^c| \geq 2|\mathcal{S}_k| + 2|\mathcal{S}_k^c| = 2^{k+1}$. Hence the previous inequality is actually an equality. It follows that $|\mathcal{S}_{k+1}| = 2|\mathcal{S}_k|$ and $|\mathcal{S}_{k+1}^c| = 2|\mathcal{S}_k^c|$.

¹⁵For linear maps A and B between appropriate linear spaces, one has $\text{rank}(BA) + \text{null}(B) \geq \text{rank}(A)$, with equality if and only if $\mathcal{N}(B) \subseteq \mathcal{R}(A)$. This can be obtained by considering the linear map $L : \mathcal{R}(A) \rightarrow \mathcal{R}(B)$ defined at $y \in \mathcal{R}(A)$ by $Ly = By$ and using the rank-nullity theorem on L .

with $P \in \{0, 1\}^{p \times p}$ is a permutation matrix, $Q \in \mathbb{R}^{n \times n}$ is orthogonal and $R \in \mathbb{R}^{n \times p}$ is upper triangular with $R_{[r+1:n],:} = 0$. To simplify the presentation, let us assume that P is the identity matrix, in which case the first r vectors v_1, \dots, v_r (or columns of V) are linearly independent, and let us note $V_r := V_{:, [1:r]}$, $Q_r := Q_{:, [1:r]}$ and $R_r := R_{[1:r], [1:r]}$. By proposition 4.5(2),

$$\mathcal{S}_r = \{\pm 1\}^r.$$

To launch the recursive algorithm 4.2, one still need to compute a witness point x_s associated with any $s \in \mathcal{S}_r$. For this purpose, one computes a point $\hat{x} \in \cap_{i=1}^r H_i$, hence verifying $V_r^\top \hat{x} = \tau_{[1:r]}$, by $\hat{x} := V_r(V_r^\top V_r)^{-1} \tau_{[1:r]}$. Next, for any $s \in \{\pm 1\}^r$, one computes $d_s := Q_r R_r^{-\top} s \in \mathbb{R}^n$. Let us show that $x_s := \hat{x} + d_s$ is a witness point of the considered s . One has

$$V_r^\top x_s - \tau_{[1:r]} = V_r^\top [V_r(V_r^\top V_r)^{-1} \tau_{[1:r]} + Q_r R_r^{-\top} s] - \tau_{[1:r]} = (Q_r R_r)^\top Q_r R_r^{-\top} s = s.$$

1037 Therefore, $s \cdot (V_r^\top x_s - \tau_{[1:r]}) = s \cdot s = e > 0$, as desired.

1038 **4.2.2. B - Handling of a hyperplane proximity.** In the description of algo-
1039 rithm 4.2, it is shown why a witness point x of a sign vector $s \in \mathcal{S}_k$ that belongs to
1040 H_{k+1} , i.e., $v_{k+1}^\top x = \tau_{k+1}$, allows the algorithm to certify that both $(s, +1)$ and $(s, -1)$
1041 are in \mathcal{S}_{k+1} , without having to solve a LOP. We show with the next proposition that
1042 this is still true when x is near H_{k+1} , in the sense (4.9). Note that this proxim-
1043 ity to H_{k+1} is measured by strict inequalities, which is more stable with respect to
1044 numerical perturbations than an equality.

1045 **PROPOSITION 4.7** (two children without LOP). *Let $s \in \mathcal{S}_k$ and $x \in \mathbb{R}^n$ verifying*
1046 *$s \cdot (V_{:, [1:k]}^\top x - \tau_{[1:k]}) > 0$. Suppose that $v_{k+1} \neq 0$ and*

$$1047 \quad (4.9) \quad \underbrace{\max_{s_i v_i^\top v_{k+1} > 0} \frac{\tau_i - v_i^\top x}{v_i^\top v_{k+1}}}_{=: t_{\min}} < \underbrace{\frac{\tau_{k+1} - v_{k+1}^\top x}{\|v_{k+1}\|^2}}_{=: t_0} < \underbrace{\min_{s_i v_i^\top v_{k+1} < 0} \frac{\tau_i - v_i^\top x}{v_i^\top v_{k+1}}}_{=: t_{\max}}.$$

1048 *Then, for $t_- \in (t_{\min}, t_0)$, $x_- := x + t_- v_{k+1}$ is a witness point of $(s, -1)$ and, for*
1049 *$t_+ \in (t_0, t_{\max})$, $x_+ := x + t_+ v_{k+1}$ is a witness point of $(s, +1)$.*

1050 *Proof.* Note first that, in (4.9), the arguments of the maximum are negative and
1051 the arguments of the minimum are positive. Therefore, both inequalities are verified
1052 if $v_{k+1}^\top x = \tau_{k+1}$ ($t_0 = 0$), that is, when $x \in H_{k+1}$. One has, for $i \in [1:k]$,

$$1053 \quad s_i(v_i^\top(x + t v_{k+1}) - \tau_i) > 0 \iff \begin{cases} t < \frac{s_i(\tau_i - v_i^\top x)}{s_i v_i^\top v_{k+1}} & \text{if } s_i v_i^\top v_{k+1} < 0 \\ t \in \mathbb{R} & \text{if } s_i v_i^\top v_{k+1} = 0 \\ t > \frac{s_i(\tau_i - v_i^\top x)}{s_i v_i^\top v_{k+1}} & \text{if } s_i v_i^\top v_{k+1} > 0. \end{cases}$$

1054 Since the conditions imposed on t in the right-hand side of the equivalence above are
1055 satisfied by any $t \in (t_{\min}, t_{\max})$, it follows that x_\pm are witness points of s . One has

$$1056 \quad t > t_0 := \frac{\tau_{k+1} - v_{k+1}^\top x}{\|v_{k+1}\|^2} \iff v_{k+1}^\top(x + t v_{k+1}) - \tau_{k+1} > 0,$$

$$1057 \quad t < t_0 := \frac{\tau_{k+1} - v_{k+1}^\top x}{\|v_{k+1}\|^2} \iff v_{k+1}^\top(x + t v_{k+1}) - \tau_{k+1} < 0.$$

1059 Since t_+ (resp. t_-) verifies the condition in the left-hand side of the first (resp. second)
1060 equivalence above, it follows that x_+ (resp. x_-) is a witness point of $(s, +1)$ (resp.
1061 $(s, -1)$). \square

4.2.3. C - Choosing the order of the vectors. Every inner node of the \mathcal{S} -tree has one or two children and this number is sometimes detected in algorithm 4.2 by solving a LOP, which is a time consuming operation. Therefore, a way of decreasing the computation time is to reduce the number of inner nodes of the \mathcal{S} -tree. This property can be obtained by choosing wisely the order in which the pairs (v_i, τ_i) , or hyperplanes H_i , are taken into account when constructing the branches of the \mathcal{S} -tree (this order can be different from one branch to another), with the goal of placing the nodes with a single child close to the root of the tree. This strategy has been investigated in [17, § 5.2.4.C] and we adapt the heuristic to the present context of affine arrangements.

Denote by $T_s := \{i_1^s, \dots, i_k^s\}$ the set of the indices of the hyperplanes selected to reach node $s \in \mathcal{S}_k$ (T_s depends on s). At this node, the algorithm must choose the next hyperplane to consider, whose index is among the index set $T_s^c := [1:p] \setminus T_s$. With the goal of preventing, as much as possible, the node s from having two children, a natural idea is to ignore the indices of T_s^c , for which proposition 4.7 ensures two children. In the remaining index set, denote it by T_s^b , the chosen index is the one maximizing the quantity $|v_i^\top x - \tau_i| / \|[v_i; \tau_i]\|$ for $i \in T_s^b$ (x is the witness point associated by the algorithm with the current node s), since the larger this quantity is, the further x is from the chosen hyperplane, which should increase the chances that s will have only one child.

5. Chamber computation - Dual approaches. The enumeration of the chambers of an arrangement $\mathcal{A}(V, \tau)$ can be tackled by an approach different from those presented in section 4 (algorithm 4.1 and its improvements A, B and C), sometimes (or always) replacing optimization phases by algebra techniques. More specifically, we say that an algorithm has a *dual aspect* when it uses the concept of stem vector (definition 3.12) by means of proposition 3.16. Such a dual approach was introduced in [17, §§ 5.2.2-5.2.3] for linear arrangements.

Section 5.1 deals with algorithms computing $\mathcal{S}(V, \tau)$ that assume the availability of the full stem vector set $\mathfrak{S}(V, \tau)$ and do not use optimization. A method for computing $\mathfrak{S}(V, \tau)$ is presented in section 5.1.1. Section 5.1.2 describes a crude dual approach for computing $\mathcal{S}(V, \tau)$ that is only efficient for small arrangements. The algorithm proposed in section 5.1.3 has the structure of algorithm 4.1, in the sense that it constructs the \mathcal{S} -tree, but its optimization phases are replaced by the duality technique mentioned above.

In section 5.2, we present a way of obtaining a circuit, hence stem vectors, from a linear optimization problem that is associated with an infeasible sign vector (proposition 5.6). This leads to an algorithm having primal and dual aspects.

This section also assumes (4.1), that is, V has no zero column.

5.1. Algorithms using all the stem vectors. Proposition 3.16 establishes a link between the *infeasible* sign vectors, those in $\mathcal{S}(V, \tau)^c$, and the stem vectors. This section presents two algorithms that start with the computation of the complete stem vector set $\mathfrak{S}(V, \tau)$ (section 5.1.1). The first one uses these stem vectors to compute $\mathcal{S}(V, \tau)^c$, from which the feasible sign vector set $\mathcal{S}(V, \tau) = \{\pm 1\}^p \setminus \mathcal{S}(V, \tau)^c$ can be deduced (section 5.1.2). The second one computes directly $\mathcal{S}(V, \tau)$ like in the \mathcal{S} -tree primal algorithm of section 4.1, but without solving linear optimization problems and without computing witness points (section 5.1.3).

5.1.1. Stem vector computation. Let us start with the presentation of a plain algorithm that computes the disjoint sets $\mathfrak{S}_s(V, \tau)$ and $\mathfrak{S}_a(V, \tau)$ of the symmetric

and asymmetric stem vectors; recall that the set of all stem vectors is $\mathfrak{S}(V, \tau) = \mathfrak{S}_s(V, \tau) \cup \mathfrak{S}_a(V, \tau)$. This algorithm is based on the detection of the circuits of V and remark 3.13(3). It is rudimentary (the one used in [28] is valid for an arbitrary matroid and yields an interesting complexity property, but, in our experience with vector matroids, it is much less efficient than the method used in algorithm 5.1; see also [38] and the pieces of software mentioned in the introduction). The algorithm can be significantly improved in particular cases, see [39].

ALGORITHM 5.1 ($\text{stem_vectors}(\mathfrak{S}_s, \mathfrak{S}_a)$). // stem vector calculation

1. $\mathfrak{S}_s = \emptyset$ and $\mathfrak{S}_a = \emptyset$
 2. **for** $i \in [1:p]$ **do**
 3. $\text{stem_vectors_rec}(\mathfrak{S}_s, \mathfrak{S}_a, \{i\})$
 4. **endfor**
 5. Remove duplicate stem vectors in \mathfrak{S}_s and \mathfrak{S}_a
-

ALGORITHM 5.2 ($\text{stem_vectors_rec}(\mathfrak{S}_s, \mathfrak{S}_a, I_0)$).

1. **for** $i \in [\max(I_0) + 1:p]$ **do**
 2. $I := I_0 \cup \{i\}$
 3. **if** $(\mathcal{N}(V, I) \neq \{0\})$
 4. Let $\eta_I \in \mathcal{N}(V, I) \setminus \{0\}$ and $J := \{i \in I : \eta_i \neq 0\}$ // $J \in \mathcal{C}(V)$
 5. **if** $(\tau_J^T \eta_J = 0)$
 6. $\mathfrak{S}_s := \mathfrak{S}_s \cup \{\text{sgn}(\eta_J) \cup \{-\text{sgn}(\eta_J)\}\}$
 7. **else**
 8. $\mathfrak{S}_a := \mathfrak{S}_a \cup \{\text{sgn}(\tau_J^T \eta_J) \text{sgn}(\eta_J)\}$
 9. **endif**
 10. **else**
 11. $\text{stem_vectors_rec}(\mathfrak{S}_s, \mathfrak{S}_a, I)$
 12. **endif**
 13. **endfor**
-

Here are some explanations and observations on algorithm 5.1. Unless otherwise stated, the line numbers refer to algorithm 5.2.

