Primal and dual approaches for the chamber enumeration of real hyperplane arrangements – The full report*

Jean-Pierre Dussault†, Jean Charles Gilbert‡ and Baptiste Plaquevent-Jourdain§

Hyperplane arrangements is a problem that appears in various theoretical and applied mathematical contexts. This paper focuses on the enumeration of the chambers of an arrangement, a task that most often requires algebraic or numerical computation. Among the recent numerical methods, Rada and Černý's recursive algorithm outperforms previous approches, by relying on a specific tree structure and on linear optimization. This paper presents modifications and improvements to this algorithm. It also introduces a dual approach solely grounded on matroid circuits and its associated concepts of *stem vectors*, thus avoiding the need to solve linear optimization problems. Along the way, theoretical properties of arrangements, such as their cardinality and conditions for their symmetry, completeness and connectivity, as well as properties of their various stem vector sets are presented with an analytic viewpoint. It is shown, in particular, that the set of the chambers of an affine arrangement is located between those of two related linear arrangements. This leads to compact forms of the algorithms, which solve less subproblems. The proposed methods have been implemented in Julia and their efficiency is assessed on various instances of arrangements; for the best of them, this efficiency manifests itself by speedup ratios in the range $[1.4, 19.3]$ with an average value of 3.9.

Keywords: Duality · Hyperplane arrangement · Matroid circuit · Motzkin's alternative · Schläfli's bound · Stem vector · Strict linear inequality system · Tree algorithm · Winder's formula.

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*This is an extended version of the paper [17]. It contains additional material, including proofs and comments. The added text is written in dark blue.

†Département d'Informatique, Faculté des Sciences, Université de Sherbrooke, Québec, Canada (e-mail: `Jean-Pierre.Dussault@Usherbrooke.ca`). ORCID 0000-0001-7253-7462.

‡Inria Paris, Serena team, 48 rue Barrault, CS 61534, 75647 Paris Cedex, France (e-mail: `Jean-Charles.Gilbert@inria.fr`) and Département de Mathématiques, Faculté des Sciences, Université de Sherbrooke, Québec, Canada. ORCID 0000-0002-0375-4663.

§Département de Mathématiques, Faculté des Sciences, Université de Sherbrooke, Québec, Canada (e-mail: `Baptiste.Plaquevent-Jourdain@USherbrooke.ca`) and Inria Paris, Serena team, 48 rue Barrault, CS 61534, 75647 Paris Cedex, France (e-mail: `Baptiste.Plaquevent-Jourdain@inria.fr`. ORCID 0000-0001-7055-4568).

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1 Introduction

A *hyperplane* of $\mathbb{R}^n$ is a set of the form $H := \{x \in \mathbb{R}^n : v^\mathsf{T} x = \tau\}$, where $v \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $v^\mathsf{T} x = \sum_{i=1}^n v_i x_i$ denotes the Euclidean scalar product of v and x. Usually, v is asked to be nonzero, but we have allowed v to vanish to make some formulas more compact below. For $v_1, \dots, v_p \in \mathbb{R}^n$ and $\tau_1, \dots, \tau_p \in \mathbb{R}$, consider the collection of hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^\mathsf{T} x = \tau_i\}$ for $i \in [1:p]$ (this is our notation for the set of the first p positive integers). The connected parts of the complement of their union, that is $\mathbb{R}^n \setminus (\cup_{i=1}^p H_i)$, are open polyhedrons, called *chambers* or *cells*. *Hyperplane arrangements* is the name given to the discipline that describes this structure [20, 47]. Its study started at least in the 19th century [40, 43, 49] and has continued until the present with theoretical contributions [1, 32, 51, 54], algorithmic developments [16, 18, 26, 37] as well as applications [6]; see also the references therein. Arrangements can also be stated for complex numbers [31] or over finite fields [10]; arrangements of circles on a sphere is also a subject of interest, with application in biology [8]. A powerful tool to study arrangements is the characteristic polynomial, which contains much information and provides one way of computing the number of chambers (see for instance [2, 46, 54]; see the proof of formula (3.28a) below, for a different analytic approach).

This paper focuses on the numerical *enumeration* of the chambers of an arrangement. Several approaches have been designed for that purpose. The algorithm of Bieri and Nef [4] recursively sweeps the space with hyperplanes, decreasing the dimension of the current space in order to explore arrangements in affine spaces of smaller dimension. Edelsbrunner, O'Rourke and Seidel [19] have designed an asymptotically optimal algorithm. The approach of Avis, Fukuda and Sleumer [3, 44] starts with an arbitrary chamber and moves from chamber to neighboring chamber, using a "reverse search" paradigm, thanks to the connectivity of the graph structure of the chambers (proposition 3.13 below). Rada and Černý [37] use a more efficient tree, called the $\mathcal{S}$-tree below, obtained by adding hyperplanes incrementally, whereas in previous approaches all hyperplanes are considered from the start. This tree algorithm possesses various interesting properties such as output-polynomiality, meaning that each individual chamber is obtained in polynomial time, and compactness, meaning that the required memory storage is low.

Several pieces of software in algebra or combinatorics, able to deal with arrangements, have been developed: `polymake` [27], `Sagemath` [50], `Macaulay2` [24], `OSCAR` [33]. Some related works, such as the package `CountingChambers.jl` [7], focus on the use of combinatorial symmetries, eventually alongside the deletion-restriction paradigm (see also [52]), to treat arrangements with underlying symmetries and many more hyperplanes. Similar considerations also appear in `TOPCOM` [38, 39] and yield very good results on particular instances.

Improvements to Rada and Černý's algorithm are proposed and benchmarked in [16]. The authors first present heuristics to bypass some computations. Then, they introduce a dual approach based on Gordan's theorem of the alternative [22], by introducing the notion of *stem vector*, closely related to the *circuits* of a *vector matroid* associated with the arrangement. These modifications allow the authors to significantly reduce the number of linear optimization problems (LOPs) to solve, therefore lowering the computing time, or even to completely remove the need of linear optimization. This paper extends the scope of [16] to arrangements with hyperplanes not necessarily containing the origin. We shall see that the heuristics introduced in [16] have natural extensions in this general case. The same is true for the dual approach, which is here grounded on Motzkin's alternative [29]; this one is indeed naturally associated with affine arrangements. These modifications are compared in the penultimate section of the paper.

This contribution is organized as follows. Section 2 presents some notation used throughout the paper as well as Motzkin's theorem of the alternative [29], crucial in this paper, which contributes to both theoretical and algorithmic aspects. Section 3 starts with the introduction of the concept of *hyperplane arrangement*. Then, it gives conditions ensuring some properties of the associated *sign vector* set, like its symmetry and its connectivity. Next, the section introduces the notion of *stem vector*, describes its set, gives its properties and shows how the stem vectors can be used to detect the infeasibility of sign vectors (covering test of proposition 3.19). Finally, the role of the *augmented matrix* is discussed. It is shown, in particular, that the sign vector set of an affine arrangement is located between the sign vector sets of two linear arrangements. Information on the number of chambers is also given or recalled, in particular when this one is in *affine general position*.

The rest of the paper focuses on algorithmic issues. Section 4 first describes the algorithm of [37], its recursion process and its use of linear optimization. Then, we adapt the heuristic ideas proposed in [16] to affine arrangements, which improves the efficiency of the previous algorithm. Section 5 focuses on dual algorithms, which use the stem vectors and often require less computing time. Section 6 shows how a compact form of the algorithms can be constructed, taking advantage of the fact that only half of the symmetric sign vectors need to be stored. Often, this technique also allows the compact algorithms to save computing time. Finally, section 7 presents the instances used to test the algorithms, their features and some numerical results.

Our presentation is more based on linear algebra and (convex) analysis rather than on discrete geometry or algebra. More specifically, the notion of circuit of a vector matroid and the duality concepts of convex analysis are prominent in sections 3, 5 and 6. In some places, new proofs to known results are proposed with these points of view. This allows the readers with an analytic bent to have easier access to these results.

An abridged version of this paper can be found in [17].

2 Background

This section begins with a paragraph on the notation. Next, it recalls a few known results that are useful in the paper.

One denotes by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ the sets of integers, nonnegative integers and real numbers and one sets $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ and $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$ ($r \in \mathbb{R}$ is said to be *positive* if $r > 0$ and *nonnegative* if $r \geqslant 0$). For two integers $n_1 \leqslant n_2$, $[n_1 : n_2] := \{n_1, \ldots, n_2\}$ is the set of the integers between n_1 and n_2. We denote by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geqslant 0\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x > 0\}$ the nonnegative and positive orthants, where the inequalities apply componentwise. For a set S, one denotes by $|S|$ its cardinality, by S^c its complement in a set that will be clear from the context and by S^J, for an index set $J \subseteq \mathbb{N}^*$, the set of vectors, whose elements are in S and are indexed by the indices in J. The vector e denotes the vector of all ones, whose size depends on the context. The Hadamard product of u and $v \in \mathbb{R}^n$ is the vector $u \cdot v \in \mathbb{R}^n$, whose ith component is $u_i v_i$. The sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is defined by $\operatorname{sgn}(t) = +1$ if $t > 0$, $\operatorname{sgn}(t) = -1$ if $t < 0$ and $\operatorname{sgn}(0) = 0$. The sign of a vector x or a matrix M is defined componentwise: $\operatorname{sgn}(x)_i = \operatorname{sgn}(x_i)$ and $[\operatorname{sgn}(M)]_{i,j} = \operatorname{sgn}(M_{i,j})$ for i and j. For $u \in \mathbb{R}^n$, $|u| \in \mathbb{R}^n$ is the vector defined by $|u|_i = |u_i|$ for all $i \in [1 : n]$. The dimension of a space $\mathbb{E}$ is denoted by $\dim(\mathbb{E})$, the range space of a matrix $A \in \mathbb{R}^{m\times n}$ by $\mathcal{R}(A)$, its null space by $\mathcal{N}(A)$, its rank by $\operatorname{rank}(A) := \dim \mathcal{R}(A)$ and its nullity by $\operatorname{null}(A) := \dim \mathcal{N}(A) = n - \operatorname{rank}(A)$ thanks to the rank-nullity theorem. The ith row (resp. column) of A is denoted by $A_{i,:}$ (resp. $A_{:,i}$). Transposition operates after a row and/or column selection: $A_{i,:}^{\mathsf{T}}$ is a short notation for $(A_{i,:})^{\mathsf{T}}$ for instance. The vertical concatenation of matrices $A \in \mathbb{R}^{n_1 \times m}$ and $B \in \mathbb{R}^{n_2 \times m}$ is denoted by $[A; B] \in \mathbb{R}^{(n_1+n_2) \times m}$. For $u \in \mathbb{R}^n$, $\operatorname{Diag}(u) \in \mathbb{R}^{n \times n}$ is the square diagonal matrix with $\operatorname{Diag}(u)_{i,i} = u_i$. The orthogonal of a subspace $Z \subseteq \mathbb{R}^n$ is denoted by $Z^{\perp} := \{x \in \mathbb{R}^n : x^{\mathsf{T}} z = 0, \text{ for all } z \in Z\}$.

This article makes extensive use of the so-called (there have been many contributors) Motzkin theorem of the alternative [29] [25, theorem 3.17], abbreviated as *Motzkin's alternative* below, whose following simplified expression is appropriate for our purpose (the general version also includes affine equalities and non strict affine inequalities). Let us write it as an equivalence, rather than an alternative: for a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $a \in \mathbb{R}^m$,

$$\exists\, x \in \mathbb{R}^n : \ Ax > a \quad \Longleftrightarrow \quad \nexists\, \alpha \in \mathbb{R}^m_+ \setminus \{0\} : \ A^{\mathsf{T}} \alpha = 0, \ a^{\mathsf{T}} \alpha \geqslant 0. \tag{2.1}$$

Gordan's theorem of the alternative [22, 1873] is recovered when $a = 0$:

$$\exists\, x \in \mathbb{R}^n : \ Ax > 0 \quad \Longleftrightarrow \quad \nexists\, \alpha \in \mathbb{R}^m_+ \setminus \{0\} : \ A^{\mathsf{T}} \alpha = 0. \tag{2.2}$$

The latter equivalence satisfies the needs in [16] because the inequality systems encountered in that paper are homogeneous. It will also be helpful below.

The next lemma will be applied several times. It is taken from [16, lemma 2.6] and is a refinement of [53, lemma 2.1]. It is useful to get a discriminating property by a small perturbation of a point.

Lemma 2.1 (discriminating covectors) *Suppose that $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ is a Euclidean vector space, $p \in \mathbb{N}^*$ and $v_1, \ldots, v_p$ are p distinct vectors of $\mathbb{E}$. Then, the set of vectors $\xi \in \mathbb{E}$ such that $|\{\langle \xi, v_i\rangle : i \in [1:p]\}| = p$ is dense in $\mathbb{E}$.*

3 Hyperplane arrangements

3.1 Presentation

Let $n \in \mathbb{N}^*$. A *hyperplane* of $\mathbb{R}^n$ is a set of the form $H := \{x \in \mathbb{R}^n : v^{\mathsf{T}} x = \tau\}$, where $v \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. This hyperplane H is said to be *proper* if $v \neq 0$ and *improper* otherwise. A proper hyperplane H partitions $\mathbb{R}^n$ into three subsets: H itself and its negative and positive open halfspaces, respectively defined by

$$H^- := \{x \in \mathbb{R}^n : v^{\mathsf{T}} x < \tau\} \qquad \text{and} \qquad H^+ := \{x \in \mathbb{R}^n : v^{\mathsf{T}} x > \tau\}.$$

If H is improper and $\tau = 0$, then $H = \mathbb{R}^n$ and $H^- = H^+ = \varnothing$. If H is improper and $\tau \neq 0$, then $H = \varnothing$ and $H^+ = \mathbb{R}^n$ or $\varnothing$, while $H^- = (H^+)^c$.

A *hyperplane arrangement* is a collection of $p \in \mathbb{N}^*$ hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^{\mathsf{T}} x = \tau_i\}$, for $i \in [1:p]$, where $v_1, \ldots, v_p \in \mathbb{R}^n$ and $\tau_1, \ldots, \tau_p \in \mathbb{R}$. It is denoted by $\mathcal{A}(V, \tau)$, where $V := [v_1 \ \cdots \ v_p] \in \mathbb{R}^{n \times p}$ is the matrix made of the vectors v_i's and $\tau := [\tau_1; \ldots; \tau_p] \in \mathbb{R}^{p \times 1}$. The arrangement $\mathcal{A}(V, \tau)$ is said to be *proper* if V has no zero column (i.e., its hyperplanes are proper) and *improper* otherwise (in proposition 4.6, a construction may yield a harmless improper arrangement, which is the reason why we introduce this concept). The arrangement is said to be *linear* if $\tau = 0$ and *affine* in general (therefore, a linear arrangement is just a particular affine arrangement). The arrangement is said to be *centered* if all the hyperplanes have a point in common [2], which is the case if and only if $\tau \in \mathcal{R}(V^{\mathsf{T}})$ (proposition 3.5).

Whilst a proper hyperplane divides $\mathbb{R}^n$ into two nonempty open halfspaces, a proper hyperplane arrangement splits $\mathbb{R}^n$ into nonempty polyhedral convex open sets, called *chambers* (the precise definition of a chamber is given below). This is illustrated in figure 3.1 by two elementary examples that will accompany us throughout the paper. Enumerating the chambers

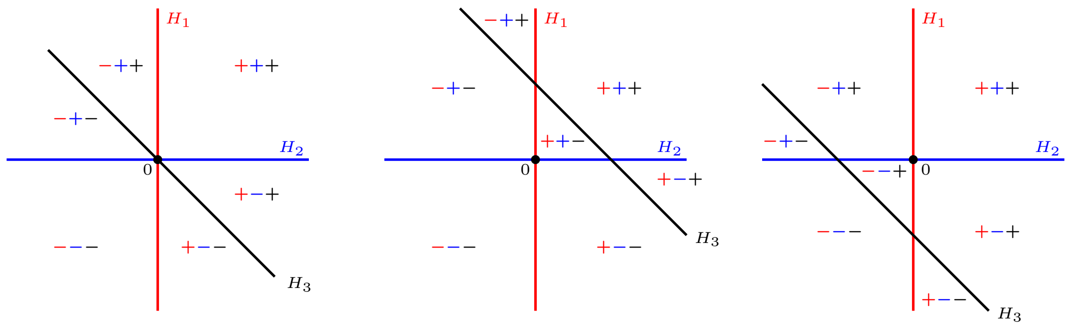


Figure 3.1: Arrangements in $\mathbb{R}^2$ specified by the hyperplanes $H_1 := \{x \in \mathbb{R}^2 : x_1 = 0\}$, $H_2 := \{x \in \mathbb{R}^2 : x_2 = 0\}$, $H_3(\text{left}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$, $H_3(\text{middle}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ and $H_3(\text{right}) := \{x \in \mathbb{R}^2 : x_1 + x_2 = -1\}$. The origin is contained in all the hyperplanes but in $H_3(\text{middle})$ and $H_3(\text{right})$, so that the arrangement in the left-hand side is *linear* with 6 chambers and the other ones are *affine* with 7 chambers.

is the problem at hand in this paper and it can be made precise in the following way. Let

$$\mathfrak{B}([1\!:\!p]) := \{(I_+, I_-) \in [1\!:\!p]^2 : I_+ \cap I_- = \varnothing,\ I_+ \cup I_- = [1\!:\!p]\}$$

be the collection of *bipartitions* (i.e., partitions into two subsets) of $[1\!:\!p]$. With each bipartition $(I_+, I_-) \in \mathfrak{B}([1\!:\!p])$, one can associate the set

$$C(I_+, I_-) := (\cap_{i \in I_+} H_i^+) \cap (\cap_{i \in I_-} H_i^-). \tag{3.1}$$

Some of these 2^p sets may be empty, while we are interested in enumerating the nonempty ones, which are called the *chambers* of the arrangement. The collection of these chambers, indexed by the bipartitions of $[1\!:\!p]$, is denoted by

$$\mathfrak{C}(V, \tau) := \{(I_+, I_-) \in \mathfrak{B}([1\!:\!p]) : C(I_+, I_-) \neq \varnothing\}. \tag{3.2}$$

When $\mathfrak{C}(V, \tau) = \mathfrak{B}([1\!:\!p])$, the arrangement $\mathcal{A}(V, \tau)$ is said to be *complete*.

As shown by the following proposition, this problem is equivalent to determining the sign vectors $s \in \{\pm 1\}^p$ that make a set of strict inequalities feasible. The collection of these sign vectors is denoted by

$$\mathcal{S}(V, \tau) := \{s \in \{\pm 1\}^p : s \cdot (V^\mathsf{T} x - \tau) > 0 \text{ for some } x \in \mathbb{R}^n\}, \tag{3.3}$$

where "$\cdot$" denotes the Hadamard product. A sign vector $s \in \{\pm 1\}^p$ in $\mathcal{S}(V, \tau)$ is said to be *feasible*, while it said to be *infeasible* if it is in the complementary set

$$\mathcal{S}(V, \tau)^c := \{\pm 1\}^p \setminus \mathcal{S}(V, \tau).$$

For $s \in \mathcal{S}(V, \tau)$, a point x verifying the system of strict inequalities in (3.3) is called a *witness point* of s [37]. It is often more convenient to work with these sign vectors $s \in \{\pm 1\}^p$ rather than with the bipartitions (I_+, I_-) of $[1\!:\!p]$ and we shall do so in the rest of the paper. To establish the correspondence between the bipartitions (I_+, I_-) of $[1:p]$ and the sign vectors s of $\{\pm 1\}^p$, one uses the following bijection

$$\phi : (I_+, I_-) \in \mathfrak{B}([1:p]) \mapsto s \in \{\pm 1\}^p, \quad \text{where } s_i = \begin{cases} +1 & \text{if } i \in I_+ \\ -1 & \text{if } i \in I_-, \end{cases} \tag{3.4}$$

whose inverse is given by

$$\phi^{-1} : s \in \{\pm 1\}^p \mapsto (\{i \in [1:p] : s_i = +1\}, \{i \in [1:p] : s_i = -1\}) \in \mathfrak{B}([1:p]).$$

Proposition 3.1 (chambers and sign vectors) *The map ϕ given by (3.4) is a bijection between the chamber set $\mathfrak{C}(V, \tau)$ and the sign vector set $\mathcal{S}(V, \tau)$.*

Proof. Let $(I_+, I_-) \in \mathfrak{B}([1:p])$ and $s := \phi((I_+, I_-))$. One has

$$\begin{aligned} (I_+, I_-) \in \mathfrak{C}(V, \tau) \quad &\iff \quad \exists\, x \in \mathbb{R} :\ v_i^{\mathsf{T}} x > \tau_i \text{ for } i \in I_+ \text{ and } v_i^{\mathsf{T}} x < \tau_i \text{ for } i \in I_- \\ &\iff \quad \exists\, x \in \mathbb{R} :\ s \cdot (V^{\mathsf{T}} x - \tau) > 0 \\ &\iff \quad s \in \mathcal{S}(V, \tau). \end{aligned}$$

This proves the bijectivity of $\phi : \mathfrak{C}(V, \tau) \to \mathcal{S}(V, \tau)$ and concludes the proof. $\square$

A consequence of this proposition is that it is equivalent to determine the chamber set $\mathfrak{C}(V, \tau)$ (geometric viewpoint) or the sign vector set $\mathcal{S}(V, \tau)$ (analytic viewpoint).

By this proposition, an arrangement $\mathcal{A}(V, \tau)$ is complete if and only if $\mathcal{S}(V, \tau) = \{\pm 1\}^p$. Note that the proposition does not assume that the hyperplanes are different. Observe also that an arrangement with identical hyperplanes is not complete.

When the hyperplanes are linear (i.e., $\tau = 0$), the description of $\mathcal{S}(V, \tau)$ has various reformulations, sometimes in very different areas of mathematics: see [16] for some of them and the references therein for others.

3.2 Properties

Hyperplane arrangements benefit from a myriad of properties. In this section, we mention and prove some of them, those that are relevant for the enumeration of chambers. Most of these properties extend, with adjustments, to affine arrangements those that are valid for linear arrangements, in particular those presented in [16] and in the references therein. For further developments and different viewpoints, see for instance [1, 32, 47, 54].

Let $V = [v_1 \ \cdots \ v_p] \in \mathbb{R}^{n \times p}$, $\tau = (\tau_1, \ldots, \tau_p) \in \mathbb{R}^p$ and $r := \operatorname{rank}(V)$. In the sequel, we consider the arrangement $\mathcal{A}(V, \tau)$ formed by the p hyperplanes $H_i := \{x \in \mathbb{R}^n : v_i^{\mathsf{T}} x = \tau_i\}$ for $i \in [1:p]$.

The next proposition gives conditions characterizing the fact that two hyperplanes are parallel or identical (two hyperplanes H and $\tilde{H}$ are said to be *parallel* if they have the same parallel subspace, that is, if $H - H = \tilde{H} - \tilde{H}$). Discarding identical hyperplanes is important to simplify the task of the algorithms enumerating the chambers and the proposition explains how to detect them from the columns of the matrix $[V; \tau^{\mathsf{T}}]$. Identical hyperplanes prevent an arrangement from being connected (proposition 3.13) and from being in general position (definitions 3.28 and 3.34). Below, we say that two vectors v and $\tilde{v} \in \mathbb{R}^n$ are *colinear* if there is an $\alpha \in \mathbb{R}^*$ such that $\tilde{v} = \alpha v$ (hence v and $\tilde{v}$ vanish simultaneously).

Proposition 3.2 (parallel and identical hyperplanes) *Let $H = \{x \in \mathbb{R}^n : v^\mathsf{T} x = \tau\}$ and $\tilde{H} = \{x \in \mathbb{R}^n : \tilde{v}^\mathsf{T} x = \tilde{\tau}\}$ be two nonempty hyperplanes. Then,*

1) *H and $\tilde{H}$ are parallel if and only if v and $\tilde{v}$ are colinear in $\mathbb{R}^n$,*
2) *$H = \tilde{H}$ if and only if (v, τ) and $(\tilde{v}, \tilde{\tau})$ are colinear in $\mathbb{R}^n \times \mathbb{R}$.*

Proof. 1) Clearly, $H - H = v^\perp$, so that H and $\tilde{H}$ are parallel if and only if $v^\perp = \tilde{v}^\perp$ or v and $\tilde{v}$ are colinear.

2) If $\tilde{H} = H$, $\tilde{v} = \alpha v$ for some $\alpha \in \mathbb{R}^*$ by point 1. Now, since $\tau v / \|v\|^2 \in H$, one has $\tau v / \|v\|^2 \in \tilde{H}$ or $\tau \tilde{v}^\mathsf{T} v / \|v\|^2 = \tilde{\tau}$ or $\alpha \tau = \tilde{\tau}$. Hence (v, τ) and $(\tilde{v}, \tilde{\tau})$ are colinear. Reciprocally, if (v, τ) and $(\tilde{v}, \tilde{\tau})$ are colinear, clearly $H = \tilde{H}$. □

The next proposition identifies some modifications of (V, τ) that have no effect on the sign vector set $\mathcal{S}(V, \tau)$. For a matrix M, we denote by $\operatorname{sgn}(M)$ the matrix defined by $[\operatorname{sgn}(M)]_{i,j} = \operatorname{sgn}(M_{i,j})$ for all i, j. Point 1 of the next proposition is related to proposition 3.2(2).

Proposition 3.3 (equivalent arrangements) *Let $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$.*

1) *If $D \in \mathbb{R}^{p \times p}$ is a nonsingular diagonal matrix, then $\mathcal{S}(VD, D\tau) = \operatorname{sgn}(D)\mathcal{S}(V, \tau)$.*
2) *If $M \in \mathbb{R}^{m \times n}$, then $\mathcal{S}(MV, \tau) \subseteq \mathcal{S}(V, \tau)$ with equality if M is injective.*

Proof. 1) This is because $s \cdot ((VD)^\mathsf{T} x - (D\tau)) > 0$ for some $x \in \mathbb{R}^n$ if and only if $(\operatorname{sgn}(D)s) \cdot (V^\mathsf{T} x - \tau) > 0$.

2) If $s \in \mathcal{S}(MV, \tau)$, then $s \cdot (V^\mathsf{T} M^\mathsf{T} \tilde{x} - \tau) > 0$ for some $\tilde{x} \in \mathbb{R}^m$, so that $s \cdot (V^\mathsf{T} x - \tau) > 0$ for some $x = M^\mathsf{T} \tilde{x} \in \mathbb{R}^n$ or $s \in \mathcal{S}(V, \tau)$. The reciprocal holds when M^T is surjective. □

A consequence of proposition 3.3(2) is that, as far as the sign vector set $\mathcal{S}(V, \tau)$ is concerned, the rank r of $V \in \mathbb{R}^{n \times p}$ is more relevant than its row dimension n. Indeed, $r \leqslant n$ and when $r < n$, one can ignore $n - r$ dependent rows of V, without modifying $\mathcal{S}(V, \tau)$. More specifically, assuming that the last $n - r$ rows of V are linearly dependent of its first r rows, one can write

$$V = \begin{bmatrix} I_r \\ A \end{bmatrix} V_{[1:r],:},$$

for some matrix $A \in \mathbb{R}^{(n-r) \times p}$. Since $[I_r; A]$ is injective, one has $\mathcal{S}(V, \tau) = \mathcal{S}(V_{[1:r],:}, \tau)$ by proposition 3.3(2). Now, the dimensions of $V_{[1:r],:}$ do not involve n, so that this presentation indicates that the role of n is not very relevant and explains why many results below show r instead of n.

Now that we have identified the chamber set $\mathfrak{C}(V, \tau)$ with the sign vector set $\mathcal{S}(V, \tau)$ (proposition 3.1), we are led to the introduction of the notion of symmetry, which naturally presents itself in $\{\pm 1\}^p$.

Definitions 3.4 (symmetric sign vector set) A set of sign vectors $S \subseteq \{\pm 1\}^p$ with $p \in \mathbb{N}^*$ is said to be *symmetric* if $-S = S$; otherwise, it is said *asymmetric*. For a given set $S \subseteq \{\pm 1\}^p$, one says that $s \in \{\pm 1\}^p$ is *symmetric in S* if $\pm s \in S$. □

This notion of symmetry intervenes in the design of the algorithms computing $\mathcal{S}(V,\tau)$. In particular, when this set is symmetric, only half of it need to be computed. Using the definition (3.3) of $\mathcal{S}(V,\tau)$, it follows immediately that

$$\mathcal{S}(V,0) \text{ is symmetric.} \tag{3.5}$$

Indeed, when $s \in \mathcal{S}(V,0)$, one has $s \cdot (Vx) > 0$ for some x, next $-s \cdot (V(-x)) > 0$, which shows that $-s \in \mathcal{S}(V,0)$. The next proposition shows that this symmetry property occurs for $\mathcal{S}(V,\tau)$ if and only if the arrangement is centered.

Proposition 3.5 (symmetry characterization) *Let $\mathcal{A}(V,\tau)$ be a proper arrangement. Then, the following properties are equivalent:*

(*i*) *$\mathcal{S}(V,\tau)$ is symmetric,*

(*ii*) *$\tau \in \mathcal{R}(V^{\mathsf{T}})$,*

(*iii*) *the arrangement is centered.*

Proof. $[(i) \Rightarrow (ii)]$ One can decompose τ as follows:

$$\tau = \tau^0 + V^{\mathsf{T}}\hat{x}, \tag{3.6a}$$

where $\tau^0 \in \mathcal{N}(V)$ and $\hat{x} \in \mathbb{R}^n$. We pursue by contraposition, assuming that $\tau^0 \neq 0$. Hence $I := \{i \in [1\!:\!p] : \tau_i^0 \neq 0\}$ is nonempty. Define $s \in \{\pm 1\}^p$ by

$$s_I := \operatorname{sgn}(\tau_I^0), \tag{3.6b}$$

while s_{I^c} is defined below in order to get

$$s \notin \mathcal{S}(V,\tau) \qquad \text{and} \qquad -s \in \mathcal{S}(V,\tau).$$

These properties suffice to prove the implication "$(i) \Rightarrow (ii)$". Set $S_I := \operatorname{Diag}(s_I)$.

To prove that $s \notin \mathcal{S}(V,\tau)$, whatever $s_{I^c} \in \{\pm 1\}^{I^c}$ is, observe that $\alpha_I := |\tau_I^0| \in \mathbb{R}_+^I \setminus \{0\}$ verifies

$$V_{:,I}S_I\alpha_I = 0 \qquad \text{and} \qquad (\tau_I^0)^{\mathsf{T}} S_I\alpha_I = \|\tau_I^0\|_2^2 \geqslant 0.$$

By Motzkin's alternative (2.1) with $A = S_I V_{:,I}^{\mathsf{T}}$ and $a = S_I\tau_I^0$, this is equivalent to

$$\nexists\, x \in \mathbb{R}^n : \quad S_I V_{:,I}^{\mathsf{T}} x > S_I \tau_I^0.$$

Hence, whatever $s_{I^c} \in \{\pm 1\}^{I^c}$ is, there is no $x \in \mathbb{R}^n$ such that $s \cdot (V^{\mathsf{T}}x - \tau^0) > 0$. Now, using (3.6a), we see that there is no $x \in \mathbb{R}^n$ such that $s \cdot (V^{\mathsf{T}}x - \tau) > 0$, which proves $s \notin \mathcal{S}(V,\tau)$.

Let us now show that $-s \in \mathcal{S}(V,\tau)$, for some s_{I^c} to specify. Observe that there is no $\alpha_I \in \mathbb{R}_+^I \setminus \{0\}$ such that

$$-V_{:,I}S_I\alpha_I = 0 \qquad \text{and} \qquad -|\tau_I^0|^{\mathsf{T}}\alpha_I \geqslant 0$$

because the last inequality, with $\alpha_I \geqslant 0$ and $|\tau_I^0| > 0$, implies that $\alpha_I = 0$. By Motzkin's alternative (2.1) with $A = -S_I V_{:,I}^{\mathsf{T}}$ and $a = -|\tau_I^0| = -S_I\tau_I^0$, this is equivalent to

$$\exists\, x \in \mathbb{R}^n : \quad -S_I V_{:,I}^{\mathsf{T}} x > -S_I \tau_I^0. \tag{3.6c}$$

Since the columns of V are nonzero, a small perturbation of x can maintain (3.6c) and ensures that the components of $V_{:,I^c}^{\mathsf{T}}x$ are nonzero (use, for example, the discriminating lemma 2.1 with the zero vector and the distinct v_i's with $i \in I^c$). Next, choosing $s_{I^c} := -\operatorname{sgn}(V_{:,I^c}^{\mathsf{T}}x)$ and setting $S_{I^c} := \operatorname{Diag}(s_{I^c})$ leads to

$$-S_{I^c}V_{:,I^c}^{\mathsf{T}}x > 0 = -S_{I^c}\tau_{I^c}^0. \tag{3.6d}$$

Thanks to (3.6c) and (3.6d), there is an $x \in \mathbb{R}^n$ such that $-s \cdot (V^{\mathsf{T}}x - \tau^0) > 0$. Now, using (3.6a), we see that there is an $x \in \mathbb{R}^n$ such that $-s \cdot (V^{\mathsf{T}}x - \tau) > 0$, which proves that $-s \in \mathcal{S}(V, \tau)$.

[$(ii) \Leftrightarrow (iii)$] Property (ii) is equivalent to the existence of $\hat{x} \in \mathbb{R}^n$ such that $V^{\mathsf{T}}\hat{x} = \tau$, which is itself equivalent to the fact that the hyperplanes have the point $\hat{x}$ in common, which means that the arrangement is centered.

[$(ii) \Rightarrow (i)$] Let $\hat{x} \in \mathbb{R}^n$ be such that $V^{\mathsf{T}}\hat{x} = \tau$. If $s \in \mathcal{S}(V, \tau)$, then $s \cdot (V^{\mathsf{T}}x - \tau) > 0$ for some $x \in \mathbb{R}^n$, hence $-s \cdot (V^{\mathsf{T}}(2\hat{x} - x) - \tau) = -s \cdot (V^{\mathsf{T}}(-x) + \tau) > 0$, implying that $-s \in \mathcal{S}(V, \tau)$. $\square$

Example 3.6 (an asymmetric sign vector set) Let $\mathcal{S}(V, \tau)$ be the sign vector set of the arrangement shown in the middle pane of figure 3.1, which is defined by

$$V := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \tau := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The unique decomposition of $\tau = \tau^0 + V^{\mathsf{T}}\hat{x}$ with $\tau^0 \in \mathcal{N}(V)$, similar to (3.6a) in the proof of proposition 3.5, is given by

$$\tau^0 = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad V^{\mathsf{T}}\hat{x} = \frac{1}{3}V^{\mathsf{T}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

The sign vector defined by (3.6b) is $s = \operatorname{sgn}(\tau^0) = (-1, -1, +1)$. The claim that $s \notin \mathcal{S}(V, \tau)$ made in the proof of proposition 3.5 is confirmed here since $s \cdot (V^{\mathsf{T}}x - \tau) > 0$ is equivalent to the system $x_1 < 0$, $x_2 < 0$ and $x_1 + x_2 > 1$, which is infeasible. One also see that $-s \in \mathcal{S}(V, \tau)$, since $-s \cdot (V^{\mathsf{T}}x - \tau) > 0$ is equivalent to the system $x_1 > 0$, $x_2 > 0$ and $x_1 + x_2 < 1$, which is feasible. $\square$

It is short to prove that

$$\mathcal{S}(V, -\tau) = -\mathcal{S}(V, \tau). \tag{3.7}$$

Indeed, if $s \in \mathcal{S}(V, -\tau)$, $s \cdot (V^{\mathsf{T}}x + \tau) > 0$ for some $x \in \mathbb{R}^n$. Therefore, $-s \cdot (V^{\mathsf{T}}(-x) - \tau) > 0$, showing that $-s \in \mathcal{S}(V, \tau)$ or $s \in -\mathcal{S}(V, \tau)$. We have shown the inclusion $\mathcal{S}(V, -\tau) \subseteq -\mathcal{S}(V, \tau)$. By changing τ into $-\tau$, one gets $\mathcal{S}(V, \tau) \subseteq -\mathcal{S}(V, -\tau) \subseteq \mathcal{S}(V, \tau)$, hence (3.7).