- The function $\text{stem_vectors_rec}(\mathfrak{S}_s, \mathfrak{S}_a, I_0)$ adds to \mathfrak{S}_s and/or \mathfrak{S}_a stem vectors σ such that $\mathfrak{J}(\sigma)$ is contained in the set formed of I_0 and indices larger than $\max(I_0)$.
- On entry in algorithm 5.2, $V_{:,I_0}$ is assumed to be injective: this is the case in line 3 of algorithm 5.1 (recall the assumption (4.1)) and in line 11 of algorithm 5.2 (since there $\mathcal{N}(V, I) = \{0\}$). Therefore, in line 4 of algorithm 5.2, $\text{null}(V, I) = 1$ and J is a circuit of V (lemma 3.11).
- The algorithm does not explore all the subsets of $[1:p]$ since, once a circuit $J \subseteq I$ has been found in line 4, an index set $I' \supseteq I$ satisfying $\text{null}(V, I') = 1$ contains no circuit different from J (lemma 3.11). This explains why there is no recursive call to stem_vectors_rec when $\mathcal{N}(V, I) \neq \{0\}$ (lines 4.9).
- In line 4, η_I is obtained by a null space computation, so that algorithm 5.2 is sensitive to rounding errors. One can use exact arithmetic linear algebra to compute the set of sign vectors when the data is rational or integer, at the expense of slower computation.
- Line 6 corresponds to remark 3.13(3.a) and line 7 to remark 3.13(3.b). These lines are symbolically written, since, in addition to $\pm \text{sgn} \eta_J$ one also has to store J . In practice, one can only store half of the symmetric stem vectors in \mathfrak{S}_s and obtain

the full set, if needed, by gathering the stem vectors of \mathfrak{S}_s and $-\mathfrak{S}_s$.

- The loop 2.4 of algorithm 5.1 may find several times the same stem vector. This is the case, for instance, if $V = [e_1, e_2, e_2]$: the circuit $J = \{2, 3\}$ is found twice by the loop of algorithm 5.1 (once with $i = 1$ and again with $i = 2$), as well as the associated stem vectors. This justifies the final elimination of duplicates in line 5 of algorithm 5.1.

5.1.2. Crude dual algorithm. The algorithm described in this section, algorithm 5.3, is a “crude” way of obtaining the sign vector set $\mathcal{S}(V, \tau)$ from the stem vector set $\mathfrak{S}(V, \tau)$. It uses the characterization of proposition 3.16. For each stem vector $\sigma \in \mathfrak{S}(V, \tau)$ with associated circuit $J = \mathfrak{J}(\sigma) \subseteq [1:p]$, the algorithm generates all the infeasible sign vectors $s \in \mathcal{S}(V, \tau)^c$ satisfying $s_J = \sigma$ and $s_{J^c} \in \{\pm 1\}^{J^c}$. This generation is made by the function `stem_to_infeas_sign_vectors` in a straightforward manner (the precise computation is not detailed). Once $\mathcal{S}(V, \tau)^c$ is computed, $\mathcal{S}(V, \tau)$ is obtained by $\{\pm 1\}^p \setminus \mathcal{S}(V, \tau)^c$.

This `stem_to_infeas_sign_vectors` function can produce duplicated sign vectors, which justifies the cleaning operation in line 5 (this one could be done simultaneously with the union in line 4). For example, with $V = [1, 1, 1]$ and $\tau^\top = [1, 0, 1]$, the stem vectors $(1, -1) \in \{\pm 1\}^{\{1,2\}}$ and $(-1, 1) \in \{\pm 1\}^{\{2,3\}}$ produce the same infeasible sign vector $(1, -1, 1)$.

ALGORITHM 5.3 (`crude_dual(S)`). // crude dual algorithm

1. $\text{Sc} = \emptyset$ // initialization of $\mathcal{S}(V, \tau)^c$
 2. `stem_vectors`($\mathfrak{S}_s, \mathfrak{S}_a$) // algorithm 5.1
 3. **for** $\sigma \in \mathfrak{S}_s \cup \mathfrak{S}_a$ **do**
 4. $\text{Sc} = \text{Sc} \cup \text{stem_to_infeas_sign_vectors}(\sigma, p)$
 5. Remove duplicates in Sc
 6. **endfor**
 7. $\mathcal{S} := \{\pm 1\}^p \setminus \text{Sc}$
-

Despite its simplicity, algorithm 5.3 is usually not very attractive. Indeed, each stem vector $\sigma \in \{\pm 1\}^J$ produces the exponential number $2^{|J^c|}$ of sign vectors s with $s_{J^c} \in \{\pm 1\}^{J^c}$. As a result, for large p , the algorithm handles a large amount of data, which can take much computing time.

5.1.3. Dual \mathcal{S} -tree algorithm. Another possibility is to use the \mathcal{S} -tree structure introduced in section 4.1. Here is the main idea. Assume that a sign vector s in \mathcal{S}_k has been computed (the set \mathcal{S}_k is defined by (4.2)). Then, algorithm 4.2 determines whether (s, s_{k+1}) belongs to \mathcal{S}_{k+1} , for $s_{k+1} \in \{\pm 1\}$. As explained in the description of algorithm 4.2, the belonging of (s, s_{k+1}) to \mathcal{S}_{k+1} can be revealed by solving a linear optimization problem. Algorithm 5.4–5.5 below does this differently. It uses the computed stem vector set $\mathfrak{S}(V, \tau)$ and is based on the fact that¹⁶

$$\mathfrak{S}(V_{:, [1:k]}, \tau_{[1:k]}) = \{\sigma \in \mathfrak{S}(V, \tau) : \mathfrak{J}(\sigma) \subseteq [1:k]\}.$$

Therefore, according to proposition 3.16, to determine whether (s, s_{k+1}) is in \mathcal{S}_{k+1} , it suffices to see whether it covers a stem vector $\sigma \in \mathfrak{S}(V, \tau)$ such that $\mathfrak{J}(\sigma) \subseteq [1:k+1]$.

¹⁶[\subseteq] Let $\sigma \in \mathfrak{S}(V_{:, [1:k]}, \tau_{[1:k]})$ and $J = \mathfrak{J}(\sigma)$. Then, $J \subseteq [1:k]$ and $J \in \mathcal{C}(V_{:, [1:k]}) \subseteq \mathcal{C}(V)$. In addition, the second condition in (3.13) is identical for $V_{:, [1:k]}$ and V , so that $\sigma \in \mathfrak{S}(V, \tau)$.

[\supseteq] Let $\sigma \in \mathfrak{S}(V, \tau)$ with $J := \mathfrak{J}(\sigma) \subseteq [1:k]$. It follows from $J \in \mathcal{C}(V)$ and $J \subseteq [1:k]$ that $J \in \mathcal{C}(V_{:, [1:k]})$. In addition, the second condition in (3.13) is identical for $V_{:, [1:k]}$ and V , so that $\sigma \in \mathfrak{S}(V_{:, [1:k]}, \tau_{[1:k]})$.

If so $(s, s_{k+1}) \in \mathcal{S}_{k+1}^c$ and any $\tilde{s} \in \{\pm 1\}^p$, extending (s, s_{k+1}) by ± 1 , will be in $\mathcal{S}(V, \tau)^c$, so that the \mathcal{S} -tree may be pruned at (s, s_{k+1}) . Otherwise $(s, s_{k+1}) \in \mathcal{S}_{k+1}$ and the recursive exploration of the \mathcal{S} -tree is pursued below (s, s_{k+1}) .

```

ALGORITHM 5.4 (d_stree).      // dual  $\mathcal{S}$ -tree algorithm
1. stem_vectors( $\mathfrak{S}_s, \mathfrak{S}_a$ )      // get the stem vectors by algorithm 5.1
2. d_stree_rec( $\emptyset, \mathfrak{S}_s \cup \mathfrak{S}_a$ )

```

```

ALGORITHM 5.5 (d_stree_rec( $s \in \{\pm 1\}^k, \mathfrak{S}$ )).
1. if ( $k = p$ )
2.   Output  $s$  and return      //  $s$  is a leaf of the  $\mathcal{S}$ -tree; end the recursion
3. endif
4. if ( $[s; +1]$  covers a stem vector of  $\mathfrak{S}$ )
5.   d_stree_rec( $[s; -1], \mathfrak{S}$ )
6. else
7.   d_stree_rec( $[s; +1], \mathfrak{S}$ )
8.   if ( $[s; -1]$  does not contain a stem vector of  $\mathfrak{S}$ )
9.     d_stree_rec( $[s; -1], \mathfrak{S}$ )
10.  endif
11. endif

```

Here are some explanations and observations on the recursive algorithm 5.5.

- If the test in line 4 holds, proposition 3.16 tells us that $[s; +1]$ is an infeasible sign vector for the arrangement $\mathcal{A}(V_{:, [1:k+1]}, \tau_{[1:k+1]})$, so that, for any $\tilde{s} \in \{\pm 1\}^{p-k-1}$, the sign vectors $[s; +1; \tilde{s}]$ is also infeasible for the arrangement $\mathcal{A}(V, \tau)$. This has two consequences:
 - there is no point in exploring the descendants of $[s; +1]$ in the \mathcal{S} -tree, which explains why there is no recursive call to `d_stree_rec($[s; +1], \mathfrak{S}$)` in that case and
 - $[s; -1]$ is necessarily a feasible sign vector for the arrangement $\mathcal{A}(V_{:, [1:k+1]}, \tau_{[1:k+1]})$ since each node of the \mathcal{S} -tree has at least one child (proposition 4.3), which explains why there is a call to `d_stree_rec($[s; -1], \mathfrak{S}$)` in line 5.
- Line 7 is justified since at that point, $[s; +1]$ is a feasible sign vector for the arrangement $\mathcal{A}(V_{:, [1:k+1]}, \tau_{[1:k+1]})$.
- Line 9 is justified since at that point, $[s; -1]$ is a feasible sign vector for the arrangement $\mathcal{A}(V_{:, [1:k+1]}, \tau_{[1:k+1]})$.
- The algorithm does not use witness points, unlike the primal \mathcal{S} -tree algorithm 4.1–4.2.

Let us emphasize the fact that algorithm 5.4 does not require to solve linear optimization problems. While this might look enticing, since the LOPs are the main cost of the primal \mathcal{S} -tree algorithm 4.1, one must be aware of two facts. First, the computation of all the circuits of V by algorithm 5.1 can be time consuming, since it requires the exploration of a tree, whose nodes at level k may have up to $p - k$ descendants. Second, determining whether a sign vector covers a stem vector can also take much computing time when the number of stem vectors is large, which is usually the case when p is large (see remark 3.13(6)).

5.2. Algorithms using some stem vectors. Instead of computing the stem vectors exhaustively like algorithm 5.1 does, which is generally a time consuming task, one can get a few stem vectors from the optimal dual variables of some linear

optimization problems (LOPs) encountered in algorithm 4.1, those that are associated with an infeasible sign vector. This device is described in section 5.2.1. Then, one can design a kind of primal-dual algorithm for computing $\mathcal{S}(V, \tau)$. This one builds the \mathcal{S} -tree, but, in order to save running time, it makes use of the stem vectors collected during its construction to prune some unfruitful branches of the \mathcal{S} -tree, which avoids having to solve some LOPs. This algorithm is presented in section 5.2.2.

5.2.1. Getting stem vectors from linear optimization. In line 11 of algorithm 4.2, one has to decide whether $(s, -s_{k+1})$ is in \mathcal{S}_{k+1} and it is suggested, after the description of the algorithm, to determine this belonging by solving the linear optimization problem (LOP) (4.5). The Lagrangian dual of this problem [5, 21, 17] reads

$$(5.1) \quad \begin{aligned} \max_{(\lambda, \mu) \in \mathbb{R}^{k+1} \times \mathbb{R}} \quad & \sum_{i \in [1:k]} \lambda_i s_i \tau_i - \lambda_{k+1} s_{k+1} \tau_{k+1} - \mu \\ \text{s.t.} \quad & \lambda \geq 0 \\ & \mu \geq 0 \\ & \sum_{i \in [1:k]} \lambda_i s_i v_i = \lambda_{k+1} s_{k+1} v_{k+1} \\ & \sum_{i \in [1:k+1]} \lambda_i + \mu = 1, \end{aligned}$$

where $\lambda \in \mathbb{R}^{k+1}$ is the dual variable associated with the first $k+1$ constraints of (4.5) and μ the dual variable associated with its last constraint.

The next proposition gives conditions ensuring that a circuit of V can be obtained from a specific solution to the dual problem (5.1) when $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$. We denote by $\text{val}(4.5)$ (resp. $\text{val}(5.1)$) the optimal value of the primal (resp. dual) optimization problem (4.5) (resp. (5.1)). By strong duality in linear optimization [5, 21] and the fact that problem (4.5) has a solution, one has $\text{val}(4.5) = \text{val}(5.1)$.

PROPOSITION 5.6 (matroid circuit detection from optimization).

- 1) Problem (5.1) has a solution, say $(\lambda, \mu) \in \mathbb{R}_+^{k+1} \times \mathbb{R}_+$.
- 2) If $s \in \mathcal{S}_k$ and $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$, then $\text{val}(4.5) \geq 0$, $\lambda_{k+1} > 0$ and $\mu = 0$.
- 3) If, in addition, (λ, μ) is an extreme point of the feasible set of (5.1), then
 - $J := \{i \in [1:k+1] : \lambda_i > 0\} \in \mathcal{C}(V)$,
 - if $\text{val}(4.5) = 0$, $\pm(s, -s_{k+1})_J$ are the two stem vectors associated with J ,
 - if $\text{val}(4.5) > 0$, $(s, -s_{k+1})_J$ is the unique stem vector associated with J .

Proof. 1) By strong duality in linear optimization [42, 5, 21], the fact that the primal problem (4.5) has a solution implies that the dual problem (5.1) has also a solution, say (λ, μ) .

2) Suppose that $s \in \mathcal{S}_k$ and that $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$. Let (x, α) be a solution to (4.5) ($\alpha = \text{val}(4.5)$ is uniquely determined). Let us show that

$$(5.2a) \quad \lambda_{k+1} > 0 \quad \text{and} \quad \mu = 0.$$

The optimal multiplier μ is associated with the constraint $\alpha \geq -1$ of the optimization problem (4.5), which is inactive ($\alpha \geq 0$ when $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$), so that it vanishes. We show that $\lambda_{k+1} > 0$ by contradiction, assuming that $\lambda_{k+1} = 0$. Then, strong duality would imply that $0 \leq \alpha = \text{val}(4.5) = \text{val}(5.1) = \sum_{i \in [1:k]} \lambda_i s_i \tau_i$, while the third constraint of (5.1) would read $\sum_{i \in [1:k]} \lambda_i s_i v_i = 0$. Then, Motzkin's alternative (2.2) would imply that there is no $x \in \mathbb{R}^n$ such that $s_i(v_i^\top x - \tau_i) > 0$, for $i \in [1:k]$, in contradiction with the assumption $s \in \mathcal{S}_k$.

3) Let $I := \{i \in [1:k], \lambda_i > 0\}$. By assumption and $\mu = 0$, $(\lambda, 0)$ is an extreme

point of the feasible set of problem (5.1), which implies that the vectors $[9, 42, 21]^{17}$

$$(5.2b) \quad \left\{ \begin{pmatrix} s_i v_i \\ 1 \end{pmatrix}_{i \in I}, \begin{pmatrix} -s_{k+1} v_{k+1} \\ 1 \end{pmatrix} \right\} \text{ are linearly independent,}$$

where we used the fact that $\lambda_{k+1} > 0$ and $\mu = 0$ by (5.2a).

One can deduce from this property that the vectors

$$(5.2c) \quad \{s_i v_i\}_{i \in I} \text{ are linearly independent.}$$

Suppose indeed that $\sum_{i \in I} \alpha_i s_i v_i = 0$ for some real numbers $(\alpha_i)_{i \in I}$. It suffices to show that these numbers vanish and we do so in two steps.

- We first show by contradiction that $\sum_{i \in I} \alpha_i = 0$. If this were not the case, one could find $t \in \mathbb{R}$ such that $\sum_{i \in I} (\lambda_i + t\alpha_i) + \lambda_{k+1} = 0$. Now, using the third constraint of problem (5.1), we would have that $\eta := ((\lambda_i + t\alpha_i)_{i \in I}, \lambda_{k+1})$ is in the null space of the nonsingular matrix whose columns are the vectors in (5.2b), which would imply that $\eta = 0$, in contradiction with $\lambda_{k+1} > 0$ imposed by (5.2a).
- Using $\sum_{i \in I} \alpha_i = 0$ and $\sum_{i \in I} \alpha_i s_i v_i = 0$, we have that the vector $((\alpha_i)_{i \in I}, 0)$ is in the null space of the nonsingular matrix whose columns are the vectors in (5.2b). Hence all the α_i 's vanish¹⁸.

Now, set $J := \{i \in [1 : k+1] : \lambda_i > 0\}$, which is $I \cup \{k+1\}$ by the definition of I and (5.2a), and introduce the diagonal matrix $D \in \mathbb{R}^{J \times J}$ defined by $D_{i,i} = s_i$ if $i \in I$ and $D_{k+1,k+1} = -s_{k+1}$. Using (5.2c), we see that¹⁹

$$\text{null}(V_{:,J}D) = 1.$$

By the third constraint of (5.1), we have that $\lambda_J \in \mathcal{N}(V_{:,J}D) \setminus \{0\}$. Since $\lambda_J > 0$, proposition 3.11 tells us that J is a circuit of $V_{:,J}D$, hence a circuit of V .

Since $\eta := D\lambda_J \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ is such that $\tau_J^T \eta = \text{val}(4.5)$, we see that the number of stem vectors associated with J is governed by $\text{val}(4.5)$, as described in remark 3.13(3). In addition, $\text{sgn}(\eta) = (s, -s_{k+1})_J$, because $\lambda_J > 0$, showing that $(s, -s_{k+1})_J$ is a stem vector. \square

A solution to problem (5.1) that is an extreme point of its feasible set can be obtained by the dual-simplex algorithm. Note that, since $\lambda_{k+1} > 0$, $k+1$ always belongs to the selected circuit J of V .

5.2.2. Primal-dual \mathcal{S} -tree algorithm. Proposition 5.6(3) shows how circuits and their associated stem vectors can be obtained when the \mathcal{S} -tree primal algorithm 4.1 solves a LOP (4.5) with an appropriate solver and observes that the sign vector $(s, -s_{k+1})$ is infeasible. Now, with the *partial* list of stem vectors so computed, which grows throughout the iterations, the algorithm can detect *some* infeasible sign vectors by using proposition 3.16, like in the crude dual algorithm 5.3 or in the \mathcal{S} -tree

¹⁷We use the fact that the i th columns of the matrix of the equality constraints of (5.1), namely

$$\begin{pmatrix} s_1 v_1 & \cdots & s_k v_k & -s_{k+1} v_{k+1} & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix},$$

corresponding to $\lambda_i > 0$ are linearly independent (recall that $\lambda_{k+1} > 0$ and $\mu = 0$).

¹⁸The vector $((\alpha_i)_{i \in I}, 0)$ is in the null space of the nonsingular matrix whose columns are the vectors in (5.2b).

¹⁹Let $(\alpha, \alpha_{k+1}) \in \mathbb{R}^I \times \mathbb{R}$ be in $\mathcal{N}(V_{:,J}D) \setminus \{0\}$: $\sum_{i \in I} \alpha_i s_i v_i - \alpha_{k+1} s_{k+1} v_{k+1} = 0$. Then, $\alpha_{k+1} \neq 0$ by (5.2c) and $\alpha \neq 0$. In addition, α/α_{k+1} is uniquely determined, again by (5.2c). Hence $\text{null}(V_{:,J}) = 1$.

dual algorithm 5.4, but without having to solve a LOP. In practice, this technique saves much computing time. Here is this *primal-dual \mathcal{S} -tree algorithm*, based on the just presented idea, which has many similarities with the primal \mathcal{S} -tree algorithm 4.1.

```

1326  ALGORITHM 5.7 (pd_stree).      // primal-dual  $\mathcal{S}$ -tree algorithm
1327      1.  $\mathfrak{S} = \emptyset$ 
1328      2. pd_stree_rec(+1,  $x_+$ ,  $\mathfrak{S}$ )      //  $x_+$  given by (4.3)1
1329      3. pd_stree_rec(-1,  $x_-$ ,  $\mathfrak{S}$ )      //  $x_-$  given by (4.3)2
1330  
```

```

1333  ALGORITHM 5.8 (pd_stree_rec( $s \in \{\pm 1\}^k$ ,  $x \in \mathbb{R}^n$ ,  $\mathfrak{S}$ )).
1334      1. if ( $k = p$ )
1335      2.   Output  $s$  and return      //  $s$  is a leaf of the  $\mathcal{S}$ -tree; end the recursion
1336      3. endif
1337      4. if ( $v_{k+1}^\top x = \tau_{k+1}$ )
1338      5.   pd_stree_rec(( $s, +1$ ),  $x + \varepsilon v_{k+1}$ ,  $\mathfrak{S}$ )      // ( $s, +1$ )  $\in \mathcal{S}_{k+1}$ 
1339      6.   pd_stree_rec(( $s, -1$ ),  $x - \varepsilon v_{k+1}$ ,  $\mathfrak{S}$ )      // ( $s, -1$ )  $\in \mathcal{S}_{k+1}$ 
1340      7.   return
1341      8. endif
1342      9.  $s_{k+1} := \text{sgn}(v_{k+1}^\top x - \tau_{k+1})$       // ( $s, s_{k+1}$ )  $\in \mathcal{S}_{k+1}$ 
1343     10. pd_stree_rec(( $s, s_{k+1}$ ),  $x$ ,  $\mathfrak{S}$ )
1344     11. if (( $s, -s_{k+1}$ ) covers a stem vector of  $\mathfrak{S}$ )
1345     12.   return
1346     13. elseif (( $s, -s_{k+1}$ ) is feasible with witness point  $\tilde{x}$ )
1347     14.   pd_stree_rec(( $s, -s_{k+1}$ ),  $\tilde{x}$ ,  $\mathfrak{S}$ )      // ( $s, -s_{k+1}$ )  $\in \mathcal{S}_{k+1}$ 
1348     15. else
1349     16.   Add one or two stem vectors to  $\mathfrak{S}$ 
1350     17. endif
1351  
```

We only comment some instructions of the primal-dual \mathcal{S} -tree algorithm 5.7 that differ from those of the primal \mathcal{S} -tree algorithm 4.1.

- Unlike algorithm 5.4, which computes all the stem vectors at first, algorithm 5.7 initializes the list of stem vectors \mathfrak{S} to the empty set in line 1. This list is next gradually filled by algorithm 5.8.
- For more efficiency, one could adapt line 4 of algorithm 5.8 and its lines 5..6 by using the improvement described in section 4.2.2.
- Lines 11..12 are new with respect to algorithm 4.1. They are used to check whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}^c$, using the stem vectors collected in \mathfrak{S} and proposition 3.16, without having to solve a LOP.
- Lines 15..16 are also new with respect to algorithm 4.1. They use proposition 5.6(3) to detect a new circuit, hence one or two new stem vectors (they are new, since otherwise the test in line 11 would have been successful and line 16 would not have been executed), which are put in \mathfrak{S} . For this, it is necessary to solve the LOP in line 13 by a method computing an extreme point of the dual feasible set of this problem.

Algorithm 5.7 can be improved by introducing the modifications A, B and C of sections 4.2.1–4.2.3.