Let us now define the *symmetric* and *asymmetric parts* of $\mathcal{S}(V, \tau)$ by

$$\mathcal{S}_s(V, \tau) := \mathcal{S}(V, \tau) \cap \mathcal{S}(V, -\tau) \qquad \text{and} \qquad \mathcal{S}_a(V, \tau) := \mathcal{S}(V, \tau) \setminus \mathcal{S}_s(V, \tau). \tag{3.8}$$

Clearly, by (3.7), $\pm s \in \mathcal{S}(V, \tau)$ when $s \in \mathcal{S}_s(V, \tau)$, while $-s \notin \mathcal{S}(V, \tau)$ when $s \in \mathcal{S}_a(V, \tau)$. This justifes the names given to $\mathcal{S}_s(V, \tau)$ and $\mathcal{S}_a(V, \tau)$. One also observes that

$$\mathcal{S}_s(V, -\tau) = -\mathcal{S}_s(V, \tau) = \mathcal{S}_s(V, \tau) \qquad \text{and} \qquad \mathcal{S}_a(V, -\tau) = -\mathcal{S}_a(V, \tau). \tag{3.9}$$

Indeed,

$$\begin{aligned}
\mathcal{S}_s(V,-\tau) &= \mathcal{S}(V,-\tau) \cap \mathcal{S}(V,\tau) = \mathcal{S}_s(V,\tau), \\
-\mathcal{S}_s(V,\tau) &= [-\mathcal{S}(V,\tau)] \cap [-\mathcal{S}(V,-\tau)] = \mathcal{S}(V,-\tau) \cap \mathcal{S}(V,\tau) = \mathcal{S}_s(V,\tau), \\
\mathcal{S}_a(V,-\tau) &= \mathcal{S}(V,-\tau) \setminus \mathcal{S}_s(V,-\tau) = [-\mathcal{S}(V,\tau)] \setminus [-\mathcal{S}_s(V,\tau)] = -\mathcal{S}_a(V,\tau).
\end{aligned}$$

Proposition 3.7 (symmetry in $\mathcal{S}(V,\tau)$) 1) $\mathcal{S}(V,0) \subseteq \mathcal{S}(V,\tau)$, *with equality if and only if* $\mathcal{S}(V,\tau)$ *is symmetric,*
2) $\mathcal{S}_s(V,\tau) = \mathcal{S}(V,0)$.

Proof. 1a) Let us first show that $\mathcal{S}(V,0) \subseteq \mathcal{S}(V,\tau)$. Let $s \in \mathcal{S}(V,0)$, so that $s \cdot (V^{\mathsf{T}}x) > 0$ for some $x \in \mathbb{R}^n$. Then, $s \cdot (V^{\mathsf{T}}(tx) - \tau) > 0$ for t large enough, implying that $s \in \mathcal{S}(V,\tau)$.

2) [$\subseteq$] If $s \in \mathcal{S}_s(V,\tau)$, one has $\pm s \in \mathcal{S}(V,\tau)$ and there are points x and $\tilde{x}$ such that $s \cdot (V^{\mathsf{T}}x - \tau) > 0$ and $-s \cdot (V^{\mathsf{T}}\tilde{x} - \tau) > 0$. After adding these inequalities side by side, one gets $s \cdot (V^{\mathsf{T}}(x - \tilde{x})) > 0$, i.e., $s \in \mathcal{S}(V,0)$. [$\supseteq$] By point 1a, with τ and $-\tau$.

1b) If $\mathcal{S}(V,0) = \mathcal{S}(V,\tau)$, $\mathcal{S}(V,\tau)$ is symmetric since so is $\mathcal{S}(V,0)$ by (3.5). Conversely, if $\mathcal{S}(V,\tau)$ is symmetric, then $\mathcal{S}(V,\tau) = -\mathcal{S}(V,\tau) = \mathcal{S}(V,-\tau)$ by (3.7), so that $\mathcal{S}(V,\tau) = \mathcal{S}_s(V,\tau)$; next $\mathcal{S}(V,\tau) = \mathcal{S}(V,0)$ follows from point 2. $\square$

As a corollary of the previous proposition, one has for a matrix $V \in \mathbb{R}^{n\times p}$ of rank r and a vector $\tau \in \mathbb{R}^p$ such that $\mathcal{S}(V,\tau) \neq \varnothing$:

$$2^r \leqslant |\mathcal{S}(V,\tau)| \leqslant 2^{\check{p}}, \tag{3.10}$$

where $\check{p} := |\{i \in [1\,{:}\,p] : V_{:,i} \neq 0\}|$.

Proof. Let $I := \{i \in [1\,{:}\,p] : V_{:,i} = 0\}$. By the assumption $\mathcal{S}(V,\tau) \neq \varnothing$, τ_I has no zero component and any $s \in \mathcal{S}(V,\tau)$ verifies $s_I := -\operatorname{sgn}(\tau_I)$. Therefore, one can only consider the components of the sign vectors that are not in I, which amounts to assuming that $I = \varnothing$. For the lower bound, one has $\mathcal{S}(V,\tau) \supseteq \mathcal{S}(V,0)$ by proposition 3.7(1) and $|\mathcal{S}(V,0)| \geqslant 2^r$ by [16, (4.6a)]. The upper bound is clear since, by definition, $\mathcal{S}(V,\tau)$ is included in $\{\pm 1\}^p$, which has cardinality 2^p. $\square$

More precise lower and upper bounds on $|\mathcal{S}(V,0)|$ and $|\mathcal{S}(V,\tau)|$ are given in propositions 3.29 and 3.36, as well as in (4.8) below.

Proposition 3.7 tells us that $\mathcal{S}(V,\tau)$ is symmetric if and only if it is invariant with respect to $\tau \in \mathbb{R}^p$ (since it is then equal to $\mathcal{S}(V,0)$ whatever τ is). In the same spirit, one can give yet another characterization of the symmetry of $\mathcal{S}(V,\tau)$, now in terms of the invariance of the chamber existence with respect to $\tau \in \mathbb{R}^p$ and $s \in \mathcal{S}(V,\tau)$, provided that V has no vanishing column. Let us introduce the possibly empty sets, associated with $s \in \{\pm 1\}^p$, denoted by

$$C_\tau(s) := \{x \in \mathbb{R}^n : s \cdot (V^{\mathsf{T}}x - \tau) > 0\},$$

which are in one to one correspondence with the sets denoted by $C(I_1, I_2)$ in (3.1), thanks to the map ϕ defined in proposition 3.1. Proposition 3.9 also shows that the symmetry of $\mathcal{S}(V,\tau)$ is equivalent to the nonemptiness of the interior of the asymptotic cones $C_\tau(s)^\infty$ of $C_\tau(s)$ for all $s \in \mathcal{S}(V,\tau)$. Recall that the asymptotic cone of a nonempty *closed* convex set K

of a vector space $\mathbb{E}$ is defined by $K^\infty := \{d \in \mathbb{E} : x + \mathbb{R}_+ d \subseteq K\}$ and does not depend on the point x chosen in K [41, theorem 8.3]. Although a chamber is a nonempty *open* set, its asymptotic cone $C_\tau(s)^\infty := \{d \in \mathbb{R}^n : x + \mathbb{R}_+ d \subseteq C_\tau(s)\}$ is also independent of $x \in C_\tau(s)$ and reads

$$C_\tau(s)^\infty = \{d \in \mathbb{R}^n : s \cdot (V^\mathsf{T} d) \geqslant 0\}.$$

Writing $C_\tau(s)^\infty = \cap_{i \in [1:p]} \{d \in \mathbb{R}^n : s_i v_i^\mathsf{T} d \geqslant 0\}$ and using the fact that the interior operator, denoted by "int" below, commutes with a finite intersection, one has

$$\operatorname{int} C_\tau(s)^\infty = \{d \in \mathbb{R}^n : s \cdot (V^\mathsf{T} d) > 0\} = C_0(s), \tag{3.11}$$

By the identity $\operatorname{int} C_\tau(s)^\infty = C_0(s)$, the notion of asymptotic cone is not essential in the statement of proposition 3.9. We have preferred to keep it since it is geometrically expressive. Proposition 3.9 is actually a straightforward consequence of the following lemma.

Lemma 3.8 (chamber existence invariance) *Let $\mathcal{A}(V, \tau)$ be a proper arrangement and $s \in \mathcal{S}(V, \tau)$. Then, the following properties are equivalent:*

(i) $-s \in \mathcal{S}(V, \tau)$,
(ii) $\operatorname{int} C_\tau(s)^\infty \neq \varnothing$,
(iii) $C_0(s) \neq \varnothing$,
(iv) $C_{\tau'}(s) \neq \varnothing$ *for all* $\tau' \in \mathbb{R}^p$.

Proof. Since $s \in \mathcal{S}(V, \tau)$, there is an $x \in \mathbb{R}^n$ such that $s \cdot (V^\mathsf{T} x - \tau) > 0$.

$[(i) \Rightarrow (ii)]$ Since $-s \in \mathcal{S}(V, \tau)$, there is an $\tilde{x} \in \mathbb{R}^n$ such that $-s \cdot (V^\mathsf{T} \tilde{x} - \tau) > 0$. It results that $d := x - \tilde{x}$ satisfies $s \cdot (V^\mathsf{T} d) > 0$ (hence, necessarily, V has no zero column).

$[(ii) \Leftrightarrow (iii)]$ By (3.11).

$[(ii) \Rightarrow (iv)]$ By (ii), there is a $d \in \mathbb{R}^n$ verifying $s \cdot (V^\mathsf{T} d) > 0$. For $\tau' \in \mathbb{R}^p$, one has $s \cdot (V^\mathsf{T}(td) - \tau') > 0$ for t sufficiently large, so that $td \in C_{\tau'}(s)$, which is nonempty.

$[(iv) \Rightarrow (i)]$ If $C_{\tau'}(s) \neq \varnothing$ for all $\tau' \in \mathbb{R}^p$, one can find $\tilde{x} \in C_{-\tau}(s)$ or $s \cdot (V^\mathsf{T} \tilde{x} - (-\tau)) > 0$ or $-s \cdot (V^\mathsf{T}(-\tilde{x}) - \tau) > 0$. Therefore, $-s \in \mathcal{S}(V, \tau)$. □

Proposition 3.9 (symmetry and chamber existence invariance) *Let $\mathcal{A}(V, \tau)$ be a proper arrangement. Then, the following properties are equivalent:*

(i) $\mathcal{S}(V, \tau)$ *is symmetric,*
(ii) $\operatorname{int} C_\tau(s)^\infty \neq \varnothing$ *for all* $s \in \mathcal{S}(V, \tau)$,
(iii) $C_0(s) \neq \varnothing$ *for all* $s \in \mathcal{S}(V, \tau)$,
(iv) $C_{\tau'}(s) \neq \varnothing$ *for all* $\tau' \in \mathbb{R}^p$ *and all* $s \in \mathcal{S}(V, \tau)$.

Note that if a chamber $C_\tau(s)$ is nonempty and bounded (like the one specified by $s = (+1, +1, -1)$ in middle pane of figure 3.1), then $C_\tau(s)^\infty = \{0\}$ and $\mathcal{S}(V, \tau)$ is asymmetric by the implication "$(i) \Rightarrow (ii)$" of the previous proposition. The converse is not true. For instance, if V in the previous example is replaced by $[V; 0_{1\times p}] \in \mathbb{R}^{(n+1)\times p}$, $\mathcal{S}(V, \tau)$ is not modified (proposition 3.3(2)), hence asymmetric, although all the chambers are now unbounded.

It is possible to detect the symmetry of a sign vector $s \in \mathcal{S}(V, \tau)$ by determining the unboundedness of the following linear optimization problem (LOP)

$$\begin{array}{rl} \inf_{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}} & \alpha \\ \text{s.t.} & s \cdot (V^\mathsf{T} x - \tau) + \alpha e \geqslant 0, \end{array} \tag{3.12}$$

where $e \in \mathbb{R}^p$ is the vector of all ones. We say that this problem is *bounded* if its optimal value is finite and *unbounded* if its optimal value is $-\infty$. Detecting the unboundedness of a LOP is possible by the primal simplex algorithm.

Proposition 3.10 (symmetry detection by LO) *Let $V \in \mathbb{R}^{n\times p}$ without zero column, $\tau \in \mathbb{R}^p$ and $s \in \{\pm 1\}^p$. Then, $s \in \mathcal{S}(V, 0)$ if and only if the LOP (3.12) is unbounded.*

Proof. [$\Rightarrow$] If $s \in \mathcal{S}(V, 0)$, there is a direction $d \in \mathbb{R}^n$ such that $s \cdot (V^\mathsf{T} d) > 0$. Define $\beta := \min\{s_i v_i^\mathsf{T} d : i \in [1:p]\} > 0$. Let (x, α) be feasible for (3.12), which is possible by taking α sufficiently large. Then, $(x + td, \alpha - t\beta)$ is also feasible for (3.12), for any $t > 0$. Since $\beta > 0$, this implies that problem (3.12) is unbounded.

[$\Leftarrow$] By assumption, for any negative α, one can find an $x_\alpha \in \mathbb{R}^n$ such that $s\cdot(V^\mathsf{T} x_\alpha - \tau) + \alpha e \geqslant 0$, showing, in particular, that $s \in \mathcal{S}(V, \tau)$. Next, $-s \cdot (V^\mathsf{T}(-x_\alpha) - \tau) - 2s \cdot \tau + \alpha e \geqslant 0$. For α sufficiently negative, one has $\alpha e - 2s \cdot \tau < 0$, hence $-s \cdot (V^\mathsf{T}(-x_\alpha) - \tau) > 0$, showing that $-s \in \mathcal{S}(V, \tau)$. Hence $s \in \mathcal{S}_s(V, \tau) = \mathcal{S}(V, 0)$. □

The notions of adjacency and connectivity presented below are crucial in some approaches for computing $\mathcal{S}(V, \tau)$ [3, 44] and are related to *graph theory*.

Definition 3.11 (adjacency in $\{\pm 1\}^p$) Two sign vectors s^1 and $s^2 \in \{\pm 1\}^p$ are said to be *adjacent* if they differ by a single component. □

Definition 3.12 (connectivity in $\{\pm 1\}^p$) A *path* of *length* l in $S \subseteq \{\pm 1\}^p$ is a finite set of sign vectors $s^0, \ldots, s^l \in S$ such that s^k and s^{k+1} are adjacent for all $k \in [0:l-1]$; in which case the path is said to be joining s^0 to s^l in S. One says that a subset $S \subseteq \{\pm 1\}^p$ is connected if any pair of elements of S can be joined by a path in S. □

One can transfer the notions of adjacency and connectivity in $\{\pm 1\}^p$ (resp. $\mathcal{S}(V, \tau)$) to $\mathfrak{B}([1:p])$ (resp. $\mathfrak{C}(V, \tau)$), thanks to the bijection ϕ defined by (3.4) and proposition 3.1, thus providing a geometric point of view: two chambers are adjacent if their sets I_+ (or I_-) differ by a single index, which means that they are on either side of (or separated by) a single hyperplane; while connectivity means that one can join any two chambers by a continuous path in $\mathbb{R}^n$ that never crosses two or more hyperplanes simultaneously.

The next proposition indicates that, provided the hyperplanes are all different (see proposition 3.2(2) for an anytical expression of this property), the sign vectors of an arrangement form a connected set.

Proposition 3.13 (connectivity of $\mathcal{S}$) *The set $\mathcal{S}(V, \tau)$ of sign vectors of a proper affine arrangement is connected if and only if its hyperplanes are all different. In this case, any elements s and $\tilde{s}$ of $\mathcal{S}(V, \tau)$ can be joined by a path in $\mathcal{S}(V, \tau)$ of length $l := \sum_{i\in[1:p]} |\tilde{s}_i - s_i|/2 \leqslant p$ and there is no path in $\mathcal{S}(V, \tau)$ joining s and $\tilde{s}$ of smaller length.*

Proof. The fact that any path joining s and $\tilde{s}$ in $\mathcal{S} \equiv \mathcal{S}(V, \tau)$ is of length $\geqslant l$ is due to the fact that s and $\tilde{s}$ have l different components and that two adjacent sign vectors differ by a single component.

$[\Rightarrow]$ We prove the contrapositive. Suppose that, for some $i \neq j$ in $[1:p]$, the hyperplanes H_i and H_j are identical. Then, by proposition 3.2(2), the nonzero pairs (v_i, τ_i) and (v_j, τ_j) are colinear in $\mathbb{R}^n \times \mathbb{R}$: $(v_j, \tau_j) = \alpha(v_i, \tau_i)$, for some $\alpha \in \mathbb{R}^*$. Assume that $\alpha > 0$ (resp. $\alpha < 0$). For any $\tilde{s} \in \mathcal{S}$, there is an $\tilde{x} \in \mathbb{R}^n$ such that $\tilde{s} \cdot (V^\mathsf{T}\tilde{x} - \tau) > 0$, which implies that one must have $\tilde{s}_i = \tilde{s}_j$ (resp. $\tilde{s}_i = -\tilde{s}_j$). Take any $s \in \mathcal{S}(V, 0)$ ($\neq \varnothing$ by (3.10)), so that $-s \in \mathcal{S}(V, 0)$ by the symmetry of $\mathcal{S}(V, 0)$, asserted in (3.5). We claim that one cannot find a path in $\mathcal{S}$ joining $s \in \mathcal{S}$ and $-s \in \mathcal{S}$. Indeed, all the components of s must change their sign. Now, the components i and j of any sign vector $\tilde{s}$ on such a path are necessarily equal (resp. opposite), so that they would change simultaneously, while adjacency imposes to change only a single sign between two consecutive sign vectors of a path.

$[\Leftarrow]$ Let s and $\tilde{s} \in \mathcal{S}$, which are assumed to be distinct (otherwise the result is straightforward). One has to show that there is a path of the given length l in $\mathcal{S}$ joining s to $\tilde{s}$. By the definition of $\mathcal{S}$, one can find x and $\tilde{x}' \in \mathbb{R}^n$ such that

$$s \cdot (V^\mathsf{T}x - \tau) > 0 \qquad \text{and} \qquad \tilde{s} \cdot (V^\mathsf{T}\tilde{x}' - \tau) > 0.$$

The desired path in $\mathcal{S}$ is determined by the sign vectors of the chambers, given by ϕ in (3.4), that are visited along the segment joining x to a small modification $\tilde{x}$ of $\tilde{x}'$. The modification is introduced so that $\tilde{x}$ of $\tilde{x}'$ belong to the same chamber and the segment joining them does not cross two or more hyperplanes simultaneously.

Here is how the modification $\tilde{x}$ of $\tilde{x}'$ is obtained. Let $d' := \tilde{x}' - x$, which is nonzero, since $s \neq \tilde{s}$. By proposition 3.2(2), since the hyperplanes of the arrangement are all different, the vectors $\{(v_i, \tau_i) \in \mathbb{R}^n \times \mathbb{R} : i \in [1:p]\}$ are not colinear, so that the vectors $\{v_i/(v_i^\mathsf{T}x - \tau_i) \in \mathbb{R}^n : i \in [1:p]\}$ are distinct. Now, for a d arbitrary close to d', hence for an

$$\tilde{x} := x + d \tag{3.13a}$$

arbitrary close to $\tilde{x}'$, one can guarantee the inequality $\tilde{s} \cdot (V^\mathsf{T}\tilde{x} - \tau) > 0$ and, by lemma 2.1,

$$|\{(v_i^\mathsf{T}d)/(v_i^\mathsf{T}x - \tau_i) : i \in [1:p]\}| = p. \tag{3.13b}$$

Since, when applying lemma 2.1, one could have added $v_0 = 0$ to the list of distinct nonzero vectors $v_i/(v_i^\mathsf{T}x - \tau_i)$'s, one can also guarantee that

$$v_i^\mathsf{T}d \neq 0, \quad \forall\, i \in [1:p]. \tag{3.13c}$$

To determine the sign vectors of the chambers that are visited along the path $t \mapsto x + td$ in $\mathbb{R}^n$, we first determine the t_i's at which this path encounters a hyperplane. By (3.13c), one can define $t_i := -(v_i^\mathsf{T}x - \tau_i)/(v_i^\mathsf{T}d)$, for $i \in [1:p]$, which are p distinct values by (3.13b). It results that the following equivalent expressions hold for all $i \in [1:p]$: $(v_i^\mathsf{T}x - \tau_i) + t_i(v_i^\mathsf{T}d) = 0$ or, using (3.13a),

$$(1 - t_i)(v_i^\mathsf{T}x - \tau_i) + t_i(v_i^\mathsf{T}\tilde{x} - \tau_i) = 0 \quad \text{or} \quad v_i^\mathsf{T}[(1 - t_i)x + t_i\tilde{x}] - \tau_i = 0. \tag{3.13d}$$

By the last expression in (3.13d), the point $z^i := (1 - t_i)x + t_i\tilde{x} = x + t_i d$ belongs to the ith hyperplane, as announced. Now, by the first expression in (3.13d), $t_i \in (0, 1)$ (i.e., z^i is in the relative interior of $[x, \tilde{x}]$) if and only if $(v_i^\mathsf{T}x - \tau_i)$ and $(v_i^\mathsf{T}\tilde{x} - \tau_i)$ have opposite signs, which also reads $s_i\tilde{s}_i = -1$. Therefore, the number of t_i's in $(0, 1)$ is equal to $l = \sum_{i \in [1:p]} |\tilde{s}_i - s_i|/2 \leqslant p$. Let us denote them by

$$0 < t_{i_1} < \cdots < t_{i_l} < 1.$$

By definition of the t_i's, for $t \in (t_{i_j}, t_{i_{j+1}})$, the sign vector $s^{i_j} := \operatorname{sgn}(V^\mathsf{T}[(1-t)x + t\tilde{x}] - \tau)$ is constant, which also reads

$$s^{i_j} \cdot (V^\mathsf{T}[(1-t)x + t\tilde{x}] - \tau) > 0, \quad \text{for } t \in (t_{i_j}, t_{i_{j+1}}).$$

Furthermore, when $t \in (0,1)$ crosses a $t_{i_j} \in (0,1)$, a single component of $V^\mathsf{T}[(1-t)x + t\tilde{x}] - \tau$ changes its sign. Therefore, we have defined a path of length $l \leqslant p$ in $\mathcal{S}$, namely $s^{i_0} = s$, s^{i_1}, ..., $s^{i_l} = \tilde{s}$, joining s to $\tilde{s}$. This proves the implication. □

3.3 Stem vectors

The notion of *stem vector* has been rediscovered in [16] for a linear arrangement $\mathcal{A}(V, 0)$ (a similar notion is indeed presented in [55, § 6.2] under the name of *signed circuit*) and it is extended in this section to an affine arrangement $\mathcal{A}(V, \tau)$. It is based on the notion of *circuit* of the vector matroid formed by the columns of V and its subsets of linearly independent columns. It is useful to determine algebraically the complement

$$\mathcal{S}^c := \{\pm 1\}^p \setminus \mathcal{S}$$

of the sign vector set $\mathcal{S} \equiv \mathcal{S}(V, \tau)$ in $\{\pm 1\}^p$. Indeed, as we shall see, a stem vector is a particular sign vector in $\{\pm 1\}^J$ for some $J \subseteq [1:p]$ and proposition 3.19 below will tell us that a sign vector s is in $\mathcal{S}^c$ if and only if s_J is a stem vector for some $J \subseteq [1:p]$. This property is used throughout sections 5 and 6. It also results immediately that, knowing the stem vectors, it is possible to generate completely $\mathcal{S}^c$ (by the crude dual algorithm 5.3). Here are the details.

Recall that a *circuit* of the *vector matroid* defined by the columns of $V \in \mathbb{R}^{n \times p}$ and its subsets of linear independent columns [34, proposition 1.1.1] is formed of the indices of a set of columns of V that are linearly dependent, whose strict subsets are the indices of linearly independent columns of V [34, proposition 1.3.5(*iii*)] (in short, it is an index set of linearly dependent columns of minimal size). In compact mathematical terms, the collection $\mathcal{C} \equiv \mathcal{C}(V)$ of the circuits associated with the matrix $V \in \mathbb{R}^{n \times p}$ is defined by

$$\mathcal{C}(V) := \{J \subseteq [1:p] : J \neq \varnothing, \operatorname{null}(V_{:,J}) = 1, \operatorname{null}(V_{:,J_0}) = 0 \text{ for all } J_0 \subsetneq J\}, \tag{3.14}$$

where "null" denotes the nullity (i.e., the dimension of the null space) and "$\subsetneq$" denotes strict inclusion. The stem vectors are defined from the circuits of V, with the desire to validate proposition 3.19 below. Recall that, with our notation, a sign vector $\sigma \in \{\pm 1\}^J$ for some $J := \{j_1, \ldots, j_{|J|}\} \subseteq [1:p]$ is a vector $(\sigma_{j_1}, \ldots, \sigma_{j_{|J|}})$ where the σ_j's are in $\{-1, +1\}$.

Note that an index set $J \subseteq [1:p]$ verifying $\operatorname{null}(V_{:,J}) = 1$ is not necessarily a circuit of V (for example, $J = [1:3]$ is not a circuit of $V = [e_1, e_2, e_2]$ although $\operatorname{null}(V) = 1$), but we have, nevertheless, the following property (see [16, proposition 3.11], for instance), which will be used several times.

Lemma 3.14 (matroid circuit detection) *Suppose that* $I \subseteq [1:p]$ *is such that* $\operatorname{null}(V_{:,I}) = 1$ *and that* $\alpha \in \mathcal{N}(V_{:,I}) \setminus \{0\}$. *Then,* $J := \{i \in I : \alpha_i \neq 0\}$ *is a matroid circuit of* V *and the unique one included in* I.

Definitions 3.15 (stem vector) A *stem vector* of the arrangement $\mathcal{A}(V,\tau)$ is a sign vector $\sigma \in \{\pm 1\}^J$ for some $J \subseteq [1:p]$ satisfying

$$\begin{cases} J \in \mathcal{C}(V) \\ \sigma = \operatorname{sgn}(\eta) \text{ for some } \eta \in \mathbb{R}^J \text{ verifying } \eta \in \mathcal{N}(V_{:,J}) \setminus \{0\} \text{ and } \tau_J^\mathsf{T} \eta \geqslant 0. \end{cases} \tag{3.15}$$

A stem vector is said to be *symmetric* if $\tau_J^\mathsf{T}\eta = 0$ and *asymmetric* otherwise (these properties do not depend on the chosen vector η, as shown in remark 3.16(3) below). We denote respectively by

$$\mathfrak{S}(V,\tau), \qquad \mathfrak{S}_s(V,\tau) \qquad \text{and} \qquad \mathfrak{S}_a(V,\tau) := \mathfrak{S}(V,\tau) \setminus \mathfrak{S}_s(V,\tau)$$

the sets of stem vectors, symmetric stem vectors and asymmetric stem vectors of the arrangement $\mathcal{A}(V,\tau)$. We denote by $\mathfrak{J} : \mathfrak{S}(V,\tau) \to \mathcal{C}(V)$ the map that associates with a stem vector $\sigma \in \{\pm 1\}^J$ its circuit $J := \mathfrak{J}(\sigma)$. □

These definitions deserve some explanations and comments.

Remarks 3.16 1) When the arrangement is linear ($\tau = 0$), one recovers definition 3.9 in [16].

2) The circuits are defined from V, while the stem vectors are defined from $[V; \tau^\mathsf{T}]$; the latter depend on τ, which is not the case of the former.

3) *A calculation method from* $\mathcal{C}(V)$. One can associate with a circuit $J \in \mathcal{C}(V)$, either one asymmetric stem vector or two symmetric stem vectors (there are no other possibilities). Take indeed a circuit $J \in \mathcal{C}(V)$. Then, by (3.14), $\operatorname{null}(V_{:,J}) = 1$ and any $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ has no zero component (since $\operatorname{null}(V_{:,J_0}) = 0$ for all $J_0 \subsetneq J$). Therefore, $\operatorname{sgn}(\eta) \in \{\pm 1\}^J$ for any $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$. Now, there may be two complementary cases.

- (a) Either $\tau_J \in \mathcal{N}(V_{:,J})^\perp$, in which case $\tau_J^\mathsf{T}\eta = 0$ for all $\eta \in \mathcal{N}(V_{:,J})$ and, according to (3.15), there are two symmetric and opposite stem vectors associated with J, namely $\pm \operatorname{sgn}(\eta_0)$ for some arbitrary $\eta_0 \in \mathcal{N}(V_{:,J}) \setminus \{0\}$.
- (b) Or $\tau_J \notin \mathcal{N}(V_{:,J})^\perp$, in which case $\tau_J^\mathsf{T}\eta \neq 0$ for some $\eta \in \mathcal{N}(V_{:,J})$ and, actually, for all $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ since $\operatorname{null}(V_{:,J}) = 1$. In this case, there is a single asymmetric stem vector associated with J, namely $\operatorname{sgn}(\eta_+)$, for some $\eta_+ \in \mathcal{N}(V_{:,J})$ such that $\tau_J^\mathsf{T}\eta_+ > 0$.

We have shown, in particular, that the symmetry (resp. asymmetry) property of a stem vector does not depend on the choice of $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ (resp. satisfying $\tau_J^\mathsf{T}\eta > 0$).

4) The stem vectors may have different sizes, because the circuits may have different sizes.

5) The sets $\mathfrak{S}(V,\tau)$, $\mathfrak{S}_s(V,\tau)$ and $\mathfrak{S}_a(V,\tau)$ are neither vector spaces nor groups. However, given a stem vector $\sigma \in \{\pm 1\}^J$, one can consider $-\sigma$ as the opposite of σ in $\{\pm 1\}^J$, so that $-\sigma \in \{\pm 1\}^J$ (with the same J). With this meaning given to $-\sigma$, one defines

$$-\mathfrak{S}(V,\tau) := \{-\sigma \in \{\pm 1\}^J : \sigma \in \mathfrak{S}(V,\tau) \text{ and } J := \mathfrak{J}(\sigma)\}. \tag{3.16}$$

Proposition 3.17(1) below claims that $\sigma \in \mathfrak{S}_s(V,\tau)$ when $\pm\sigma \in \mathfrak{S}(V,\tau)$, which justifies a posteriori the qualifier "symmetric" given to the stem vectors in $\mathfrak{S}_s(V,\tau)$.

6) A matrix $V \in \mathbb{R}^{n \times p}$ of rank r has at most $\binom{p}{r+1}$ circuits and this bound is reached if and only if the columns of V are in linear general position (definition 3.28 below) [12]; in that case, the circuits are exactly the selections of $r+1$ columns of V. This number can be exponential in p.

7) For $j \in [1:p]$, one has $\{j\} \in \mathcal{C}(V)$ if and only if $V_{:,j} = 0$. If $J \in \mathcal{C}(V)$ and $|J| \geqslant 2$, $V_{:,J}$ has no zero column. □

It is easy to see that

$$-\,\mathfrak{S}(V,\tau)=\mathfrak{S}(V,-\tau) \qquad \text{and} \qquad -\,\mathfrak{S}_a(V,\tau)=\mathfrak{S}_a(V,-\tau). \tag{3.17}$$

Proof. Indeed, let $\sigma\in\mathfrak{S}(V,\tau)$ with $J=\mathfrak{J}(\sigma)$. Then, $J\in\mathcal{C}(V)$ and $\sigma=\operatorname{sgn}(\eta)$ for some $\eta\in\mathcal{N}(V_{:,J})\setminus\{0\}$ satisfying $\tau_J^{\mathsf{T}}\eta\geqslant 0$. Since $J\in\mathcal{C}(V)$ and $-\sigma=\operatorname{sgn}(-\eta)$ for some $-\eta\in\mathcal{N}(V_{:,J})\setminus\{0\}$ satisfying $(-\tau_J)^{\mathsf{T}}(-\eta)\geqslant 0$, it follows that $-\sigma\in\mathfrak{S}(V,-\tau)$. We have shown that $-\mathfrak{S}(V,\tau)\subseteq\mathfrak{S}(V,-\tau)$. Changing τ into $-\tau$ and using (3.16) yield $\mathfrak{S}(V,-\tau)\subseteq-\mathfrak{S}(V,\tau)$. The desired identity $-\mathfrak{S}(V,\tau)=\mathfrak{S}(V,-\tau)$ follows.

One proceeds similarly to show that $-\mathfrak{S}_s(V,\tau)=\mathfrak{S}_s(V,-\tau)$, by changing the inequalities "$\geqslant$" into equalities (actually, proposition 3.17(1) below will show that $-\mathfrak{S}_s(V,\tau)=\mathfrak{S}_s(V,\tau)$). Now, the last formula in (3.17) is deduced directly from the definition $\mathfrak{S}_a(V,\tau):=\mathfrak{S}(V,\tau)\setminus\mathfrak{S}_s(V,\tau)$. □

Here are some more properties of the stem vectors, which are direct consequences of their definition. The properties stated in proposition 3.17 can be symbolically represented like in figure 3.2. In the next proposition, we use the symbol "$\uplus$" for the disjoint union of sets.

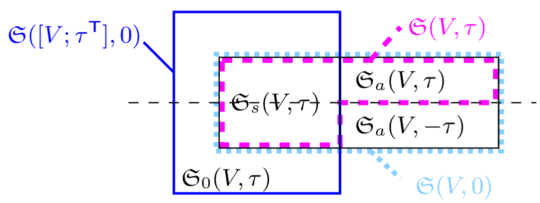


Figure 3.2: Symbolic representation of the sets $\mathfrak{S}(V,\tau)$, $\mathfrak{S}_s(V,\tau)$, $\mathfrak{S}_a(V,\tau)$, $\mathfrak{S}(V,0)$, $\mathfrak{S}_0(V,\tau)$ and $\mathfrak{S}([V;\tau^{\mathsf{T}}],0)$, respecting propositions 3.17, 3.24 and 3.26. The horizontal dashed line aims at representing the reflexion between a stem vector σ and its opposite $-\sigma$: $\mathfrak{S}_s(V,\tau)$, $\mathfrak{S}(V,0)$, $\mathfrak{S}_0(V,\tau)$ and $\mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ are symmetric in the sense that $-\mathfrak{S}_s(V,\tau)=\mathfrak{S}_s(V,\tau)$, $-\mathfrak{S}(V,0)=\mathfrak{S}(V,0)$, $-\mathfrak{S}_0(V,\tau)=\mathfrak{S}_0(V,\tau)$ and $-\mathfrak{S}([V;\tau^{\mathsf{T}}],0)=\mathfrak{S}([V;\tau^{\mathsf{T}}],0)$. By propositions 3.18 and 3.24, the diagram simplifies when $\tau\in\mathcal{R}(V^{\mathsf{T}})$, since then $\mathfrak{S}_a(V,\tau)=\mathfrak{S}_a(V,-\tau)=\mathfrak{S}_0(V,\tau)=\varnothing$ and there is only one set left.

Proposition 3.17 (stem vector properties) *Let $V\in\mathbb{R}^{n\times p}$ and $\tau\in\mathbb{R}^p$. Then,*

1) $\mathfrak{S}(V,\tau)\cap\mathfrak{S}(V,-\tau)=\mathfrak{S}_s(V,\tau)=\mathfrak{S}_s(V,-\tau)$,
2) $\mathfrak{S}(V,\tau)\cup\mathfrak{S}(V,-\tau)=\mathfrak{S}(V,0)$,
3) $\mathfrak{S}_s(V,\tau)\uplus\mathfrak{S}_a(V,\tau)\uplus\mathfrak{S}_a(V,-\tau)=\mathfrak{S}(V,\tau)\uplus\mathfrak{S}_a(V,-\tau)=\mathfrak{S}(V,0)$.

Proof. 1) The last equality can be deduced from the first one, so that only the latter needs to be proved.

[$\subseteq$] Let $\sigma\in\mathfrak{S}(V,\tau)\cap\mathfrak{S}(V,-\tau)$. Then, on the one hand, $\sigma=\operatorname{sgn}(\eta)\in\{\pm1\}^J$ for some $J\in\mathcal{C}(V)$ and some $\eta\in\mathcal{N}(V_{:,J})\setminus\{0\}$ verifying $\tau_J^{\mathsf{T}}\eta\geqslant 0$ and, on the other hand, $-\sigma=\operatorname{sgn}(\tilde\eta)\in\{\pm1\}^J$ (the same J, see remark 3.16(5)) for some $\tilde\eta\in\mathcal{N}(V_{:,J})\setminus\{0\}$ verifying $\tau_J^{\mathsf{T}}\tilde\eta\geqslant 0$. Since $\operatorname{null}(V_{:,J})=1$ by (3.14), $\tilde\eta=\alpha\eta$ for some $\alpha\in\mathbb{R}^*$. Then, $-\sigma=\operatorname{sgn}(\tilde\eta)=\operatorname{sgn}(\alpha)\operatorname{sgn}(\eta)=\operatorname{sgn}(\alpha)\sigma$ shows that $\operatorname{sgn}(\alpha)=-1$, so that $0\leqslant\tau_J^{\mathsf{T}}\tilde\eta=\alpha(\tau_J^{\mathsf{T}}\eta)$. Hence $\tau_J^{\mathsf{T}}\eta\leqslant 0$, so that $\tau_J^{\mathsf{T}}\eta=0$ and σ is symmetric.

[$\supseteq$] Since $\mathfrak{S}_s(V,\tau) \subseteq \mathfrak{S}(V,\tau)$, it suffices to show that $\mathfrak{S}_s(V,\tau) \subseteq \mathfrak{S}(V,-\tau)$ or $-\mathfrak{S}_s(V,\tau) \subseteq \mathfrak{S}(V,\tau)$ by (3.17). If $\sigma \in \mathfrak{S}_s(V,\tau)$ and $J := \mathfrak{J}(\sigma)$, one has $J \in \mathcal{C}(V)$, $\sigma = \operatorname{sgn}(\eta)$ for some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^{\mathsf{T}}\eta = 0$. Then, clearly, $-\sigma = \operatorname{sgn}(-\eta)$ with $-\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^{\mathsf{T}}(-\eta) = 0$. Therefore, $-\sigma \in \mathfrak{S}(V,\tau)$.

2) [$\subseteq$] It suffices to show that $\mathfrak{S}(V,\tau) \subseteq \mathfrak{S}(V,0)$ for an arbitrary τ, which is quite clear since a stem vector $\sigma := \operatorname{sgn}(\eta) \in \mathfrak{S}(V,\tau)$ with $J := \mathfrak{J}(\sigma)$ must satisfy one more property (namely $\tau_J^{\mathsf{T}}\eta \geqslant 0$) than those in $\mathfrak{S}(V,0)$.

[$\supseteq$] Let $\sigma \in \mathfrak{S}(V,0)$ and $J = \mathfrak{J}(\sigma)$. Then, $\sigma := \operatorname{sgn}(\eta) \in \{\pm 1\}^J$ for some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$. We see that $\sigma \in \mathfrak{S}(V,\tau)$ if $\tau_J^{\mathsf{T}}\eta > 0$, $\sigma \in \mathfrak{S}(V,-\tau)$ if $\tau_J^{\mathsf{T}}\eta < 0$ and both sets $\mathfrak{S}(V,\tau) \cap \mathfrak{S}(V,-\tau) = \mathfrak{S}_s(V,\tau)$ if $\tau_J^{\mathsf{T}}\eta = 0$.

3) Let us first show that the sets in the left-hand side are disjoint. The sets $\mathfrak{S}_s(V,\tau)$ and $\mathfrak{S}_a(V,\tau)$ are disjoint by their definition. By point 1, $\mathfrak{S}_s(V,-\tau) = \mathfrak{S}_s(V,\tau)$, so that $\mathfrak{S}_s(V,\tau)$ and $\mathfrak{S}_a(V,-\tau)$ are disjoint by their definition. Next, one cannot find an $s \in \mathfrak{S}_a(V,\tau) \cap \mathfrak{S}_a(V,-\tau)$, since s would be in $\mathfrak{S}_s(V,\tau)$ by point 1, which is in contradiction with $s \in \mathfrak{S}_a(V,\tau)$.

Now, the first equality follows from $\mathfrak{S}(V,\tau) = \mathfrak{S}_s(V,\tau) \uplus \mathfrak{S}_a(V,\tau)$ and the second equality follows from $\mathfrak{S}_s(V,\tau) \cup \mathfrak{S}_a(V,\tau) = \mathfrak{S}(V,\tau)$, $\mathfrak{S}_s(V,\tau) \cup \mathfrak{S}_a(V,-\tau) = \mathfrak{S}_s(V,-\tau) \cup \mathfrak{S}_a(V,-\tau) = \mathfrak{S}(V,-\tau)$ and point 2. □

In complement to the characterizations of the centered arrangements in propositions 3.5 (symmetry of $\mathcal{S}$) and 3.9 (chamber existence invariance), the following characterization is given in terms of the absence of asymmetric stem vector.

Proposition 3.18 (centered arrangement and symmetric stem vector set) *For an affine hyperplane arrangement, the following properties are equivalent:*

(i) the arrangement is centered,

(ii) all the stem vectors are symmetric.

Proof. For $J \subseteq [1\!:\!p]$ and $\eta \in \mathbb{R}^J$, we denote by $\bar{\eta} \in \mathbb{R}^p$ the extended vector associated with η that is defined by $\bar{\eta}_j = \eta_j$ for $j \in J$ and $\bar{\eta}_j = 0$ for $j \notin J$.

[$(i) \Rightarrow (ii)$] If the arrangement is centered, one has $\tau \in \mathcal{R}(V^{\mathsf{T}}) = \mathcal{N}(V)^{\perp}$ by proposition 3.5. A stem vector is of the form $\operatorname{sgn}(\eta)$ with $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^{\mathsf{T}}\eta \geqslant 0$ for some $J \in \mathcal{C}(V)$. Then, the extended vector $\bar{\eta}$ is in $\mathcal{N}(V)$, so that $\tau_J^{\mathsf{T}}\eta = \tau^{\mathsf{T}}\bar{\eta} = 0$ (since $\tau \in \mathcal{N}(V)^{\perp}$ and $\bar{\eta} \in \mathcal{N}(V)$), showing that the stem vector is symmetric.

[$(ii) \Rightarrow (i)$] If all the stem vectors are symmetric, then $\tau_J^{\mathsf{T}}\eta = 0$ for all $J \in \mathcal{C}(V)$ and $\eta \in \mathcal{N}(V_{:,J})$. If $\bar{\eta}$ extends such an η and if we show that these $\bar{\eta}$'s generate $\mathcal{N}(V)$, we will have $\tau \in \mathcal{N}(V)^{\perp} = \mathcal{R}(V^{\mathsf{T}})$, implying that the arrangement is centered (proposition 3.5).

Let $r := \operatorname{rank}(V)$, so that $\dim \mathcal{N}(V) = p - r$. To conclude the proof, it suffices to find $p-r$ vectors $\eta \in \mathcal{N}(V_{:,J})$, with $J \in \mathcal{C}(V)$, so that their extensions $\bar{\eta}$ are linearly independent.

By definition of the rank, one can find a set of r linearly independent columns of V; let J_0 denote the set of their indices. Denote the other column indices by $\{j_1, \ldots, j_{p-r}\} := [1\!:\!p] \setminus J_0$ and set $J_i' := J_0 \cup \{j_i\}$ for $i \in [1\!:\!p-r]$. From lemma 3.14, J_i' contains a unique circuit, which is denoted by $J_i \in \mathcal{C}(V)$, and $\eta_i \in \mathcal{N}(V_{:,J_i}) \setminus \{0\}$ has no zero component. Necessarily, j_i is in J_i (since, otherwise, $J_i \subseteq J_0$, in which case $V_{:,J_i}$ would be injective and J_i would not be a circuit) and j_i is not in $J_{i'}$ for $i' \in [1\!:\!p-r] \setminus \{i\}$ (j_i is not in $J_{i'}'$ by construction, hence not in $J_{i'} \subseteq J_{i'}'$). Hence, the vectors $\bar{\eta}_i$ extending $\eta_i \in \mathcal{N}(V_{:,J_i})$, $i \in [1\!:\!p]$, are linearly independent in $\mathbb{R}^p$. □

The previous proposition can be rephrased in many ways, in particular as the following equivalence

$$\tau \in \mathcal{R}(V^\mathsf{T}) \qquad \iff \qquad \mathfrak{S}(V,\tau) = \mathfrak{S}(V,0). \tag{3.18}$$

Indeed, on the one hand, by proposition 3.5, $\tau \in \mathcal{R}(V^\mathsf{T})$ is equivalent to saying that the arrangement is centered. On the other hand, by proposition 3.17(3), $\mathfrak{S}(V,\tau) = \mathfrak{S}(V,0)$ amounts to saying that all the stem vectors are symmetric.

The following proposition extends naturally [16, proposition 3.10] to the affine arrangements considered in this paper. The possibility of having the equivalence (3.19) was a certificate for the appropriateness of the proposed definition 3.15 of stem vector. The role of this equivalence is important in the design of algorithms having a dual aspect, like those developed in section 5. The proof of the proposition is grounded on duality, via Motzkin's alternative (2.1).

Proposition 3.19 (generating $\boldsymbol{\mathcal{S}^c}$ from the stem vectors) *For $s \in \{\pm 1\}^p$,*

$$s \in \mathcal{S}(V,\tau)^c \quad \iff \quad s_J \in \mathfrak{S}(V,\tau) \text{ for some } J \subseteq [1\!:\!p]. \tag{3.19}$$

Proof. [$\Rightarrow$] Take $s \in \mathcal{S}(V,\tau)^c$. Our goal is to show that the index set $J \subseteq [1\!:\!p]$ in the right-hand side of (3.19) can be determined as one satisfying the following two properties:

$$\{x \in \mathbb{R}^n : s_j(v_j^\mathsf{T} x - \tau_j) > 0 \text{ for all } j \in J\} = \varnothing, \tag{3.20a}$$

$$\forall\, J_0 \subsetneq J,\ \{x \in \mathbb{R}^n : s_j(v_j^\mathsf{T} x - \tau_j) > 0 \text{ for all } j \in J_0\} \neq \varnothing. \tag{3.20b}$$

To show that a J satisfying (3.20a) and (3.20b) exists, let us start with $J = [1\!:\!p]$, which verifies (3.20a), since $s \in \mathcal{S}(V,\tau)^c$. Next, remove one index j from $[1\!:\!p]$ if (3.20a) holds for $J = [1\!:\!p] \setminus \{j\}$. Pursuing the elimination of indices j in this way, one finally obtain an index set J satisfying (3.20a) and $\{x \in \mathbb{R}^n : s_j(v_j^\mathsf{T} x - \tau_j) > 0 \text{ for all } j \in J \setminus \{j_0\}\} \neq \varnothing$ for all $j_0 \in J$. Then, (3.20b) clearly holds.

We claim that, for a J satisfying (3.20a) and (3.20b), s_J is a stem vector, which will conclude the proof of the implication.

To show that s_J, with J verifying (3.20a)-(3.20b), is a stem vector, we stick to definition 3.15 and start by showing that J is a matroid circuit. By (3.20a), $J \neq \varnothing$. Next, by Motzkin's alternative (2.1) with $A := \operatorname{Diag}(s_J) V_{:,J}^\mathsf{T}$ and $a := s_J \cdot \tau_J$, (3.20a) and (3.20b) read

$$\exists\, \alpha \in \mathbb{R}^J_+ \setminus \{0\} \text{ such that } V_{:,J}(s_J \cdot \alpha) = 0,\ \tau_J^\mathsf{T}(s_J \cdot \alpha) \geqslant 0, \tag{3.20c}$$

$$\forall\, J_0 \subsetneq J,\ \nexists\, \alpha' \in \mathbb{R}^{J_0}_+ \setminus \{0\} \text{ such that } V_{:,J_0}(s_{J_0} \cdot \alpha') = 0,\ \tau_{J_0}^\mathsf{T}(s_{J_0} \cdot \alpha') \geqslant 0. \tag{3.20d}$$

From these properties, one deduces that $\alpha > 0$ ($\alpha \geqslant 0$ by (3.20c) and α has no zero component since otherwise (3.20d) would not hold) and that $\operatorname{null}(V_{:,J}) \geqslant 1$ ($s_J \cdot \alpha \in \mathcal{N}(V_{:,J}) \setminus \{0\}$). To show that $\operatorname{null}(V_{:,J}) = 1$, we proceed by contradiction. Suppose that there is a nonzero $\alpha'' \in \mathbb{R}^J$ that is not colinear with α and that verifies $V_{:,J}(s_J \cdot \alpha'') = 0$. Since α and α'' are nonzero and not colinear, they have at least two components and one can find $r \in \mathbb{R}$ such that $\beta := \alpha'' - r\alpha \in \mathbb{R}^J$ has at least one positive and one negative component (take for instance $r := (r_1 + r_2)/2$, where $r_1 := \max\{r \in \mathbb{R} : r\alpha \leqslant \alpha''\} < r_2 := \min\{r \in \mathbb{R} : \alpha'' \leqslant r\alpha\}$). One can also assume that $\tau_J^\mathsf{T}(s_J \cdot \beta) \geqslant 0$ (otherwise replace β by $-\beta$, which also has at least one positive and one negative component; one can check below that this sign inversion has no unpleasant impact on the reasoning). Now, set $t := 1/\max\{-\beta_j/\alpha_j : j \in J\}$, which is positive, and $J_0 := \{j \in J : \alpha_j + t\beta_j > 0\}$, so that $J \setminus J_0 = \{j \in J : \alpha_j + t\beta_j = 0\}$. Using the fact that β has positive components and the definition of t, we see that $\varnothing \neq J_0 \subsetneq J$. Let us introduce $\alpha' := \alpha + t\beta \geqslant 0$, which verifies $\alpha'_j > 0$ for $j \in J_0 \neq \varnothing$ and $\alpha'_j = 0$ for $j \in J \setminus J_0 \neq \varnothing$. Therefore,

$$
\begin{aligned}
&V_{:,J_0}(s_{J_0} \cdot \alpha'_{J_0}) \\
&\quad = V_{:,J}(s_J \cdot \alpha') \qquad [\alpha'_{J\setminus J_0} = 0] \\
&\quad = V_{:,J}(s_J \cdot \alpha) + t\, V_{:,J}(s_J \cdot \beta) \qquad [\alpha' := \alpha + t\beta] \\
&\quad = tV_{:,J}(s_J \cdot \alpha'') - rt\, V_{:,J}(s_J \cdot \alpha) \qquad [V_{:,J}(s_J \cdot \alpha) = 0,\ \beta = \alpha'' - r\alpha] \\
&\quad = 0 \qquad [V_{:,J}(s_J \cdot \alpha'') = V_{:,J}(s_J \cdot \alpha) = 0]
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_{J_0}^{\mathsf{T}}(s_{J_0} \cdot \alpha'_{J_0}) &= \tau_J^{\mathsf{T}}(s_J \cdot \alpha') \qquad [\alpha'_{J\setminus J_0} = 0] \\
&= \tau_J^{\mathsf{T}}(s_J \cdot \alpha) + t\, \tau_J^{\mathsf{T}}(s_J \cdot \beta) \qquad [\alpha' := \alpha + t\beta] \\
&\geqslant 0 \qquad [\tau_J^{\mathsf{T}}(s_J \cdot \alpha) \geqslant 0,\ \tau_J^{\mathsf{T}}(s_J \cdot \beta) \geqslant 0,\ t > 0].
\end{aligned}
$$

These last two outcomes are in contradiction with (3.20d), as expected.

To show that $J \in \mathcal{C}$ defined by (3.14), we still have to prove that $V_{:,J_0}$ is injective when $J_0 \subsetneq J$. Equivalently, it suffices to show that any $\beta \in \mathcal{N}(V_{:,J})$ with some zero component vanishes. We proceed by contradiction. If there is a $\beta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ with a zero component, $s_J \cdot \alpha$ and β would be two linearly independent vectors in $\mathcal{N}(V_{:,J})$ (since $s_J \cdot \alpha$ has no zero component), contradicting $\operatorname{null}(V_{:,J}) = 1$.

Now, since $s_J = \operatorname{sgn}(s_J \cdot \alpha)$, since $s_J \cdot \alpha \in \mathcal{N}(V_{:,J})$ and $\tau_J^{\mathsf{T}}(s_J \cdot \alpha) \geqslant 0$ by (3.20c) and since J is a matroid circuit of V, s_J is a stem vector (definition 3.15).

$[\Leftarrow]$ Since s_J is a stem vector, it reads $s_J := \operatorname{sgn}(\eta) \in \{\pm 1\}^J$ for some $J \in \mathcal{C}$ and some $\eta \in \mathbb{R}^J$ satisfying $V_{:,J}\eta = 0$ and $\tau_J^{\mathsf{T}}\eta \geqslant 0$. Then, $\alpha := s_J \cdot \eta = |\eta|$ is in $\mathbb{R}^J_+ \setminus \{0\}$ and verifies $V_{:,J}(s_J \cdot \alpha) = 0$ and $\tau_J^{\mathsf{T}}(s_J \cdot \alpha) \geqslant 0$. By Motzkin's alternative (2.1), there is no $x \in \mathbb{R}^n$ such that $s_J \cdot (V_{:,J}^{\mathsf{T}} x - \tau_J) > 0$. Hence, there is certainly no $x \in \mathbb{R}^n$ such that $s \cdot (V^{\mathsf{T}} x - \tau) > 0$. This means that $s \in \mathcal{S}(V, \tau)^c$. □

We say that $s \in \mathcal{S}(V, \tau)^c$ *covers* a sign vector $\sigma \in \{\pm 1\}^J$ for some $J \subseteq [1\!:\!p]$ if $s_J = \sigma$. Given a set of stem vecors $\mathfrak{S}$, checking whether a sign vector s covers some $\sigma \in \mathfrak{S}$ is called below a *covering test*. This operation is an essential step of the dual algorithms of section 5.

Remark 3.20 One might wonder whether having a sign vector $s \in \{\pm 1\}^p$ such that $\pm s \in \mathcal{S}(V, \tau)^c$ would imply that one has $\pm s_J \in \mathfrak{S}(V, \tau)$ for some $J \in [1\!:\!p]$. This implication does not hold. Equivalently, for a given $s \in \{\pm 1\}^p$, the two nonempty sets $\{J \subseteq [1\!:\!p] : s_J \in \mathfrak{S}(V, \tau)\}$ and $\{J \subseteq [1\!:\!p] : s_J \in -\mathfrak{S}(V, \tau)\}$ may have an empty intersection (note that its union may also differ from $\mathcal{C}$). This is the case, for example, when $V := \left[\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} := [0\ 0\ 1\ 2]$ (add one hyperplane perpendicular to $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ in figure 3.1(middle)). More is said on this situation in the equivalence (3.22) below.

Let us analyze the given example. The set of circuits of V is

$$
\mathcal{C} = \{\{3, 4\},\ \{1, 2, 3\},\ \{1, 2, 4\}\}.
$$

The sign vector $s := (-1, -1, +1, -1)$ is such that $\pm s \in \mathcal{S}(V, \tau)^c$, since there is no $x \in \mathbb{R}^2$ such that $x_1 < 0$, $x_2 < 0$, $x_1 + x_2 > 1$ and $x_1 + x_2 < 2$, as well as no $x \in \mathbb{R}^2$ such that $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 < 1$ and $x_1 + x_2 > 2$. Nevertheless, the two sets

$$\{J \subseteq [1:p] : s_J \in \mathfrak{S}(V, \tau)\} = \{\{1, 2, 3\}\} \quad \text{and} \quad \{J \subseteq [1:p] : s_J \in -\mathfrak{S}(V, \tau)\} = \{\{3, 4\}\}$$

have no circuit in common (note that their union is not $\mathcal{C}$).

We anticipate (3.22), which points out that $\pm s \in \mathcal{S}(V, \tau)^c$ if and only if $s_J \in \mathfrak{S}([V; \tau^{\mathsf{T}}], 0)$ for some $J \in [1:p]$. This is what we observe here with $s := (-1, -1, +1, -1)$. One has $\mathcal{C}([V; \tau^{\mathsf{T}}]) = \{\{1, 2, 3, 4\}\}$ and $s \in \mathfrak{S}([V; \tau^{\mathsf{T}}], 0)$ since $s = \operatorname{sgn}(\eta)$ with $\eta = (-1, -1, 2, -1) \in \mathcal{N}([V; \tau^{\mathsf{T}}])$. □

3.4 Augmented matrix

According to definition 3.3, verifying whether $s \in \mathcal{S}(V, \tau)$ amounts to checking whether there is an $x \in \mathbb{R}^n$ such that $s \cdot (V^{\mathsf{T}} x - \tau) > 0$ or, equivalently, whether there is a pair $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$s \cdot ([V; \tau^{\mathsf{T}}]^{\mathsf{T}} [x; \xi]) > 0 \qquad \text{and} \qquad \xi = -1.$$

The first condition above reads $s \in \mathcal{S}([V; \tau^{\mathsf{T}}], 0)$ and refers to the linear arrangement in $\mathbb{R}^{n+1}$ governed by the *augmented matrix* $[V; \tau^{\mathsf{T}}]$. This presentation of the problem shows that there must be links between the following sign vector sets and between the following stem vector sets

$$\mathcal{S}(V, 0), \qquad \mathcal{S}(V, \tau) \qquad \text{and} \qquad \mathcal{S}([V; \tau^{\mathsf{T}}], 0), \tag{3.21a}$$

$$\mathfrak{S}(V, 0), \qquad \mathfrak{S}(V, \tau) \qquad \text{and} \qquad \mathfrak{S}([V; \tau^{\mathsf{T}}], 0). \tag{3.21b}$$

For example, we already know the inclusions $\mathcal{S}(V, 0) \subseteq \mathcal{S}(V, \tau)$ and $\mathfrak{S}(V, \tau) \subseteq \mathfrak{S}(V, 0)$ from propositions 3.7 and 3.17(2).

This section aims at identifying a few properties where the augmented matrix $[V; \tau^{\mathsf{T}}]$ intervenes. In section 3.4.1, some links between the sets in (3.21a) are highlighted. Section 3.4.2 establishes some connections between the circuits of V and $[V; \tau^{\mathsf{T}}]$, as well as between the stem vector sets in (3.21b). In section 3.4.3, one observes that the identity obtained in proposition 3.21(4) makes it easy to deduce a formula for $|\mathcal{S}(V, \tau)|$ (proposition 3.27) and a known bound on $|\mathcal{S}(V, \tau)|$ (proposition 3.36), which is reached if and only if the arrangement is in *affine general position* (proposition 3.33 and definition 3.34). The role of the augmented matrix in the specification of the notion of *affine general position* is also pointed out.

Viewing an affine arrangement in $x \in \mathbb{R}^n$ as the intersection of a linear arrangement in $(x, \xi) \in \mathbb{R}^{n+1}$ with the affine space $\{(x, \xi) \in \mathbb{R}^{n+1} : \xi = -1\}$ is called the *method of coning* in [32, definition 1.15].

3.4.1 Sign vectors of the augmented matrix

Recall the definition of $\mathcal{S}_s(V, \tau)$ and $\mathcal{S}_a(V, \tau)$ in (3.8) and the properties (3.9). Some of the properties stated in proposition 3.21(1) can be symbolically represented like in figure 3.3. In the next proposition, we use the symbol "$\uplus$" for the disjoint union of sets.

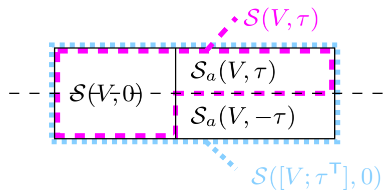


Figure 3.3: Symbolic representation of the sets $\mathcal{S}(V,0)$, $\mathcal{S}(V,\tau)$, $\mathcal{S}_a(V,\tau)$ and $\mathcal{S}([V;\tau^{\mathsf{T}}],0)$, respecting (3.7), (3.8), (3.9) and propositions 3.7 and 3.21. The horizontal dashed line aims at representing the reflection between a sign vector s and its opposite $-s$: $\mathcal{S}(V,0)$, $\mathcal{S}([V;\tau^{\mathsf{T}}],0)$ and $\mathcal{S}([V;\tau^{\mathsf{T}}],0)^c$ are symmetric in the sense of definition 3.4.

Proposition 3.21 (properties with $\boldsymbol{\mathcal{S}([V;\tau^{\mathsf{T}}],0)}$) *Let $\mathcal{A}(V,\tau)$ be an arrangement with $V\in\mathbb{R}^{n\times p}$ and $\tau\in\mathbb{R}^p$. Then, the following properties hold.*

1) $\mathcal{S}(V,\tau)\cap\mathcal{S}(V,-\tau)=\mathcal{S}(V,0)\subseteq\mathcal{S}(V,\tau)\subseteq\mathcal{S}([V;\tau^{\mathsf{T}}],0)=\mathcal{S}(V,\tau)\cup\mathcal{S}(V,-\tau)$.
2) $\mathcal{S}(V,\tau)^c\cup\mathcal{S}(V,-\tau)^c=\mathcal{S}(V,0)^c\supseteq\mathcal{S}(V,\tau)^c\supseteq\mathcal{S}([V;\tau^{\mathsf{T}}],0)^c=\mathcal{S}(V,\tau)^c\cap\mathcal{S}(V,-\tau)^c$.
3) $\mathcal{S}(V,0)\uplus\mathcal{S}_a(V,\tau)\uplus\mathcal{S}_a(V,-\tau)=\mathcal{S}([V;\tau^{\mathsf{T}}],0)$.
4) $2|\mathcal{S}(V,\tau)|=|\mathcal{S}(V,0)|+|\mathcal{S}([V;\tau^{\mathsf{T}}],0)|$.
5) $2|\mathcal{S}(V,\tau)^c|=|\mathcal{S}(V,0)^c|+|\mathcal{S}([V;\tau^{\mathsf{T}}],0)^c|$.

Proof. 1) The first equality repeats proposition 3.7(2), using (3.8), the first inclusion repeats proposition 3.7(1) and the second inclusion is straightforward: if $s\in\mathcal{S}(V,\tau)$, one has $s\cdot(V^{\mathsf{T}}x-\tau)>0$ for some $x\in\mathbb{R}^n$ or $s\cdot([V;\tau^{\mathsf{T}}]^{\mathsf{T}}[x;-1])>0$, implying that $s\in\mathcal{S}([V;\tau^{\mathsf{T}}],0)$.

Consider now the last equality. [$\subseteq$] Let $s\in\mathcal{S}([V;\tau^{\mathsf{T}}],0)$, so that $s\cdot(V^{\mathsf{T}}x+\tau\xi)>0$ for some $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}$. By homogeneity, it follows that $s\in\mathcal{S}(V,\tau)$ if $\xi<0$, that $s\in\mathcal{S}(V,-\tau)$ if $\xi>0$ and that $s\in\mathcal{S}(V,0)=\mathcal{S}(V,\tau)\cap\mathcal{S}(V,-\tau)$ if $\xi=0$. [$\supseteq$] By the second inclusion, $\mathcal{S}(V,\tau)\subseteq\mathcal{S}([V;\tau^{\mathsf{T}}],0)$ and $\mathcal{S}(V,-\tau)=-\mathcal{S}(V,\tau)\subseteq-\mathcal{S}([V;\tau^{\mathsf{T}}],0)=\mathcal{S}([V;\tau^{\mathsf{T}}],0)$.

2) Take the complement of the sets in point 1.

3) Let us first show that the sets are disjoint. By (3.8) and proposition 3.7(2), one has $\mathcal{S}_a(V,\pm\tau)=\mathcal{S}(V,\pm\tau)\setminus\mathcal{S}(V,0)$, so that $\mathcal{S}(V,0)\cap\mathcal{S}_a(V,\pm\tau)=\varnothing$. Use the same arguments to get that

$$\begin{aligned}\mathcal{S}_a(V,\tau)\cap\mathcal{S}_a(V,-\tau)&=[\mathcal{S}(V,\tau)\cap\mathcal{S}(V,0)^c]\cap[\mathcal{S}(V,-\tau)\cap\mathcal{S}(V,0)^c]\\&=[\mathcal{S}(V,\tau)\cap\mathcal{S}(V,-\tau)]\cap\mathcal{S}(V,0)^c\\&=\mathcal{S}(V,0)\cap\mathcal{S}(V,0)^c\\&=\varnothing.\end{aligned}$$

Consider now the identity:

$$\begin{aligned}\mathcal{S}(V,0)\cup\mathcal{S}_a(V,\tau)\cup\mathcal{S}_a(V,-\tau)&=\mathcal{S}(V,0)\cup[\mathcal{S}(V,\tau)\cap\mathcal{S}(V,0)^c]\cup[\mathcal{S}(V,-\tau)\cap\mathcal{S}(V,0)^c]\\&=\mathcal{S}(V,0)\cup\mathcal{S}(V,\tau)\cup\mathcal{S}(V,-\tau)\\&=\mathcal{S}([V;\tau^{\mathsf{T}}],0)\qquad\text{[point 1]}.\end{aligned}$$

4) The identity results from

$$\begin{aligned}|\mathcal{S}([V;\tau^{\mathsf{T}}],0)|&=|\mathcal{S}(V,\tau)|+|\mathcal{S}(V,-\tau)|-|\mathcal{S}(V,\tau)\cap\mathcal{S}(V,-\tau)|\qquad\text{[point 1]}\\&=2|\mathcal{S}(V,\tau)|-|\mathcal{S}(V,0)|\qquad\text{[(3.7), (3.8), proposition 3.7(2)]}.\end{aligned}$$

5) Each set in point 4 is a part of $\{\pm 1\}^p$ of cardinality 2^p. Hence, the identity in point 4 gives

$$2(2^p - |\mathcal{S}(V,\tau)^c|) = (2^p - |\mathcal{S}(V,0)^c|) + (2^p - |\mathcal{S}([V;\tau^\mathsf{T}],0)^c|).$$

Point 5 follows after subtracting 2^{p+1} from both sides. $\square$

Remarks 3.22 1) Let us emphasize the meaning of the inclusions in proposition 3.21(1): the *affine* arrangement $\mathcal{A}(V,\tau)$ has its sign vectors (in bijection with its chambers, by proposition 3.1) containing those of the *linear* arrangement $\mathcal{A}(V,0)$ and contained in those of the *linear* arrangement $\mathcal{A}([V;\tau^\mathsf{T}],0)$.

2) As a corollary of proposition 3.21(2), one has (see remark 3.20):

$$\pm\, s \in \mathcal{S}(V,\tau)^c \qquad \Longleftrightarrow \qquad s_J \in \mathfrak{S}([V;\tau^\mathsf{T}],0) \text{ for some } J \subseteq [1\!:\!p]. \tag{3.22}$$

Indeed $\pm s \in \mathcal{S}(V,\tau)^c$ if and only if $s \in \mathcal{S}(V,\tau)^c \cap \mathcal{S}(V,-\tau)^c = \mathcal{S}([V;\tau^\mathsf{T}],0)^c$ (proposition 3.21(2)), which is equivalent to $s_J \in \mathfrak{S}([V;\tau^\mathsf{T}],0)$ for some $J \in [1\!:\!p]$ (proposition 3.19). Note that $\mathfrak{S}([V;\tau^\mathsf{T}],0)$ is symmetric, so that the properties in (3.22) are also equivalent to the fact that $s_J \in -\mathfrak{S}([V;\tau^\mathsf{T}],0)$ for some $J \subseteq [1\!:\!p]$. $\square$

3.4.2 Circuits and stem vectors of the augmented matrix

The next propositions highlight connections between the circuits and the stem vectors of V and those of the augmented matrix $[V;\tau^\mathsf{T}]$. Recall from proposition 3.5 that an arrangement is centered if and only if $\tau \in \mathcal{R}(V^\mathsf{T})$. Note also that

$$\operatorname{rank}([V;\tau^\mathsf{T}]) = \begin{cases} \operatorname{rank}(V) & \text{if } \tau \in \mathcal{R}(V^\mathsf{T}) \\ \operatorname{rank}(V)+1 & \text{otherwise,} \end{cases} \tag{3.23a}$$

$$\operatorname{null}([V;\tau^\mathsf{T}]) = \begin{cases} \operatorname{null}(V) & \text{if } \tau \in \mathcal{R}(V^\mathsf{T}) \\ \operatorname{null}(V)-1 & \text{otherwise.} \end{cases} \tag{3.23b}$$

The formula of $\operatorname{rank}([V;\tau^\mathsf{T}])$ is clear and the one of $\operatorname{null}([V;\tau^\mathsf{T}])$ can be deduced from (3.23a) by the rank-nullity theorem.

Proposition 3.23 (circuits of V and $[V;\tau^\mathsf{T}]$) *Let $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$. Then, the following properties are equivalent:*

(i) $\mathcal{C}(V) = \mathcal{C}([V;\tau^\mathsf{T}])$,

(ii) $\mathcal{C}(V) \subseteq \mathcal{C}([V;\tau^\mathsf{T}])$,

(iii) $\tau \in \mathcal{R}(V^\mathsf{T})$, *meaning that that the arrangement $\mathcal{A}(V,\tau)$ is centered.*

Proof. $[(i) \Rightarrow (ii)]$ Clear.

$[(ii) \Rightarrow (iii)]$ Let $J \in \mathcal{C}(V)$. Then, $\operatorname{null}(V_{:,J}) = 1$ by (3.14). By assumption, $J \in \mathcal{C}([V;\tau^\mathsf{T}])$, so that $\operatorname{null}([V;\tau^\mathsf{T}]_{:,J}) = 1$, as well. By (3.23b), $\tau_J \in \mathcal{R}(V_{:,J}^\mathsf{T}) = \mathcal{N}(V_{:,J})^\perp$. According to remark 3.16(3.a), the stem vectors associated with J are symmetric. Since J is arbitrary in $\mathcal{C}(V)$, one has $\mathfrak{S}(V,\tau) = \mathfrak{S}_s(V,\tau)$, implying that the arrangement is centered (proposition 3.18).

$[(iii) \Rightarrow (i)]$ Let $J \subseteq [1\!:\!p]$ and $J_0 \subsetneq J$. When $\tau \in \mathcal{R}(V^\mathsf{T})$, one has $\tau_J \in \mathcal{R}(V_{:,J}^\mathsf{T})$ and $\tau_{J_0} \in \mathcal{R}(V_{:,J_0}^\mathsf{T})$, so that (3.23b) yields

$$\operatorname{null}([V;\tau^\mathsf{T}]_{:,J}) = \operatorname{null}(V_{:,J}) \qquad \text{and} \qquad \operatorname{null}([V;\tau^\mathsf{T}]_{:,J_0}) = \operatorname{null}(V_{:,J_0}).$$

It follows that $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J}) = 1$ and $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J_0}) = 0$ for all $J_0 \subsetneq J$ if and only if $\operatorname{null}(V_{:,J}) = 1$ and $\operatorname{null}(V_{:,J_0}) = 0$ for all $J_0 \subsetneq J$. In other words, $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$ if and only if $J \in \mathcal{C}(V)$. We have shown that $\mathcal{C}([V;\tau^{\mathsf{T}}]) = \mathcal{C}(V)$. □

The implication $(ii) \Rightarrow (i)$ of lemma 3.23 is not based on the fact that one would always have $\mathcal{C}(V) \supseteq \mathcal{C}([V;\tau^{\mathsf{T}}])$, which is not true. As a counter-example, take $V = \left[\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [0\ 0\ 1\ 2]$ (this is the same example as in remark 3.20), in which case one has $\mathcal{C}(V) = \{\{3,4\}, \{1,2,3\}, \{1,2,4\}\}$, while $\mathcal{C}([V;\tau^{\mathsf{T}}]) = \{\{1,2,3,4\}\}$. Actually, the property $\mathcal{C}(V) \supseteq \mathcal{C}([V;\tau^{\mathsf{T}}])$, which is therefore weaker than those in lemma 3.23, has various other equivalent interesting formulations, including $\mathfrak{S}_s(V,\tau) = \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$, as shown by the following proposition. Recall figure 3.2 for a symbolic representation of the stem vector sets.

Proposition 3.24 (stem vectors of $\mathcal{A}(V,\tau)$ and $\mathcal{A}([V;\tau^{\mathsf{T}}],0)$) Consider an arrangement $\mathcal{A}(V,\tau)$. Then,

$$\mathfrak{S}_a(V,\tau) \cap \mathfrak{S}([V;\tau^{\mathsf{T}}],0) = \varnothing \qquad \text{and} \qquad \mathfrak{S}_s(V,\tau) \subseteq \mathfrak{S}([V;\tau^{\mathsf{T}}],0), \tag{3.24}$$

with equality in the last inclusion if the arrangement is centered. More precisely, the following properties are equivalent:

(i) $\mathfrak{S}_s(V,\tau) = \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$,
(ii) $\mathfrak{S}_s(V,\tau) \supseteq \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$,
(iii) $\mathcal{C}(V) \supseteq \mathcal{C}([V;\tau^{\mathsf{T}}])$,
(iv) $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$, for all $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$.

Proof. 1) $[(3.24)_1]$ Let $\sigma \in \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ and $J := \mathfrak{J}(\sigma)$. Then, $\sigma = \operatorname{sgn}(\eta)$ for some $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}$. This η verifies $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^{\mathsf{T}}\eta = 0$. It follows that $\sigma \notin \mathfrak{S}_a(V,\tau)$, either because $J \notin \mathcal{C}(V)$ or because $J \in \mathcal{C}(V)$, in which case $\sigma \in \mathfrak{S}_s(V,\tau)$ by the properties of η.