6. Compact version of the algorithms. All the algorithms computing the sign vector set $\mathcal{S}(V, \tau)$ presented so far, except algorithm 5.3, recursively construct

1372 the \mathcal{S} -tree introduced in algorithm 4.1, namely (recall the definition (4.2) of $\mathcal{S}_k(V, \tau)$)

$$1373 \quad (6.1) \quad \mathcal{T}(V, \tau) := \bigcup_{k \in [1:p]} \mathcal{S}_k(V, \tau).$$

When the arrangement is not centered (equivalently, $\tau \notin \mathcal{R}(V^\top)$), some sets $\mathcal{S}_k(V, \tau)$ are asymmetric (proposition 3.5), so that the sign vectors of the two subtrees $\mathcal{T}^+(V, \tau) := \{s \in \mathcal{T}(V, \tau) : s_1 = +1\}$ and $\mathcal{T}^-(V, \tau) := \{s \in \mathcal{T}(V, \tau) : s_1 = -1\}$ of the \mathcal{S} -tree, rooted at the nodes $\{+1\}$ and $\{-1\}$, respectively, are not opposite to each other. Therefore, one cannot just compute $\mathcal{T}^+(V, \tau)$ or $\mathcal{T}^-(V, \tau)$ to get all $\mathcal{T}(V, \tau)$ (recall that when the arrangement is centered, $\mathcal{S}(V, \tau) = \mathcal{S}(V, 0)$ and only half of the sign vectors needs to be computed [17]). Nevertheless, these two subtrees have some opposite sign vectors, the symmetric ones, those in $\mathcal{T}(V, 0) = \bigcup_{k \in [1:p]} \mathcal{S}_k(V, 0)$. The set of asymmetric sign vectors in $\mathcal{T}(V, \tau)$ is denoted by

$$\mathcal{T}_a(V, \tau) := \bigcup_{k \in [1:p]} \mathcal{S}_{a,k}(V, \tau),$$

1374 where $\mathcal{S}_{a,k}(V, \tau) := \mathcal{S}_k(V, \tau) \setminus \mathcal{S}_{s,k}(V, \tau)$. Therefore, it is natural to look for a way
 1375 to avoid as much as possible repeating the costly operations (linear optimization
 1376 problems or stem vector coverings) common to the construction of the two subtrees
 1377 $\mathcal{T}^+(V, \tau)$ and $\mathcal{T}^-(V, \tau)$. The goal of this section is to propose algorithms having that
 1378 property; they can have a primal or dual nature.

6.1. The compact \mathcal{S} -tree. For an arrangement $\mathcal{A}(V, \tau)$, with $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$, and for $k \in [1:p]$, we denote the arrangement associated with the first k columns of V and the first k components of τ by

$$\mathcal{A}_k(V, \tau) := \mathcal{A}(V_{:, [1:k]}, \tau_{[1:k]}).$$

1379 By proposition 3.18, we have that

$$1380 \quad (6.2a) \quad \mathcal{S}_k(V, 0) \subseteq \mathcal{S}_k(V, \tau) \subseteq \mathcal{S}_k([V; \tau^\top], 0)$$

$$1381 \quad (6.2b) \quad \mathcal{S}_k([V; \tau^\top], 0) \setminus \mathcal{S}_k(V, 0) = \mathcal{S}_{a,k}(V, \tau) \cup \mathcal{S}_{a,k}(V, -\tau).$$

1383 The algorithms described in this section are based on the following considerations.
 1384 By (3.8), the set $\mathcal{T}(V, \tau)$ of the feasible sign vectors of the \mathcal{S} -tree can be written
 1385 $\mathcal{T}(V, 0) \cup \mathcal{T}_a(V, \tau)$. Taking the intersection with $\mathcal{T}^+(V, \tau)$ and $\mathcal{T}^-(V, \tau)$ provides a
 1386 partition of $\mathcal{T}(V, \tau)$ into four sets:

$$1387 \quad (6.3) \quad \mathcal{T}(V, 0) \cap \mathcal{T}^+(V, \tau), \mathcal{T}_a(V, \tau) \cap \mathcal{T}^+(V, \tau), \mathcal{T}(V, 0) \cap \mathcal{T}^-(V, \tau), \mathcal{T}_a(V, \tau) \cap \mathcal{T}^-(V, \tau).$$

1388 Since²⁰

$$1389 \quad (6.4) \quad \mathcal{T}(V, 0) \cap \mathcal{T}^-(V, \tau) = -[\mathcal{T}(V, 0) \cap \mathcal{T}^+(V, \tau)],$$

only two sets must be computed to be able to retrieve all the sign vectors of $\mathcal{T}(V, \tau)$, namely the union of the first two sets of the partition (6.3) and the last one:

$$\mathcal{T}^+(V, \tau) \quad \text{and} \quad \mathcal{T}_a(V, \tau) \cap \mathcal{T}^-(V, \tau).$$

²⁰Since $\mathcal{T}^-(V, \tau) \neq -\mathcal{T}^+(V, \tau)$, let us be precise. Let us show the inclusion “ \subseteq ” in (6.4). If $s \in \mathcal{T}(V, 0) \cap \mathcal{T}^-(V, \tau)$, then $-s \in -\mathcal{T}(V, 0) = \mathcal{T}(V, 0)$; furthermore $-s \in \mathcal{T}(V, \tau)$ (by $-s \in \mathcal{T}(V, 0)$, just proven) and $(-s)_1 = +1$, so that $-s \in \mathcal{T}^+(V, \tau)$. The reverse inclusion can be proven similarly.

The principle of the algorithms described in this section consists in computing the subtree $\mathcal{T}^+(V, \tau)$ rooting at $s_1 = +1$ and in grafting to it the subtrees of

$$-[\mathcal{T}_a(V, \tau) \cap \mathcal{T}^-(V, \tau)],$$

which is in $\mathcal{T}_a(V, -\tau)$ ²¹. This forms what we call the *compact S-tree*. More precisely, if $s \in \mathcal{S}_k(V, 0) \cap \mathcal{T}^+(V, \tau)$ and $(-s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \tau) \cap \mathcal{T}^-(V, \tau)$ for some $s_{k+1} \in \{\pm 1\}$, the subtree of $\mathcal{T}^-(V, \tau)$ rooting at $(-s, s_{k+1})$ is grafted at s in the compact tree (with its sign vectors multiplied by -1 , so that $(s, -s_{k+1})$ can be a child of s). As a result, the nodes of the level k of the compact \mathcal{S} -tree are in one of the sets

$$(6.5) \quad \mathcal{S}_k(V, 0), \quad \mathcal{S}_{a,k}(V, \tau) \quad \text{or} \quad \mathcal{S}_{a,k}(V, -\tau).$$

Eventually, a sign vector $s \in \mathcal{S}_a(V, -\tau)$ must be multiplied by -1 to get it in $-\mathcal{S}_a(V, -\tau) = \mathcal{S}_a(V, \tau) \subseteq \mathcal{S}(V, \tau)$. This principle is illustrated in figure 6.1. House-

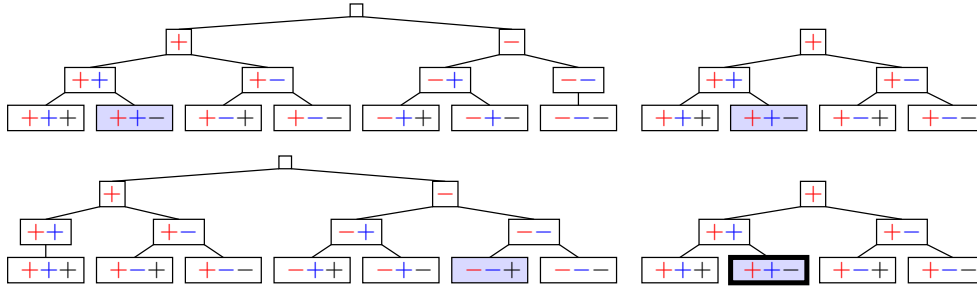


FIG. 6.1. *Standard S-trees (left) and compact S-trees (right) of the arrangements in the middle pane (above, compare with figure 4.1) and the right-hand side pane (below) of figure 3.1. The sign vectors in the white boxes are in $\mathcal{T}(V, 0)$, those in the blue/gray boxes are in $\mathcal{S}_a(V, \tau)$ and the one in the blue/gray box with bold edges is in $\mathcal{S}_a(V, -\tau)$; this last sign vector must be multiplied by -1 to get a sign vector in $-\mathcal{S}_a(V, -\tau) = \mathcal{S}_a(V, \tau) \subseteq \mathcal{S}(V, \tau)$.*

keeping is done by attaching a flag \boxtimes to each node s of the resulting tree, in order to specify which of the sign vector sets listed in (6.5) s belongs. As claimed in point 5 of the next proposition, the grafting process does not introduce nodes with two different flags: if $(s, s_{k+1}) \in -[\mathcal{S}_{a,k+1}(V, \tau) \cap \mathcal{T}^-(V, \tau)]$ is grafted to the compact \mathcal{S} -tree, then (s, s_{k+1}) is not in $\mathcal{T}^+(V, \tau)$.

The proof of the next proposition is given in [16].

PROPOSITION 6.1 (compact S-tree). *Let $k \in [1 : p - 1]$ and let $s \in \mathcal{S}_k(V, 0) \cup \mathcal{S}_{a,k}(V, \tau) \cup \mathcal{S}_{a,k}(V, -\tau)$ be a sign vector of the compact S-tree. Set $\mathcal{S}_k^\pm(V, \tau) := \mathcal{S}_k(V, \tau) \cap \mathcal{T}^\pm(V, \tau)$.*

- 1) *If $s \in \mathcal{S}_k(V, 0)$, one child of s in the compact S-tree is in $\mathcal{S}_{k+1}(V, 0)$.*
- 2) *If $s \in \mathcal{S}_{a,k}(V, \tau)$, the children of s in the compact S-tree are in $\mathcal{S}_{a,k+1}(V, \tau)$.*
- 3) *If $s \in \mathcal{S}_{a,k}(V, -\tau)$, the children of s in the compact S-tree are in $\mathcal{S}_{a,k+1}(V, -\tau)$.*
- 4) *If $(s, s_{k+1}) \in -[\mathcal{S}_{a,k+1}(V, \tau) \cap \mathcal{T}^-(V, \tau)]$ with $s_{k+1} \in \{\pm 1\}$, then $(s, s_{k+1}) \notin \mathcal{T}^+(V, \tau)$.*
- 5) *Level k of the compact S-tree is formed of $\mathcal{S}_k^+(V, \tau) \cup (-[\mathcal{S}_{a,k}(V, \tau) \cap \mathcal{T}^-(V, \tau)])$.*

6.2. Compact primal S-tree algorithm. In accordance with the presentation of section 6.1, the *compact primal S-tree algorithm*, whose reasoned description

²¹The inclusion is clear since $\mathcal{T}_a(V, \tau) \cap \mathcal{T}^-(V, \tau) \subseteq \mathcal{T}_a(V, \tau)$ and $-\mathcal{T}_a(V, \tau) = \mathcal{T}_a(V, -\tau)$.

is given below, ignores the subtree $\mathcal{T}^-(V, \tau)$ rooting at $\{+1\}$, constructs the subtree $\mathcal{T}^+(V, \tau)$ rooting at $\{+1\}$ and grafts to it the opposite of the sign vectors in the subtrees of $\mathcal{T}_a(V, \tau) \cap \mathcal{T}^-(V, \tau)$. Note that $s_1 \in \mathcal{S}_1(V, 0)$. Let us describe this algorithm. Its formal statement is given afterwards.

The algorithm identifies each node at level k of the compact \mathcal{S} -tree by a triplet (s, x, \boxplus) , where $s \in \{\pm 1\}^k$ is the sign vector of the node, $\boxplus \in \{-1, 0, +1\}$ is a flag specifying to which sign set s belongs and x is some witness point. More specifically,

$$(6.6) \quad \begin{cases} s \in \mathcal{S}_k(V, 0), x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V, 0) & \text{if } \boxplus = 0, \\ s \in \mathcal{S}_{a,k}(V, \tau), x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V, \tau) & \text{if } \boxplus = +1, \\ s \in \mathcal{S}_{a,k}(V, -\tau), x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V, -\tau) & \text{if } \boxplus = -1. \end{cases}$$

The flag \boxplus is used below as a scalar, hence $\boxplus s$ is the vector whose i th component is $\boxplus s_i$. The initialization of the algorithm is done as follows.

0. Take $s_1 = +1 \in \mathcal{S}_1(V, 0)$ and v_1 as witness point for s_1 in $\mathcal{A}_1(V, 0)$.

Consider now a node at level k of the compact \mathcal{S} -tree, which is specified by a triplet (s, x, \boxplus) satisfying (6.6). We just have to specify how the algorithm determines the children of that node.

1. Suppose that $s \in \mathcal{S}_k(V, 0)$ with x as witness point in $\mathcal{A}_k(V, 0)$, i.e., $\boxplus = 0$.

Using proposition 4.7 with $\tau = 0$, the algorithm can detect whether s has two easily computable children in $\mathcal{A}(V, 0)$ and can find associated witness points. If such is the case, the algorithm pursues recursively from $(s, +1)$ and $(s, -1)$, with appropriate witness points. It returns afterwards.

Otherwise, $v_{k+1}^\top x \neq 0$, implying that $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$ for $s_{k+1} := \text{sgn}(v_{k+1}^\top x)$ and that the algorithm can pursue recursively from (s, s_{k+1}) with x as witness point in $\mathcal{A}_{k+1}(V, 0)$.

Now, the algorithm has to specify to what set $(s, -s_{k+1})$ belongs: $\mathcal{S}_{k+1}(V, 0)$, $\mathcal{S}_{a,k+1}(V, \tau)$, $\mathcal{S}_{a,k+1}(V, -\tau)$ or $\mathcal{S}_{k+1}([V; \tau^\top], 0)^c$ (there are no other possibilities, see proposition 3.18 and figure 3.3). For this purpose, the compact primal \mathcal{S} -tree algorithm starts by solving the LOP (4.5) with $\tau = 0$, to see whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$. Denote by (x_0, α_0) a solution to this LOP.

1.1. If $\alpha_0 < 0$, then $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$ and the algorithm pursues recursively from $(s, -s_{k+1})$ with x_0 as witness point in $\mathcal{A}_{k+1}(V, 0)$.

1.2. Otherwise, $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V, 0)$ and the algorithm determines whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}([V; \tau^\top], 0)$ by solving the following LOP, which is similar to (4.5) with $\tau = 0$, but for the arrangement $\mathcal{A}([V; \tau^\top], 0)$ instead of $\mathcal{A}(V, 0)$:

$$(6.7) \quad \begin{aligned} \min_{(x, \xi, \alpha) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}} \quad & \alpha \\ \text{s.t.} \quad & s_i(v_i^\top x + \tau_i \xi) + \alpha \geq 0, \quad \text{for } i \in [1 : k] \\ & -s_{k+1}(v_{k+1}^\top x + \tau_{k+1} \xi) + \alpha \geq 0 \\ & \alpha \geq -1. \end{aligned}$$

Denote by (x_1, ξ_1, α_1) a solution to this problem.

1.2.1. If $\alpha_1 \geq 0$, then $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}([V; \tau^\top], 0)$, hence $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V, \tau) \cup \mathcal{S}_{k+1}(V, -\tau)$ by (6.2a) and $(s, -s_{k+1})$ can be discarded from the generated compact tree.

1.2.2. Otherwise, $\alpha_1 < 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{k+1}([V; \tau^\top], 0) \setminus \mathcal{S}_{k+1}(V, 0) = \mathcal{S}_{a,k+1}(V, \tau) \cup \mathcal{S}_{a,k+1}(V, -\tau)$ by (6.2b). Note that one cannot have

1454 $\xi_1 = 0$ in that case, since then one would have $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$,
 1455 which is excluded in the considered case. Therefore,
 1456 – either $\xi_1 < 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \tau)$, by (6.7), with $-x_1/\xi_1$
 1457 as witness point in $\mathcal{A}_{k+1}(V, \tau)$,
 1458 – or $\xi_1 > 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, -\tau)$, by (6.7), with x_1/ξ_1 as
 1459 witness point in $\mathcal{A}_{k+1}(V, -\tau)$.
 1460 In these last two cases, the algorithm can pursue recursively from $(s,$
 1461 $-s_{k+1})$.

1462 2. Suppose that $s \in \mathcal{S}_{a,k}(V, \tau)$ with x as witness point in $\mathcal{A}_k(V, \tau)$, i.e., $\boxminus = +1$.

1463 Using proposition 4.7, the algorithm can detect whether s has two easily com-
 1464 putable children in $\mathcal{A}(V, \tau)$ and can find associated witness points. If such is the
 1465 case, the algorithm pursues recursively from $(s, +1)$ and $(s, -1)$, with appropriate
 1466 witness points. It returns afterwards.

1467 Otherwise, $v_{k+1}^\top x \neq \tau_{k+1}$, implying that $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, \tau)$ for $s_{k+1} :=$
 1468 $\text{sgn}(v_{k+1}^\top x - \tau_{k+1})$ and that the algorithm can pursue recursively from (s, s_{k+1})
 1469 with x as witness point in $\mathcal{A}_{k+1}(V, \tau)$.

1470 Now, the algorithm has to determine whether $(s, -s_{k+1})$ is infeasible or is in
 1471 $\mathcal{S}_{a,k+1}(V, \tau)$ (there are no other possibilities, according to proposition 6.1(2)).
 1472 For this purpose, the algorithm solves (4.5). Let (x_2, α_2) be a solution.

1473 – If $\alpha_2 < 0$, then $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \tau)$ and the compact algorithm can
 1474 pursue recursively from $(s, -s_{k+1})$.
 1475 – Otherwise, $(s, -s_{k+1})$ is infeasible in $\mathcal{A}_{k+1}(V, \tau)$ and the algorithm can prune
 1476 the compact \mathcal{S} -tree at that node.

1477 3. The last case, when $s \in \mathcal{S}_{a,k}(V, -\tau)$, with x as witness point in $\mathcal{A}_k(V, -\tau)$, i.e.,
 1478 $\boxminus = -1$, is similar to case 2 and is detailed in [16].

1479 One can now present schematically the compact form of the primal \mathcal{S} -tree algo-
 1480 rithm 4.1. To shorten its statement and the one of the next algorithm 6.7, we introduce
 1481 the following two functions: **output_s**, which outputs sign vectors of $\mathcal{S}(V, \tau)$ at a leaf
 1482 of the compact \mathcal{S} -tree (its behavior is more complex than for the standard algorithms
 1483 and depends on the type \boxminus of the leaf node s , see (6.6)), and **c_p_two_children**,
 1484 which detects whether s has the two children that are given by proposition 4.7; if this
 1485 is the case, it pursues the compact \mathcal{S} -tree construction at $(s, \pm 1)$ and returns **true**;
 1486 otherwise, it returns **false**.

1487 **ALGORITHM 6.2** (**output_s**(s, \boxminus)).

1489 It is assumed that $s \in \{\pm 1\}^p$ and that $\boxminus \in \{-1, 0, +1\}$.

```

1491 1. if ( $\boxminus = 0$ )
1492 2.   output  $\pm s$       //  $s \in \mathcal{S}(V, 0)$ 
1493 3. else
1494 4.   output  $\boxminus s$       //  $s \in \mathcal{S}_a(V, \boxminus \tau)$ 
1495 5. endif

```

1496 **ALGORITHM 6.3** (**c_p_two_children**(s, x, \boxminus)).

1497 It is assumed that $s \in \{\pm 1\}^k$ and that (s, x, \boxminus) satisfies (6.6).

```

1501 1. if ( $v_{k+1}^\top x \simeq \boxminus \tau_{k+1}$ )      // two easy children in  $\mathcal{A}_{k+1}(V, \boxminus \tau)$ 
1502 2.   c_p_stree_rec( $(s, +1), x + t_+ v_{k+1}, \boxminus$ ) for some  $t_+ \in (t_0, t_{\max})$ 
1503 3.   c_p_stree_rec( $(s, -1), x + t_- v_{k+1}, \boxminus$ ) for some  $t_- \in (t_{\min}, t_0)$ 

```

```

1504     4.  return true
1505     5.  else
1506     6.  return false
1507     7.  endif
1508

```

1509 We have chosen to present the cases when $\boxplus = \pm 1$ jointly in lines 22..25, to save
1510 space. An expanded presentation is given in [16].

1513 ALGORITHM 6.4 (`c_p_stree`). Let be given $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$.

```

1514     1. c_p_stree_rec(+1, v1, 0)
1515

```

1518 ALGORITHM 6.5 (`c_p_stree_rec(s, x, \boxplus)`).

1519 It is assumed that $s \in \{\pm 1\}^k$ and that (s, x, \boxplus) satisfies (6.6).

```

1520     1. if (k = p)      // s is a leaf of the compact S-tree
1521     2.   output_s(s,  $\boxplus$ )
1522     3.   return
1523     4. endif
1524     5. if (c_p_two_children(s, x,  $\boxplus$ ))      // two easy children in  $\mathcal{A}_{k+1}(V, \boxplus \tau)$ 
1525     6.   return
1526     7. endif
1527     8.  $s_{k+1} := \text{sgn}(v_{k+1}^\top x - \boxplus \tau_{k+1})$       //  $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, \boxplus \tau)$ 
1528     9. c_p_stree_rec((s, sk+1), x,  $\boxplus$ )
1529    10. if ( $\boxplus = 0$ )      //  $s \in \mathcal{S}_k(V, 0)$  with  $x$  as witness point in  $\mathcal{A}_k(V, 0)$ 
1530    11.   Solve (4.5) with  $\tau = 0$ ; let  $(x_0, \alpha_0)$  be a solution
1531    12.   if ( $\alpha_0 < 0$ )      //  $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$ 
1532    13.     c_p_stree_rec((s, -sk+1), x0, 0)
1533    14.   else
1534    15.     Solve (6.7); let  $(x_1, \xi_1, \alpha_1)$  be a solution      // here  $\xi_1 \neq 0$ 
1535    16.     if ( $\alpha_1 < 0$ )      //  $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, -\text{sgn}(\xi_1)\tau)$ 
1536    17.       c_p_stree_rec((s, -sk+1), x1/| $\xi_1$ |, -sgn( $\xi_1$ ))
1537    18.     endif
1538    19.   endif
1539    20.   return
1540    21. endif      // here  $\boxplus \in \{\pm 1\}$ , hence  $s \in \mathcal{S}_{a,k}(V, \boxplus \tau)$ 
1541    22. Solve (4.5) with  $\tau \curvearrowright \boxplus \tau$ ; let  $(x_2, \alpha_2)$  be a solution
1542    23. if ( $\alpha_2 < 0$ )      //  $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \boxplus \tau)$ 
1543    24.   c_p_stree_rec((s, -sk+1), x2,  $\boxplus$ )
1544    25. endif
1545

```

1546 Observe that, as claimed by proposition 6.1(2-3), once $s \in \mathcal{S}_{a,k}(V, \tau)$ (resp.
1547 $s \in \mathcal{S}_{a,k}(V, -\tau)$), its descendants in the compact \mathcal{S} -tree are all in $\mathcal{S}_{a,l}(V, \tau)$ (resp.
1548 $\mathcal{S}_{a,l}(V, -\tau)$) for some $l \in [k+1:p]$. In these cases, the compact algorithm solves
1549 at most one LOP per sign vector (in step 22), like in the standard version of the
1550 algorithm, which solves at most one LOP in $\mathcal{T}^+(V, \tau)$ or $\mathcal{T}^-(V, \tau)$, not both since
1551 an asymmetric sign vector only appears in one of these subtrees. When $s \in \mathcal{S}_k(V, 0)$
1552 has one child in $\mathcal{S}_{a,k+1}(V, \pm \tau)$, the compact algorithm solves two LOPs (in steps 11
1553 and 15), like in the standard algorithm (one LOP in $\mathcal{T}^\pm(V, \tau)$ to accept the child
1554 in $\mathcal{S}_{a,k+1}(V, \tau)$ and one in $\mathcal{T}^\mp(V, \tau)$ to reject a child in $\mathcal{S}_{a,k+1}(V, \tau)$). The only sign
1555 vectors at which the compact algorithm solves less LOPs than the standard algorithm
1556 are those in $\mathcal{S}(V, 0)$ with two symmetric children. In this case, the compact algorithm

solves a single LOP (in step 11), while the standard algorithm solves two LOPs (one in each subtree $\mathcal{T}^+(V, \tau)$ and $\mathcal{T}^-(V, \tau)$). Therefore, the compact algorithm 6.4 is all the more advantageous with respect to the standard algorithm 4.1 as $|\mathcal{T}(V, 0)|/|\mathcal{T}(V, \tau)|$ is large (it is always ≤ 1); see [35].

6.3. Compact primal-dual \mathcal{S} -tree algorithm. There are several ways of using the stem vectors, in order to avoid having to solve all or part of the LOPs of the standard primal \mathcal{S} -tree algorithms 4.1, most of them having a compact form. In this section, we only consider a compact version of the primal-dual \mathcal{S} -tree algorithm 5.7. The statement of the algorithm is immediate as soon as we know how it collects the stem vectors and how it uses them. This is essentially what we clarify in this section, leaving a more complete description to [16].