$[(3.24)_2]$ Let $\sigma \in \mathfrak{S}_s(V,\tau)$ and $J := \mathfrak{J}(\sigma)$. Then, $J \in \mathcal{C}(V)$ and $\sigma = \operatorname{sgn}(\eta)$ for some $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ verifying $\tau_J^{\mathsf{T}}\eta = 0$. It follows that $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}$. To show that $\sigma \in \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$, one still has to verify that $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$ (definition 3.14). First, $J \neq \varnothing$, since $J \in \mathcal{C}(V)$. Next, $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J}) = 1$, since $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}$ and $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J}) \leqslant \operatorname{null}(V_{:,J}) = 1$ (because $J \in \mathcal{C}(V)$). Finally, for all $J_0 \subsetneq J$, one has $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J_0}) = 0$, since $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J_0}) \leqslant \operatorname{null}(V_{:,J_0}) = 0$ (because $J \in \mathcal{C}(V)$).

2) Suppose now that the arrangement $\mathcal{A}(V,\tau)$ is centered (or $\tau \in \mathcal{R}(V^{\mathsf{T}})$) and let us show that $\mathfrak{S}_s(V,\tau) \supseteq \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ (this could also be viewed as a consequence of the implication $(iv) \Rightarrow (i)$ proved below). Let $\sigma \in \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ and $J := \mathfrak{J}(\sigma)$. Then, $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$ and $\sigma = \operatorname{sgn}(\eta)$ for some $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}$. Then, $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau_J^{\mathsf{T}}\eta = 0$. We see that it suffices now to observe that $J \in \mathcal{C}(V)$, which results from the implication $(iii) \Rightarrow (i)$ of proposition 3.23.

3) Consider now the equivalences (i)-(iv).

$[(i) \Leftrightarrow (ii)]$ By (3.24).

$[(ii) \Rightarrow (iii)]$ Let $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$. By remark 3.16(3.a), there is a stem vector $\sigma \in \{\pm 1\}^J$ that is in $\mathfrak{S}([V;\tau^{\mathsf{T}}],0)$, hence in $\mathfrak{S}_s(V,\tau)$ by (ii). This latter fact implies that $J \in \mathcal{C}(V)$.

$[(iii) \Rightarrow (iv)]$ Let $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$. By (iii), $J \in \mathcal{C}(V)$, so that $\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J}) = \operatorname{null}(V_{:,J})$ (these nullities $= 1$). Then, (3.23b) implies that $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$.

$[(iv) \Rightarrow (ii)]$ Let $\sigma = \operatorname{sgn}(\eta) \in \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ and $J := \mathfrak{J}(\sigma)$. Since $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\} \subseteq \mathcal{N}(V_{:,J}) \setminus \{0\}$ and $\tau^{\mathsf{T}}\eta = 0$, it suffices to show that $J \in \mathcal{C}(V)$. This property follows from $J \in \mathcal{C}([V;\tau^{\mathsf{T}}]_{:,J})$, since $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$, from $\mathcal{C}([V;\tau^{\mathsf{T}}]_{:,J}) = \mathcal{C}(V_{:,J})$, by $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$ and the implication $(iii) \Rightarrow (i)$ of proposition 3.23, and from $\mathcal{C}(V_{:,J}) \subseteq \mathcal{C}(V)$. $\square$

Examples 3.25 1) An example in which the equivalent properties of proposition 3.24 hold, but not those of proposition 3.23, is given by $V = \left[\begin{smallmatrix}1&0&1&1\\0&1&1&1\end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [0\ 0\ 0\ 1]$ (the same V as in remark 3.20, but a different τ). One has

$$\mathcal{C}(V) = \{\{3,4\},\{1,2,3\},\{1,2,4\}\} \qquad \text{and} \qquad \mathcal{C}([V;\tau^{\mathsf{T}}]) = \{\{1,2,3\}\}.$$

We see that $\mathcal{C}([V;\tau^{\mathsf{T}}])$ is included in $\mathcal{C}(V)$ but is not equal to it. Note also that $\tau \notin \mathcal{R}(V^{\mathsf{T}})$,

$$\mathfrak{S}_s(V,\tau) = \mathfrak{S}([V;\tau^{\mathsf{T}}],0) = \left\{ \pm \begin{bmatrix}1\\1\\-1\end{bmatrix} \in \{\pm1\}^{\{1,2,3\}} \right\} \quad \text{and}$$

$$\mathfrak{S}_a(V,\tau) = \left\{ \begin{bmatrix}-1\\1\end{bmatrix} \in \{\pm1\}^{\{3,4\}}, \begin{bmatrix}-1\\-1\\1\end{bmatrix} \in \{\pm1\}^{\{1,2,4\}} \right\}.$$

All this is in agreement with propositions 3.23 and 3.24.

2) Property (iv) of proposition 3.24 does not imply that $\tau_K \in \mathcal{R}(V_{:,K}^{\mathsf{T}})$, for $K := \cup\{J \in \mathcal{C}([V;\tau^{\mathsf{T}}])\}$. To see this, take $V = \left[\begin{smallmatrix}1&0&1&1&0&1\\0&1&1&0&1&1\end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [0\ 0\ 1\ 0\ 0\ 1]$. One has $\mathcal{C}([V;\tau^{\mathsf{T}}]) = \{\{1,4\},\{2,5\},\{3,6\}\}$, $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$ for all $J \in \mathcal{C}([V;\tau^{\mathsf{T}}])$ and $K = [1\!:\!6]$, but $\tau \notin \mathcal{R}(V^{\mathsf{T}})$. $\square$

Recall $(3.24)_2$. In the algorithm of section 6.3.1, it will be interesting to partition $\mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ in $\mathfrak{S}_s(V,\tau)$ and $\mathfrak{S}_0(V,\tau)$, where

$$\mathfrak{S}_0(V,\tau) := \mathfrak{S}([V;\tau^{\mathsf{T}}],0) \setminus \mathfrak{S}_s(V,\tau).$$

The stem vectors of $\mathfrak{S}_0(V,\tau)$ can be recognized thanks to the following proposition.

Proposition 3.26 ($\mathfrak{S}_0(V,\tau)$ and $\mathfrak{S}_s(V,\tau)$ characterizations) *Let $\mathcal{A}(V,\tau)$ be an arrangement with $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$. Let $\sigma \in \mathfrak{S}([V;\tau^{\mathsf{T}}],0)$ and $J = \mathfrak{J}(\sigma)$. Then,*

$$\sigma \in \mathfrak{S}_0(V,\tau) \quad \Longleftrightarrow \quad \begin{cases} J \in \mathcal{C}([V;\tau^{\mathsf{T}}]) \setminus \mathcal{C}(V) \\ \sigma = \operatorname{sgn}(\eta) \textit{ for some } \eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}. \end{cases} \tag{3.25a}$$

$$\sigma \in \mathfrak{S}_s(V,\tau) \iff J \in \mathcal{C}(V) \iff \operatorname{null}(V_{:,J}) = 1 \iff \tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}}). \tag{3.25b}$$

Proof. By definition 3.15 with $[V;\tau^{\mathsf{T}}]$ replacing V, the definitions of σ and J yield

$$\begin{cases} J \in \mathcal{C}([V;\tau^{\mathsf{T}}]) \\ \sigma = \operatorname{sgn}(\eta) \text{ for some } \eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}. \end{cases} \tag{3.26}$$

[(3.25a), $\Rightarrow$] By (3.26), $\tau_J^{\mathsf{T}}\eta = 0$. Then $J \in \mathcal{C}(V)$ would have $\sigma \in \mathfrak{S}_s(V,\tau)$.
[(3.25a), $\Leftarrow$] If $\sigma \in \mathfrak{S}_s(V,\tau)$, then $J \in \mathcal{C}(V)$, a contradiction.
[(3.25b), $\sigma \in \mathfrak{S}_s(V,\tau) \Leftrightarrow J \in \mathcal{C}(V)$] By (3.25a) and (3.26).

[(3.25b), $J \in \mathcal{C}(V) \Rightarrow \operatorname{null}(V_{:,J}) = 1$] By the definition 3.14 of a matroid circuit.

[(3.25b), $\operatorname{null}(V_{:,J}) = 1 \Leftrightarrow \tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$] By (3.23b) and $\operatorname{null}([V_{:,J}; \tau_J^{\mathsf{T}}]) = 1$.

[(3.25b), $\operatorname{null}(V_{:,J}) = 1$ and $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}}) \Rightarrow J \in \mathcal{C}(V)$] By the definition 3.14 of a matroid circuit, it remains to show that $\operatorname{null}(V_{:,J_0}) = 0$ for $J_0 \subsetneq J$. Let $J_0 \subsetneq J$. It follows from $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$ that $\tau_{J_0} \in \mathcal{R}(V_{:,J_0}^{\mathsf{T}})$, Then, by (3.23b), $\operatorname{null}(V_{:,J_0}) = \operatorname{null}([V_{:,J_0}; \tau_{J_0}^{\mathsf{T}}]) = 0$ since $J \in \mathcal{C}([V; \tau^{\mathsf{T}}])$ by (3.26). □

3.4.3 Sign vector set cardinality

Proposition 3.21(4) and Winder's formula of $|\mathcal{S}(V,0)|$ (linear arrangement) make it possible to give expressions of $|\mathcal{S}(V,\tau)|$ and $|\mathcal{S}_a(V,\tau)|$ having the flavor of Winder's. Formula (3.28a) below is given by Zaslavsky [54, corollary 5.9, page 68], who makes its connection with a cardinality formula using a characteristic polynomial of the arrangement [54, theorem A, page 18]; an approach that looks rather different from ours.

Recall that, for a matrix $V \in \mathbb{R}^{n\times p}$ without zero column, *Winder's formula* of the cardinality of $\mathcal{S}(V,0)$ reads [52, 1966] (see also [15, § 4.2.1])

$$|\mathcal{S}(V,0)| = \sum_{J\subseteq[1:p]} (-1)^{\operatorname{null}(V_{:,J})}, \tag{3.27}$$

where the term in the right-hand side corresponding to $J = \varnothing$ is 1 (one takes the convention that $\operatorname{null}(V_{:,\varnothing}) = 0$).

Proposition 3.27 (cardinality of $\mathcal{S}(V,\tau)$) *Consider a proper affine arrangement $\mathcal{A}(V,\tau)$, with $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$. Then,*

$$|\mathcal{S}(V,\tau)| = \sum_{\substack{J\subseteq[1:p]\\ \tau_J\in\mathcal{R}(V_{:,J}^{\mathsf{T}})}} (-1)^{\operatorname{null}(V_{:,J})}, \tag{3.28a}$$

$$|\mathcal{S}_a(V,\tau)| = \sum_{\substack{J\subseteq[1:p]\\ \tau_J\notin\mathcal{R}(V_{:,J}^{\mathsf{T}})}} (-1)^{\operatorname{null}(V_{:,J})-1}, \tag{3.28b}$$

where the term in the right-hand side of (3.28a) corresponding to $J = \varnothing$ is 1 (it is considered that $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$ for $J = \varnothing$).

Proof. Using proposition 3.21(4), Winder's formula (3.27) and (3.23b), one gets

$$\begin{aligned}
2|\mathcal{S}(V,\tau)| &= |\mathcal{S}(V,0)| + |\mathcal{S}([V;\tau^{\mathsf{T}}],0)| \\
&= \sum_{J\subseteq[1:p]} (-1)^{\operatorname{null}(V_{:,J})} + \sum_{J\subseteq[1:p]} (-1)^{\operatorname{null}([V;\tau^{\mathsf{T}}]_{:,J})} \\
&= 2 \sum_{\substack{J\subseteq[1:p]\\ \tau_J\in\mathcal{R}(V_{:,J}^{\mathsf{T}})}} (-1)^{\operatorname{null}(V_{:,J})} + \sum_{\substack{J\subseteq[1:p]\\ \tau_J\notin\mathcal{R}(V_{:,J}^{\mathsf{T}})}} \Big[(-1)^{\operatorname{null}(V_{:,J})} + (-1)^{\operatorname{null}(V_{:,J})-1}\Big] \\
&= 2 \sum_{\substack{J\subseteq[1:p]\\ \tau_J\in\mathcal{R}(V_{:,J}^{\mathsf{T}})}} (-1)^{\operatorname{null}(V_{:,J})},
\end{aligned}$$

since $(-1)^{\text{null}(V_{:,J})} + (-1)^{\text{null}(V_{:,J})-1} = 0$ (note that $\text{null}(V_{:,J}) > 0$ if $\tau_J \notin \mathcal{R}(V_{:,J}^{\mathsf{T}})$). Formula (3.28a) follows. Formula (3.28b) of $|\mathcal{S}_a(V,\tau)|$ comes from $(3.8)_2$, $\mathcal{S}_s(V,\tau) = \mathcal{S}(V,0)$ and proposition 3.21(1), which implies that $|\mathcal{S}_a(V,\tau)| = |\mathcal{S}(V,\tau)| - |\mathcal{S}(V,0)|$. □

Note that one recovers (3.27) from (3.28a) when the arrangement is centered (i.e., $\tau \in \mathcal{R}(V^{\mathsf{T}})$). By proposition 3.21(1), $\mathcal{S}(V,0) \subseteq \mathcal{S}(V,\tau)$, so that $|\mathcal{S}(V,0)| \leqslant |\mathcal{S}(V,\tau)|$, but this inequality is not easy to deduce from (3.27) and (3.28a), since the terms of the sums in the right-hand sides of these formulas may be negative and positive.

Formula (3.28a) is usually not easy to evaluate because the number of terms in the sum can be large. It is therefore sometimes useful to have a bound on $|\mathcal{S}(V,\tau)|$ that is easier to compute than the exact formula. Proposition 3.21(4), joined to Schläfli's bound on $|\mathcal{S}(V,0)|$ (linear arrangement), makes it possible to recover a known bound on $|\mathcal{S}(V,\tau)|$ and to clarify the conditions under which this bound is reached. Recall that, for a matrix $V \in \mathbb{R}^{n\times p}$ of rank r, *Schläfli's bound* on the cardinality of $\mathcal{S}(V,0)$ reads [43, p. 211] (see also [15, proposition 4.15])

$$|\mathcal{S}(V,0)| \leqslant 2 \sum_{i\in[0:r-1]} \binom{p-1}{i}. \tag{3.29}$$

Winder [52, 1966, corollary] showed that the upper bound in (3.29) is reached if the arrangement $\mathcal{A}(V,0)$ is in *linear general position*, a concept defined as follows.

Definition 3.28 (linear general position) Let be given $V \in \mathbb{R}^{n\times p}$ of rank r, without zero column. The linear arrangement $\mathcal{A}(V,0)$ is (or the columns of V are) said to be in *linear general position* if the following equivalent properties hold

$$\begin{aligned} &\forall\, I \subseteq [1:p]: \quad \dim(\cap_{i\in I} H_i^0) = n - \min(|I|, r), \\ &\forall\, I \subseteq [1:p]: \quad \text{rank}(V_{:,I}) = \min(|I|, r), \end{aligned}$$

where $H_i^0 := \{x \in \mathbb{R}^n : V_{:,i}^{\mathsf{T}} x = 0\}$ for $i \in [1:p]$. □

The first condition has a geometric nature, while the second one has an algebraic flavor. Their equivalence comes from the fact that $\dim(\cap_{i\in I} H_i^0) = n - \text{rank}(V_{:,I})$. Observe that the inequality $\text{rank}(V_{:,I}) \leqslant \min(|I|, r)$ always holds. This linear general position property is clearly less restrictive than the injectivity of V, since it holds for an injective V but does not impose $p \leqslant n$. In [16, proposition 4.10], it is shown analytically that the linear general position is also necessary to have equality in (3.29). Let us summarize these facts in a proposition, which will be useful below.

Proposition 3.29 (bound on $|\mathcal{S}(V,0)|$) Consider a proper linear arrangement $\mathcal{A}(V,0)$, with $V \in \mathbb{R}^{n\times p}$ of rank r. Then, (3.29) holds. Furthermore, equality holds in (3.29) if and only if the arrangement is in linear general position.

For instance, the arrangement on the left-hand side pane in figure 3.1 is in linear general position and verifies (3.29) with equality. Also, linear general position generally occurs for a matrix V of rank r that is randomly generated, since then one usually has $\text{rank}(V_{:,I}) = \min(|I|, r)$ for all $I \subseteq [1:p]$.

The general position for an affine arrangement, like the one in the middle and right-hand side panes of figure 3.1, is different from that specified by definition 3.28. It is usually given a definition in geometric terms. We are also going to give it an algebraic expression (definition 3.34), since this one easily provides necessary and sufficient conditions to have equality in a bound on $|\mathcal{S}(V,\tau)|$ (proposition 3.36). All this requires some preliminary analysis.

Applying proposition 3.29 to the proper linear arrangement $\mathcal{A}([V;\tau^{\mathsf{T}}],0)$, with the augmented matrix $[V;\tau^{\mathsf{T}}] \in \mathbb{R}^{(n+1)\times p}$ of rank $\bar{r}$, yields

$$|\mathcal{S}([V;\tau^{\mathsf{T}}],0)| \leqslant 2 \sum_{i\in[0\,:\,\bar{r}-1]} \binom{p-1}{i}, \tag{3.30}$$

with equality if and only if

$$\forall\, I \subseteq [1\,:\,p]: \quad \operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|,\bar{r}). \tag{3.31}$$

By (3.23a), $\bar{r} = r$ if and only if $\tau \in \mathcal{R}(V^{\mathsf{T}})$, meaning that the arrangement $\mathcal{A}(V,\tau)$ is centered (proposition 3.5), and $\bar{r} = r+1$ otherwise. Nevertheless, it is not the upper bound in (3.30) that plays a role in the definition of the *affine general position* for the affine arrangement $\mathcal{A}(V,\tau)$, but this one is rather grounded on the following lemma, which deals with a property slightly different from (3.31), but actually stronger (see the comment after the lemma).

Lemma 3.30 (contribution to the affine general position) Consider a proper affine arrangement $\mathcal{A}(V,\tau)$, with $V \in \mathbb{R}^{n\times p}$ of rank r and $\tau \in \mathbb{R}^p$. Then, the following two conditions are equivalent:

(i) $|\mathcal{S}([V;\tau^{\mathsf{T}}],0)| = 2\sum_{i\in[0\,:\,r]} \binom{p-1}{i}$,

(ii) $\forall\, I \subseteq [1\,:\,p]:$ $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|,r+1)$.

Proof. We split the proof of the equivalence in two complementary cases.

1) If $\operatorname{rank}([V;\tau^{\mathsf{T}}]) = r+1$, the equivalence follows immediately from the equivalence in proposition 3.29 with $[V;\tau^{\mathsf{T}}]$ playing the role of V.

2) Suppose now that $\operatorname{rank}([V;\tau^{\mathsf{T}}]) = r$.

Let us first show that $r = p$ when (i) or (ii) holds. On the one hand, Schläfli's bound (3.29) gives $|\mathcal{S}([V;\tau^{\mathsf{T}}],0)| \leqslant 2\sum_{i\in[0\,:\,r-1]} \binom{p-1}{i}$, so that (i) shows that one must have $\binom{p-1}{r} = 0$ or $r \geqslant p$ or $r = p$ (since one always has $r \leqslant p$). On the other hand, (ii) with $I = [1\,:\,p]$ readily yields $r = p$.

Next, for all $I \subseteq [1\,:\,p]$, $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) \leqslant r$, which implies that (ii) is equivalent to

$$\forall\, I \subseteq [1\,:\,p]: \quad \operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|,r).$$

Next, the equivalence in proposition 3.29 with $[V;\tau^{\mathsf{T}}]$ playing the role of V indicates that the above condition is equivalent to

$$|\mathcal{S}([V;\tau^{\mathsf{T}}],0)| = 2 \sum_{i\in[0\,:\,r-1]} \binom{p-1}{i}.$$

This is clearly equivalent to (i) since $\binom{p-1}{r} = 0$ when $p = r$. $\square$

Lemma 3.30(ii) implies (3.31) since, when lemma 3.30(ii) holds, one must have $p = r$ (take $I = [1\,:\,p]$), so that $\bar{r} = r$ and, for any $I \subseteq [1\,:\,p]$, one has $|I| \leqslant p = r$ and $\min(|I|,r+1) = \min(|I|,r) = \min(|I|,\bar{r})$. But the converse is not true in general: if $p > r$, one may have the situation where (3.31) holds but not lemma 3.30(ii): for example when $V = \left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, $\tau = 0$ (see figure 3.1(left)) and $I = [1:3]$.

The notion of affine general position for the arrangement $\mathcal{A}(V,\tau)$ is well known, but we shall need another expression of this one. It is based on the following easy lemmas and proposition. The goal of the equivalences stated in the lemmas is to transform *geometric* conditions on the intersection $\cap_{i\in I} H_i$ of the hyperplanes $H_i := \{x \in \mathbb{R}^n : V_{:,i}^{\mathsf{T}} x = \tau_i\}$ into *algebraic* conditions on the rank of $V_{:,I}$ or $[V;\tau^{\mathsf{T}}]_{:,I}$. The lemmas can consider the index sets $I \subseteq [1:p]$ individually, while the proposition must consider them simultaneously, for one of its implications.

Lemma 3.31 (case $|I| \leqslant r$) *Let be given $V \in \mathbb{R}^{n\times p}$ of rank r, without zero column, $\tau \in \mathbb{R}^p$, $I \subseteq [1:p]$ with $|I| \leqslant n$ and $H_i := \{x \in \mathbb{R}^n : V_{:,i}^{\mathsf{T}} x = \tau_i\}$ for $i \in I$. Then, the following properties are equivalent:*

(i) $\cap_{i\in I} H_i \neq \varnothing$ *and* $\dim(\cap_{i\in I} H_i) = n - |I|$,

(ii) $\operatorname{rank}(V_{:,I}) = |I|$.

In these cases, one necessarily has $|I| \leqslant r$.

Proof. In both cases (i) or (ii), one can find an $\hat{x} \in \mathbb{R}^n$ such that $V_{:,I}^{\mathsf{T}} \hat{x} = \tau_I$ (in case (i), this is an $\hat{x} \in \cap_{i\in I} H_i \neq \varnothing$; in case (ii), this comes from the surjectivity of $V_{:,I}^{\mathsf{T}}$). Then,

$$x \in \cap_{i\in I} H_i \qquad \Longleftrightarrow \qquad x - \hat{x} \in \mathcal{N}(V_{:,I}^{\mathsf{T}}) = \mathcal{R}(V_{:,I})^{\perp}.$$

Therefore, $\dim(\cap_{i\in I} H_i) = \dim \mathcal{R}(V_{:,I})^{\perp} = n - \operatorname{rank}(V_{:,I})$. The equivalence follows.

For the last claim, write $|I| = \operatorname{rank}(V_{:,I}) \leqslant \operatorname{rank}(V) = r$. □

Lemma 3.32 (case $|I| \geqslant r+1$) *Let be given $V \in \mathbb{R}^{n\times p}$ of rank r, without zero column, $\tau \in \mathbb{R}^p$, $I \subseteq [1:p]$ and $H_i := \{x \in \mathbb{R}^n : V_{:,i}^{\mathsf{T}} x = \tau_i\}$ for $i \in I$. Then, the following properties are equivalent:*

(i) $\cap_{i\in I} H_i = \varnothing$ *and* $\operatorname{rank}(V_{:,I}) = r$,

(ii) $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$.

In these cases, one necessarily has $|I| \geqslant r+1$ and $\tau_I \notin \mathcal{R}(V_{:,I}^{\mathsf{T}})$.

Proof. $[(i) \Rightarrow (ii)]$ If $\cap_{i\in I} H_i = \varnothing$, then $\tau_I \notin \mathcal{R}(V_{:,I}^{\mathsf{T}})$, so that $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \operatorname{rank}(V_{:,I}^{\mathsf{T}}) + 1 = r+1$.

$[(ii) \Rightarrow (i)]$ Since $\operatorname{rank}(V) = r$, one has $\operatorname{rank}(V_{:,I}) \leqslant r$. Now, if $\operatorname{rank}(V_{:,I}) < r$, one has $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) < r+1$, contradicting (ii). Therefore, $\operatorname{rank}(V_{:,I}) = r$.

Assumption (ii) and $\operatorname{rank}(V_{:,I}) = r$ imply that $\tau_I \notin \mathcal{R}(V_{:,I}^{\mathsf{T}})$ or $\cap_{i\in I} H_i = \varnothing$.

[Last claim] By (ii), the number $|I|$ of columns of $([V;\tau^{\mathsf{T}}]_{:,I})$ must be $\geqslant r+1$ in order to have $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$. By (i), $\cap_{i\in I} H_i = \varnothing$, which implies that $\tau_I \notin \mathcal{R}(V_{:,I}^{\mathsf{T}})$. □

Proposition 3.33 (affine general position) *Let be given $V \in \mathbb{R}^{n\times p}$ of rank r, without zero column, and $\tau \in \mathbb{R}^p$. Set $H_i := \{x \in \mathbb{R}^n : V_{:,i}^{\mathsf{T}} x = \tau_i\}$ for $i \in [1\!:\!p]$. Then, the following properties are equivalent:*

$$\forall\, I \subseteq [1\!:\!p]: \quad \begin{cases} \cap_{i\in I} H_i \neq \varnothing \text{ and } \dim(\cap_{i\in I} H_i) = n - |I| & \text{if } |I| \leqslant r \\ \cap_{i\in I} H_i = \varnothing & \text{if } |I| \geqslant r+1, \end{cases} \tag{3.32a}$$

$$\forall\, I \subseteq [1\!:\!p]: \quad \begin{cases} \operatorname{rank}(V_{:,I}) = |I| & \text{if } |I| \leqslant r \\ \operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1 & \text{if } |I| \geqslant r+1, \end{cases} \tag{3.32b}$$

$$\forall\, I \subseteq [1\!:\!p]: \quad \begin{cases} \operatorname{rank}(V_{:,I}) = \min(|I|, r) \\ \operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|, r+1). \end{cases} \tag{3.32c}$$

Proof. [(3.32a) $\Rightarrow$ (3.32b)] 1) The case "$|I| \leqslant r$" results from the implication $(i) \Rightarrow (ii)$ of lemma 3.31.

2) Consider now the case "$|I| \geqslant r+1$". Let $I' \subseteq I$ with $|I'| = r$. Then, $\operatorname{rank}(V_{:,I'}) \leqslant \operatorname{rank}(V_{:,I}) \leqslant r$ and, by the point 1 just proven, $\operatorname{rank}(V_{:,I'}) = |I'| = r$. Therefore, $\operatorname{rank}(V_{:,I}) = r$. Now, applying the implication $(i) \Rightarrow (ii)$ of lemma 3.32, one gets $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$.

[(3.32b) $\Rightarrow$ (3.32a)] The proof can be done for each $I \subseteq [1\!:\!p]$ individually. The case "$|I| \leqslant r$" results from the implication $(ii) \Rightarrow (i)$ of lemma 3.31. The case "$|I| \geqslant r+1$" results from the implication $(ii) \Rightarrow (i)$ of lemma 3.32.

[(3.32b) $\Rightarrow$ (3.32c)] The proof can be done for each $I \subseteq [1\!:\!p]$ individually.

1) [$\operatorname{rank}(V_{:,I}) = \min(|I|, r)$] This identity clearly holds from (3.32b) if $|I| \leqslant r$. If $|I| \geqslant r+1$, (3.32b) yields $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$, which implies that $\operatorname{rank}(V_{:,I}) = \operatorname{rank}(V_{:,I}^{\mathsf{T}}) \geqslant r$. Since $\operatorname{rank}(V_{:,I}) \leqslant \operatorname{rank}(V) \leqslant r$, one gets $\operatorname{rank}(V_{:,I}) = r$, as desired.

2) [$\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|, r+1)$] If $|I| \leqslant r$, one has $\operatorname{rank}(V_{:,I}) = |I|$ by the first part of this proof, so that $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) \geqslant |I|$. Now, by its number of columns, $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) \leqslant |I|$, so that $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = |I|$. If $|I| \geqslant r+1$, $(3.32b)_2$ tells us that $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$.

[(3.32c) $\Rightarrow$ (3.32b)] If $|I| \leqslant r$, $(3.32c)_1$ shows that $\operatorname{rank}(V_{:,I}) = |I|$. If $|I| \geqslant r+1$, $(3.32c)_2$ shows that $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1$. □

Definition 3.34 (affine general position) A proper affine arrangement $\mathcal{A}(V,\tau)$ is said to be in *affine general position* if the equivalent properties of proposition 3.33 hold. □

Remarks 3.35 1) The two conditions in (3.32c) are independent of each other. For the arrangement in the left-hand side pane of figure 3.1, the first condition in (3.32c) holds (linear general position) but not the second one (take $I = [1\!:\!3]$). For the arrangement defined by $V = \left[\begin{smallmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [0\;0\;1]$, the second condition in (3.32c) holds but not the first one. In the latter example, one has $r = 2$, the first condition in (3.32c) fails for $I = \{1,3\}$, since $\operatorname{rank}(V_{:,I}) = 1 < \min(|I|, r) = 2$, while the second condition in (3.32c) holds, so that the arrangement is not in affine general position (note also that $|\mathcal{S}(V;\tau^{\mathsf{T}})| = 6$ and not 7 as it should be for an arrangement in affine general position with $r = 2$, according to proposition 3.36).

2) Even for a linear proper arrangement, we make a difference between *linear general position* (definition 3.28) and *affine general position* (definition 3.34), since an arrangement $\mathcal{A}(V,0)$ can be considered as a linear arrangement or an affine arrangement $\mathcal{A}(V,\tau)$ with $\tau = 0$. We see on (3.32c) and from the first remark that the notion of affine general position is more

restrictive than the notion of linear general position since it requires one more independent condition. This nuance is not felt by the sign vector sets, since the sign vector set $\mathcal{S}(V,0)$ of the linear arrangement $\mathcal{A}(V,0)$ is identical to the sign vector set $\mathcal{S}(V,\tau)$ of the affine arrangement $\mathcal{A}(V,\tau)$ with $\tau=0$ or even $\tau\in\mathcal{R}(V^{\mathsf{T}})$: (3.27) is identical to (3.28a) when $\tau=0$.

3) A centered proper arrangement $\mathcal{A}(V,\tau)$, with $V\in\mathbb{R}^{n\times p}$ of rank r, can be in affine general position, only if $p=r$ (take $I=[1:p]$ in $(3.32c)_2$).

4) Affine general position usually holds when V of rank r and τ are randomly generated. Indeed, $\operatorname{rank}(V_{:,I})=\min(|I|,r)$ generally holds for a randomly generated V of rank r and $\operatorname{rank}([V;\tau^{\mathsf{T}}]_{:,I})=\min(|I|,r+1)$ generally holds for randomly generated data (V of rank r and τ), since then $[V;\tau^{\mathsf{T}}]$ is generally of rank $r+1$. □

Condition (3.32a) is the one that is usually given to define the affine general position of an affine arrangement $\mathcal{A}(V,\tau)$ [48, p. 287]; it has a geometric nature. Condition (3.32c) is the form that suits the needs of the proof of the following proposition. We have not found elsewhere the fact that the affine general position is necessary to have equality in (3.33) (for equality in (3.33) when the arrangement is in general position, see [54, $(5.7)_1$]). The right-hand side of (3.33) is sequence A008949 in [30] (see also [45]).

Proposition 3.36 (bound on $|\mathcal{S}(V,\tau)|$) *Consider a proper arrangement $\mathcal{A}(V,\tau)$, with $V\in\mathbb{R}^{n\times p}$ of rank r and $\tau\in\mathbb{R}^p$. Then,*

$$|\mathcal{S}(V,\tau)|\leqslant\sum_{i\in[0:r]}\binom{p}{i}, \tag{3.33}$$

with equality if and only if the arrangement is in affine general position, in the sense of definition 3.34.

Proof. Observe first that $\operatorname{rank}([V;\tau^{\mathsf{T}}])\leqslant r+1$. Then, by (3.29), one has

$$|\mathcal{S}(V,0)|\leqslant 2\sum_{i\in[0:r-1]}\binom{p-1}{i}, \tag{3.34a}$$

$$|\mathcal{S}([V;\tau^{\mathsf{T}}],0)|\leqslant 2\sum_{i\in[0:r]}\binom{p-1}{i}. \tag{3.34b}$$

Using these estimates in proposition 3.21(4) provides

$$\begin{aligned}|\mathcal{S}(V,\tau)| &= \frac{1}{2}\,|\mathcal{S}(V,0)|+\frac{1}{2}\,|\mathcal{S}([V;\tau^{\mathsf{T}}],0)|\\ &\leqslant \binom{p-1}{0}+\binom{p-1}{1}+\cdots+\binom{p-1}{r-1}\\ &\qquad \underbrace{\phantom{\binom{p-1}{0}}}_{\binom{p}{0}}+\underbrace{\binom{p-1}{0}}_{\binom{p}{1}}+\cdots+\underbrace{\binom{p-1}{r-2}}_{\binom{p}{r-1}}+\underbrace{\binom{p-1}{r-1}+\binom{p-1}{r}}_{\binom{p}{r}} \qquad (3.34c)\\ &= \sum_{i\in[0:r]}\binom{p}{i},\end{aligned}$$

which is the bound (3.33).

By the previous reasoning, equality holds in (3.33) if and only if equalities hold in (3.34a) and (3.34b). By proposition 3.29 and lemma 3.30, these last equalities are equivalent to the affine general position condition (3.32c). □

For instance, the arrangements in the middle and right-hand side panes in figure 3.1 are in affine general position and verifies (3.33) with equality ($p = 3$ and $r = 2$).

Observe from (3.34c) that $2\sum_{i\in[0:r-1]}\binom{p-1}{i} = \sum_{i\in[0:r]}\binom{p}{i} - \binom{p-1}{r}$, so that the bound (3.29) on $|\mathcal{S}(V,0)|$ is lower that the bound (3.33) on $|\mathcal{S}(V,\tau)|$, unless $r = p$, in which case the affine arrangement is centered (necessarily $\tau \in \mathcal{R}(V^{\mathsf{T}})$) and can be viewed as a translated linear arrangement.

4 Chamber computation - Primal approaches

This section starts the algorithmic part of the paper, which focuses on the computation of the sign vector set $\mathcal{S} \equiv \mathcal{S}(V,\tau)$, defined by (3.3), of the considered arrangement $\mathcal{A}(V,\tau)$. By proposition 3.1, the bijection ϕ, defined by (3.4), establishes a one-to-one correspondence between these sign vectors and the chambers of the arrangement. In this section, we assume that the arrangement is proper, which means that V has only nonzero columns:

$$\forall\, j \in [1:p]: \quad V_{:,j} \neq 0. \tag{4.1}$$

Section 6 describes compact versions of some algorithms. Finally, section 7 compares these different algorithms on various instances of arrangements.

Many algorithms have been designed to list the chambers of an arrangement (see the introduction). Most of them adopt a *primal* strategy, in the sense that they focus on the realization of the inequality system $s \cdot (V^{\mathsf{T}}x - \tau) > 0$ in (3.3), by trying to compute *witness points* $x \in \mathbb{R}^n$. Section 4.1 describes this $\mathcal{S}$-tree mechanism of [37], while section 4.2 adapts to affine arrangements some of the enhancements brought to this algorithm in [16] for linear arrangements.