As shown in section 5, stem vectors can be used to detect sign vectors that are not in $\mathcal{S}(V, \tau)$, using proposition 3.16. Our goal in this section is to apply this technique to construct the compact primal-dual \mathcal{S} -tree, by “dualizing” the compact primal \mathcal{S} -tree algorithm 6.4 (we have found this approach easier than “compacting” the standard primal-dual \mathcal{S} -tree algorithm 5.7). The principle is simple. The algorithm manages subsets $\tilde{\mathcal{S}}_s$ of $\mathcal{S}_s(V, \tau)$, $\tilde{\mathcal{S}}_a$ of $\mathcal{S}_a(V, \tau)$ and $\tilde{\mathcal{S}}_0$ of $\mathcal{S}_0(V, \tau)$, named *collectors*, which are initially empty and are progressively filled during the iterations (this is explained below). Then, each group of statements in algorithm 6.5 (the lines given by the first

Algorithm 6.5		Algorithm 6.8		
Lines	Lines	Sign vector set	Stem vector set	Collectors
11..14	11..19	$\mathcal{S}_{k+1}(V, 0)$	$\mathcal{S}(V, 0)$	$\tilde{\mathcal{S}}_s \cup \tilde{\mathcal{S}}_a \cup (-\tilde{\mathcal{S}}_a)$
15..18	20..27	$\mathcal{S}_{k+1}([V; \tau^\top], 0)$	$\mathcal{S}([V; \tau^\top], 0)$	$\tilde{\mathcal{S}}_s \cup \tilde{\mathcal{S}}_0$
22..25	30..39	$\mathcal{S}_{a,k+1}(V, \boxplus \tau)$	$\mathcal{S}(V, \boxplus \tau)$	$\tilde{\mathcal{S}}_s \cup (\boxplus \tilde{\mathcal{S}}_a)$

TABLE 6.1
Corresponding lines in algorithms 6.5 and 6.8.

column of table 6.1), dealing with a LOP and its consequences, is replaced by other lines in algorithm 6.8 (those given by the second column of table 6.1). These latter lines are organized as follows.

- So as to avoid having to solve certain LOPs, a covering test is run to see whether the sign vector $(s, -s_{k+1})$ is in the set in the third column of table 6.1 (recall definition (4.2)). Appropriate stem vectors must be used to realize that operation, namely those in the collectors in the fifth column of table 6.1, which are contained in the sets in the fourth column of table 6.1 (see propositions 3.14 and 3.21, and figure 3.2).
- If the covering test succeeds (i.e., $(s, -s_{k+1})$ covers an appropriate stem vector), then $(s, -s_{k+1})$ is not in the sign vector set in the third column of table 6.1 (proposition 3.16) and the compact \mathcal{S} -tree is pruned.
- Otherwise, because there is no equality between the stem vector sets in the fourth column of table 6.1 and their collectors in the fifth column of table 6.1, a LOP is solved like in algorithm 6.4.
- If this LOP has a negative optimal value, $(s, -s_{k+1})$ is in the sign vector set in the third column of table 6.1 and the recursion is proceeded from that node.
- Otherwise, one or two stem vectors are added to the appropriate collectors in the fifth column of table 6.1.

This yields the following algorithm. One first adapts the `c_p_two_children` algorithm 6.3, so that it calls the appropriate procedures of the present framework.

ALGORITHM 6.6 (`c_pd_two_children`(s, x, \mathbb{S})).

It is assumed that $s \in \{\pm 1\}^k$ and (s, x, \mathbb{S}) satisfies (6.6).

1. **if** $(v_{k+1}^\top x \simeq \mathbb{S} \tau_{k+1})$ // two easy children in $\mathcal{A}_{k+1}(V, \mathbb{S} \tau)$
 2. `c_pd_stree_rec`(($s, +1$), $x + t_+ v_{k+1}$, \mathbb{S}) for some $t_+ \in (t_0, t_{\max})$
 3. `c_pd_stree_rec`(($s, -1$), $x + t_- v_{k+1}$, \mathbb{S}) for some $t_- \in (t_{\min}, t_0)$
 4. **return true**
 5. **else**
 6. **return false**
 7. **endif**
-

We can now present the result of the adaptation of algorithm 6.4 along the principle described above.

ALGORITHM 6.7 (`c_pd_stree`). Let be given $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$.

1. $\tilde{\mathfrak{S}}_s = \emptyset, \tilde{\mathfrak{S}}_a = \emptyset, \tilde{\mathfrak{S}}_0 = \emptyset$ // initial empty collectors
 2. `c_pd_stree_rec`($+1, v_1, 0, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0$)
-

ALGORITHM 6.8 (`c_pd_stree_rec`($s, x, \mathbb{S}, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0$)).

It is assumed that $s \in \{\pm 1\}^k$, that (x, s, \mathbb{S}) satisfies (6.6) and that $\tilde{\mathfrak{S}}_s \subseteq \mathfrak{S}_s(V, \tau), \tilde{\mathfrak{S}}_a \subseteq \mathfrak{S}_a(V, \tau), \tilde{\mathfrak{S}}_0 \subseteq \mathfrak{S}_0(V, \tau)$.

1. **if** $(k = p)$ // s is a leaf of the compact \mathcal{S} -tree
2. `output_s`(s, \mathbb{S})
3. **return**
4. **endif**
5. **if** (`c_pd_two_children`(s, x, \mathbb{S})) // two easy children in $\mathcal{A}_{k+1}(V, \mathbb{S} \tau)$
6. **return**
7. **endif**
8. $s_{k+1} := \text{sgn}(v_{k+1}^\top x - \mathbb{S} \tau_{k+1})$ // $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, \mathbb{S} \tau)$
9. `c_pd_stree_rec`((s, s_{k+1}), $x, \mathbb{S}, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0$)
10. **if** ($\mathbb{S} = 0$) // $s \in \mathcal{S}_k(V, 0)$ with x as witness point in $\mathcal{A}_k(V, 0)$
11. **if** $((s, -s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_a \cup (-\tilde{\mathfrak{S}}_a)$)
12. Solve (4.5) with $\tau = 0$; let (x_0, α_0) be a solution
13. **if** $(\alpha_0 < 0)$ // $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$
14. `c_pd_stree_rec`(($s, -s_{k+1}$), $x_0, 0, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0$)
15. **return**
16. **else**
17. Add two or one stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_a$, respectively
18. **endif**
19. **endif** // here $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V, 0)$, check if it $\in \mathcal{S}_{k+1}([V; \tau^\top], 0)$
20. **if** $((s, -s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_0$)
21. Solve (6.7); let (x_1, ξ_1, α_1) be a solution // here $\xi_1 \neq 0$
22. **if** $(\alpha_1 < 0)$ // $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, -\text{sgn}(\xi_1)\tau)$
23. `c_pd_stree_rec`(($s, -s_{k+1}$), $x_1/|\xi_1|, -\text{sgn}(\xi_1), \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0$)
24. **else**
25. Add two stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_0$
26. **endif**
27. **endif**

```

1649 28. return
1650 29. endif // here  $\mathbb{S} \in \{\pm 1\}$ , hence  $s \in \mathcal{S}_{a,k}(V, \mathbb{S}\tau)$ 
1651 30. if  $((s, -s_{k+1})$  does not cover a stem vector of  $\tilde{\mathfrak{S}}_s \cup (\mathbb{S}\tilde{\mathfrak{S}}_a)$ )
1652 31. Solve (4.5) with  $\tau \curvearrowright \mathbb{S}\tau$ ; let  $(x_2, \alpha_2)$  be a solution
1653 32. if  $(\alpha_2 < 0)$ 
1654 33. c_pd_stree_rec $((s, -s_{k+1}), x_2, \mathbb{S}, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0)$ 
1655 34. else
1656 35. Add two or one stem vectors to  $\tilde{\mathfrak{S}}_s$  or  $\tilde{\mathfrak{S}}_a$ , respectively
1657 36. endif
1658 39. endif
1659

```

1660 **7. Numerical results.** The goal of this section is to assess the efficiency of a
1661 selection of algorithms enumerating the chambers of a choice of hyperplane arrange-
1662 ments, among the algorithms introduced in sections 4, 5 and 6. Section 7.1 lists and
1663 briefly describes the chosen arrangement instances. The considered algorithms are
1664 specified in section 7.2. Section 7.3 details and discusses the results of this evaluation.

1665 **7.1. Arrangement instances.** This section describes the arrangements that
1666 form the test bed for the evaluation of the selected algorithms presented in the next
1667 section. These arrangements $\mathcal{A}(V, \tau)$ are specified by their matrix $V \in \mathbb{R}^{n \times p}$ and
1668 vector $\tau \in \mathbb{R}^p$ (see section 3.1). One always has $p > n$ and $r := \text{rank}(V) = n$. The
1669 instance features are given in tables 7.1 (theoretical values, for some of them) and 7.2
1670 (numerical values). More is said on these problems in [16].

1671 The given five problems are affine, with $\tau \neq 0$. Some of them were examined
1672 in [37]. Their linear version, obtained by setting $\tau = 0$, were considered in [17].
1673 Random numbers are generated with the Julia function `rand`.

- 1674 • **rand-n-p:** $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$ are randomly generated in $[-5 : +5]$.
- 1675 • **srand-n-p-q:** One has $V_{:, [1:n]} = I_n$ and each of the remaining $p - n$ columns has q
1676 nonzero random integer elements, randomly positioned. Each element of $\tau_{[n+1:p]}$
1677 has a $1/2$ probability of being a random integer; it vanishes otherwise; $\tau_{[1:n]} = 0$.
1678 Random integers are taken in $[-10 : +10] \setminus \{0\}$.
- 1679 • **2d-n-p:** The matrix V is such that: $V_{[1:2], [1:n-2]} = 0$ and $V_{[3:n], [n-1:p]} = 0$. Its
1680 remaining elements and τ are randomly generated integers in $[-20 : +20]$ [37, 17].
- 1681 • **perm-n:** This problem refers to the hyperplane arrangements that are called *per-*
1682 *mutahedron* in [37]: one has $p = n(n+1)/2$, $V_{:, [1:n]}$ is the identity matrix and
1683 $V_{:, [n+1:p]}$ is a Coxeter matrix [36] (each column is of the form $e_i - e_j$ for some
1684 $i < j$ in $[1:n]$, where e_k is the k th basis vector of \mathbb{R}^n). The vector τ is defined by
1685 $\tau_i = 1$ for $i \in [1:n]$ and $\tau_i = 0$ for $i \in [n+1:p]$. Since $(1, \dots, 1)$ belongs to all the
1686 hyperplanes, the arrangement is centered.
- 1687 • **ratio-n-p-t:** $V_{:, [1:n]}$, $\tau_{[1:n]}$ are randomly generated in $[-50 : +50]$ and $\mathfrak{t} \in [0, 1]$.
1688 Then, the remaining columns of $[V; \tau^\top]$ can either be random with probability $1 - \mathfrak{t}$
1689 or randomly generated linear combinations in $[-4 : +4]$ of the previous vectors. One
1690 recovers problem **rand-n-p** when $\mathfrak{t} = 0$.

1691 Some cardinality formulas are gathered in table 7.1. The numerical values of several
1692 cardinalities of the considered instances are given in table 7.2.

1693 **REMARKS 7.1** (on table 7.2).

- 1694 1) As expected, the randomly generated arrangements **rand-*** are in affine general
1695 position (definition 3.29). This is revealed in table 7.2 by a number $|\mathfrak{S}_s(V, \tau)|/2 +$

Problems	Circuits	Stem vectors			Chambers
	$ \mathcal{C}(V) $	$ \mathfrak{S}_s(V, \tau) /2$	$ \mathfrak{S}_a(V, \tau) $	$ \mathfrak{S}([V; \tau^\top], 0) /2$	$ \mathcal{S}(V, \tau) $
rand-n-p	$\binom{p}{n+1}$	0	$\binom{p}{n+1}$	$\binom{p}{n+2}$	$\sum_{i=0}^n \binom{p}{i}$
2d-n-p	$\binom{p-n+2}{3}$	0	$\binom{p-n+2}{3}$	$\binom{p-n+2}{4}$	$2^{n-2} \sum_{i=0}^2 \binom{p-n+2}{i}$
perm-n	$\sum_{i=3}^{n+1} \frac{i!}{2i} \binom{n+1}{i}$	$\sum_{i=3}^{n+1} \frac{i!}{2i} \binom{n+1}{i}$	0	$\sum_{i=3}^{n+1} \frac{i!}{2i} \binom{n+1}{i}$	$(n+1)!$

TABLE 7.1

Cardinality formulas for some instances, when $p > n$ and $\text{rank}(V) = n$.