4.1 Primal $\mathcal{S}$-tree algorithm

For $k \in [1:p]$, define the partial sign vector set $\mathcal{S}_k \subseteq \{\pm 1\}^k$ and its complement $\mathcal{S}_k^c$ in $\{\pm 1\}^k$ by

$$\mathcal{S}_k \equiv \mathcal{S}_k(V,\tau) := \mathcal{S}(V_{:,[1:k]}, \tau_{[1:k]}) \qquad \text{and} \qquad \mathcal{S}_k^c := \{\pm 1\}^k \setminus \mathcal{S}_k. \tag{4.2}$$

Hence, $\mathcal{S}_k$ is the sign vector set of the arrangement associated with the matrix $V_{:,[1:k]} \in \mathbb{R}^{n\times k}$ and the vector $\tau_{[1:k]} \in \mathbb{R}^k$. Let us denote by v_i the ith column of V, by τ_i the ith component of τ and by $H_i := \{x \in \mathbb{R}^n : v_i^{\mathsf{T}}x = \tau_i\}$ the ith hyperplane. The $\mathcal{S}$*-tree* is a tree structure, whose k level contains the sign vectors in $\mathcal{S}_k$. Therefore, in addition to its empty root, the complete $\mathcal{S}$-tree has p levels and the bottom one is $\mathcal{S}_p = \mathcal{S}(V,\tau)$. The first level is $\mathcal{S}_1 = \{+1,-1\}$, because the inequalities $(+1)(v_1^{\mathsf{T}}x_+ - \tau_1) > 0$ and $(-1)(v_1^{\mathsf{T}}x_- - \tau_1) > 0$ are satisfied by the following two witness points, located on either side of the hyperplane H_1:

$$x_+ := (\tau_1 + 1)v_1/\|v_1\|^2 \qquad \text{and} \qquad x_- := (\tau_1 - 1)v_1/\|v_1\|^2. \tag{4.3}$$

The level $k+1$ is obtained by considering the additional pair $(v_{k+1}, \tau_{k+1}) \in \mathbb{R}^n \times \mathbb{R}$, which defines the hyperplane H_{k+1}. It can be constructed from the level k as follows. By the general assumption (4.1), every node $s \in \mathcal{S}_k$ may have one or two children, namely $(s, +1)$ and/or $(s, -1)$. Geometrically, there are two children if and only if the chamber associated with s is divided in two parts by the hyperplane H_{k+1}, but this geometric view is not easy to detect algebraically in terms of sign vectors (see below). Figure 4.1 shows the three levels of the $\mathcal{S}$-tree corresponding to the arrangement in the middle pane of figure 3.1. Now, instead of

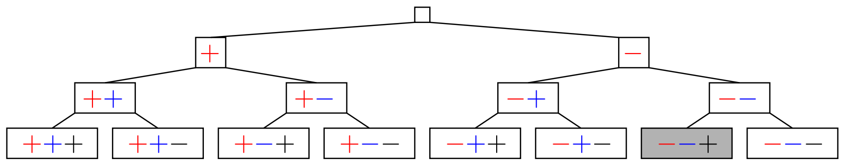


Figure 4.1: $\mathcal{S}$-tree of the arrangement in the middle pane of figure 3.1. The gray node is actually absent from the tree, since there is no chamber associated with $s = (-1, -1, +1)$ (no x such that $s \cdot (V^\mathsf{T} x - \tau) > 0$).

searching the children of every $s \in \mathcal{S}_k$ in order to obtain $\mathcal{S}_{k+1}$, the $\mathcal{S}$-tree will be constructed by a depth-first search [37], in order to avoid having to keep $\mathcal{S}_k$ in memory, which can be large. In this approach, at most p nodes along a path from the root node to a leaf node must be stored at a time. Note that, in the case of a linear arrangement (i.e., $\tau = 0$) or, more generally, a centered arrangement (i.e., $\tau \in \mathcal{R}(V^\mathsf{T})$), $\mathcal{S}(V, \tau)$ is symmetric (proposition 3.5) and only half of the sign vectors must be computed.

In the algorithm descriptions, it is assumed that the problem data (V, τ) is known and we do not repeat this data on entry of the functions. A function can modify its arguments. Let us now outline the algorithm exploring the $\mathcal{S}$-tree, called `p_stree` (algorithm 4.1, "`p`" for "primal"), which uses for this purpose a recursive procedure called `p_stree_rec` (algorithm 4.2).

Algorithm 4.1 (`p_stree`) // primal $\mathcal{S}$-tree algorithm

`p_stree_rec`$(+1, x_+)$ // x_+ given by $(4.3)_1$
`p_stree_rec`$(-1, x_-)$ // x_- given by $(4.3)_2$

Algorithm 4.2 (`p_stree_rec`$(s \in \{\pm 1\}^k$, $x \in \mathbb{R}^n)$)

`if` $(k = p)$
  Output s and `return` // s is a leaf of the $\mathcal{S}$-tree; end the recursion
`endif`
`if` $(v_{k+1}^\mathsf{T} x = \tau_{k+1})$
  `p_stree_rec`$((s, +1), x + \varepsilon v_{k+1})$ // $(s, +1) \in \mathcal{S}_{k+1}$
  `p_stree_rec`$((s, -1), x - \varepsilon v_{k+1})$ // $(s, -1) \in \mathcal{S}_{k+1}$
  `return`
`endif`
$s_{k+1} := \operatorname{sgn}(v_{k+1}^\mathsf{T} x - \tau_{k+1})$ // $(s, s_{k+1}) \in \mathcal{S}_{k+1}$
`p_stree_rec`$((s, s_{k+1}), x)$
`if` $((s, -s_{k+1})$ is feasible with witness point $\tilde{x})$
  `p_stree_rec`$((s, -s_{k+1}), \tilde{x})$ // $(s, -s_{k+1}) \in \mathcal{S}_{k+1}$
`endif`

The algorithm `p_stree` executes the recursive algorithm `p_stree_rec` for constructing the descendants of the nodes "+1" and "−1" of the first level of the $\mathcal{S}$-tree. For its part, the algorithm `p_stree_rec` constructs the descendants of a node $s \in \mathcal{S}_k$, knowing a witness point, that is a point $x \in \mathbb{R}^n$ in the chamber associated with s, hence $s_i(v_i^\mathsf{T} x - \tau_i) > 0$ for $i \in [1:k]$. Let us examine its instructions.

- If $k = p$ (instructions 1..3), the node s is a leaf of the $\mathcal{S}$-tree and has no child. Then, `p_stree_rec` just outputs s (it prints it or stores it, depending on the user's wish) and returns to the calling procedure.
- Instructions 4..8 consider the case when x is exactly in the hyperplane H_{k+1}, that is when $v_{k+1}^\mathsf{T} x = \tau_{k+1}$ (in section 4.2.2, the mechanism used in that case will also be applied when x is sufficiently closed to H_{k+1}): then s has two children $(s, \pm 1)$, since, for an easily computable sufficiently small $\varepsilon > 0$, $x_\pm^\varepsilon := x \pm \varepsilon v_{k+1}$ satisfies $s_i(v_i^\mathsf{T} x_\pm^\varepsilon - \tau_i) > 0$, for all $i \in [1:k]$ and $\pm(v_{k+1}^\mathsf{T} x_\pm^\varepsilon - \tau_{k+1}) > 0$. Note that if $x \in H_{k+1}$, then H_{k+1} is not identical to a previous hyperplane H_i, for $i \in [1:k]$, since x does not belong to any of these H_i's.
- In the sequel $v_{k+1}^\mathsf{T} x \neq \tau_{k+1}$, so that $s_{k+1} := \operatorname{sgn}(v_{k+1}^\mathsf{T} x - \tau_{k+1}) \in \{\pm 1\}$ and $(s, s_{k+1}) \in \mathcal{S}_{k+1}$ with witness point x. The instructions 9..10 deal with that situation, asking to compute the descendants of (s, s_{k+1}).
- Instructions 11..13 examine whether $(s, -s_{k+1})$ is also a child of s, which amounts to determining whether the following system has a solution $\tilde{x} \in \mathbb{R}^n$ (see below how this can be done):
$$\begin{cases} s_i(v_i^\mathsf{T} \tilde{x} - \tau_i) > 0, & \text{for } i \in [1:k] \\ -s_{k+1}(v_{k+1}^\mathsf{T} \tilde{x} - \tau_{k+1}) > 0. \end{cases} \tag{4.4}$$
If this is the case, the descendants of $(s, -s_{k+1})$ are searched using `p_stree_rec`.

To determine whether the strict inequalities (4.4) are compatible, one can, like in [37], recast the problem as a linear optimization problem (LOP) and check whether its optimal value is negative. The linear optimization problem reads

$$\begin{array}{rl} \min_{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}} & \alpha \\ \text{s.t.} & s_i(v_i^\mathsf{T} x - \tau_i) + \alpha \geqslant 0, \quad \text{for } i \in [1:k] \\ & -s_{k+1}(v_{k+1}^\mathsf{T} x - \tau_{k+1}) + \alpha \geqslant 0 \\ & \alpha \geqslant -1. \end{array} \tag{4.5}$$

This optimization problem is feasible (by taking an arbitrary $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ sufficiently large) and bounded (i.e., its optimal value is bounded below, here by -1), so that it has a solution [5, theorem 19.1]. Denote it by $(\bar{x}, \bar{\alpha})$. It is clear that (4.4) is feasible if and only if $\bar{\alpha} < 0$. This equivalence can then be used as a feasibility criterion for (4.4).

For future reference, we quote in a proposition an observation, which is deduced from the algorithm.

Proposition 4.3 (binary $\mathcal{S}$-tree) *Let $\mathcal{A}(V, \tau)$ be a proper arrangement. Then, the $\mathcal{S}$-tree is a binary tree.*

Proof. This fact is obtained by construction of the $\mathcal{S}$-tree in algorithm 4.1–4.2. Note that to have two children in lines 5..6 of algorithm 4.2, one must have $v_{k+1} \neq 0$, hence the assumption of a *proper* arrangement. If $v_{k+1} = 0$, either s has no child (if $\tau_{k+1} = 0$) or the single child $(s, -\operatorname{sgn}(\tau_{k+1}))$. □

Can one detect the symmetry of $(s, -s_{k+1})$ in $\mathcal{S}_{k+1}$, in the sense of definition 3.4, by checking whether the optimal value $\bar{\alpha}$ of problem (4.5) is -1? Proposition 4.4 and counter-example 4.5 answer negatively. Their meaning is as follows. Problem (4.5) tries to find a point $\bar{x}$ as far as possible from the boundary of the chamber associated with $(s, -s_{k+1})$ and, if the constraint $\alpha \geqslant -1$ was not present, the optimal value $\bar{\alpha}$ would reflect the distance from $\bar{x}$ (if this one exists) to the chamber boundary. Now, when $(s, -s_{k+1})$ is symmetric in $\mathcal{S}_{k+1}$, the interior of the asymptotic cone of its associated chamber is not empty (lemma 3.8(ii)), so that, without the constraint $\alpha \geqslant -1$, the optimal value would be $-\infty$. This implies that the optimal value of problem (4.5) is $\bar{\alpha} = -1$ when $(s, -s_{k+1})$ is symmetric in $\mathcal{S}_{k+1}$ (proposition 4.4). For an asymmetric $(s, -s_{k+1})$, that distance will be large if, for the given τ, the feasible set is sufficiently large; in that case, one also has $\bar{\alpha} = -1$. Therefore, an optimal value $\bar{\alpha} = -1$ cannot be used to detect the symmetric of $(s, -s_{k+1})$ in $\mathcal{S}_{k+1}$ (counter-example 4.5).

Proposition 4.4 (a symmetric sign vector yields $\bar{\alpha} = -1$) *If* $\pm(s, -s_{k+1}) \in \mathcal{S}_{k+1}$, *then the optimal value of the LOP* (4.5) *is* $\bar{\alpha} = -1$.

Proof. By the implication (i) $\Rightarrow$ (ii) of lemma 3.8, $\pm(s, -s_{k+1}) \in \mathcal{S}_{k+1}$ implies that there exists a $d \in \mathbb{R}^n$ such that $(s, -s_{k+1}) \cdot (V_{:,[1:k+1]}^{\mathsf{T}} d) > 0$. Therefore, the constraints of (4.5) can be satisfied with $\alpha = -1$ and x replaced by $x + td$ with t sufficiently large. $\square$

Counter-example 4.5 (asymmetric sign vector with $\bar{\alpha} = -1$) Consider the arrangement $\mathcal{A}(V, \tau)$ with $V = \left[\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [t \quad -t \quad t \quad -t]$ for some $t > 0$. The sign vector $s = (-1, +1, -1, +1)$ is in $\mathcal{S}_a(V, \tau)$ and its associated chamber $\{x \in \mathbb{R}^2 : -t < x_1 < t, -t < x_2 < t\}$ is bounded. A solution to problem (4.5) is $\bar{x} = 0$ and $\bar{\alpha} = \max(-t, -1)$. Hence, the optimal value is -1 as soon as $t \geqslant 1$. $\square$

Nevertheless, the detection of the symmetry of $(s, -s_{k+1})$ is possible if the lower constraint on α in (4.5) is discarded; see proposition 3.10.

By proposition 4.3, any sign vector in $\mathcal{S}_k$, with $k \in [1 : p-1]$, has either one or two children in $\mathcal{S}_{k+1}$. The next proposition characterizes the sign vectors of $\mathcal{S}_k$ that have two children in $\mathcal{S}_{k+1}$. It extends to affine arrangements proposition 4.9 in [15], which is there used to give an analytic version of Winder's proof of (3.27), giving the cardinality of $\mathcal{S}(V, 0)$. Below, proposition 4.6 will be useful to get the lower bound (4.8) on $|\mathcal{S}(V, \tau)|$, improving (3.10). In the statement of proposition 4.6, $\mathrm{P}_{k+1} : \mathbb{R}^n \to H_{k+1} - H_{k+1}$ denotes the orthogonal projector on the subspace $v_{k+1}^{\perp}$ that is parallel to the affine space $H_{k+1} := \{x \in \mathbb{R}^n : v_{k+1}^{\mathsf{T}} x = \tau_{k+1}\}$; while $\hat{x}_{k+1} := \tau_{k+1} v_{k+1} / \|v_{k+1}\|^2$ is the unique point in $\mathcal{N}(\mathrm{P}_{k+1}) \cap H_{k+1}$. We also denote by P_{k+1} the transformation matrix of the projector, so that $\mathrm{P}_{k+1} V_{:,[1:k]}$ can be viewed as the product of two matrices (its jth column is $\mathrm{P}_{k+1} v_j$, for $j \in [1 : k]$). Note that in (4.6b), $\tilde{V} := \mathrm{P}_{k+1} V_{:,[1:k]}$ may have zero columns, in which case, by its definition (3.3), the set $\mathcal{S}(\tilde{V}, \tilde{\tau})$ will be nonempty if the corresponding components of $\tilde{\tau}$ do not vanish.

Proposition 4.6 (two child criterion) *Let* $V \in \mathbb{R}^{n \times p}$, $s \in \{\pm 1\}^k$ *for some* $k \in [1 : p-1]$

and $\hat{x}_{k+1}$ be the unique point in $\mathcal{N}(\mathrm{P}_{k+1}) \cap H_{k+1}$. Then,

$$
\begin{aligned}
&(s, +1) \text{ and } (s, -1) \in \mathcal{S}_{k+1} \\
&\iff \exists\, x \in \mathbb{R}^n : s_i(v_i^{\mathsf{T}} x - \tau_i) > 0, \text{ for } i \in [1\!:\!k], \text{ and } v_{k+1}^{\mathsf{T}} x - \tau_{k+1} = 0 \qquad (4.6a) \\
&\iff s \in \mathcal{S}(\mathrm{P}_{k+1} V_{:,[1:k]}, \tau_{[1:k]} - V_{:,[1:k]}^{\mathsf{T}} \hat{x}_{k+1}). \qquad (4.6b)
\end{aligned}
$$

Proof. To simplify the notation, set $V_k := V_{:,[1:k]}$. The following equivalences prove the result ((4.7a) and (4.7b) are justified afterwards):

$$
\begin{aligned}
&(s, +1) \text{ and } (s, -1) \in \mathcal{S}(V_{k+1}, \tau_{[1:k+1]}) \\
&\iff \begin{cases} \exists\, x_+ \in \mathbb{R}^n : s_i(v_i^{\mathsf{T}} x_+ - \tau_i) > 0, \text{ for } i \in [1\!:\!k], \text{ and } +(v_{k+1}^{\mathsf{T}} x_+ - \tau_{k+1}) > 0 \\ \exists\, x_- \in \mathbb{R}^n : s_i(v_i^{\mathsf{T}} x_- - \tau_i) > 0, \text{ for } i \in [1\!:\!k], \text{ and } -(v_{k+1}^{\mathsf{T}} x_- - \tau_{k+1}) > 0 \end{cases} \\
&\iff \exists\, x \in \mathbb{R}^n : s_i(v_i^{\mathsf{T}} x - \tau_i) > 0, \text{ for } i \in [1\!:\!k], \text{ and } v_{k+1}^{\mathsf{T}} x - \tau_{k+1} = 0 \qquad (4.7a) \\
&\iff \exists\, x \in \mathbb{R}^n : s_i([\mathrm{P}_{k+1} v_i]^{\mathsf{T}} x - [\tau_i - v_i^{\mathsf{T}} \hat{x}_{k+1}]) > 0, \text{ for } i \in [1\!:\!k] \qquad (4.7b) \\
&\iff s \in \mathcal{S}(\mathrm{P}_{k+1} V_k, \tau_{[1:k]} - V_k^{\mathsf{T}} \hat{x}_{k+1}).
\end{aligned}
$$

The equivalence in (4.7a) is shown as follows.

[⇒] Define $t_- := +(v_{k+1}^{\mathsf{T}} x_+ - \tau_{k+1}) > 0$, $t_+ := -(v_{k+1}^{\mathsf{T}} x_- - \tau_{k+1}) > 0$, $\alpha_- := t_-/(t_- + t_+) \in (0,1)$ and $\alpha_+ := t_+/(t_- + t_+) \in (0,1)$. Then, $x = \alpha_+ x_+ + \alpha_- x_-$ is appropriate since

$$
\begin{aligned}
\text{for } i \in [1\!:\!k]: \quad s_i(v_i^{\mathsf{T}} x - \tau_i) &= \alpha_+ s_i[v_i^{\mathsf{T}} x_+ - \tau_i] + \alpha_- s_i[v_i^{\mathsf{T}} x_- - \tau_i] > 0, \\
v_{k+1}^{\mathsf{T}} x - \tau_{k+1} &= \alpha_+[v_{k+1}^{\mathsf{T}} x_+ - \tau_{k+1}] + \alpha_-[v_{k+1}^{\mathsf{T}} x_- - \tau_{k+1}] = \alpha_+ t_- - \alpha_- t_+ = 0.
\end{aligned}
$$

[⇐] Take $x_\pm = x \pm \varepsilon v_{k+1}$ for a sufficiently small $\varepsilon > 0$.

The equivalence in (4.7b) is shown as follows.

[⇒] The point x in (4.7a) satisfies $x \in H_{k+1}$, so that $x = \mathrm{P}_{k+1} x + \hat{x}_{k+1}$, since $x - \mathrm{P}_{k+1} x \in \mathcal{N}(\mathrm{P}_{k+1}) \cap H_{k+1} = \{\hat{x}_{k+1}\}$. Furthermore, by (4.7a), one has for $i \in [1\!:\!k]$:

$$
0 < s_i(v_i^{\mathsf{T}} x - \tau_i) = s_i(v_i^{\mathsf{T}} [\mathrm{P}_{k+1} x + \hat{x}_{k+1}] - \tau_i) = s_i([\mathrm{P}_{k+1} v_i]^{\mathsf{T}} x - [\tau_i - v_i^{\mathsf{T}} \hat{x}_{k+1}]),
$$

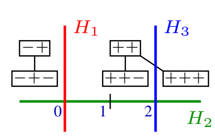
where we have used $\mathrm{P}_{k+1}^{\mathsf{T}} = \mathrm{P}_{k+1}$ (P_{k+1} is an orthogonal projector).

[⇐] Take $x := \mathrm{P}_{k+1} x_0 + \hat{x}_{k+1} \in H_{k+1}$ in (4.7a), where x_0 is the x given by (4.7b). □

Note that if the equivalences in proposition 4.6 hold, then $s \in \mathcal{S}_k$.

Example 4.7 (two child criterion) Consider the arrangement $\mathcal{A}(V, \tau)$ with $V = \left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]$ and $\tau^{\mathsf{T}} = [0\ 0\ 2]$ (see the picture below) and suppose that $k = 2$ in proposition 4.6. One has $V_{:,[1:2]} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\mathrm{P}_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\hat{x}_3^{\mathsf{T}} = [2\ 0]$. Since $V_{:,[1:2]}$ is injective, $\mathcal{S}_2 = \{\pm 1\}^2$. The criterion (4.6b) tells us that $s \in \mathcal{S}_2$ has two children if and only if there exists an $x \in \mathbb{R}^n$ that satisfies

$$
s \cdot \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) > 0.
$$



This condition holds if and only if $s_1 = +1$, which is what can be observed in the picture. □

The next proposition improves and extends to affine arrangement proposition 4.6 in [16]. We denote by $\operatorname{vect}\{v_1, \ldots, v_k\}$ the vector space spanned by the vectors $v_1, \ldots, v_k$.

Proposition 4.8 (incrementation) *Let $V = [v_1 \cdots v_p] \in \mathbb{R}^{n\times p}$.*

1) *If $s \in \mathcal{S}_k^c$, then $(s, \pm 1) \in \mathcal{S}_{k+1}^c$. Consequently, $|\mathcal{S}_{k+1}^c| \geqslant 2|\mathcal{S}_k^c|$.*
2) *If $v_{k+1} \notin \operatorname{vect}\{v_1, \ldots, v_k\}$, then, $(s, \pm 1) \in \mathcal{S}_{k+1}$ for all $s \in \mathcal{S}_k$, $|\mathcal{S}_{k+1}| = 2|\mathcal{S}_k|$ and $|\mathcal{S}_{k+1}^c| = 2|\mathcal{S}_k^c|$.*
3) *If $v_{k+1} \in \operatorname{vect}\{v_1, \ldots, v_k\}$, $V_{:,[1:k+1]}$ has no zero column, $H_{k+1} \neq H_i$ for $i \in [1:k]$, and $r_k := \dim \operatorname{vect}\{v_1, \ldots, v_k\}$, then $|\mathcal{S}_{k+1}| \geqslant |\mathcal{S}_k| + 2^{r_k - 1}$.*

Proof. 1) If $s \in \mathcal{S}_k^c$, there is no $x \in \mathbb{R}^n$ such that $s_i(v_i^{\mathsf{T}} x - \tau_i) > 0$ for $i \in [1:k]$. Therefore, there is certainly no $x \in \mathbb{R}^n$ satisfying $s_i(v_i^{\mathsf{T}} x - \tau_i) > 0$ for $i \in [1:k+1]$, with $s_{k+1} \in \{\pm 1\}$. Therefore, $(s, \pm 1) \in \mathcal{S}_{k+1}^c$. This observation implies that $|\mathcal{S}_{k+1}^c| \geqslant 2|\mathcal{S}_k^c|$.

2) Let Q be the orthogonal projector on $\operatorname{vect}\{v_1, \ldots, v_k\}^{\perp}$ for the Euclidean scalar product. By assumption, $\mathrm{Q}\, v_{k+1} \neq 0$. Let $s \in \mathcal{S}_k$, so that there is an $x \in \mathbb{R}^n$ such that $s_i(v_i^{\mathsf{T}} x - \tau_i) > 0$ for $i \in [1:k]$. For any $t \in \mathbb{R}$ and $i \in [1:k]$, the points $x_\pm := x \pm t\, \mathrm{Q}\, v_{k+1}$ verify $s_i(v_i^{\mathsf{T}} x_\pm - \tau_i) = s_i(v_i^{\mathsf{T}} x - \tau_i) > 0$ (because $v_i^{\mathsf{T}} \mathrm{Q}\, v_{k+1} = 0$). In addition, for $t > 0$ sufficiently large, one has $\pm(v_{k+1}^{\mathsf{T}} x_\pm - \tau_{k+1}) = \pm(v_{k+1}^{\mathsf{T}} x - \tau_{k+1}) + t\|\mathrm{Q}\, v_{k+1}\|^2 > 0$ (because $\mathrm{Q}^2 = \mathrm{Q}$ and $\mathrm{Q}^{\mathsf{T}} = \mathrm{Q}$). We have shown that both $(s, +1)$ and $(s, -1)$ are in $\mathcal{S}_{k+1}$. Therefore, $|\mathcal{S}_{k+1}| \geqslant 2|\mathcal{S}_k|$.

Now, $|\mathcal{S}_k| + |\mathcal{S}_k^c| = 2^k$, $|\mathcal{S}_{k+1}| + |\mathcal{S}_{k+1}^c| = 2^{k+1}$ and $|\mathcal{S}_{k+1}^c| \geqslant 2|\mathcal{S}_k^c|$ by point 1. Therefore, one must have $|\mathcal{S}_{k+1}| = 2|\mathcal{S}_k|$ and $|\mathcal{S}_{k+1}^c| = 2|\mathcal{S}_k^c|$.

3) One has $\operatorname{null}(\mathrm{P}_{k+1}) = 1$ (since $\mathcal{N}(\mathrm{P}_{k+1}) = \mathbb{R} v_{k+1}$) and $\operatorname{rank}(V_{:,[1:k]}) = r_k$ (by definition). Then, $\operatorname{rank}(\mathrm{P}_{k+1} V_{:,[1:k]}) \geqslant r_k - 1$.

To apply (3.10) to the arrangement $\mathcal{A}(\mathrm{P}_{k+1} V_{:,[1:k]}, \tau_{[1:k]} - V_{:,[1:k]}^{\mathsf{T}} \hat{x}_{k+1})$, one must show that $v_i^{\mathsf{T}} \hat{x}_{k+1} \neq \tau_i$ when $i \in [1:k]$ and $\mathrm{P}_{k+1} v_i = 0$ (i.e., v_{k+1} and v_i are colinear or H_{k+1} and H_i are parallel by proposition 3.2(1)). This is indeed the case, since $v_i^{\mathsf{T}} \hat{x}_{k+1} = \tau_i$ would imply that $H_{k+1} = H_i$ (because then $\hat{x}_{k+1}$ would belong to both H_i and H_{k+1}, which are parallel), in contradiction with the assumption. By (3.10), $|\mathcal{S}(\mathrm{P}_{k+1} V_{:,[1:k]}, \tau_{[1:k]} - V_{:,[1:k]}^{\mathsf{T}} \hat{x}_{k+1})| \geqslant 2^{r_k - 1}$.

By proposition 4.6, there are at least $2^{r_k - 1}$ sign vectors in $\mathcal{S}_k$ with two children. Since any $s \in \mathcal{S}_k$ has at least one child when $v_{k+1} \neq 0$, one gets $|\mathcal{S}_{k+1}| \geqslant |\mathcal{S}_k| + 2^{r_k - 1}$. □

As a corollary of the previous proposition, one gets the following improvement of the lower bound of $|\mathcal{S}(V, \tau)|$ given by (3.10).

Corollary 4.9 (lower bound of $|\mathcal{S}(V, \tau)|$) *For a matrix V of rank r without zero column and a vector τ such that all the hyperplanes H_i are different, one has*

$$2^r + 2^{r-1}(p - r) \leqslant |\mathcal{S}(V, \tau)|. \tag{4.8}$$

Proof. By a possible change of column order, which only affects the chamber numbering, not the cardinality of $\mathcal{S}(V, \tau)$, one can assume that the first r columns of V are linearly independent. Then, by proposition 4.8(2), $|\mathcal{S}_r| = 2^r$. Next, for $k \in [r : p-1]$, $\dim \operatorname{vect}\{v_1, \ldots, v_k\} = r$, so that proposition 4.8(3) implies that $|\mathcal{S}_{k+1}| \geqslant |\mathcal{S}_k| + 2^{r-1}$. By induction, one gets (4.8). □

4.2 Preventing some computations

The main computation cost of algorithm 4.1 comes from solving the LOPs (4.5) at some inner nodes. This section describes three ways of bypassing LOPs. They are adapted from [16], where only linear arrangements are considered, and are identified by the letters A, B and C (the letters appearing in the section titles), which will also be used to label more efficient variants of algorithm 4.1. These variants significantly speed up the algorithm by reducing the numbers of LOPs to solve (see section 7).

4.2.1 A - Rank of the arrangement

Instead of starting the $\mathcal{S}$-tree with the two nodes of $\mathcal{S}_1 = \{+1, -1\}$, like in algorithm 4.1, one can start it with the 2^r nodes of $\mathcal{S}_r = \{\pm 1\}^r$, by considering first a selection of $r := \operatorname{rank}(V)$ linearly independent vectors whose $\mathcal{S}$-tree is easy to construct without having to solve any LOP. Here are the details.

Numerically, r linearly independent vectors can be found by a QR factorization of V:

$$VP = QR,$$

with $P \in \{0,1\}^{p\times p}$ is a permutation matrix, $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{n\times p}$ is upper triangular with $R_{[r+1\,:\,n],:} = 0$. To simplify the presentation, let us assume that P is the identity matrix, in which case the first r vectors $v_1, \ldots, v_r$ (or columns of V) are linearly independent, and let us note $V_r := V_{:,[1\,:\,r]}$, $Q_r := Q_{:,[1\,:\,r]}$ and $R_r := R_{[1\,:\,r],[1\,:\,r]}$. By proposition 4.8(2),

$$\mathcal{S}_r = \{\pm 1\}^r.$$

To launch the recursive algorithm 4.2, one still need to compute a witness point x_s associated with any $s \in \mathcal{S}_r$. For this purpose, one computes a point $\hat{x} \in \cap_{i=1}^r H_i$, hence verifying $V_r^{\mathsf{T}}\hat{x} = \tau_{[1\,:\,r]}$, by $\hat{x} := V_r(V_r^{\mathsf{T}}V_r)^{-1}\tau_{[1\,:\,r]}$. Next, for any $s \in \{\pm 1\}^r$, one computes $d_s := Q_r R_r^{-\mathsf{T}} s \in \mathbb{R}^n$. Let us show that $x_s := \hat{x} + d_s$ is a witness point of the considered s. One has

$$V_r^{\mathsf{T}}x_s - \tau_{[1\,:\,r]} = V_r^{\mathsf{T}}[V_r(V_r^{\mathsf{T}}V_r)^{-1}\tau_{[1\,:\,r]} + Q_r R_r^{-\mathsf{T}}s] - \tau_{[1\,:\,r]} = (Q_r R_r)^{\mathsf{T}} Q_r R_r^{-\mathsf{T}} s = s.$$

Therefore, $s \cdot (V_r^{\mathsf{T}}x_s - \tau_{[1\,:\,r]}) = s \cdot s = e > 0$, as desired.

4.2.2 B - Handling of a hyperplane proximity

In the description of algorithm 4.2, it is shown why a witness point x of a sign vector $s \in \mathcal{S}_k$ that belongs to H_{k+1}, i.e., $v_{k+1}^{\mathsf{T}}x = \tau_{k+1}$, allows the algorithm to certify that both $(s, +1)$ and $(s, -1)$ are in $\mathcal{S}_{k+1}$, without having to solve a LOP. We show with the next proposition that this is still true when x is near H_{k+1}, in the sense (4.9). Note that this proximity to H_{k+1} is measured by strict inequalities, which is more stable with respect to numerical perturbations than an equality.

Proposition 4.10 (two children without LOP) *Let $s \in \mathcal{S}_k$ and $x \in \mathbb{R}^n$ verifying*

$s \cdot (V_{:,[1:k]}^{\mathsf{T}} x - \tau_{[1:k]}) > 0$. Suppose that $v_{k+1} \neq 0$ and

$$\underbrace{\max_{s_i v_i^{\mathsf{T}} v_{k+1} > 0} \frac{\tau_i - v_i^{\mathsf{T}} x}{v_i^{\mathsf{T}} v_{k+1}}}_{=: t_{\min}} < \underbrace{\frac{\tau_{k+1} - v_{k+1}^{\mathsf{T}} x}{\|v_{k+1}\|^2}}_{=: t_0} < \underbrace{\min_{s_i v_i^{\mathsf{T}} v_{k+1} < 0} \frac{\tau_i - v_i^{\mathsf{T}} x}{v_i^{\mathsf{T}} v_{k+1}}}_{=: t_{\max}}. \tag{4.9}$$

Then, for $t_- \in (t_{\min}, t_0)$, $x_- := x + t_- v_{k+1}$ is a witness point of $(s, -1)$ and, for $t_+ \in (t_0, t_{\max})$, $x_+ := x + t_+ v_{k+1}$ is a witness point of $(s, +1)$.

Proof. Note first that, in (4.9), the arguments of the maximum are negative and the arguments of the minimum are positive. Therefore, both inequalities are verified if $v_{k+1}^{\mathsf{T}} x = \tau_{k+1}$ ($t_0 = 0$), that is, when $x \in H_{k+1}$. One has, for $i \in [1:k]$,

$$s_i(v_i^{\mathsf{T}}(x + t v_{k+1}) - \tau_i) > 0 \quad \Longleftrightarrow \quad \begin{cases} t < \frac{s_i(\tau_i - v_i^{\mathsf{T}} x)}{s_i v_i^{\mathsf{T}} v_{k+1}} & \text{if } s_i v_i^{\mathsf{T}} v_{k+1} < 0 \\ t \in \mathbb{R} & \text{if } s_i v_i^{\mathsf{T}} v_{k+1} = 0 \\ t > \frac{s_i(\tau_i - v_i^{\mathsf{T}} x)}{s_i v_i^{\mathsf{T}} v_{k+1}} & \text{if } s_i v_i^{\mathsf{T}} v_{k+1} > 0. \end{cases}$$

Since the conditions imposed on t in the right-hand side of the equivalence above are satisfied by any $t \in (t_{\min}, t_{\max})$, it follows that $x_\pm$ are witness points of s. One has

$$\begin{aligned} t > t_0 := \tfrac{\tau_{k+1} - v_{k+1}^{\mathsf{T}} x}{\|v_{k+1}\|^2} \quad &\Longleftrightarrow \quad v_{k+1}^{\mathsf{T}}(x + t v_{k+1}) - \tau_{k+1} > 0, \\ t < t_0 := \tfrac{\tau_{k+1} - v_{k+1}^{\mathsf{T}} x}{\|v_{k+1}\|^2} \quad &\Longleftrightarrow \quad v_{k+1}^{\mathsf{T}}(x + t v_{k+1}) - \tau_{k+1} < 0. \end{aligned}$$

Since t_+ (resp. t_-) verifies the condition in the left-hand side of the first (resp. second) equivalence above, it follows that x_+ (resp. x_-) is a witness point of $(s, +1)$ (resp. $(s, -1)$). □

4.2.3 C - Choosing the order of the vectors

Every inner node of the $\mathcal{S}$-tree has one or two children and this number is sometimes detected in algorithm 4.2 by solving a LOP, which is a time consuming operation. Therefore, a way of decreasing the computation time is to reduce the number of inner nodes of the $\mathcal{S}$-tree. This property can be obtained by choosing wisely the order in which the pairs (v_i, τ_i), or hyperplanes H_i, are taken into account when constructing the branches of the $\mathcal{S}$-tree (this order can be different from one branch to another), with the goal of placing the nodes with a single child close to the root of the tree. This strategy has been investigated in [16, § 5.2.4.C] and we adapt the heuristic to the present context of affine arrangements.

Denote by $T_s := \{i_1^s, \ldots, i_k^s\}$ the set of the indices of the hyperplanes selected to reach node $s \in \mathcal{S}_k$ (T_s depends on s). At this node, the algorithm must choose the next hyperplane to consider, whose index is among the index set $T_s^c := [1:p] \setminus T_s$. With the goal of preventing, as much as possible, the node s from having two children, a natural idea is to ignore the indices of T_s^c, for which proposition 4.10 ensures two children. In the remaining index set, denote it by T_s^b, the chosen index is the one maximizing the quantity $|v_i^{\mathsf{T}} x - \tau_i| / \|[v_i; \tau_i]\|$ for $i \in T_s^b$ (x is the witness point associated by the algorithm with the current node s), since the larger this quantity is, the further x is from the chosen hyperplane, which should increase the chances that s will have only one child.

5 Chamber computation- Dual approaches

The enumeration of the chambers of an arrangement $\mathcal{A}(V,\tau)$ can be tackled by an approach different from those presented in section 4 (algorithm 4.1 and its improvements A, B and C), sometimes (or always) replacing optimization phases by algebra techniques. More specifically, we say that an algorithm has a *dual aspect* when it uses the concept of stem vector (definition 3.15) by means of proposition 3.19. Such a dual approach was introduced in [16, §§ 5.2.2-5.2.3] for linear arrangements.

Section 5.1 deals with algorithms computing $\mathcal{S}(V,\tau)$ that assume the availability of the full stem vector set $\mathfrak{S}(V,\tau)$ and do not use optimization. A method for computing $\mathfrak{S}(V,\tau)$ is presented in section 5.1.1. Section 5.1.2 describes a crude dual approach for computing $\mathcal{S}(V,\tau)$ that is only efficient for small arrangements. The algorithm proposed in section 5.1.3 has the structure of algorithm 4.1, in the sense that it constructs the $\mathcal{S}$-tree, but its optimization phases are replaced by the duality technique mentioned above.

In section 5.2, we present a way of obtaining a circuit, hence stem vectors, from a linear optimization problem that is associated with an infeasible sign vector (proposition 5.6). This leads to an algorithm having primal and dual aspects.

This section also assumes (4.1), that is, V has no zero column.

5.1 Algorithms using all the stem vectors

Proposition 3.19 establishes a link between the *infeasible* sign vectors, those in $\mathcal{S}(V,\tau)^c$, and the stem vectors. This section presents two algorithms that start with the computation of the complete stem vector set $\mathfrak{S}(V,\tau)$ (section 5.1.1). The first one uses these stem vectors to compute $\mathcal{S}(V,\tau)^c$, from which the feasible sign vector set $\mathcal{S}(V,\tau)=\{\pm 1\}^p\setminus\mathcal{S}(V,\tau)^c$ can be deduced (section 5.1.2). The second one computes directly $\mathcal{S}(V,\tau)$ like in the $\mathcal{S}$-tree primal algorithm of section 4.1, but without solving linear optimization problems and without computing witness points (section 5.1.3).