Problems	n	p	Circuits of V	Stem vectors of $\mathcal{A}(V, \tau)$		Stem vectors of $\mathcal{A}([V; \tau^\top], 0)$		Chambers	
			Bound	$ \mathfrak{S}_s /2$	$ \mathfrak{S}_a $	$ \mathfrak{S} /2$	Bound	$ \mathcal{S}(V, \tau) $	Bound
rand-2-8	2	8	56	0	56	70	70	37	37
rand-4-8	4	8	56	0	56	28	28	163	163
rand-4-9	4	9	126	0	126	84	84	256	256
rand-5-10	5	10	210	0	210	120	120	638	638
rand-4-11	4	11	462	0	462	462	462	562	562
rand-6-12	6	12	792	0	792	495	495	2510	2510
rand-5-13	5	13	1716	0	1716	1716	1716	2380	2380
rand-7-14	7	14	3003	0	3003	2002	2002	9908	9908
rand-7-15	7	15	6435	0	6435	5005	5005	16384	16384
rand-8-16	8	16	11440	0	11440	8008	8008	39203	39203
rand-9-17	9	17	19448	0	19448	12376	12376	89846	89846
srand-8-20-2	8	20	167960	56	321	987	184756	36225	263950
srand-8-20-4	8	20	167960	1185	70650	94534	184756	213467	263950
srand-8-20-6	8	20	167960	20413	123909	105345	184756	245396	263950
2d-4-20	4	20	15504	1	815	3046	38760	684	6196
2d-5-20	5	20	38760	0	680	2380	77520	1232	21700
2d-6-20	6	20	77520	1	559	1808	125970	2176	60460
2d-7-20	7	20	125970	0	443	1365	167960	3840	137980
2d-8-20	8	20	167960	0	364	1001	184756	6784	263950
perm-5	5	15	5005	197	0	197	6435	720	4944
perm-6	6	21	116280	1172	0	1172	203490	5040	82160
perm-7	7	28	3108105	8018	0	8018	6906900	40320	1683218
perm-8	8	36	94143280	62814	0	62814	254186856	362880	40999516
ratio-3-20-0.7	3	20	4845	19	4614	14043	15504	1119	1351
ratio-3-20-0.9	3	20	4845	118	4550	12993	15504	1176	1351
ratio-4-20-0.7	4	20	15504	102	15271	36781	38760	6015	6196
ratio-4-20-0.9	4	20	15504	2327	11908	19882	38760	4600	6196
ratio-5-20-0.7	5	20	38760	97	33945	61452	77520	15136	21700
ratio-5-20-0.9	5	20	38760	23514	10954	23514	77520	11325	21700
ratio-6-20-0.7	6	20	77250	238	76595	120663	125970	59519	60640
ratio-6-20-0.9	6	20	77250	345	71861	106115	125970	53795	60460
ratio-7-20-0.7	7	20	125970	125	123792	159956	167960	135064	137980
ratio-7-20-0.9	7	20	125970	154	123731	159636	167960	135039	137980

TABLE 7.2

Description of the 33 considered arrangements. The first column gives the problem names. The next two columns specify the dimensions of $V \in \mathbb{R}^{n \times p}$. The 4th column gives the upper bound on the number of circuits of V , recalled in remark 3.13(6); by remark 3.13(3), it is also an upper bound on $|\mathfrak{S}_s|/2 + |\mathfrak{S}_a|$, where $|\mathfrak{S}_s|$ (resp. $|\mathfrak{S}_a|$) is the number of symmetric (resp. asymmetric) stem vectors (definition 3.12) of the arrangement $\mathcal{A}(V, \tau)$; $|\mathfrak{S}_s|/2$ and $|\mathfrak{S}_a|$ are given in columns 5 and 6. Columns 7 and 8 give half the number of stem vectors of the arrangement $\mathcal{A}([V; \tau^\top], 0)$ and its Schläfli upper bound, derived from (3.26). The last two columns give the number $|\mathcal{S}(V, \tau)|$ of chambers of the arrangement $\mathcal{A}(V, \tau)$ and its upper bound given by (3.28).

1696 $|\mathfrak{S}_a(V, \tau)|$ of circuits of V (5th and 6th columns, see remark 3.13(3)) that reaches
 1697 its maximum (4th column), see remark 3.13(6); by a number $|\mathfrak{S}([V; \tau^\top], 0)|/2$ of
 1698 circuits of $[V; \tau^\top]$ (7th column, see [17, after definition 3.9]) that reaches its maxi-
 1699 mum (8th column), see remark 3.13(6); and by a number $|\mathcal{S}(V, \tau)|$ of sign vectors
 1700 (9th column) that reaches its upper bound (10th column), see proposition 3.31.
 1701 2) Half the number of stem vectors of the linear arrangement $\mathcal{A}([V; \tau^\top], 0)$ (7th col-
 1702 umn) is also the number $|\mathcal{C}([V; \tau^\top])|$ of circuits of $[V; \tau^\top]$ (see [17, after definition
 1703 3.9]) and we see that this one is unrelated to the number of circuits of V (sum
 1704 of columns 5 and 6). This confirms the observation made after proposition 3.20,
 1705 according to which neither $\mathcal{C}(V) \subseteq \mathcal{C}([V; \tau^\top])$ nor $\mathcal{C}([V; \tau^\top]) \subseteq \mathcal{C}(V)$ must hold. \square

1706 **7.2. Assessed algorithms.** In the next section, the following algorithms have
 1707 been evaluated on the problem instances listed in the previous section. These algo-
 1708 rithms are identified by the following labels.

1709 RC: the original RC algorithm [37].

1710 P: the primal \mathcal{S} -tree algorithm 4.1.

1711 PD: the primal-dual \mathcal{S} -tree algorithm 5.7.

1712 D: the dual \mathcal{S} -tree algorithm 5.4.

1713 RC/C: the compact version of the RC algorithm (see below).

1714 P/C: the compact primal \mathcal{S} -tree algorithm 6.4.

1715 PD/C: the compact primal-dual \mathcal{S} -tree algorithm 6.7.

1716 D/C: the compact version of the D algorithm (see below).

1717 All the algorithms, but RC and RC/C, benefit from the enhancements A (section 4.2.1),
 1718 B (section 4.2.2) and C (section 4.2.3). By want of space, the algorithms RC, RC/C and
 1719 D/C have not been presented in sections 4 and 6. Briefly, algorithm RC is algorithm 4.2
 1720 (with its header 4.1) without its steps 4-8; algorithm RC/C is algorithm 6.5 (with its
 1721 header 6.4) without its steps 5-7; algorithm D/C is obtained from algorithm 5.4 using
 1722 the compaction principles described in section 6.

1723 **7.3. Numerical results.** To evaluate the algorithms listed in the previous sec-
 1724 tion, we have implemented them in a Julia code named `isf.jl`, which extends the Matlab
 1725 code `isf.m` [14, 13], from linear to general affine arrangements. The implementation
 1726 has been done in Julia (version 1.8.5) on a MacBookPro18,2/10cores (parallelism is
 1727 not implemented however) with the system macOS Monterey, version 12.6.1.

1728 All the solvers, but D and D/C, need to solve linear optimization problems (LOPs).
 1729 The linear optimization solver used in the Julia code is Gurobi. This one appears to be
 1730 more efficient than the Matlab solver Linprog used in [14, 13]. Since the improvement
 1731 is obtained by a reduction of the number of LOPs, which are solved much faster in
 1732 the Julia version, we observe a less important improvement (wrt the RC algorithm) in
 1733 computing time in the present study (Julia code) than reported in [17].

1734 The main computational burden of the “pure primal” variants P and P/C is
 1735 the solution of the LOPs while, for the “pure dual” variants D and D/C, it is the
 1736 computation of the stem vectors and their use in the covering tests. These are not
 1737 comparable. Therefore, counting the number of LOPs or the number of covering
 1738 tests is not a relevant criterion for comparing the solvers. For this reason, we rely on
 1739 computing time. Since the RC algorithm was shown in [37] to have better performance
 1740 in time than earlier methods, a comparison is often made with the RC algorithm. Since
 1741 this algorithm is implemented in Python, we avoid biases due to the programming
 1742 language by making the comparison with our Julia version of the RC algorithm, which
 1743 can be easily simulated from algorithm 4.1, as mentioned above.

For ease of reading, the comparison of the solvers' efficiency is carried out by using *performance profiles* [11] (tables with precise numbers are also given in appendix A): these are curves in a graph with the *relative efficiency* on the x -axis (sometimes in logarithmic scale) and a *percentage of problems* on the y -axis. There is one graph per performance, which is the computing time in our case, and there is one curve per solver in that graph: a point (e, f) of the curve of a solver tells us that the efficiency of this solver is never worse than e times that of the best solver (this one depends on the considered problem) on a fraction f of the problems. As a result, *the solver with the highest curve, if any, can be legitimately considered as the most effective one*, while the ranking of the other solvers by the position of their curve in the graph should be taken with caution [23]. The performance profiles only depend on the *relative* performance of the solvers, that is, for a particular problem, their performance divided by the one of the best solver for that problem. Therefore taking the *computing time* or the *computing time per chamber* as performance yield the same performance profiles.

7.3.1. Standard solvers. Let us first compare the standard solvers RC, P, PD and D with each other, on the selected arrangements described in table 7.2. The comparison is made on the computing times reported in table A.1 and the associated performance profiles are given in figure 7.1.

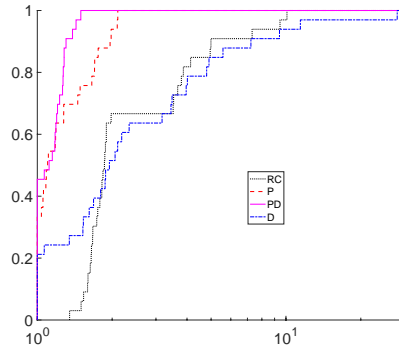


FIG. 7.1. Performance profiles of the RC, P, PD and D algorithms, for the computing time.

One observes that the PD algorithm is generally the most efficient one when the *computing time* is taken as a reference (it has the “highest” curve in figure 7.1). The speedup with respect to the RC algorithm can reach 10: this can be observed in table A.1 (ratio 10.14 of the PD algorithm on instance `perm-8`) or on figure 7.1 (by the abscissa of the rightmost change in the curve of the RC algorithm, whose relative performance is there given relatively to the PD algorithm).

7.3.2. Compact solvers. To show the interest of the compact versions of the algorithms, introduced in section 6, we compare each solver RC, P, PD and D to its compact version RC/C, P/C, PD/C and D/C. The computing times are given in table A.2 and the performance profiles are shown in figure 7.2. Recall that, by construction of the compact versions, the improvement ratios are bounded above by 2, approximately.

We observe indeed on table A.2 that the compact versions improve their standard version on the computing time, particularly for the PD/C, for which the mean (resp. median) improvement is 1.35 (resp. 1.35). The improvement bound 2 is obtained by D/C on the instances `srnd-8-20-4` and `perm-7`. This improvement can also be observed on figure 7.2, with more ambiguity, since it is not indicated which algorithm

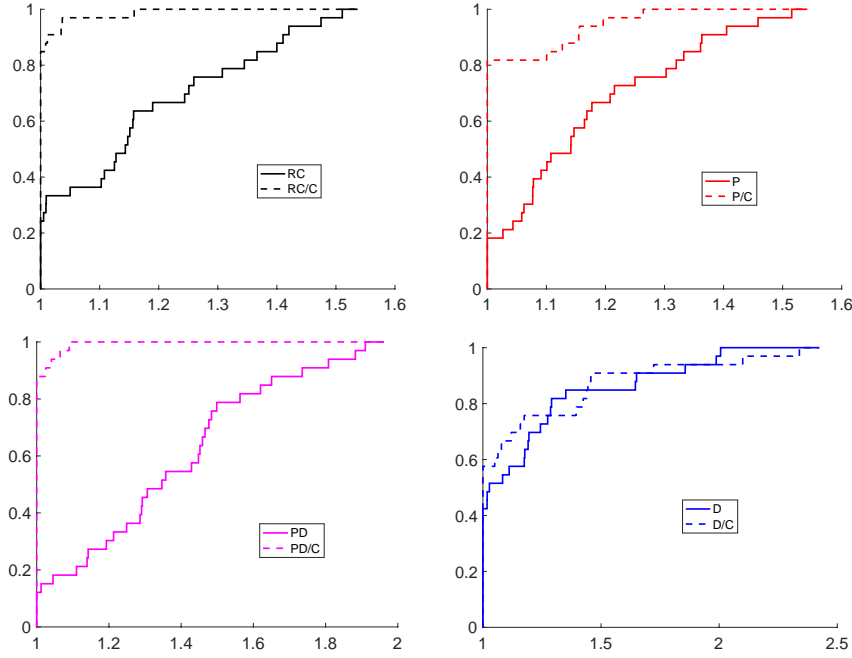


FIG. 7.2. Performance profiles of the RC vs. RC/C, P vs. P/C, PD vs. PD/C and D vs. D/C algorithms, for the computing time. The dashed lines refer to the compact versions of the algorithms.

is the best for each problem instance (for example the x -axis larger than 2 for the performance profiles D vs. D/C is not due to a performance of D/C that is 2.34 times better than D on some problem, but the opposite: it is D that is 2.34 times faster than D/C on some problem). Nevertheless, it is indeed algorithm A/C that generally outperforms algorithm A when $A = \text{RC}, P$ or PD (their curve is higher).