5.1.1 Stem vector computation

Let us start with the presentation of a plain algorithm that computes the disjoint sets $\mathfrak{S}_s(V,\tau)$ and $\mathfrak{S}_a(V,\tau)$ of the symmetric and asymmetric stem vectors; recall that the set of all stem vectors is $\mathfrak{S}(V,\tau)=\mathfrak{S}_s(V,\tau)\cup\mathfrak{S}_a(V,\tau)$. This algorithm is based on the detection of the circuits of V and remark 3.16(3). It is rudimentary (the one used in [28] is valid for an arbitrary matroid and yields an interesting complexity property, but, in our experience with vector matroids, it is much less efficient than the method used in algorithm 5.1; see also [38] and the pieces of software mentioned in the introduction). The algorithm can be significantly improved in particular cases, see [39].

Algorithm 5.1 (`stem_vectors(`$\boldsymbol{\mathfrak{S}_s}$`, `$\boldsymbol{\mathfrak{S}_a}$`)`) // stem vector calculation

1. $\mathfrak{S}_s=\varnothing$ and $\mathfrak{S}_a=\varnothing$
2. `for` $i\in[1\!:\!p]$ `do`
3. `stem_vectors_rec(`$\mathfrak{S}_s$, $\mathfrak{S}_a$, $\{i\}$`)`
4. `endfor`
5. Remove duplicate stem vectors in $\mathfrak{S}_s$ and $\mathfrak{S}_a$

Algorithm 5.2 (`stem_vectors_rec(`$\mathfrak{S}_s$, $\mathfrak{S}_a$, I_0`)`)

1. `for` $i \in [\max(I_0)+1:p]$ `do`
2. $\quad I := I_0 \cup \{i\}$
3. $\quad$ `if` $(\mathcal{N}(V_{:,I}) \neq \{0\})$
4. $\quad\quad$ Let $\eta_I \in \mathcal{N}(V_{:,I}) \setminus \{0\}$ and $J := \{i \in I : \eta_i \neq 0\}$ $\qquad // \; J \in \mathcal{C}(V)$
5. $\quad\quad$ `if` $(\tau_J^{\mathsf{T}} \eta_J = 0)$
6. $\quad\quad\quad \mathfrak{S}_s := \mathfrak{S}_s \cup \{\operatorname{sgn}(\eta_J\} \cup \{-\operatorname{sgn}(\eta_J)\}$
7. $\quad\quad$ `else`
8. $\quad\quad\quad \mathfrak{S}_a := \mathfrak{S}_a \cup \{\operatorname{sgn}(\tau_J^{\mathsf{T}} \eta_J) \operatorname{sgn}(\eta_J)\}$
9. $\quad\quad$ `endif`
10. $\quad$ `else`
11. $\quad\quad$ `stem_vectors_rec(`$\mathfrak{S}_s$, $\mathfrak{S}_a$, I`)`
12. $\quad$ `endif`
13. `endfor`

Here are some explanations and observations on algorithm 5.1. Unless otherwise stated, the line numbers refer to algorithm 5.2.

- The function `stem_vectors_rec(`$\mathfrak{S}_s$`, `$\mathfrak{S}_a$`, `I_0`)` adds to $\mathfrak{S}_s$ and/or $\mathfrak{S}_a$ stem vectors σ such that $\mathfrak{J}(\sigma)$ is contained in the set formed of I_0 and indices larger that $\max(I_0)$.
- On entry in algorithm 5.2, $V_{:,I_0}$ is assumed to be injective: this is the case in line 3 of algorithm 5.1 (recall the assumption (4.1)) and in line 11 of algorithm 5.2 (since there $\mathcal{N}(V_{:,I}) = \{0\}$). Therefore, in line 4 of algorithm 5.2, $\operatorname{null}(V_{:,I}) = 1$ and J is a circuit of V (lemma 3.14).
- The algorithm does not explore all the subsets of $[1:p]$ since, once a circuit $J \subseteq I$ has been found in line 4, an index set $I' \supseteq I$ satisfying $\operatorname{null}(N_{:,I'}) = 1$ contains no circuit different from J (lemma 3.14). This explains why there is no recursive call to `stem_vectors_rec` when $\mathcal{N}(V_{:,I}) \neq \{0\}$ (lines 4..9).
- In line 4, η_I is obtained by a null space computation, so that algorithm 5.2 is sensitive to rounding errors. One can use exact arithmetic linear algebra to compute the set of sign vectors when the data is rational or integer, at the expense of slower computation. This is particularly easy to implement in Julia and requires minimal modifications, thanks to built-in exact linear algebra methods.
- Line 6 corresponds to remark 3.16(3.a) and line 7 to remark 3.16(3.b). These lines are symbolically written, since, in addition to $\pm \operatorname{sgn} \eta_J$ one also has to store J. In practice, one can only store half of the symmetric stem vectors in $\mathfrak{S}_s$ and obtain the full set, if needed, by gathering the stem vectors of $\mathfrak{S}_s$ and $-\mathfrak{S}_s$.
- The loop 2..4 of algorithm 5.1 may find several times the same stem vector. This is the case, for instance, if $V = [e_1, e_2, e_2]$: the circuit $J = \{2, 3\}$ is found twice by the loop of algorithm 5.1 (once with $i = 1$ and again with $i = 2$), as well as the associated stem vectors. This justifies the final elimination of duplicates in line 5 of algorithm 5.1.

5.1.2 Crude dual algorithm

The algorithm described in this section, algorithm 5.3, is a "crude" way of obtaining the sign vector set $\mathcal{S}(V, \tau)$ from the stem vector set $\mathfrak{S}(V, \tau)$. It uses the characterization of proposition 3.19. For each stem vector $\sigma \in \mathfrak{S}(V, \tau)$ with associated circuit $J = \mathfrak{J}(\sigma) \subseteq [1:p]$, the algorithm generates all the infeasible sign vectors $s \in \mathcal{S}(V,\tau)^c$ satisfying $s_J = \sigma$ and $s_{J^c} \in \{\pm1\}^{J^c}$. This generation is made by the function `stem_to_infeas_sign_vectors` in a straightforward manner (the precise computation is not detailed). Once $\mathcal{S}(V,\tau)^c$ is computed, $\mathcal{S}(V,\tau)$ is obtained by $\{\pm1\}^p \setminus \mathcal{S}(V,\tau)^c$.

This `stem_to_infeas_sign_vectors` function can produce duplicated sign vectors, which justifies the cleaning operation in line 5 (this one could be done simultaneously with the union in line 4). For example, with $V = [1,1,1]$ and $\tau^{\mathsf{T}} = [1,0,1]$, the stem vectors $(1,-1) \in \{\pm1\}^{\{1,2\}}$ and $(-1,1) \in \{\pm1\}^{\{2,3\}}$ produce the same infeasible sign vector $(1,-1,1)$.

Algorithm 5.3 (`crude_dual`($\mathcal{S}$)) // crude dual algorithm

1. `Sc` $= \varnothing$ // initialization of $\mathcal{S}(V,\tau)^c$
2. `stem_vectors`($\mathfrak{S}_s$, $\mathfrak{S}_a$) // algorithm 5.1
3. `for` $\sigma \in \mathfrak{S}_s \cup \mathfrak{S}_a$ `do`
4. `Sc = Sc` $\cup$ `stem_to_infeas_sign_vectors`(σ`, p)`
5. Remove duplicates in `Sc`
6. `endfor`
7. $\mathcal{S} := \{\pm1\}^p \setminus$ `Sc`

Despite its simplicity, algorithm 5.3 is usually not very attractive. Indeed, each stem vector $\sigma \in \{\pm1\}^J$ produces the exponential number $2^{|J^c|}$ of sign vectors s with $s_{J^c} \in \{\pm1\}^{J^c}$. As a result, for large p, the algorithm handles a large amount of data, which can take much computing time.

5.1.3 Dual $\mathcal{S}$-tree algorithm

Another possibility is to use the $\mathcal{S}$-tree structure introduced in section 4.1. Here is the main idea. Assume that a sign vector s in $\mathcal{S}_k$ has been computed (the set $\mathcal{S}_k$ is defined by (4.2)). Then, algorithm 4.2 determines whether (s, s_{k+1}) belongs to $\mathcal{S}_{k+1}$, for $s_{k+1} \in \{\pm1\}$. As explained in the description of algorithm 4.2, the belonging of (s, s_{k+1}) to $\mathcal{S}_{k+1}$ can be revealed by solving a linear optimization problem. Algorithm 5.4–5.5 below does this differently. It uses the computed stem vector set $\mathfrak{S}(V,\tau)$ and is based on the fact that

$$\mathfrak{S}(V_{:,[1:k]}, \tau_{[1:k]}) = \{\sigma \in \mathfrak{S}(V,\tau) : \mathfrak{J}(\sigma) \subseteq [1:k]\}.$$

Therefore, according to proposition 3.19, to determine whether (s, s_{k+1}) is in $\mathcal{S}_{k+1}$, it suffices to see whether it covers a stem vector $\sigma \in \mathfrak{S}(V,\tau)$ such that $\mathfrak{J}(\sigma) \subseteq [1:k+1]$. If so $(s, s_{k+1}) \in \mathcal{S}_{k+1}^c$ and any $\tilde{s} \in \{\pm1\}^p$, extending (s, s_{k+1}) by ±1, will be in $\mathcal{S}(V,\tau)^c$, so that the $\mathcal{S}$-tree may be pruned at (s, s_{k+1}). Otherwise $(s, s_{k+1}) \in \mathcal{S}_{k+1}$ and the recursive exploration of the $\mathcal{S}$-tree is pursued below (s, s_{k+1}).

Algorithm 5.4 (`d_stree`) // dual $\mathcal{S}$-tree algorithm

1. `stem_vectors`($\mathfrak{S}_s$, $\mathfrak{S}_a$) // get the stem vectors by algorithm 5.1
2. `d_stree_rec`($\varnothing$,$\mathfrak{S}_s \cup \mathfrak{S}_a$)

Algorithm 5.5 (`d_stree_rec(s` $\in \{\pm 1\}^k$`,`$\mathfrak{S}$`)`)

1. `if` $(k = p)$
2.   Output s and `return`   // s is a leaf of the $\mathcal{S}$-tree; end the recursion
3. `endif`
4. `if` $([s;+1]$ covers a stem vector of $\mathfrak{S})$
5.   `d_stree_rec([s;-1],`$\mathfrak{S}$`)`
6. `else`
7.   `d_stree_rec([s;+1],`$\mathfrak{S}$`)`
8.   `if` $([s;-1]$ does not contain a stem vector of $\mathfrak{S})$
9.   `d_stree_rec([s;-1],`$\mathfrak{S}$`)`
10.   `endif`
11. `endif`

Here are some explanations and observations on the recursive algorithm 5.5.

- If the test in line 4 holds, proposition 3.19 tells us that $[s;+1]$ is an infeasible sign vector for the arrangement $\mathcal{A}(V_{:,[1:k+1]}, \tau_{[1:k+1]})$, so that, for any $\tilde{s} \in \{\pm 1\}^{p-k-1}$, the sign vectors $[s;+1;\tilde{s}]$ is also infeasible for the arrangement $\mathcal{A}(V,\tau)$. This has two consequences:
  - there is no point in exploring the descendants of $[s;+1]$ in the $\mathcal{S}$-tree, which explains why there is no recursive call to `d_stree_rec([s;+1],`$\mathfrak{S}$`)` in that case and
  - $[s;-1]$ is necessarily a feasible sign vector for the arrangement $\mathcal{A}(V_{:,[1:k+1]}, \tau_{[1:k+1]})$ since each node of the $\mathcal{S}$-tree has at least one child (proposition 4.3), which explains why there is a call to `d_stree_rec([s;-1],`$\mathfrak{S}$`)` in line 5.
- Line 7 is justified since at that point, $[s;+1]$ is a feasible sign vector for the arrangement $\mathcal{A}(V_{:,[1:k+1]}, \tau_{[1:k+1]})$.
- Line 9 is justified since at that point, $[s;-1]$ is a feasible sign vector for the arrangement $\mathcal{A}(V_{:,[1:k+1]}, \tau_{[1:k+1]})$.
- The algorithm does not use witness points, unlike the primal $\mathcal{S}$-tree algorithm 4.1–4.2.

Let us emphasize the fact that algorithm 5.4 does not require to solve linear optimization problems. While this might look enticing, since the LOPs are the main cost of the primal $\mathcal{S}$-tree algorithm 4.1, one must be aware of two facts. First, the computation of all the circuits of V by algorithm 5.1 can be time consuming, since it requires the exploration of a tree, whose nodes at level k may have up to $p-k$ descendants. Second, determining whether a sign vector covers a stem vector can also take much computing time when the number of stem vectors is large, which is usually the case when p is large (see remark 3.16(6)).

5.2 Algorithms using some stem vectors

Instead of computing the stem vectors exhaustively like algorithm 5.1 does, which is generally a time consuming task, one can get a few stem vectors from the optimal dual variables of some linear optimization problems (LOPs) encountered in algorithm 4.1, those that are associated with an infeasible sign vector. This device is described in section 5.2.1. Then, one can design a kind of primal-dual algorithm for computing $\mathcal{S}(V,\tau)$. This one builds the $\mathcal{S}$-tree, but, in order to save running time, it makes use of the stem vectors collected during its construction to prune some unfruitful branches of the $\mathcal{S}$-tree, which avoids having to solve some LOPs. This algorithm is presented in section 5.2.2.

5.2.1 Getting stem vectors from linear optimization

In line 11 of algorithm 4.2, one has to decide whether $(s, -s_{k+1})$ is in $\mathcal{S}_{k+1}$ and it is suggested, after the description of the algorithm, to determine this belonging by solving the linear optimization problem (LOP) (4.5). The Lagrangian dual of this problem [5, 16, 21] reads

$$\begin{array}{rl} \max_{(\lambda,\mu)\in\mathbb{R}^{k+1}\times\mathbb{R}} & \sum_{i\in[1:k]} \lambda_i s_i \tau_i - \lambda_{k+1} s_{k+1} \tau_{k+1} - \mu \\ \text{s.t.} & \lambda \geqslant 0 \\ & \mu \geqslant 0 \\ & \sum_{i\in[1:k]} \lambda_i s_i v_i = \lambda_{k+1} s_{k+1} v_{k+1} \\ & \sum_{i\in[1:k+1]} \lambda_i + \mu = 1, \end{array} \tag{5.1}$$

where $\lambda \in \mathbb{R}^{k+1}$ is the dual variable associated with the first $k+1$ constraints of (4.5) and μ the dual variable associated with its last constraint. To see this, use the Lagrangian of problem (4.5), which is the function $\ell : (x, \alpha, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{R}$, defined at $(x, \alpha, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}$ by

$$\ell(x,\alpha,\lambda,\mu) = \alpha - \sum_{i\in[1:k]} \lambda_i \left[s_i(v_i^\mathsf{T} x - \tau_i) + \alpha\right] - \lambda_{k+1}\left[-s_{k+1}(v_{k+1}^\mathsf{T} x - \tau_{k+1}) + \alpha\right] - \mu(\alpha+1).$$

Then problem (4.5) reads $\inf_{(x,\alpha)} \sup_{(\lambda,\mu)\geqslant 0} \ell(x,\alpha,\lambda,\mu)$, so that the dual problem reads

$$\begin{aligned} &\sup_{(\lambda,\mu)\geqslant 0} \inf_{(x,\alpha)} \ell(x,\alpha,\lambda,\mu) \\ &= \sup_{(\lambda,\mu)\geqslant 0} \begin{cases} \sum_{i\in[1:k]} \lambda_i s_i \tau_i - \lambda_{k+1} s_{k+1}\tau_{k+1} - \mu & \text{if } \sum_{i\in[1:k]} \lambda_i s_i v_i = \lambda_{k+1} s_{k+1} v_{k+1} \\ & \quad \text{and } \sum_{i\in[1:k+1]} \lambda_i + \mu = 1 \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} \sup_{(\lambda,\mu)} \sum_{i\in[1:k]} \lambda_i s_i \tau_i - \lambda_{k+1} s_{k+1}\tau_{k+1} - \mu \\ \lambda \geqslant 0 \\ \mu \geqslant 0 \\ \sum_{i\in[1:k]} \lambda_i s_i v_i = \lambda_{k+1} s_{k+1} v_{k+1} \\ \sum_{i\in[1:k+1]} \lambda_i + \mu = 1, \end{cases} \end{aligned}$$

which is (5.1). When $\tau = 0$, one does not recover problem [16, (5.3)] since the latter is derived from a primal problem [16, (5.2)] that has been written differently than our (4.5) with $\tau = 0$.

The next proposition gives conditions ensuring that a circuit of V can be obtained from a specific solution to the dual problem (5.1) when $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$. We denote by val (4.5) (resp. val (5.1)) the optimal value of the primal (resp. dual) optimization problem (4.5) (resp. (5.1)). By strong duality in linear optimization [5, 21] and the fact that problem (4.5) has a solution, one has val (4.5) = val (5.1).

Proposition 5.6 (matroid circuit detection from optimization)

1) *Problem* (5.1) *has a solution, say* $(\lambda, \mu) \in \mathbb{R}^{k+1}_+ \times \mathbb{R}_+$.
2) *If* $s \in \mathcal{S}_k$ *and* $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$, *then* val (4.5) $\geqslant 0$, $\lambda_{k+1} > 0$ *and* $\mu = 0$.
3) *If, in addition,* (λ, μ) *is an extreme point of the feasible set of* (5.1), *then*
   - $J := \{i \in [1:k+1] : \lambda_i > 0\} \in \mathcal{C}(V)$,

- *if* val (4.5) $= 0$, $\pm(s, -s_{k+1})_J$ *are the two symmetric stem vectors associated with* J,
- *if* val (4.5) > 0, $(s, -s_{k+1})_J$ *is the unique asymmetric stem vector associated with* J.

Proof. 1) By strong duality in linear optimization [5, 21, 42], the fact that the primal problem (4.5) has a solution implies that the dual problem (5.1) has also a solution, say (λ, μ).

2) Suppose that $s \in \mathcal{S}_k$ and that $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$. Let (x, α) be a solution to (4.5) ($\alpha = \operatorname{val}(4.5)$ is uniquely determined). Let us show that

$$\lambda_{k+1} > 0 \qquad \text{and} \qquad \mu = 0. \tag{5.2a}$$

The optimal multiplier μ is associated with the constraint $\alpha \geqslant -1$ of the optimization problem (4.5), which is inactive ($\alpha \geqslant 0$ when $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}$), so that it vanishes. We show that $\lambda_{k+1} > 0$ by contradiction, assuming that $\lambda_{k+1} = 0$. Then, strong duality would imply that $0 \leqslant \alpha = \operatorname{val}(4.5) = \operatorname{val}(5.1) = \sum_{i \in [1:k]} \lambda_i s_i \tau_i$, while the third constraint of (5.1) would read $\sum_{i \in [1:k]} \lambda_i s_i v_i = 0$. Then, Motzkin's alternative (2.1) would imply that there is no $x \in \mathbb{R}^n$ such that $s_i(v_i^{\mathsf{T}} x - \tau_i) > 0$, for $i \in [1:k]$, in contradiction with the assumption $s \in \mathcal{S}_k$.

3) Let $I := \{i \in [1:k],\ \lambda_i > 0\}$. By assumption and $\mu = 0$, $(\lambda, 0)$ is an extreme point of the feasible set of problem (5.1), which implies that the vectors [9, 21, 42]

$$\left\{ \begin{pmatrix} s_i v_i \\ 1 \end{pmatrix}_{i \in I}, \begin{pmatrix} -s_{k+1} v_{k+1} \\ 1 \end{pmatrix} \right\} \text{ are linearly independent,} \tag{5.2b}$$

where we used the fact that $\lambda_{k+1} > 0$ and $\mu = 0$ by (5.2a).

One can deduce from this property that the vectors

$$\{s_i v_i\}_{i \in I} \text{ are linearly independent.} \tag{5.2c}$$

Suppose indeed that $\sum_{i \in I} \alpha_i s_i v_i = 0$ for some real numbers $(\alpha_i)_{i \in I}$. It suffices to show that these numbers vanish and we do so in two steps.

- We first show by contradiction that $\sum_{i \in I} \alpha_i = 0$. If this were not the case, one could find $t \in \mathbb{R}$ such that $\sum_{i \in I} (\lambda_i + t\alpha_i) + \lambda_{k+1} = 0$. Now, using the third constraint of problem (5.1), we would have that $\eta := ((\lambda_i + t\alpha_i)_{i \in I}, \lambda_{k+1})$ is in the null space of the nonsingular matrix whose columns are the vectors in (5.2b), which would imply that $\eta = 0$, in contradiction with $\lambda_{k+1} > 0$ imposed by (5.2a).
- Using $\sum_{i \in I} \alpha_i = 0$ and $\sum_{i \in I} \alpha_i s_i v_i = 0$, we have that the vector $((\alpha_i)_{i \in I}, 0)$ is in the null space of the nonsingular matrix whose columns are the vectors in (5.2b). Hence all the α_i's vanish.

Now, set $J := \{i \in [1:k+1] : \lambda_i > 0\}$, which is $I \cup \{k+1\}$ by the definition of I and (5.2a), and introduce the diagonal matrix $D \in \mathbb{R}^{J \times J}$ defined by $D_{i,i} = s_i$ if $i \in I$ and $D_{k+1,k+1} = -s_{k+1}$. Using (5.2c), we see that

$$\operatorname{null}(V_{:,J} D) = 1.$$

By the third constraint of (5.1), we have that $\lambda_J \in \mathcal{N}(V_{:,J} D) \setminus \{0\}$. Since $\lambda_J > 0$, proposition 3.14 tells us that J is a circuit of $V_{:,J} D$, hence a circuit of V.

Since $\eta := D\lambda_J \in \mathcal{N}(V_{:,J}) \setminus \{0\}$ is such that $\tau_J^{\mathsf{T}} \eta = \operatorname{val}(4.5)$, we see that the number of stem vectors associated with J is governed by val (4.5), as described in remark 3.16(3). In addition, $\operatorname{sgn}(\eta) = (s, -s_{k+1})_J$, because $\lambda_J > 0$, showing that $(s, -s_{k+1})_J$ is a stem vector. $\square$

A solution to problem (5.1) that is an extreme point of its feasible set can be obtained by the dual-simplex algorithm. Note that, since $\lambda_{k+1} > 0$, $k+1$ always belongs to the selected circuit J of V.

5.2.2 Primal-dual $\mathcal{S}$-tree algorithm

Proposition 5.6(3) shows how circuits and their associated stem vectors can be obtained when the $\mathcal{S}$-tree primal algorithm 4.1 solves a LOP (4.5) with an appropriate solver and observes that the sign vector $(s, -s_{k+1})$ is infeasible. Now, with the *partial* list of stem vectors so computed, which grows throughout the iterations, the algorithm can detect *some* infeasible sign vectors by using proposition 3.19, like in the crude dual algorithm 5.3 or in the $\mathcal{S}$-tree dual algorithm 5.4, but without having to solve a LOP. In practice, this technique saves much computing time. Here is this *primal-dual $\mathcal{S}$-tree algorithm*, based on the just presented idea, which has many similarities with the primal $\mathcal{S}$-tree algorithm 4.1.

Algorithm 5.7 (`pd_stree`) // primal-dual $\mathcal{S}$-tree algorithm

1. $\mathfrak{S} = \varnothing$
2. `pd_stree_rec`$(+1,\, x_+,\, \mathfrak{S})$ // x_+ given by $(4.3)_1$
3. `pd_stree_rec`$(-1,\, x_-,\, \mathfrak{S})$ // x_- given by $(4.3)_2$

Algorithm 5.8 (`pd_stree_rec`$(s \in \{\pm 1\}^k,\, x \in \mathbb{R}^n,\, \mathfrak{S})$) 1. **if** $(k = p)$

2. Output s and **return** // s is a leaf of the $\mathcal{S}$-tree; end the recursion
3. **endif**
4. **if** $(v_{k+1}^{\mathsf{T}} x = \tau_{k+1})$
5. `pd_stree_rec`$((s, +1), x + \varepsilon v_{k+1},\, \mathfrak{S})$ // $(s, +1) \in \mathcal{S}_{k+1}$
6. `pd_stree_rec`$((s, -1), x - \varepsilon v_{k+1},\, \mathfrak{S})$ // $(s, -1) \in \mathcal{S}_{k+1}$
7. **return**
8. **endif**
9. $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x - \tau_{k+1})$ // $(s, s_{k+1}) \in \mathcal{S}_{k+1}$
10. `pd_stree_rec`$((s, s_{k+1}), x,\, \mathfrak{S})$
11. **if** $((s, -s_{k+1})$ covers a stem vector of $\mathfrak{S})$
12. **return**
13. **elseif** $((s, -s_{k+1})$ is feasible with witness point $\tilde{x})$
14. `pd_stree_rec`$((s, -s_{k+1}),\, \tilde{x},\, \mathfrak{S})$ // $(s, -s_{k+1}) \in \mathcal{S}_{k+1}$
15. **else**
16. Add one or two stem vectors to $\mathfrak{S}$
17. **endif**

We only comment some instructions of the primal-dual $\mathcal{S}$-tree algorithm 5.7 that differ from those of the primal $\mathcal{S}$-tree algorithm 4.1.

- Unlike algorithm 5.4, which computes all the stem vectors at first, algorithm 5.7 initializes the list of stem vectors $\mathfrak{S}$ to the empty set in line 1. This list is next gradually filled by algorithm 5.8.
- For more efficiency, one could adapt line 4 of algorithm 5.8 and its lines 5..6 by using the improvement described in section 4.2.2.

- Lines 11..12 are new with respect to algorithm 4.1. They are used to check whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}^c$, using the stem vectors collected in $\mathfrak{S}$ and proposition 3.19, without having to solve a LOP.
- Lines 15..16 are also new with respect to algorithm 4.1. They use proposition 5.6(3) to detect a new circuit, hence one or two new stem vectors (they are new, since otherwise the test in line 11 would have been successful and line 16 would not have been executed), which are put in $\mathfrak{S}$. For this, it is necessary to solve the LOP in line 13 by a method computing an extreme point of the dual feasible set of this problem.

Algorithm 5.7 can be improved by introducing the modifications A, B and C of sections 4.2.1–4.2.3.

6 Compact version of the algorithms

All the algorithms computing the sign vector set $\mathcal{S}(V,\tau)$ presented so far, except algorithm 5.3, recursively construct the $\mathcal{S}$-tree introduced in algorithm 4.1, namely (recall the definition (4.2) of $\mathcal{S}_k(V,\tau)$)

$$\mathcal{T}(V,\tau) := \bigcup_{k\in[1:p]} \mathcal{S}_k(V,\tau). \tag{6.1}$$

When the arrangement is not centered (equivalently, $\tau \notin \mathcal{R}(V^{\mathsf{T}})$), some sets $\mathcal{S}_k(V,\tau)$ are asymmetric (proposition 3.5), so that the sign vectors of the two subtrees $\mathcal{T}^+(V,\tau) := \{s \in \mathcal{T}(V,\tau) : s_1 = +1\}$ and $\mathcal{T}^-(V,\tau) := \{s \in \mathcal{T}(V,\tau) : s_1 = -1\}$ of the $\mathcal{S}$-tree, rooted at the nodes $\{+1\}$ and $\{-1\}$, respectively, are not opposite to each other. Therefore, one cannot just compute $\mathcal{T}^+(V,\tau)$ or $\mathcal{T}^-(V,\tau)$ to get all $\mathcal{T}(V,\tau)$ (recall that when the arrangement is centered, $\mathcal{S}(V,\tau) = \mathcal{S}(V,0)$ and only half of the sign vectors needs to be computed [16]). Nevertheless, these two subtrees have some opposite sign vectors, the symmetric ones, those in $\mathcal{T}(V,0) = \cup_{k\in[1:p]}\mathcal{S}_k(V,0)$. The set of asymmetric sign vectors in $\mathcal{T}(V,\tau)$ is denoted by

$$\mathcal{T}_a(V,\tau) := \bigcup_{k\in[1:p]} \mathcal{S}_{a,k}(V,\tau),$$

where $\mathcal{S}_{a,k}(V,\tau) := \mathcal{S}_k(V,\tau) \setminus \mathcal{S}_{s,k}(V,\tau)$. Therefore, it is natural to look for a way to avoid as much as possible repeating the costly operations (linear optimization problems or stem vector coverings) common to the construction of the two subtrees $\mathcal{T}^+(V,\tau)$ and $\mathcal{T}^-(V,\tau)$. The goal of this section is to propose algorithms having that property; they can have a primal or dual nature.

6.1 The compact $\mathcal{S}$-tree

For an arrangement $\mathcal{A}(V,\tau)$, with $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$, and for $k \in [1:p]$, we denote the arrangement associated with the first k columns of V and the first k components of τ by

$$\mathcal{A}_k(V,\tau) := \mathcal{A}(V_{:,[1:k]}, \tau_{[1:k]}).$$

By proposition 3.21, we have that

$$\mathcal{S}_k(V,0) \subseteq \mathcal{S}_k(V,\tau) \subseteq \mathcal{S}_k([V;\tau^{\mathsf{T}}],0) \tag{6.2a}$$

$$\mathcal{S}_k([V;\tau^{\mathsf{T}}],0) \setminus \mathcal{S}_k(V,0) = \mathcal{S}_{a,k}(V,\tau) \cup \mathcal{S}_{a,k}(V,-\tau). \tag{6.2b}$$

The algorithms described in this section are based on the following considerations. By (3.8), the set $\mathcal{T}(V,\tau)$ of the feasible sign vectors of the $\mathcal{S}$-tree can be written $\mathcal{T}(V,0) \cup \mathcal{T}_a(V,\tau)$. Taking the intersection with $\mathcal{T}^+(V,\tau)$ and $\mathcal{T}^-(V,\tau)$ provides a partition of $\mathcal{T}(V,\tau)$ into four sets:

$$\mathcal{T}(V,0) \cap \mathcal{T}^+(V,\tau),\ \mathcal{T}_a(V,\tau) \cap \mathcal{T}^+(V,\tau),\ \mathcal{T}(V,0) \cap \mathcal{T}^-(V,\tau),\ \mathcal{T}_a(V,\tau) \cap \mathcal{T}^-(V,\tau). \tag{6.3}$$

Since

$$\mathcal{T}(V,0) \cap \mathcal{T}^-(V,\tau) = -[\mathcal{T}(V,0) \cap \mathcal{T}^+(V,\tau)], \tag{6.4}$$

only two sets must be computed to be able to retrieve all the sign vectors of $\mathcal{T}(V,\tau)$, namely the union of the first two sets of the partition (6.3) and the last one:

$$\mathcal{T}^+(V,\tau) \qquad \text{and} \qquad \mathcal{T}_a(V,\tau) \cap \mathcal{T}^-(V,\tau).$$

The principle of the algorithms described in this section consists in computing the subtree $\mathcal{T}^+(V,\tau)$ rooting at $s_1 = +1$ and in grafting to it the subtrees of

$$-[\mathcal{T}_a(V,\tau) \cap \mathcal{T}^-(V,\tau)],$$

which is in $\mathcal{T}_a(V,-\tau)$. This forms what we call the *compact $\mathcal{S}$-tree*. More precisely, if $s \in \mathcal{S}_k(V,0) \cap \mathcal{T}^+(V,\tau)$ and $(-s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau) \cap \mathcal{T}^-(V,\tau)$ for some $s_{k+1} \in \{\pm 1\}$, the subtree of $\mathcal{T}^-(V,\tau)$ rooting at $(-s, s_{k+1})$ is grafted at s in the compact tree (with its sign vectors multiplied by -1, so that $(s, -s_{k+1})$ can be a child of s). As a result, the nodes of the level k of the compact $\mathcal{S}$-tree are in one of the sets

$$\mathcal{S}_k(V,0), \quad \mathcal{S}_{a,k}(V,\tau) \quad \text{or} \quad \mathcal{S}_{a,k}(V,-\tau). \tag{6.5}$$

Eventually, a sign vector $s \in \mathcal{S}_a(V,-\tau)$ must be multiplied by -1 to get it in $-\mathcal{S}_a(V,-\tau) = \mathcal{S}_a(V,\tau) \subseteq \mathcal{S}(V,\tau)$. This principle is illustrated in figure 6.1. Housekeeping is done by attach-

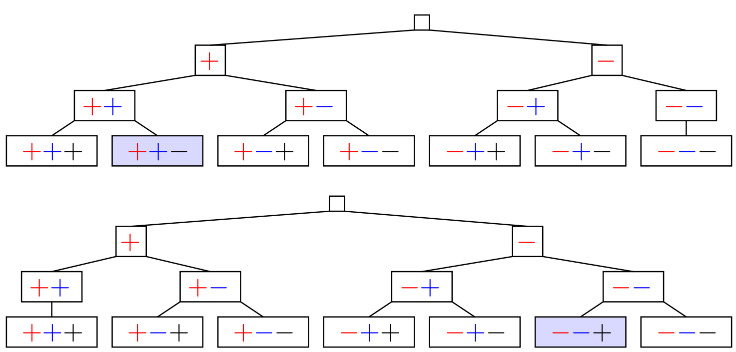


Figure 6.1: Standard $\mathcal{S}$-trees (left) and compact $\mathcal{S}$-trees (right) of the arrangements in the middle pane (above, compare with figure 4.1) and the right-hand side pane (below) of figure 3.1. The sign vectors in the white boxes are in $\mathcal{T}(V,0)$, those in the blue/gray boxes are in $\mathcal{S}_a(V,\tau)$ and the one in the blue/gray box with bold edges is in $\mathcal{S}_a(V,-\tau)$; this last sign vector must be multiplied by -1 to get a sign vector in $-\mathcal{S}_a(V,-\tau) = \mathcal{S}_a(V,\tau) \subseteq \mathcal{S}(V,\tau)$.

ing a flag $\boxed{\mathrm{S}}$ to each node s of the resulting tree, in order to specify which of the sign vector sets listed in (6.5) s belongs. As claimed in point 5 of the next proposition, the grafting process does not introduce nodes with two different flags: if $(s, s_{k+1}) \in -[\mathcal{S}_{a,k+1}(V,\tau) \cap \mathcal{T}^-(V,\tau)]$ is grafted to the compact $\mathcal{S}$-tree, then (s, s_{k+1}) is not in $\mathcal{T}^+(V,\tau)$.

Proposition 6.1 (compact $\mathcal{S}$-tree) *Let $k \in [1:p-1]$ and let $s \in \mathcal{S}_k(V,0) \cup \mathcal{S}_{a,k}(V,\tau) \cup \mathcal{S}_{a,k}(V,-\tau)$ be a sign vector of the compact $\mathcal{S}$-tree. Set $\mathcal{S}_k^{\pm}(V,\tau) := \mathcal{S}_k(V,\tau) \cap \mathcal{T}^+(V,\tau)$.*

1) *If $s \in \mathcal{S}_k(V,0)$, one child of s in the compact $\mathcal{S}$-tree is in $\mathcal{S}_{k+1}(V,0)$.*
2) *If $s \in \mathcal{S}_{a,k}(V,\tau)$, the children of s in the compact $\mathcal{S}$-tree are in $\mathcal{S}_{a,k+1}(V,\tau)$.*
3) *If $s \in \mathcal{S}_{a,k}(V,-\tau)$, the children of s in the compact $\mathcal{S}$-tree are in $\mathcal{S}_{a,k+1}(V,-\tau)$.*
4) *If $(s, s_{k+1}) \in -[\mathcal{S}_{a,k+1}(V,\tau) \cap \mathcal{T}^-(V,\tau)]$ with $s_{k+1} \in \{\pm 1\}$, then $(s, s_{k+1}) \notin \mathcal{T}^+(V,\tau)$.*
5) *Level k of the compact $\mathcal{S}$-tree is formed of $\mathcal{S}_k^+(V,\tau) \cup (-[\mathcal{S}_{a,k}(V,\tau) \cap \mathcal{T}^-(V,\tau)])$.*

Proof. Let $s \in \mathcal{S}_k(V,0) \cup \mathcal{S}_{a,k}(V,\tau) \cup \mathcal{S}_{a,k}(V,-\tau)$ be a sign vector in the compact $\mathcal{S}$-tree.

1) Suppose that $s \in \mathcal{S}_k(V,0)$ and let x be a witness point of s in $\mathcal{A}_k(V,0)$ (i.e., $s \cdot (V^\mathsf{T} x) > 0$). If $s_{k+1} := \operatorname{sgn}(v_{k+1}^\mathsf{T} x) \neq 0$, then $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$. If $v_{k+1}^\mathsf{T} x = 0$, then $x \pm \varepsilon v_{k+1}$, with $\varepsilon > 0$ small enough, is a witness point for $(s, \pm 1)$ in $\mathcal{A}_{k+1}(V,0)$ (recall the general assumption (4.1)); hence, $(s, \pm 1) \in \mathcal{S}_{k+1}(V,0)$.

2) Suppose that $s \in \mathcal{S}_{a,k}(V,\tau)$ and that (s, s_{k+1}) is in the compact $\mathcal{S}$-tree for some $s_{k+1} \in \{\pm 1\}$. One has to show that $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau)$. One cannot have $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$, since this would imply that $s \in \mathcal{S}_k(V,0) = \mathcal{S}_{s,k}(V,\tau)$ (proposition 3.7(2)), which is in contradiction with the assumption of the case. One cannot have $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,-\tau)$ either, since, otherwise $-(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau) \subseteq \mathcal{S}_{k+1}(V,\tau)$, implying that $-s \in \mathcal{S}_k(V,\tau)$, which is in contradiction with the assumption of the case. Since (s, s_{k+1}) is in the compact $\mathcal{S}$-tree, one must have $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau)$.

3) This case is akin to case 2. Suppose that $s \in \mathcal{S}_{a,k}(V,-\tau)$ and that (s, s_{k+1}) is in the compact $\mathcal{S}$-tree for some $s_{k+1} \in \{\pm 1\}$. One has to show that $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,-\tau)$. One cannot have $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$, since this would imply that $s \in \mathcal{S}_k(V,0) = \mathcal{S}_{s,k}(V,-\tau)$, which is in contradiction with the assumption of the case. One cannot have $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau)$ either, since, otherwise $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,\tau)$, implying that $s \in \mathcal{S}_k(V,\tau)$ or $-s \in \mathcal{S}_k(V,-\tau)$, which is in contradiction with the assumption of the case. Since (s, s_{k+1}) is in the compact $\mathcal{S}$-tree, one must have $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,-\tau)$.

4) Suppose that $(s, s_{k+1}) \in -[\mathcal{S}_{a,k+1}(V,\tau) \cap \mathcal{T}^-(V,\tau)]$ and that $(s, s_{k+1}) \in \mathcal{T}^+(V,\tau)$, for some $s_{k+1} \in \{\pm 1\}$. We have to show a contradiction. By $(s, s_{k+1}) \in \mathcal{T}^+(V,\tau)$, there are two possible cases.

- Either $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$. Then, $-(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$, which is in contradiction with the assumption $-(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau)$.
- Or $(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau)$. Then, $-(s, s_{k+1}) \notin \mathcal{S}_{k+1}(V,\tau)$, which is in contradiction with the assumption $-(s, s_{k+1}) \in \mathcal{S}_{a,k+1}(V,\tau) \subseteq \mathcal{S}_{k+1}(V,\tau)$.

5) The union follows from the very construction of the compact $\mathcal{S}$-tree, described before the proposition, and the disjoint union follows from point 4. □

6.2 Compact primal $\mathcal{S}$-tree algorithm

In accordance with the presentation of section 6.1, the *compact primal $\mathcal{S}$-tree algorithm*, whose reasoned description is given below, ignores the subtree $\mathcal{T}^-(V,\tau)$ rooting at $\{+1\}$, constructs the subtree $\mathcal{T}^+(V,\tau)$ rooting at $\{+1\}$ and grafts to it the opposite of the sign vectors in the subtrees of $\mathcal{T}_a(V,\tau) \cap \mathcal{T}^-(V,\tau)$. Note that $s_1 \in \mathcal{S}_1(V,0)$. Let us describe this algorithm. Its formal statement is given afterwards.

The algorithm identifies each node at level k of the compact $\mathcal{S}$-tree by a triplet $(s, x, \boxed{\mathrm{S}})$, where $s \in \{\pm 1\}^k$ is the sign vector of the node, $\boxed{\mathrm{S}} \in \{-1, 0, +1\}$ is a flag specifying to which sign set s belongs and x is some witness point. More specifically,

$$\begin{cases} s \in \mathcal{S}_k(V,0),\ x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V,0) & \text{if } \boxed{\mathrm{S}} = 0, \\ s \in \mathcal{S}_{a,k}(V,\tau),\ x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V,\tau) & \text{if } \boxed{\mathrm{S}} = +1, \\ s \in \mathcal{S}_{a,k}(V,-\tau),\ x \text{ is a witness point for } s \text{ in } \mathcal{A}_k(V,-\tau) & \text{if } \boxed{\mathrm{S}} = -1. \end{cases} \tag{6.6}$$

The flag $\boxed{\mathrm{S}}$ is used below as a scalar, hence $\boxed{\mathrm{S}}\, s$ is the vector whose ith component is $\boxed{\mathrm{S}}\, s_i$. The initialization of the algorithm is done as follows.

0. Take $s_1 = +1 \in \mathcal{S}_1(V,0)$ and v_1 as witness point for s_1 in $\mathcal{A}_1(V,0)$.

Consider now a node at level k of the compact $\mathcal{S}$-tree, which is specified by a triplet $(s, x, \boxed{\mathrm{S}})$ satisfying (6.6). We just have to specify how the algorithm determines the children of that node.

1. Suppose that $s \in \mathcal{S}_k(V,0)$ with x as witness point in $\mathcal{A}_k(V,0)$, i.e., $\boxed{\mathrm{S}} = 0$.

   Using proposition 4.10 with $\tau = 0$, the algorithm can detect whether s has two easily computable children in $\mathcal{A}(V,0)$ and can find associated witness points. If such is the case, the algorithm pursues recursively from $(s,+1)$ and $(s,-1)$, with appropriate witness points. It returns afterwards.

   Otherwise, $v_{k+1}^{\mathsf{T}} x \neq 0$, implying that $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$ for $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x)$ and that the algorithm can pursue recursively from (s, s_{k+1}) with x as witness point in $\mathcal{A}_{k+1}(V,0)$.

   Now, the algorithm has to specify to what set $(s, -s_{k+1})$ belongs: $\mathcal{S}_{k+1}(V,0)$, $\mathcal{S}_{a,k+1}(V,\tau)$, $\mathcal{S}_{a,k+1}(V,-\tau)$ or $\mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0)^c$ (there are no other possibilities, see proposition 3.21 and figure 3.3). For this purpose, the compact primal $\mathcal{S}$-tree algorithm starts by solving the LOP (4.5) with $\tau = 0$, to see whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$. Denote by (x_0, α_0) a solution to this LOP.

   1.1. If $\alpha_0 < 0$, then $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$ and the algorithm pursues recursively from $(s, -s_{k+1})$ with x_0 as witness point in $\mathcal{A}_{k+1}(V,0)$.

   1.2. Otherwise, $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V,0)$ and the algorithm determines whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0)$ by solving the following LOP, which is similar to (4.5) with $\tau = 0$, but for the arrangement $\mathcal{A}([V;\tau^{\mathsf{T}}],0)$ instead of $\mathcal{A}(V,0)$:

$$\begin{array}{rl} \min_{(x,\xi,\alpha)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}} & \alpha \\ \text{s.t.} & s_i(v_i^{\mathsf{T}} x + \tau_i \xi) + \alpha \geqslant 0, \quad \text{for } i \in [1:k] \\ & -s_{k+1}(v_{k+1}^{\mathsf{T}} x + \tau_{k+1}\xi) + \alpha \geqslant 0 \\ & \alpha \geqslant -1. \end{array} \tag{6.7}$$

   Denote by (x_1, ξ_1, α_1) a solution to this problem.

   1.2.1. If $\alpha_1 \geqslant 0$, then $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0)$, hence $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V,\tau) \cup \mathcal{S}_{k+1}(V,-\tau)$ by (6.2a) and $(s, -s_{k+1})$ can be discarded from the generated compact tree.

   1.2.2. Otherwise, $\alpha_1 < 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0) \setminus \mathcal{S}_{k+1}(V,0) = \mathcal{S}_{a,k+1}(V,\tau) \cup \mathcal{S}_{a,k+1}(V,-\tau)$ by (6.2b). Note that one cannot have $\xi_1 = 0$ in that case, since then one would have $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, 0)$, which is excluded in the considered case. Therefore,

  - either $\xi_1 < 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \tau)$, by (6.7), with $-x_1/\xi_1$ as witness point in $\mathcal{A}_{k+1}(V, \tau)$,
  - or $\xi_1 > 0$ and $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, -\tau)$, by (6.7), with x_1/ξ_1 as witness point in $\mathcal{A}_{k+1}(V, -\tau)$.

In these last two cases, the algorithm can pursue recursively from $(s, -s_{k+1})$.

2. Suppose that $s \in \mathcal{S}_{a,k}(V, \tau)$ with x as witness point in $\mathcal{A}_k(V, \tau)$, i.e., $\boxed{s} = +1$.

   Using proposition 4.10, the algorithm can detect whether s has two easily computable children in $\mathcal{A}(V, \tau)$ and can find associated witness points. If such is the case, the algorithm pursues recursively from $(s, +1)$ and $(s, -1)$, with appropriate witness points. It returns afterwards.

   Otherwise, $v_{k+1}^{\mathsf{T}} x \neq \tau_{k+1}$, implying that $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, \tau)$ for $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x - \tau_{k+1})$ and that the algorithm can pursue recursively from (s, s_{k+1}) with x as witness point in $\mathcal{A}_{k+1}(V, \tau)$.

   Now, the algorithm has to determine whether $(s, -s_{k+1})$ is infeasible or is in $\mathcal{S}_{a,k+1}(V, \tau)$ (there are no other possibilities, according to proposition 6.1(2)). For this purpose, the algorithm solves (4.5). Let (x_2, α_2) be a solution.

   - If $\alpha_2 < 0$, then $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, \tau)$ and the compact algorithm can pursue recursively from $(s, -s_{k+1})$.
   - Otherwise, $(s, -s_{k+1})$ is infeasible in $\mathcal{A}_{k+1}(V, \tau)$ and the algorithm can prune the compact $\mathcal{S}$-tree at that node.

3. One still has to consider the case when $s \in \mathcal{S}_{a,k}(V, -\tau)$, with x as witness point in $\mathcal{A}_k(V, -\tau)$, i.e., $\boxed{s} = -1$. This case is similar to case 2, because $-s \in \mathcal{S}_{a,k}(V, \tau)$.

   Using proposition 4.10, with τ changed into $-\tau$, the algorithm can detect whether s has two easily computable children in $\mathcal{A}(V, -\tau)$ and can find associated witness points. If such is the case, the algorithm pursues recursively from $(s, +1)$ and $(s, -1)$, with appropriate witness points. It returns afterwards.

   Otherwise, $v_{k+1}^{\mathsf{T}} x \neq -\tau_{k+1}$, implying that $(s, s_{k+1}) \in \mathcal{S}_{k+1}(V, -\tau)$ for $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x + \tau_{k+1})$ and that the algorithm can pursue recursively from (s, s_{k+1}) with x as witness point in $\mathcal{A}_{k+1}(V, -\tau)$.

   Now, the algorithm has to determine whether $(s, -s_{k+1})$ is infeasible or is in $\mathcal{S}_{a,k+1}(V, -\tau)$ (there are no other possibilities, according to proposition 6.1(3)). For this purpose, the algorithm solves (4.5), with τ replaced by $-\tau$, which reads

$$\begin{array}{rl} \min_{(x,\alpha)\in\mathbb{R}^n\times\mathbb{R}} & \alpha \\ \text{s.t.} & s_i(v_i^{\mathsf{T}} x + \tau_i) + \alpha \geqslant 0, \quad \text{for } i \in [1:k] \\ & -s_{k+1}(v_{k+1}^{\mathsf{T}} x + \tau_{k+1}) + \alpha \geqslant 0 \\ & \alpha \geqslant -1. \end{array} \tag{6.8}$$

   Let (x_2, α_2) be a solution.

   - If $\alpha_2 < 0$, then $(s, -s_{k+1}) \in \mathcal{S}_{a,k+1}(V, -\tau)$ and the compact algorithm can pursue recursively from $(s, -s_{k+1})$.
   - Otherwise, $(s, -s_{k+1})$ is infeasible in $\mathcal{A}_{k+1}(V, -\tau)$ and the algorithm can prune the compact $\mathcal{S}$-tree at that node.

One can now present schematically the compact form of the primal $\mathcal{S}$-tree algorithm 4.1. To shorten its statement and the one of the next algorithm 6.5, we introduce the following function `output_s`, which outputs sign vectors of $\mathcal{S}(V,\tau)$ at a leaf of the compact $\mathcal{S}$-tree (its behavior is more complex than for the standard algorithms and depends on the type $\boxed{s}$ of the leaf node s, see (6.6)).

Algorithm 6.2 (`output_s`(s, $\boxed{s}$))
It is assumed that $s \in \{\pm 1\}^p$ and that $\boxed{s} \in \{-1, 0, +1\}$.

`if` ($\boxed{s} = 0$)
    `output` $\pm s$ // $s \in \mathcal{S}(V,0)$
`else`
    `output` $\boxed{s}\, s$ // $s \in \mathcal{S}_a(V, \boxed{s}\,\tau)$
`endif`

We have chosen to present the cases when $\boxed{s} = +1$ and $\boxed{s} = -1$ separately, for clarity. A more compact presentation, closer to an implementation, is given in [17].

Algorithm 6.3 (`c_p_stree`) Let be given $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$.

`c_p_stree_rec`($+1$, v_1, 0).

Algorithm 6.4 (`c_p_stree_rec(s,` x, $\boxed{s}$))
It is assumed that $(x, s, \boxed{s})$ satisfies (6.6).

`if` ($k = p$)   // s is a leaf of the compact $\mathcal{S}$-tree
    `output_s`(`s`, $\boxed{s}$)
    `return`
`endif`

Algorithm 6.4 (continued)

`if` ($\boxed{s} = 0$)   // $s \in \mathcal{S}_k(V,0)$ with x as witness point in $\mathcal{A}_k(V,0)$
    `if` ($v_{k+1}^\mathsf{T} x \simeq 0$)   // two easy children in $\mathcal{A}_{k+1}(V,0)$
        `c_p_stree_rec`($(s,+1)$, $x + t_+ v_{k+1}$, 0) for some $t_+ \in (t_0, t_{\max})$
        `c_p_stree_rec`($(s,-1)$, $x + t_- v_{k+1}$, 0) for some $t_- \in (t_{\min}, t_0)$
        `return`
    `endif`
    $s_{k+1} := \operatorname{sgn}(v_{k+1}^\mathsf{T} x)$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V,0)$
    `c_p_stree_rec`((s, s_{k+1}), x, 0)
    Solve (4.5) with $\tau = 0$; let (x_0, α_0) be a solution
    `if` ($\alpha_0 < 0$)   // $(s, -s_{k+1})$ is a child in $\mathcal{S}_{k+1}(V,0)$
        `c_p_stree_rec`($(s, -s_{k+1})$, x_0, 0)
    `else`

Solve (6.7); let (x_1, ξ_1, α_1) be a solution
`if` $(\alpha_1 < 0)$
`if` $(\xi_1 < 0)$
`c_p_stree_rec`$((s, -s_{k+1}),\ -x_1/\xi_1,\ +1)$
`else` // here $\xi_1 > 0$
`c_p_stree_rec`$((s, -s_{k+1}),\ x_1/\xi_1,\ -1)$
`endif`
`endif`
`endif`
`return`
`endif`

Algorithm 6.4 (continued)

`if` $(\boxed{S} = +1)$ // $s \in \mathcal{S}_k(V, \tau)$ with x as witness point in $\mathcal{A}_k(V, \tau)$
`if` $(v_{k+1}^{\mathsf{T}} x \simeq \tau_{k+1})$ // two easy children in $\mathcal{A}_{k+1}(V, \tau)$
`c_p_stree_rec`$((s, +1),\ x + t_+ v_{k+1},\ +1)$ for some $t_+ \in (t_0, t_{\max})$
`c_p_stree_rec`$((s, -1),\ x + t_- v_{k+1},\ +1)$ for some $t_- \in (t_{\min}, t_0)$
`return`
`endif`
$s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x - \tau_{k+1})$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V, \tau)$
`c_p_stree_rec`$((s, s_{k+1}),\ x,\ +1)$
Solve (4.5); let (x_2, α_2) be a solution
`if` $(\alpha_2 < 0)$
`c_p_stree_rec`$((s, -s_{k+1}),\ x_2,\ +1)$
`endif`
`return`
`endif`

Algorithm 6.4 (continued)

`if` $(\boxed{S} = -1)$ // $s \in \mathcal{S}_k(V, -\tau)$ with x as witness point in $\mathcal{A}_k(V, -\tau)$
`if` $(v_{k+1}^{\mathsf{T}} x \simeq -\tau_{k+1})$ // two easy children in $\mathcal{A}_{k+1}(V, -\tau)$
`c_p_stree_rec`$((s, +1),\ x + t_+ v_{k+1},\ -1)$ for some $t_+ \in (t_0, t_{\max})$
`c_p_stree_rec`$((s, -1),\ x + t_- v_{k+1},\ -1)$ for some $t_- \in (t_{\min}, t_0)$
`return`
`endif`
$s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x + \tau_{k+1})$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V, -\tau)$
`c_p_stree_rec`$((s, s_{k+1}),\ x,\ -1)$
Solve (6.8); let (x_2, α_2) be a solution
`if` $(\alpha_2 < 0)$
`c_p_stree_rec`$((s, -s_{k+1}),\ x_2,\ -1)$
`endif`
`return`
`endif`

Observe that, as claimed by proposition 6.1(2-3), once $s \in \mathcal{S}_{a,k}(V,\tau)$ or $\boxed{\text{s}} = +1$ (resp. $s \in \mathcal{S}_{a,k}(V,-\tau)$ or $\boxed{\text{s}} = -1$), its descendants in the compact $\mathcal{S}$-tree are all in $\mathcal{S}_{a,l}(V,\tau)$ (resp. $\mathcal{S}_{a,l}(V,-\tau)$) for some $l \in [k+1:p]$. In these cases, the compact algorithm solves at most one LOP per sign vector (in step 36 or 50), like in the standard version of the algorithm, which solves at most one LOP in $\mathcal{T}^+(V,\tau)$ or $\mathcal{T}^-(V,\tau)$, not both since an asymmetric sign vector only appears in one of these subtrees. When $s \in \mathcal{S}_k(V,0)$ has one child in $\mathcal{S}_{a,k+1}(V,\pm\tau)$, the compact algorithm solves two LOPs (in steps 13 and 17), like in the standard algorithm (one LOP in $\mathcal{T}^\pm(V,\tau)$ to accept the child in $\mathcal{S}_{a,k+1}(V,\tau)$ and one in $\mathcal{T}^\mp(V,\tau)$ to reject a child in $\mathcal{S}_{a,k+1}(V,\tau)$). The only sign vectors at which the compact algorithm solves less LOPs than the standard algorithm are those in $\mathcal{S}(V,0)$ with two symmetric children. In this case, the compact algorithm solves a single LOP (in step 13), while the standard algorithm solves two LOPs (one in each subtree $\mathcal{T}^+(V,\tau)$ and $\mathcal{T}^-(V,\tau)$). Therefore, the compact algorithm 6.3 is all the more advantageous with respect to the standard algorithm 4.1 as $|\mathcal{T}(V,0)|/|\mathcal{T}(V,\tau)|$ is large (it is always $\leqslant 1$); see [35].

6.3 Compact primal-dual $\mathcal{S}$-tree algorithm

There are several ways of using the stem vectors, in order to avoid having to solve all or part of the LOPs of the standard primal $\mathcal{S}$-tree algorithms 4.1, most of them having a compact form. In this section, we only consider a compact version of the primal-dual $\mathcal{S}$-tree algorithm 5.7. The statement of the algorithm is immediate as soon as we know how it collects the stem vectors and how it uses them. These two points are clarified in section 6.3.1 and a formal description of the algorithm is given in section 6.3.2.

6.3.1 Stem vectors – "Récoltes et semailles"

As shown in section 5, stem vectors can be used to detect sign vectors that are not in $\mathcal{S}(V,\tau)$, using proposition 3.19. Our goal in this section is to apply this technique to construct the compact primal-dual $\mathcal{S}$-tree, by "dualizing" the compact primal $\mathcal{S}$-tree algorithm 6.3 (we have found this approach easier than "compacting" the standard primal-dual $\mathcal{S}$-tree algorithm 5.7).

Looking back to the compact primal $\mathcal{S}$-tree algorithm 6.3, we see that, by solving a LOP, it detects whether a sign vector belongs to $\mathcal{S}_{k+1}(V,0)$, in steps 13..16, or to $\mathcal{S}_{k+1}([V;\tau^\mathsf{T}],0)$, in steps 17..24, or to $\mathcal{S}_{k+1}(V,\tau)$, in steps 36..39 and 50..53. To be able to realize these detections, the compact primal-dual algorithm that is presented below uses proposition 3.19 and gradually spots stem vectors that are in the following sets (see propositions 3.17 and 3.24, and figure 3.2)

$$\begin{aligned}
\mathfrak{S}_k(V,\tau) &:= \mathfrak{S}(V_{:,[1:k]},\tau_{[1:k]}),\\
\mathfrak{S}_{s,k}(V,\tau) &:= \mathfrak{S}_s(V_{:,[1:k]},\tau_{[1:k]}),\\
\mathfrak{S}_{a,k}(V,\tau) &:= \mathfrak{S}_a(V_{:,[1:k]},\tau_{[1:k]}),\\
\mathfrak{S}_{0,k}(V,\tau) &:= \mathfrak{S}_0(V_{:,[1:k]},\tau_{[1:k]}).
\end{aligned}$$

This compact algorithm takes care to specify whether a detected stem vector in $\mathfrak{S}_k(V,\tau)$ is actually symmetric or asymmetric and whether a detected stem vector in $\mathfrak{S}_k([V;\tau^\mathsf{T}],0)$ is actually in $\mathfrak{S}_{s,k}(V,\tau)$ or $\mathfrak{S}_{0,k}(V,\tau)$. By doing so, a stem vector in $\mathfrak{S}_{s,k}(V,\tau) = \mathfrak{S}_k([V;\tau^\mathsf{T}],0) \cap \mathfrak{S}_k(V,\tau) \subseteq \mathfrak{S}(V,0)$ can then be used to exclude a sign vector from belonging to $\mathcal{S}_k([V;\tau^\mathsf{T}],0)$, $\mathcal{S}_k(V,\tau)$, or $\mathcal{S}_k(V,0)$.

Recall from proposition 3.17(2) and (3.24) that $\mathfrak{S}(V,\tau) \subseteq \mathfrak{S}(V,0)$ (see also figure 3.2). It is also clear, from their definition 3.15, that $\mathfrak{S}_k(V,\tau) \subseteq \mathfrak{S}_{k+1}(V,\tau) \subseteq \mathfrak{S}(V,\tau)$, $\mathfrak{S}_{s,k}(V,\tau) \subseteq$

$\mathfrak{S}_{s,k+1}(V,\tau) \subseteq \mathfrak{S}_s(V,\tau)$, $\mathfrak{S}_{a,k}(V,\tau) \subseteq \mathfrak{S}_{a,k+1}(V,\tau) \subseteq \mathfrak{S}_a(V,\tau)$ and $\mathfrak{S}_{0,k}(V,\tau) \subseteq \mathfrak{S}_{0,k+1}(V,\tau) \subseteq \mathfrak{S}_0(V,\tau)$. Therefore, the stem vectors of any type at level k of the $\mathcal{S}$-tree are also stem vectors of the same type at a higher level of the tree.

The stem vectors found in $\mathfrak{S}_s(V,\tau)$ are put in the so-called *collector* denoted by $\tilde{\mathfrak{S}}_s$, those found in $\mathfrak{S}_a(V,\tau)$ are put in the *collector* $\tilde{\mathfrak{S}}_a$ and those found in $\mathfrak{S}_0(V,\tau)$ are put in the *collector* $\tilde{\mathfrak{S}}_0$. Since, usually, $\tilde{\mathfrak{S}}_s \neq \mathfrak{S}_s(V,\tau)$, $\tilde{\mathfrak{S}}_a \neq \mathfrak{S}_a(V,\tau)$ and $\tilde{\mathfrak{S}}_0 \neq \mathfrak{S}_0(V,\tau)$, LOPs still need to be solved. Let us look at these LOPs, identified by their step number in algorithm 6.3.

- Steps 13..16: On the one hand, the LOP (4.5) with $\tau = 0$ is used to detect whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V,0)$. By proposition 3.19, if $(s, -s_{k+1})$ covers a stem vector in $\mathfrak{S}_{k+1}(V,0) = \mathfrak{S}_{s,k+1}(V,\tau) \uplus \mathfrak{S}_{a,k+1}(V,\tau) \uplus \mathfrak{S}_{a,k+1}(V,-\tau)$ (recall from (3.17) that $\mathfrak{S}_{a,k+1}(V,-\tau) = -\mathfrak{S}_{a.k+1}(V,\tau)$), $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V,0)$ and there is no need to solve the LOP (4.5) with $\tau = 0$.

  On the other hand, by [16, proposition 5.9], when $\alpha_0 \geqslant 0$, one can obtain a matroid circuit $J \subseteq [1\!:\!k+1]$ of $V_{:,[1:k+1]}$ when the solver of the LOP (4.5) with $\tau = 0$ provides a dual solution at an extreme point of the dual feasible set. We claim that this circuit J allows the algorithm to get stem vectors in $\mathfrak{S}_{s,k+1}(V,\tau)$ or in $\mathfrak{S}_{a,k+1}(V,\tau)$. Indeed, by the technique presented in remark 3.16(3), we see that, after having computed $\eta \in \mathcal{N}(V_{:,J}) \setminus \{0\}$, the algorithm gets two opposite stem vectors $\pm\operatorname{sgn}(\eta)$ in $\mathfrak{S}_{s,k+1}(V,\tau)$ if $\tau_J^{\mathsf{T}}\eta = 0$ and it gets a stem vector $\operatorname{sgn}(\tau_J^{\mathsf{T}}\eta)\operatorname{sgn}(\eta)$ in $\mathfrak{S}_{a,k+1}(V,\tau)$ if $\tau_J^{\mathsf{T}}\eta \neq 0$.

  In conclusion, this step needs as many stem vectors from $\mathfrak{S}_{s,k+1}(V,\tau)$ and $\mathfrak{S}_{a,k+1}(V,\tau)$ as possible and can add two stem vectors to $\tilde{\mathfrak{S}}_s$ or one stem vector to $\tilde{\mathfrak{S}}_a$.

- Steps 17..24: On the one hand, the LOP (6.7) is used to detect whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0)$. By proposition 3.19, if $(s, -s_{k+1})$ covers a stem vector in $\mathfrak{S}_{k+1}([V;\tau^{\mathsf{T}}],0) = \mathfrak{S}_{s,k+1}(V,\tau) \uplus \mathfrak{S}_{0,k+1}(V,\tau)$, $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}([V;\tau^{\mathsf{T}}],0)$ and there is no need to solve the LOP (6.7).

  On the other hand, by [16, proposition 5.9], when $\alpha_1 \geqslant 0$, one can obtain a matroid circuit $J \subseteq [1\!:\!k+1]$ of $[V;\tau^{\mathsf{T}}]_{:,[1:k+1]}$ when the solver of the LOP (6.7) provides a dual solution at an extreme point of the dual feasible set. We claim that this circuit J allows the algorithm to get stem vectors in $\mathfrak{S}_{s,k+1}(V,\tau)$ or $\mathfrak{S}_{0,k+1}(V,\tau)$. Indeed, let $\eta \in \mathcal{N}([V;\tau^{\mathsf{T}}]_{:,J}) \setminus \{0\}$. Then, $\pm\operatorname{sgn}(\eta) \in \mathfrak{S}_{k+1}([V;\tau^{\mathsf{T}}],0) = \mathfrak{S}_{s,k+1}(V,\tau) \uplus \mathfrak{S}_{0,k+1}(V,\tau)$. Now, to determine whether $\pm\operatorname{sgn}(\eta)$ is in $\mathfrak{S}_{s,k+1}(V,\tau)$ or in $\mathfrak{S}_{0,k+1}(V,\tau)$, the algorithm can easily check whether $\tau_J \in \mathcal{R}(V_{:,J}^{\mathsf{T}})$ and use the equivalence in (3.25b).

  In conclusion, this step needs as many stem vectors from $\mathfrak{S}_{s,k+1}(V,\tau)$ and $\mathfrak{S}_{0,k+1}(V,\tau)$ as possible and can add two stem vectors to $\tilde{\mathfrak{S}}_s$ or to $\tilde{\mathfrak{S}}_0$.

- Steps 36..39: On the one hand, the LOP (4.5) is used to detect whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V,\tau)$. By proposition 3.19, if $(s, -s_{k+1})$ covers a stem vector in $\mathfrak{S}_{k+1}(V,\tau) = \mathfrak{S}_{s,k+1}(V,\tau) \uplus \mathfrak{S}_{a,k+1}(V,\tau)$, $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V,\tau)$ and there is no need to solve the LOP (4.5).

  On the other hand, by proposition 5.6(3), when $\alpha_2 \geqslant 0$, one can obtain two stem vectors in $\mathfrak{S}_{s,k+1}(V,\tau)$ or one stem vector in $\mathfrak{S}_{a,k+1}(V,\tau)$.

  In conclusion, this step needs as many stem vectors from $\mathfrak{S}_{s,k+1}(V,\tau)$ and $\mathfrak{S}_{a,k+1}(V,\tau)$ as possible and can add two stem vectors to $\tilde{\mathfrak{S}}_s$ or one stem vector to $\tilde{\mathfrak{S}}_a$.

- Steps 50..53: This case is similar to the previous one, with the same conclusion.

  On the one hand, the LOP (6.8) is used to detect whether $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V,-\tau)$. By proposition 3.19, if $(s, -s_{k+1})$ covers a stem vector in $\mathfrak{S}_{k+1}(V,-\tau) = \mathfrak{S}_{s,k+1}(V,\tau) \uplus$

$\mathfrak{S}_{a,k+1}(V,-\tau)$, $(s,-s_{k+1}) \notin \mathcal{S}_{k+1}(V,-\tau)$ and there is no need to solve the LOP (4.5).

On the other hand, by proposition 5.6(3), when $\alpha_2 \geqslant 0$, one can obtain two stem vectors in $\mathfrak{S}_{s,k+1}(V,-\tau) = \mathfrak{S}_{s,k+1}(V,\tau)$ (by proposition 3.17(1)) or one stem vector in $\mathfrak{S}_{a,k+1}(V,-\tau)$ $= -\mathfrak{S}_{a,k+1}(V,\tau)$ (by (3.17)).

In conclusion, this step needs as many stem vectors from $\mathfrak{S}_{s,k+1}(V,\tau)$ and $\mathfrak{S}_{a,k+1}(V,-\tau) = -\mathfrak{S}_{a,k+1}(V,\tau)$ as possible and can add two stem vectors to $\tilde{\mathfrak{S}}_s$ or one stem vector to $\tilde{\mathfrak{S}}_a$.

The compact primal-dual algorithm differs from the compact primal $\mathcal{S}$-tree algorithm 6.3 only when a LOP has to be solved in the primal algorithm. In this case, as shown by the discussion above, it realizes a covering test and the LOP is solved if and only if the covering test fails. If this is the case and if the LOP has a nonnegative optimal value, stem vectors are constructed and these progressively complete some collectors $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$ or $\tilde{\mathfrak{S}}_0$.

6.3.2 Formal statement

One can now present a formal statement of the *compact primal-dual $\mathcal{S}$-tree algorithm*. As shown by the discussion in the previous section, this compact algorithm differs from algorithm 6.3 in the sense that it uses and fills in subsets $\tilde{\mathfrak{S}}_s$ of $\mathfrak{S}_s(V,\tau)$, $\tilde{\mathfrak{S}}_a$ of $\mathfrak{S}_a(V,\tau)$ and $\tilde{\mathfrak{S}}_0$ of $\mathfrak{S}_0(V,\tau)$.

Algorithm 6.5 (`c_pd_stree`) Let be given $V \in \mathbb{R}^{n\times p}$ and $\tau \in \mathbb{R}^p$.

$\tilde{\mathfrak{S}}_s = \varnothing$, $\tilde{\mathfrak{S}}_a = \varnothing$, $\tilde{\mathfrak{S}}_0 = \varnothing$ // initial empty collectors
`c_pd_stree_rec`($+1$, v_1, 0, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$)

Algorithm 6.6 (`c_pd_stree_rec`$(s, x, \boxed{s}, \tilde{\mathfrak{S}}_s, \tilde{\mathfrak{S}}_a, \tilde{\mathfrak{S}}_0)$)

It is assumed that $(x, s, \boxed{s})$ satisfies (6.6) and that $\tilde{\mathfrak{S}}_s \subseteq \mathfrak{S}_s(V,\tau)$, $\tilde{\mathfrak{S}}_a \subseteq \mathfrak{S}_a(V,\tau)$, $\tilde{\mathfrak{S}}_0 \subseteq \mathfrak{S}_0(V,\tau)$.

`if` $(k = p)$ // s is a leaf of the compact $\mathcal{S}$-tree
  `output_s(s,` $\boxed{s}$`)`
  `return`
`endif`

Algorithm 6.6 (continued)

`if` $(\boxed{s} = 0)$ // $s \in \mathcal{S}_k(V,0)$ with x as witness point in $\mathcal{A}_k(V,0)$
  `if` $(v_{k+1}^{\mathsf{T}} x \simeq 0)$ // two easy children in $\mathcal{S}_{k+1}(V,0)$
    `c_pd_stree_rec`($(s,+1)$, $x + t_+ v_{k+1}$, 0, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$) for a $t_+ \in (t_0, t_{\max})$
    `c_pd_stree_rec`($(s,-1)$, $x + t_- v_{k+1}$, 0, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$) for a $t_- \in (t_{\min}, t_0)$
    `return`
  `endif`
  $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x)$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V,0)$
  `c_pd_stree_rec`((s, s_{k+1}), x, 0, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$)

`if` $((s, -s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_a \cup (-\tilde{\mathfrak{S}}_a))$
  Solve (4.5) with $\tau = 0$; let (x_0, α_0) be a solution
  `if` $(\alpha_0 < 0)$ // $(s, -s_{k+1})$ is a child in $\mathcal{S}_{k+1}(V, 0)$
    `c_pd_stree_rec`$((s, -s_{k+1}),\ x_0,\ 0,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$
    `return`
  `else`
    Add two or one stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_a$, respectively
  `endif`
`endif`   // here $(s, -s_{k+1}) \notin \mathcal{S}_{k+1}(V, 0)$, check if $(s, -s_{k+1}) \in \mathcal{S}_{k+1}(V, \tau)$
`if` $((s, -s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_0)$
  Solve (6.7); let (x_1, ξ_1, α_1) be a solution
  `if` $(\alpha_1 < 0)$
    `if` $(\xi_1 < 0)$
      `c_pd_stree_rec`$((s, -s_{k+1}),\ -x_1/\xi_1,\ +1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$
    `else`   // here $\xi_1 > 0$
      `c_pd_stree_rec`$((s, -s_{k+1}),\ x_1/\xi_1,\ -1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$
    `endif`
  `else`
    Add two stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_0$
  `endif`
`endif`
`return`
`endif`

Algorithm 6.6 (continued)

`if` $(\boxed{s} = +1)$ // $s \in \mathcal{S}_k(V, \tau)$ with x as witness point in $\mathcal{A}_k(V, \tau)$
  `if` $(v_{k+1}^{\mathsf{T}} x \simeq \tau_{k+1})$ // two easy children in $\mathcal{S}_{k+1}(V, \tau)$
    `c_pd_stree_rec`$((s, +1),\ x + t_+ v_{k+1},\ +1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$ for a $t_+ \in (t_0, t_{\max})$
    `c_pd_stree_rec`$((s, -1),\ x + t_- v_{k+1},\ +1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$ for a $t_- \in (t_{\min}, t_0)$
    `return`
  `endif`
  $s_{k+1} := \operatorname{sgn}(v_{k+1}^{\mathsf{T}} x - \tau_{k+1})$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V, \tau)$
  `c_pd_stree_rec`$((s, s_{k+1}),\ x,\ +1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$
  `if` $((s, -s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_a)$
    Solve (4.5); let (x_2, α_2) be a solution
    `if` $(\alpha_2 < 0)$
      `c_pd_stree_rec`$((s, -s_{k+1}),\ x_2,\ +1,\ \tilde{\mathfrak{S}}_s,\ \tilde{\mathfrak{S}}_a,\ \tilde{\mathfrak{S}}_0)$
    `else`
      Add two or one stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_a$, respectively
    `endif`
  `endif`
  `return`
`endif`

Algorithm 6.6 (continued)

if ($\boxed{\mathrm{S}} = -1$) // $s \in \mathcal{S}_k(V, -\tau)$ with x as witness point in $\mathcal{A}_k(V, -\tau)$
  if ($v_{k+1}^\mathsf{T} x \simeq -\tau_{k+1}$) // two easy children in $\mathcal{S}_{k+1}(V, -\tau)$
    c_pd_stree_rec($(s, +1)$, $x{+}t_+v_{k+1}$, -1, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$) for a $t_+ \in (t_0, t_{\max})$
    c_pd_stree_rec($(s, -1)$, $x{+}t_-v_{k+1}$, -1, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$) for a $t_- \in (t_{\min}, t_0)$
    return
  endif
  $s_{k+1} := \operatorname{sgn}(v_{k+1}^\mathsf{T} x + \tau_{k+1})$ // (s, s_{k+1}) is a child in $\mathcal{S}_{k+1}(V, -\tau)$
  c_pd_stree_rec((s, s_{k+1}), x, -1, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$)
  if ($(-s, s_{k+1})$ does not cover a stem vector of $\tilde{\mathfrak{S}}_s \cup \tilde{\mathfrak{S}}_a$)
    Solve (6.8); let (x_2, α_2) be a solution
    if ($\alpha_2 < 0$)
      c_pd_stree_rec($(s, -s_{k+1})$, x_2, -1, $\tilde{\mathfrak{S}}_s$, $\tilde{\mathfrak{S}}_a$, $\tilde{\mathfrak{S}}_0$)
    else
      Add two or one stem vectors to $\tilde{\mathfrak{S}}_s$ or $\tilde{\mathfrak{S}}_a$, respectively
    endif
  endif
  return
endif

7 Numerical results

The goal of this section is to assess the efficiency of a selection of algorithms enumerating the chambers of a choice of hyperplane arrangements, among the algorithms introduced in sections 4, 5 and 6. Section 7.1 lists and briefly describes the chosen arrangement instances. The considered algorithms are specified in section 7.2. Section 7.3 details and discusses the results of this evaluation.