The performance profiles of figure 7.3 compares the most effective solver, namely PD/C, to the RC solver. The former shows a speedup that can reach 19.3.

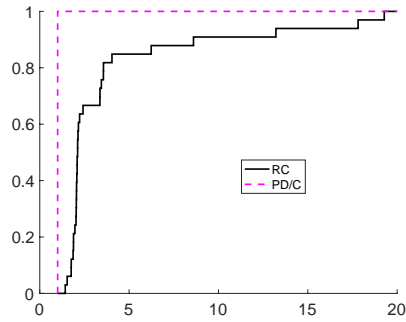


FIG. 7.3. Performance profiles of the RC vs. PD/C solvers, for the computing time.

8. Conclusion. This paper deals with the enumeration of the chambers of a hyperplane arrangement. It brings improvements to a recursive algorithm proposed by Rada and Černý, and proposes a family of new algorithms having, to various extends, dual aspects based on the Motzkin alternative, matroid circuits and the introduced

notion of *stem vector*. Most algorithms are grounded on a tree of sign vectors that are in a one-to-one correspondence with the chambers of the arrangement. Compact versions of the algorithms are also presented, which aim at reducing the size of the sign vector tree, in order to avoid duplicating costly identical operations like solving linear optimization problems and covering tests. The most efficient method of this algorithm anthology is the one that includes primal and dual ingredients, and uses the compact form of the tree, which has been named PD/C in the paper. The speedup it provides, with respect to Rada-Černý’s algorithm, much depends on the features of the considered arrangement, in particular its dimensions, and ranges between 1.4 and 19.3, with a mean value of 3.9.

These algorithms are grounded on a theory that is presented before their introduction. This one includes the structure of the sign vector sets and the stem vector sets, in particular conditions for their symmetry, their connectivity, their full cardinality and much more.

Numerous aspects of the presented algorithms can be further improved or developed, covering both conceptual and implementation aspects. Let us mention a few topics. (i) The linear optimization problems could be solved approximately, hence saving computation time when the tested sign vector is feasible. (ii) The way stem vectors are computed, stored and used could be improved, with specific structures designed for that purpose. (iii) In case the arrangements present combinatorial symmetries, the approaches presented in [39, 7] should increase significantly the algorithm performances. (iv) The proposed approaches could be extended to compute the chambers of the hyperplane arrangement and subarrangements, those recursively included in the hyperplane intersections of any smaller dimension.

Appendix A. Tables with numerical results.

This appendix gives the tables with the detailed numerical results, comparing the solvers selected in section 7.2, on which the performance profiles of figures 7.1, 7.2 and 7.3 are based. Comments on these results can be found in section 7.3. Table A.1 deals with the standard algorithms and table A.2 is related to the compact versions of these algorithms.

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Problems	RC	P		PD		D	
	time	time	ratio	time	ratio	time	ratio
rand-2-8	0.09	0.03	3.76	0.03	3.51	0.02	4.13
rand-4-8	0.12	0.07	1.71	0.08	1.47	0.06	1.86
rand-4-9	0.20	0.12	1.67	0.13	1.56	0.10	1.99
rand-5-10	0.48	0.27	1.79	0.31	1.58	0.26	1.89
rand-4-11	0.56	0.35	1.58	0.38	1.47	0.30	1.88
rand-6-12	1.89	1.18	1.59	1.34	1.41	1.09	1.73
rand-5-13	2.32	1.37	1.69	1.45	1.60	1.30	1.79
rand-7-14	6.96	4.37	1.59	5.14	1.35	4.66	1.50
rand-7-15	12.40	7.51	1.65	8.92	1.39	10.10	1.22
rand-8-16	29.20	17.50	1.67	21.10	1.38	28.30	1.03
rand-9-17	63.00	39.40	1.60	50.50	1.25	70.80	0.89
srand-8-20-2	33.30	9.70	3.43	6.67	5.00	23.00	1.45
srand-8-20-4	199.00	105.00	1.90	137.00	1.45	989.00	0.20
srand-8-20-6	238.00	131.00	1.81	196.00	1.21	947.00	0.25
2d-4-20	2.12	0.89	2.38	0.60	3.53	1.01	2.11
2d-5-20	3.50	1.58	2.22	0.95	3.67	1.86	1.88
2d-6-20	5.84	2.82	2.07	1.66	3.53	3.11	1.88
2d-7-20	9.86	4.48	2.20	2.55	3.86	5.24	1.88
2d-8-20	16.40	7.35	2.23	4.32	3.80	8.13	2.02
perm-5	1.17	0.46	2.53	0.24	4.96	0.82	1.42
perm-6	10.60	3.04	3.50	1.45	7.33	7.11	1.50
perm-7	106.00	23.60	4.49	11.20	9.50	128.00	0.83
perm-8	1070.00	210.00	5.10	106.00	10.14	2970.00	0.36
ratio-3-20-0.7	2.53	1.54	1.65	1.39	1.82	2.12	1.19
ratio-3-20-0.9	2.59	1.80	1.44	1.41	1.84	2.16	1.20
ratio-4-20-0.7	11.10	6.17	1.80	5.94	1.87	12.50	0.89
ratio-4-20-0.9	7.07	5.53	1.28	4.33	1.63	9.46	0.75
ratio-5-20-0.7	21.00	12.00	1.75	12.80	1.64	38.10	0.55
ratio-5-20-0.9	16.00	11.30	1.42	9.57	1.67	22.40	0.71
ratio-6-20-0.7	75.90	46.10	1.65	58.50	1.30	183.00	0.42
ratio-6-20-0.9	65.40	43.60	1.50	50.10	1.30	175.00	0.37
ratio-7-20-0.7	147.00	109.00	1.36	151.00	0.98	523.00	0.28
ratio-7-20-0.9	148.00	96.40	1.54	138.00	1.07	538.00	0.28
Mean			2.11		2.76		1.28
Median			1.71		1.63		1.23

TABLE A.1

Computing times (in seconds) for the standard algorithms listed in section 7.2. For each algorithm $A := P, PD$ or D , the second column gives the ratios $\text{time}(RC)/\text{time}(A)$

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Problems	RC/C			P/C			PD/C			D/C		
	time	ratio	ratio	time	ratio	ratio	time	ratio	ratio	time	ratio	ratio
rand-2-8	0.08	1.25	1.25	0.03	0.89	3.33	0.03	0.96	3.37	0.03	0.85	3.52
rand-4-8	0.08	1.37	1.37	0.05	1.33	2.28	0.05	1.45	2.14	0.05	1.24	2.32
rand-4-9	0.16	1.24	1.24	0.10	1.21	2.02	0.11	1.21	1.89	0.09	1.11	2.20
rand-5-10	0.35	1.40	1.40	0.22	1.25	2.24	0.24	1.29	2.05	0.22	1.18	2.23
rand-4-11	0.49	1.15	1.15	0.30	1.16	1.84	0.30	1.29	1.89	0.29	1.03	1.93
rand-6-12	1.34	1.41	1.41	0.87	1.36	2.18	0.90	1.48	2.09	0.92	1.19	2.07
rand-5-13	1.95	1.19	1.19	1.20	1.14	1.93	1.11	1.31	2.09	1.20	1.08	1.93
rand-7-14	4.90	1.42	1.42	3.11	1.41	2.24	3.43	1.50	2.03	3.90	1.19	1.78
rand-7-15	9.22	1.34	1.34	5.69	1.32	2.18	6.04	1.48	2.05	8.58	1.18	1.45
rand-8-16	19.80	1.47	1.47	12.00	1.46	2.43	13.50	1.56	2.16	17.20	1.65	1.70
rand-9-17	41.70	1.51	1.51	26.00	1.52	2.42	30.60	1.65	2.06	42.90	1.65	1.47
srand-8-20-2	30.20	1.10	1.10	9.45	1.03	3.52	5.34	1.25	6.24	22.60	1.02	1.47
srand-8-20-4	158.00	1.26	1.26	80.60	1.30	2.47	93.90	1.46	2.12	493.00	2.01	0.40
srand-8-20-6	182.00	1.31	1.31	96.10	1.36	2.48	121.00	1.62	1.97	701.00	1.35	0.34
2d-4-20	2.10	1.01	1.01	1.03	0.87	2.06	0.62	0.98	3.45	1.74	0.58	1.22
2d-5-20	3.54	0.99	0.99	1.89	0.84	1.85	1.04	0.92	3.37	2.71	0.69	1.29
2d-6-20	5.89	0.99	0.99	3.26	0.87	1.79	1.64	1.01	3.56	4.43	0.70	1.32
2d-7-20	9.81	1.01	1.01	4.93	0.91	2.00	2.44	1.05	4.04	7.31	0.72	1.35
2d-8-20	16.40	1.00	1.00	9.29	0.79	1.77	4.60	0.94	3.57	11.70	0.69	1.40
perm-5	1.04	1.12	1.12	0.42	1.11	2.80	0.14	1.74	8.60	0.64	1.29	1.83
perm-6	9.27	1.14	1.14	2.82	1.08	3.76	0.80	1.81	13.22	5.58	1.27	1.90
perm-7	94.00	1.13	1.13	22.30	1.06	4.75	5.95	1.88	17.82	64.40	1.99	1.65
perm-8	1070.00	1.00	1.00	195.00	1.08	5.49	55.50	1.91	19.28	1600.00	1.86	0.67
ratio-3-20-0.7	2.53	1.00	1.00	1.45	1.06	1.74	1.22	1.14	2.07	4.96	0.43	0.51
ratio-3-20-0.9	2.68	0.97	0.97	1.65	1.09	1.57	1.27	1.11	2.04	3.12	0.69	0.83
ratio-4-20-0.7	11.00	1.01	1.01	5.73	1.08	1.94	4.98	1.19	2.23	13.30	0.94	0.83
ratio-4-20-0.9	8.19	0.86	0.86	5.30	1.04	1.33	3.79	1.14	1.87	7.33	1.29	0.96
ratio-5-20-0.7	20.00	1.05	1.05	10.90	1.10	1.93	9.92	1.29	2.12	80.00	0.48	0.26
ratio-5-20-0.9	13.90	1.15	1.15	9.67	1.17	1.65	6.61	1.45	2.42	22.00	1.02	0.73
ratio-6-20-0.7	68.50	1.11	1.11	40.20	1.15	1.89	43.10	1.36	1.76	212.00	0.86	0.36
ratio-6-20-0.9	67.80	0.96	0.96	38.20	1.14	1.71	37.20	1.35	1.76	196.00	0.89	0.33
ratio-7-20-0.7	127.00	1.16	1.16	89.70	1.22	1.64	103.00	1.47	1.43	564.00	0.93	0.26
ratio-7-20-0.9	128.00	1.16	1.16	81.90	1.18	1.81	96.60	1.43	1.53	565.00	0.95	0.26
Mean		1.16	1.16		1.14	2.33		1.35	3.95		1.09	1.30
Median		1.14	1.14		1.14	2.02		1.35	2.12		1.03	1.35

TABLE A.2

Computing times (in seconds) for the compact algorithms listed in section 7.2. For each algorithm $A = RC, P, PD,$ or D , the first column gives the computing time of A/C in seconds, the second column gives the ratios $\text{time}(A)/\text{time}(A/C)$ (upper bounded by 2, approximately) and the third column gives the ratios $\text{time}(RC)/\text{time}(A/C)$.

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