7.1 Arrangement instances

This section describes the arrangements that form the test bed for the evaluation of the selected algorithms presented in the next section. These arrangements $\mathcal{A}(V, \tau)$ are specified by their matrix $V \in \mathbb{R}^{n \times p}$ and vector $\tau \in \mathbb{R}^p$ (see section 3.1). One always has $p > n$ and $r := \operatorname{rank}(V) = n$. The instance features are given in tables 7.1 (theoretical values, for some of them) and 7.2 (numerical values).

The given five problems are affine, with $\tau \neq 0$. Some of them were examined in [37]. Their linear version, obtained by setting $\tau = 0$, were considered in [16]. Random numbers are generated with the Julia function `rand`.

- `rand-n-p`: $V \in \mathbb{R}^{n \times p}$ and $\tau \in \mathbb{R}^p$ are randomly generated in $[-5\,{:}\,{+5}]$.

  These arrangements are likely to be in affine general position (definition 3.34). In this case, one has the following formulas. By remark 3.16(6), $\mathcal{C}(V) = \{J \in [1\,{:}\,p] : |J| = r + 1\}$, $|\mathcal{C}(V)| = \binom{p}{r+1}$, $\mathcal{C}([V, \tau^\mathsf{T}]) = \{J \in [1\,{:}\,p] : |J| = r + 2\}$ and $|\mathcal{C}([V, \tau^\mathsf{T}])| = \binom{p}{r+2}$. By remark 3.16(3) and randomness, any circuit of V gives rise to a single asymmetric stem vector (the scalar product $\tau_J^\mathsf{T} \eta$ is likely to be nonzero when τ is random), so that $\mathfrak{S}_s(V, \tau) = \varnothing$ and

$|\mathfrak{S}_a(V,\tau)| = |\mathcal{C}(V)| = \binom{p}{r+1}$. By [16, definition 3.9], one has $|\mathfrak{S}([V,\tau^{\mathsf{T}}],0)| = |\mathcal{C}([V,\tau^{\mathsf{T}}])| = \binom{p}{r+2}$. Finally, $|\mathcal{S}(V,\tau)| = \sum_{[0:r]} \binom{p}{i}$ by proposition 3.36.

- `srand-n-p-q`: One has $V_{:,[1:n]} = I_n$ and each of the remaining $p-n$ columns has q nonzero random integer elements, randomly positioned. Each element of $\tau_{[n+1:p]}$ has a $1/2$ probability of being a random integer; it vanishes otherwise; $\tau_{[1:n]} = 0$. Random integers are taken in $[-10:+10] \setminus \{0\}$.
- `2d-n-p`: The matrix V is such that: $V_{[1:2],[1:n-2]} = 0$ and $V_{[3:n],[n-1:p]} = 0$. Its remaining elements and τ are randomly generated integers in $[-20:+20]$ [16, 37].

  One can compute some cardinalities.

  Consider first the circuits. The special bloc form of V and the supposed linear independence of the first $n-2$ columns of V (resp. $[V;\tau^{\mathsf{T}}]$) imply that a circuit of V (resp. $[V;\tau^{\mathsf{T}}]$) contains no index in $[1:n-2]$. Therefore

$$\mathcal{C}(V) = \{(i,j,k) \in [n-1:p]^3 : i,\ j \text{ and } k \text{ are all different}\},$$
$$\mathcal{C}([V;\tau^{\mathsf{T}}]) = \{(i,j,k,l) \in [n-1:p]^4 : i,\ j,\ k \text{ and } l \text{ are all different}\}.$$

  It follows that $|\mathcal{C}(V,\tau)| = \binom{p-n+2}{3}$ and $|\mathcal{C}([V;\tau^{\mathsf{T}}],0)| = \binom{p-n+2}{4}$.
  For computing $|\mathcal{S}(V,\tau)|$, one observes that the arrangement $\mathcal{A}(V,\tau)$ can be decomposed into two "independent" arrangements

$$\mathcal{A}(V_{:,[1:n-2]}, \tau_{[1:n-2]}) \qquad \text{and} \qquad \mathcal{A}(V_{:,[n-1:p]}, \tau_{[n-1:p]})$$

  in the sense that

$$s \in \mathcal{S}(V,\tau) \qquad \iff \qquad \begin{cases} s_{[1:n-2]} \in \mathcal{S}(V_{:,[1:n-2]}, \tau_{[1:n-2]}) \\ s_{[n-1:p]} \in \mathcal{S}(V_{:,[n-1:p]}, \tau_{[n-1:p]}). \end{cases}$$

  This is because the witness points of the first arrangement is decomposed in two witness points of the last two arrangements. Since the last two arrangements are also likely to be in affine general position (definition 3.34), one gets by proposition 3.36:

$$|\mathcal{S}(V,\tau)| = |\mathcal{S}(V_{:,[1:n-2]}, \tau_{[1:n-2]})|\ |\mathcal{S}(V_{:,[n-1:p]}, \tau_{[n-1:p]})| = 2^{n-2} \sum_{i\in[0:2]} \binom{p-n+2}{i}.$$

- `perm-n`: This problem refers to the hyperplane arrangements that are called *permutahedron* in [37]: one has $p = n(n+1)/2$, $V_{:,[1:n]}$ is the identity matrix and $V_{:,[n+1:p]}$ is a Coxeter matrix [36] (each column is of the form $e_i - e_j$ for some $i < j$ in $[1:n]$, where e_k is the kth basis vector of $\mathbb{R}^n$). The vector τ is defined by $\tau_i = 1$ for $i \in [1:n]$ and $\tau_i = 0$ for $i \in [n+1:p]$. Since $(1,\ldots,1)$ belongs to all the hyperplanes, the arrangement is centered. One can find in [35, §4.4.1], a justification of the expressions of $|\mathcal{C}(V)| = |\mathfrak{S}_s(V,\tau)| = \sum_{i=3}^{n+1} \frac{i!}{2i} \binom{n+1}{i}$, $|\mathfrak{S}_a(V,\tau)| = 0$ and $|\mathcal{S}(V,\tau)| = (n+1)!$.
- `ratio-n-p-t`: $V_{:,[1:n]}$, $\tau_{[1:n]}$ are randomly generated in $[-50:+50]$ and $\mathtt{t} \in [0,1]$. Then, the remaining columns of $[V;\tau^{\mathsf{T}}]$ can either be random with probability $1-\mathtt{t}$ or randomly generated linear combinations in $[-4:+4]$ of the previous vectors. One recovers problem `rand-n-p` when $\mathtt{t} = 0$.

The cardinality formulas given above are gathered in table 7.1. The numerical values of several cardinalities of the considered instances are given in table 7.2.

Problems	Circuits $\lvert\mathcal{C}(V)\rvert$	Stem vectors $\lvert\mathfrak{S}_s(V,\tau)\rvert/2$	$\lvert\mathfrak{S}_a(V,\tau)\rvert$	$\lvert\mathfrak{S}([V;\tau^\mathsf{T}],0)\rvert/2$	Chambers $\lvert\mathcal{S}(V,\tau)\rvert$
`rand-n-p`	$\binom{p}{n+1}$	0	$\binom{p}{n+1}$	$\binom{p}{n+2}$	$\sum_{i=0}^{n}\binom{p}{i}$
`2d-n-p`	$\binom{p-n+2}{3}$	0	$\binom{p-n+2}{3}$	$\binom{p-n+2}{4}$	$2^{n-2}\sum_{i=0}^{2}\binom{p-n+2}{i}$
`perm-n`	$\sum_{i=3}^{n+1}\frac{i!}{2i}\binom{n+1}{i}$	$\sum_{i=3}^{n+1}\frac{i!}{2i}\binom{n+1}{i}$	0	$\sum_{i=3}^{n+1}\frac{i!}{2i}\binom{n+1}{i}$	$(n+1)!$

Table 7.1: Cardinality formulas for some instances, when $p > n$ and $\operatorname{rank}(V) = n$.

Remarks 7.1 (on table 7.2) 1) As expected, the randomly generated arrangements `rand-*` are in affine general position (definition 3.34). This is revealed in table 7.2 by a number $|\mathfrak{S}_s(V,\tau)|/2 + |\mathfrak{S}_a(V,\tau)|$ of circuits of V (5th and 6th columns, see remark 3.16(3)) that reaches its maximum (4th column), see remark 3.16(6); by a number $|\mathfrak{S}([V;\tau^\mathsf{T}],0)|/2$ of circuits of $[V;\tau^\mathsf{T}]$ (7th column, see [16, after definition 3.9]) that reaches its maximum (8th column), see remark 3.16(6); and by a number $|\mathcal{S}(V,\tau)|$ of sign vectors (9th column) that reaches its upper bound (10th column), see proposition 3.36.

2) Half the number of stem vectors of the linear arrangement $\mathcal{A}([V;\tau^\mathsf{T}],0)$ (7th column) is also the number $|\mathcal{C}([V;\tau^\mathsf{T}])|$ of circuits of $[V;\tau^\mathsf{T}]$ (see [16, after definition 3.9]) and we see that this one is unrelated to the number of circuits of V (sum of columns 5 and 6). This confirms the observation made after proposition 3.23, according to which neither $\mathcal{C}(V) \subseteq \mathcal{C}([V;\tau^\mathsf{T}])$ nor $\mathcal{C}([V;\tau^\mathsf{T}]) \subseteq \mathcal{C}(V)$ must hold. □

7.2 Assessed algorithms

In the next section, the following algorithms have been evaluated on the problem instances listed in the previous section. These algorithms are identified by the following labels.

- RC: the original RC algorithm [37].
- P: the primal $\mathcal{S}$-tree algorithm 4.1.
- PD: the primal-dual $\mathcal{S}$-tree algorithm 5.7.
- D: the dual $\mathcal{S}$-tree algorithm 5.4.
- RC/C: the compact version of the RC algorithm (see below).
- P/C: the compact primal $\mathcal{S}$-tree algorithm 6.3.
- PD/C: the compact primal-dual $\mathcal{S}$-tree algorithm 6.5.
- D/C: the compact version of the D algorithm (see below).

All the algorithms, but RC and RC/C, benefit from the enhancements A (section 4.2.1), B (section 4.2.2) and C (section 4.2.3). By want of space, the algorithms RC, RC/C and D/C have not been presented in sections 4 and 6. Briefly, algorithm RC is algorithm 4.2 (with its header 4.1) without its steps 4-8; algorithm RC/C is algorithm 6.4 (with its header 6.3) without its steps 6-10, 29-33 and 43-47; algorithm D/C is obtained from algorithm 5.4 using the compaction principles described in section 6.

7.3 Numerical results

To evaluate the algorithms listed in the previous section, we have implemented them in a Julia code named isf.jl, which extends the Matlab code isf.m [13, 14], from linear to general affine arrangements. The implementation has been done in Julia (version `1.8.5`) on a `MacBookPro18,2/10cores` (parallelism is not implemented however) with the system `macOS`

			Circuits of V	Stem vectors of $\mathcal{A}(V,\tau)$		Stem vectors of $\mathcal{A}([V;\tau^\mathsf{T}],0)$		Chambers	
Problems	n	p	Bound	$\lvert\mathfrak{S}_s\rvert/2$	$\lvert\mathfrak{S}_a\rvert$	$\lvert\mathfrak{S}\rvert/2$	Bound	$\lvert\mathcal{S}(V,\tau)\rvert$	Bound
`rand-2-8`	2	8	56	0	56	70	70	37	37
`rand-4-8`	4	8	56	0	56	28	28	163	163
`rand-4-9`	4	9	126	0	126	84	84	256	256
`rand-5-10`	5	10	210	0	210	120	120	638	638
`rand-4-11`	4	11	462	0	462	462	462	562	562
`rand-6-12`	6	12	792	0	792	495	495	2510	2510
`rand-5-13`	5	13	1716	0	1716	1716	1716	2380	2380
`rand-7-14`	7	14	3003	0	3003	2002	2002	9908	9908
`rand-7-15`	7	15	6435	0	6435	5005	5005	16384	16384
`rand-8-16`	8	16	11440	0	11440	8008	8008	39203	39203
`rand-9-17`	9	17	19448	0	19448	12376	12376	89846	89846
`srand-8-20-2`	8	20	167960	56	321	987	184756	36225	263950
`srand-8-20-4`	8	20	167960	1185	70650	94534	184756	213467	263950
`srand-8-20-6`	8	20	167960	20413	123909	105345	184756	245396	263950
`2d-4-20`	4	20	15504	1	815	3046	38760	684	6196
`2d-5-20`	5	20	38760	0	680	2380	77520	1232	21700
`2d-6-20`	6	20	77520	1	559	1808	125970	2176	60460
`2d-7-20`	7	20	125970	0	443	1365	167960	3840	137980
`2d-8-20`	8	20	167960	0	364	1001	184756	6784	263950
`perm-5`	5	15	5005	197	0	197	6435	720	4944
`perm-6`	6	21	116280	1172	0	1172	203490	5040	82160
`perm-7`	7	28	3108105	8018	0	8018	6906900	40320	1683218
`perm-8`	8	36	94143280	62814	0	62814	254186856	362880	40999516
`ratio-3-20-0.7`	3	20	4845	19	4614	14043	15504	1119	1351
`ratio-3-20-0.9`	3	20	4845	118	4550	12993	15504	1176	1351
`ratio-4-20-0.7`	4	20	15504	102	15271	36781	38760	6015	6196
`ratio-4-20-0.9`	4	20	15504	2327	11908	19882	38760	4600	6196
`ratio-5-20-0.7`	5	20	38760	97	33945	61452	77520	15136	21700
`ratio-5-20-0.9`	5	20	38760	23514	10954	23514	77520	11325	21700
`ratio-6-20-0.7`	6	20	77250	238	76595	120663	125970	59519	60640
`ratio-6-20-0.9`	6	20	77250	345	71861	106115	125970	53795	60460
`ratio-7-20-0.7`	7	20	125970	125	123792	159956	167960	135064	137980
`ratio-7-20-0.9`	7	20	125970	154	123731	159636	167960	135039	137980

Table 7.2: Description of the 33 considered arrangements. The first column gives the problem names. The next two columns specify the dimensions of $V \in \mathbb{R}^{n\times p}$. The 4th column gives the upper bound on the number of circuits of V, recalled in remark 3.16(6); by remark 3.16(3), it is also an upper bound on $|\mathfrak{S}_s|/2 + |\mathfrak{S}_a|$, where $|\mathfrak{S}_s|$ (resp. $|\mathfrak{S}_a|$) is the number of symmetric (resp. asymmetric) stem vectors (definition 3.15) of the arrangement $\mathcal{A}(V,\tau)$; $|\mathfrak{S}_s|/2$ and $|\mathfrak{S}_a|$ are given in columns 5 and 6. Columns 7 and 8 give half the number of stem vectors of the arrangement $\mathcal{A}([V;\tau^\mathsf{T}],0)$ and its Schläfli upper bound, derived from (3.29). The last two columns give the number $|\mathcal{S}(V,\tau)|$ of chambers of the arrangement $\mathcal{A}(V,\tau)$ and its upper bound given by (3.33).

Monterey, version 12.6.1.

All the solvers, but D and D/C, need to solve linear optimization problems (LOPs). The linear optimization solver used in the Julia code is Gurobi. This one appears to be more efficient than the Matlab solver Linprog used in [13, 14]. Since the improvement is obtained by a reduction of the number of LOPs, which are solved much faster in the Julia version, we observe a less important improvement (wrt the RC algorithm) in computing time in the present study (Julia code) than reported in [16].

The main computational burden of the "pure primal" variants P and P/C is the solution of the LOPs while, for the "pure dual" variants D and D/C, it is the computation of the stem vectors and their use in the covering tests. These are not comparable. Therefore, counting the number of LOPs or the number of covering tests is not a relevant criterion for comparing the solvers. For this reason, we rely on computing time. Since the RC algorithm was shown in [37] to have better performance in time than earlier methods, a comparison is often made with the RC algorithm. Since this algorithm is implemented in Python, we avoid biases due to the programming language by making the comparison with our Julia version of the RC algorithm, which can be easily simulated from algorithm 4.1, as mentioned above.

For ease of reading, the comparison of the solvers' efficiency is carried out by using *performance profiles* [11] (tables with precise numbers are also given in appendix A): these are curves in a graph with the *relative efficiency* on the x-axis (sometimes in logarithmic scale) and a *percentage of problems* on the y-axis. There is one graph per performance, which is the computing time in our case, and there is one curve per solver in that graph: a point (e, f) of the curve of a solver tells us that the efficiency of this solver is never worse than e times that of the best solver (this one depends on the considered problem) on a fraction f of the problems. As a result, *the solver with the highest curve, if any, can be legitimately considered as the most effective one*, while the ranking of the other solvers by the position of their curve in the graph should be taken with caution [23]. The performance profiles only depend on the *relative* performance of the solvers, that is, for a particular problem, their performance divided by the one of the best solver for that problem. Therefore taking the *computing time* or the *computing time per chamber* as performance yield the same performance profiles.

7.3.1 Standard solvers

Let us first compare the standard solvers RC, P, PD and D with each other, on the selected arrangements described in table 7.2. The comparison is made on the computing times reported in table A.1 and the associated performance profiles are given in figure 7.1.

One observes that the PD algorithm is generally the most efficient one when the *computing time* is taken as a reference (it has the "highest" curve in figure 7.1). The speedup with respect to the RC algorithm can reach 10: this can be observed in table A.1 (ratio 10.14 of the PD algorithm on instance perm-8) or on figure 7.1 (by the abscissa of the rightmost change in the curve of the RC algorithm, whose relative performance is there given relatively to the PD algorithm).

7.3.2 Compact solvers

To show the interest of the compact versions of the algorithms, introduced in section 6, we compare each solver RC, P, PD and D to its compact version RC/C, P/C, PD/C and D/C. The computing times are given in table A.2 and the performance profiles are shown in figure 7.2. Recall that, by construction of the compact versions, the improvement ratios are bounded above by 2, approximately.

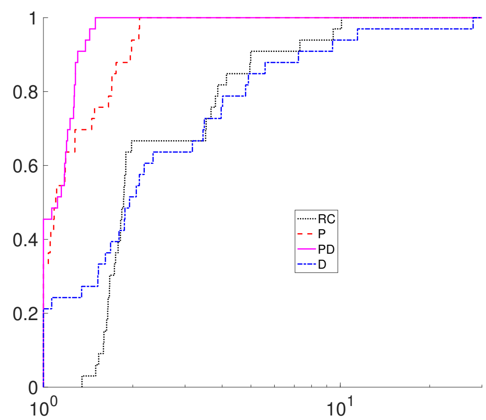


Figure 7.1: Performance profiles of the RC, P, PD and D algorithms, for the computing time.

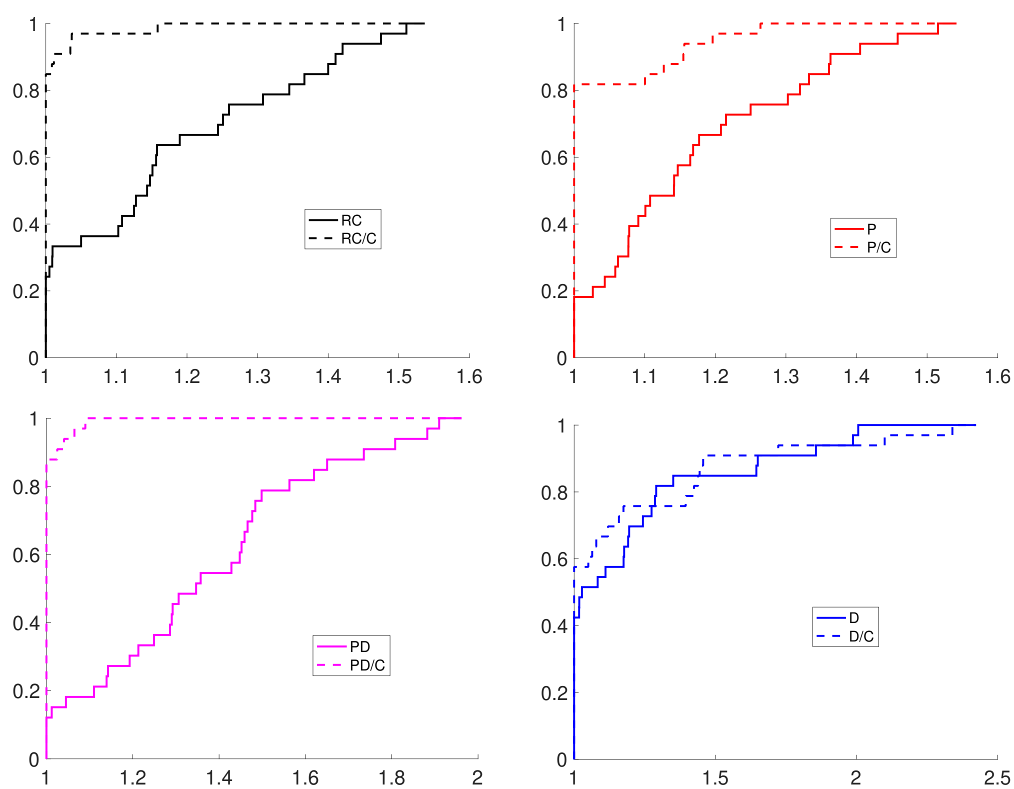


Figure 7.2: Performance profiles of the RC vs. RC/C, P vs. P/C, PD vs. PD/C and D vs. D/C algorithms, for the computing time. The dashed lines refer to the compact versions of the algorithms.

We observe indeed on table A.2 that the compact versions improve their standard version on the computing time, particularly for the PD/C, for which the mean (resp. median) improvement is 1.35 (resp. 1.35). The improvement bound 2 is obtained by D/C on the instances `srand-8-20-4` and `perm-7`. This improvement can also be observed on figure 7.2, with more ambiguity, since it is not indicated which algorithm is the best for each problem instance (for example the x-axis larger than 2 for the performance profiles D vs. D/C is not due to a performance of D/C that is 2.34 times better than D on some problem, but the opposite: it is D that is 2.34 times faster than D/C on some problem). Nevertheless, it is indeed algorithm A/C that generally outperforms algorithm A when A = RC, P or PD (their curve is higher).

The performance profiles of figure 7.3 compares the most effective solver, namely PD/C, to the RC solver. The former shows a speedup that can reach 19.3.

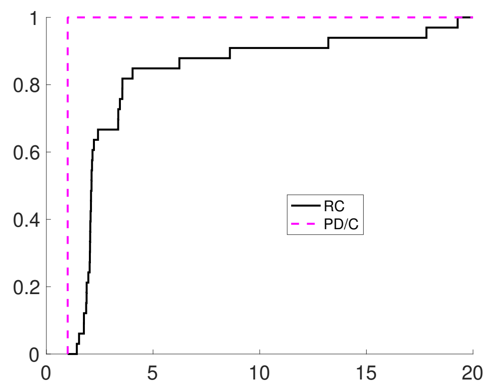


Figure 7.3: Performance profiles of the RC vs. PD/C solvers, for the computing time.

8 Conclusion

This paper deals with the enumeration of the chambers of a hyperplane arrangement. It brings improvements to a recursive algorithm proposed by Rada and Černý, and proposes a family of new algorithms having, to various extends, dual aspects based on the Motzkin alternative, matroid circuits and the introduced notion of *stem vector*. Most algorithms are grounded on a tree of sign vectors that are in a one-to-one correspondence with the chambers of the arrangement. Compact versions of the algorithms are also presented, which aim at reducing the size of the sign vector tree, in order to avoid duplicating costly identical operations like solving linear optimization problems and covering tests. The most efficient method of this algorithm anthology is the one that includes primal and dual ingredients, and uses the compact form of the tree, which has been named PD/C in the paper. The speedup it provides, with respect to Rada-Černý's algorithm, much depends on the features of the considered arrangement, in particular its dimensions, and ranges between 1.4 and 19.3, with a mean value of 3.9.

These algorithms are grounded on a theory that is presented before their introduction. This one includes the structure of the sign vector sets and the stem vector sets, in particular conditions for their symmetry, their connectivity, their full cardinality and much more.

Numerous aspects of the presented algorithms can be further improved or developed, covering both conceptual and implementation aspects. Let us mention a few topics. (i) The linear optimization problems could be solved approximately, hence saving computation time when the tested sign vector is feasible. (ii) The way stem vectors are computed, stored and used could be improved, with specific structures designed for that purpose. (iii) In case the arrangements present combinatorial symmetries, the approaches presented in [7, 39] should increase significantly the algorithm performances. (*iv*) The proposed approaches could be extended to compute the chambers of the hyperplane arrangement and subarrangements, those recursively included in the hyperplane intersections of any smaller dimension.

A Tables with numerical results

This appendix gives the tables with the detailed numerical results, comparing the solvers selected in section 7.2, on which the performance profiles of figures 7.1, 7.2 and 7.3 are based. Comments on these results can be found in section 7.3. Table A.1 deals with the standard algorithms and table A.2 is related to the compact versions of these algorithms.

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Problems	RC time	P time	P ratio	PD time	PD ratio	D time	D ratio
`rand-2-8`	0.09	0.03	3.76	0.03	3.51	**0.02**	**4.13**
`rand-4-8`	0.12	0.07	1.71	0.08	1.47	**0.06**	**1.86**
`rand-4-9`	0.20	0.12	1.67	0.13	1.56	**0.10**	**1.99**
`rand-5-10`	0.48	0.27	1.79	0.31	1.58	**0.26**	**1.89**
`rand-4-11`	0.56	0.35	1.58	0.38	1.47	**0.30**	**1.88**
`rand-6-12`	1.89	1.18	1.59	1.34	1.41	**1.09**	**1.73**
`rand-5-13`	2.32	1.37	1.69	1.45	1.60	**1.30**	**1.79**
`rand-7-14`	6.96	**4.37**	**1.59**	5.14	1.35	4.66	1.50
`rand-7-15`	12.40	**7.51**	**1.65**	8.92	1.39	10.10	1.22
`rand-8-16`	29.20	**17.50**	**1.67**	21.10	1.38	28.30	1.03
`rand-9-17`	63.00	**39.40**	**1.60**	50.50	1.25	70.80	0.89
`srand-8-20-2`	33.30	9.70	3.43	**6.67**	**5.00**	23.00	1.45
`srand-8-20-4`	199.00	**105.00**	**1.90**	137.00	1.45	989.00	0.20
`srand-8-20-6`	238.00	**131.00**	**1.81**	196.00	1.21	947.00	0.25
`2d-4-20`	2.12	0.89	2.38	**0.60**	**3.53**	1.01	2.11
`2d-5-20`	3.50	1.58	2.22	**0.95**	**3.67**	1.86	1.88
`2d-6-20`	5.84	2.82	2.07	**1.66**	**3.53**	3.11	1.88
`2d-7-20`	9.86	4.48	2.20	**2.55**	**3.86**	5.24	1.88
`2d-8-20`	16.40	7.35	2.23	**4.32**	**3.80**	8.13	2.02
`perm-5`	1.17	0.46	2.53	**0.24**	**4.96**	0.82	1.42
`perm-6`	10.60	3.04	3.50	**1.45**	**7.33**	7.11	1.50
`perm-7`	106.00	23.60	4.49	**11.20**	**9.50**	128.00	0.83
`perm-8`	1070.00	210.00	5.10	**106.00**	**10.14**	2970.00	0.36
`ratio-3-20-0.7`	2.53	1.54	1.65	**1.39**	**1.82**	2.12	1.19
`ratio-3-20-0.9`	2.59	1.80	1.44	**1.41**	**1.84**	2.16	1.20
`ratio-4-20-0.7`	11.10	6.17	1.80	**5.94**	**1.87**	12.50	0.89
`ratio-4-20-0.9`	7.07	5.53	1.28	**4.33**	**1.63**	9.46	0.75
`ratio-5-20-0.7`	21.00	**12.00**	**1.75**	12.80	1.64	38.10	0.55
`ratio-5-20-0.9`	16.00	11.30	1.42	**9.57**	**1.67**	22.40	0.71
`ratio-6-20-0.7`	75.90	**46.10**	**1.65**	58.50	1.30	183.00	0.42
`ratio-6-20-0.9`	65.40	**43.60**	**1.50**	50.10	1.30	175.00	0.37
`ratio-7-20-0.7`	147.00	**109.00**	**1.36**	151.00	0.98	523.00	0.28
`ratio-7-20-0.9`	148.00	**96.40**	**1.54**	138.00	1.07	538.00	0.28
Mean			2.11		**2.76**		1.28
Median			**1.71**		1.63		1.23

Table A.1: Computing times (in seconds) for the *standard* algorithms listed in section 7.2. For each algorithm A := P, PD or D, the second column gives the ratios time(RC)/time(A)

Problems	RC/C time	RC/C ratio	RC/C ratio	P/C time	P/C ratio	P/C ratio	PD/C time	PD/C ratio	PD/C ratio	D/C time	D/C ratio	D/C ratio
`rand-2-8`	0.08	1.25	1.25	0.03	0.89	3.33	0.03	0.96	3.37	**0.03**	0.85	**3.52**
`rand-4-8`	0.08	1.37	1.37	0.05	1.33	2.28	0.05	1.45	2.14	**0.05**	1.24	**2.32**
`rand-4-9`	0.16	1.24	1.24	0.10	1.21	2.02	0.11	1.21	1.89	**0.09**	1.11	**2.20**
`rand-5-10`	0.35	1.40	1.40	**0.22**	1.25	**2.24**	0.24	1.29	2.05	0.22	1.18	2.23
`rand-4-11`	0.49	1.15	1.15	0.30	1.16	1.84	0.30	1.29	1.89	**0.29**	1.03	**1.93**
`rand-6-12`	1.34	1.41	1.41	**0.87**	1.36	**2.18**	0.90	1.48	2.09	0.92	1.19	2.07
`rand-5-13`	1.95	1.19	1.19	**1.20**	1.14	**1.93**	1.11	1.31	2.09	1.20	1.08	1.93
`rand-7-14`	4.90	1.42	1.42	**3.11**	1.41	**2.24**	3.43	1.50	2.03	3.90	1.19	1.78
`rand-7-15`	9.22	1.34	1.34	**5.69**	1.32	**2.18**	6.04	1.48	2.05	8.58	1.18	1.45
`rand-8-16`	19.80	1.47	1.47	**12.00**	1.46	**2.43**	13.50	1.56	2.16	17.20	1.65	1.70
`rand-9-17`	41.70	1.51	1.51	**26.00**	1.52	**2.42**	30.60	1.65	2.06	42.90	1.65	1.47
`srand-8-20-2`	30.20	1.10	1.10	9.45	1.03	3.52	**5.34**	1.25	**6.24**	22.60	1.02	1.47
`srand-8-20-4`	158.00	1.26	1.26	**80.60**	1.30	**2.47**	93.90	1.46	2.12	493.00	2.01	0.40
`srand-8-20-6`	182.00	1.31	1.31	**96.10**	1.36	**2.48**	121.00	1.62	1.97	701.00	1.35	0.34
`2d-4-20`	2.10	1.01	1.01	1.03	0.87	2.06	**0.62**	0.98	**3.45**	1.74	0.58	1.22
`2d-5-20`	3.54	0.99	0.99	1.89	0.84	1.85	**1.04**	0.92	**3.37**	2.71	0.69	1.29
`2d-6-20`	5.89	0.99	0.99	3.26	0.87	1.79	**1.64**	1.01	**3.56**	4.43	0.70	1.32
`2d-7-20`	9.81	1.01	1.01	4.93	0.91	2.00	**2.44**	1.05	**4.04**	7.31	0.72	1.35
`2d-8-20`	16.40	1.00	1.00	9.29	0.79	1.77	**4.60**	0.94	**3.57**	11.70	0.69	1.40
`perm-5`	1.04	1.12	1.12	0.42	1.11	2.80	**0.14**	1.74	**8.60**	0.64	1.29	1.83
`perm-6`	9.27	1.14	1.14	2.82	1.08	3.76	**0.80**	1.81	**13.22**	5.58	1.27	1.90
`perm-7`	94.00	1.13	1.13	22.30	1.06	4.75	**5.95**	1.88	**17.82**	64.40	1.99	1.65
`perm-8`	1070.00	1.00	1.00	195.00	1.08	5.49	**55.50**	1.91	**19.28**	1600.00	1.86	0.67
`ratio-3-20-0.7`	2.53	1.00	1.00	1.45	1.06	1.74	**1.22**	1.14	**2.07**	4.96	0.43	0.51
`ratio-3-20-0.9`	2.68	0.97	0.97	1.65	1.09	1.57	**1.27**	1.11	**2.04**	3.12	0.69	0.83
`ratio-4-20-0.7`	11.00	1.01	1.01	5.73	1.08	1.94	**4.98**	1.19	**2.23**	13.30	0.94	0.83
`ratio-4-20-0.9`	8.19	0.86	0.86	5.30	1.04	1.33	**3.79**	1.14	**1.87**	7.33	1.29	0.96
`ratio-5-20-0.7`	20.00	1.05	1.05	10.90	1.10	1.93	**9.92**	1.29	**2.12**	80.00	0.48	0.26
`ratio-5-20-0.9`	13.90	1.15	1.15	9.67	1.17	1.65	**6.61**	1.45	**2.42**	22.00	1.02	0.73
`ratio-6-20-0.7`	68.50	1.11	1.11	**40.20**	1.15	**1.89**	43.10	1.36	1.76	212.00	0.86	0.36
`ratio-6-20-0.9`	67.80	0.96	0.96	38.20	1.14	1.71	**37.20**	1.35	**1.76**	196.00	0.89	0.33
`ratio-7-20-0.7`	127.00	1.16	1.16	**89.70**	1.22	**1.64**	103.00	1.47	1.43	564.00	0.93	0.26
`ratio-7-20-0.9`	128.00	1.16	1.16	**81.90**	1.18	**1.81**	96.60	1.43	1.53	565.00	0.95	0.26
Mean		1.16	1.16		1.14	2.33		1.35	**3.95**		1.09	1.30
Median		1.14	1.14		1.14	2.02		1.35	**2.12**		1.03	1.35

Table A.2: Computing times (in seconds) for the *compact* algorithms listed in section 7.2. For each algorithm A = RC, P, PD, or D, the first column gives the computing time of A/C in seconds, the second column gives the ratios time(A)/time(A/C) (upper bounded by 2, approximately) and the third column gives the ratios time(RC)/time(A/C).

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