On the computation of the B-differential of the Min C-function for the balanced linear complementarity problem

Baptiste Plaquevent-Jourdain, with
Jean-Pierre Dussault, Université de Sherbrooke
Jean Charles Gilbert, INRIA Paris

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Outline

Plan

1. Overview
2. Underlying problem
3. Incrementing the sign vectors
4. Subproblems by optimization

Form of the problems

Complementarity problems [CPS92], [FP03]

$$\begin{aligned} & 0 \leq F(x) \perp G(x) \geq 0 \\ \Leftrightarrow\ & \forall\, i, F_i(x) \geq 0, G_i(x) \geq 0, F_i(x)G_i(x) = 0 \end{aligned} \tag{1}$$

Where $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are smooth. Affine case:

$$\begin{gathered} F(x) \equiv Ax + a,\ \ G(x) \equiv Bx + b, \quad A, B \in \mathbb{R}^{n \times n}, a, b \in \mathbb{R}^n \\ 0 \leq (Ax + a) \perp (Bx + b) \geq 0 \end{gathered} \tag{2}$$

Remark: $u \geq 0, v \geq 0, uv = 0 \Leftrightarrow \min(u, v) = 0$

$(1) \Leftrightarrow \forall\, i, H_i(x) := \min(F_i(x), G_i(x)) = 0 \Leftrightarrow H(x) = 0$

Reformulation by C-functions

C-functions

$$\varphi(u, v) = 0 \Leftrightarrow u \geq 0, v \geq 0, uv = 0$$

Examples: minimum, Fischer,... $\rightarrow$ componentwise.

Fischer $\varphi_F(u, v) = \sqrt{u^2 + v^2} - (u + v)$ [Fis92]; more differentiable, less linear.
φ_F: much work already done [FS97] (Facchinei, Soares), [GK96] (Geiger-Kanzow)

C-functions are nondifferentiable $\rightarrow$ nonsmooth techniques

Nonsmooth equations - context

For scalar functions: subgradients
For systems $H(x) = 0$: semismooth Newton [QS93] (Qi, Sun)

Generalized derivatives: Bouligand differential

$$\partial_{\mathrm{B}} H(x) = \{J \in \mathbb{R}^{n \times n} : \exists\ (x_k)_k \to x, H'(x_k) \text{ exists and } \to J\} \quad (3)$$

Example: for $|\cdot|$ on $\mathbb{R}$, $\partial_B|\cdot|(0) = \{-1, +1\}$.
One element $J^0 \in \partial_B$: technique of [Qi93] (Qi)

Among other differentials: Clarke: $\partial_C H(x) = \mathrm{conv}(\partial_B H(x))$

Main difficulty

Objective of this work

Determine generalized Jacobians of $x \mapsto \min(Ax + a, Bx + b)$

- structure of the Jacobians?
- number of elements $|\partial_B H(x)|$?
- How to get them efficiently?

Plan

Definition of the B-differential: which sequences to use?

$$\partial_B H(x) = \{J : \exists\ (x_k)_k \to x, H'(x_k) \text{ defined and } \to J\}$$
$$(x_k)_k =?$$

Indices of non-differentiability
$I(x) := \{i \in [1:n] : A_{i,:}x + a_i = B_{i,:}x + b_i, A_{i,:} \neq B_{i,:}\}$

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- $(d_k)_k \subset \mathbb{R}^n \setminus \bigcup[\text{hyperplanes}]$

Main question

$$\forall\ i \in I(x), v_i^{\mathsf{T}} := B_{i,:} - A_{i,:} \quad \text{find } d_k, \forall\ i \in I(x), v_i^{\mathsf{T}} d_k \neq 0 \tag{4}$$

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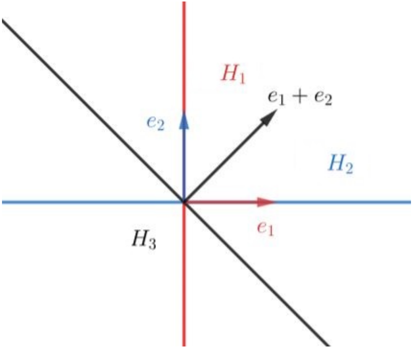
Vectors and hyperplanes - 1

Affinity: Jacobian constant on each zone: lines $\to 0$



$v_1 = e_1, v_2 = e_2, v_3 = e_1 + e_2$; the hyperplanes define different regions.

Vectors and hyperplanes - 2

If $i \notin I(x)$, component i of H is differentiable at x

$$\begin{aligned} A_{i,:} = B_{i,:} \quad &\Rightarrow \quad F_i \equiv G_i \\ A_{i,:}x + a_i < B_{i,:}x + b_i \quad &\Rightarrow \quad J_i = A_{i,:} \\ A_{i,:}x + a_i > B_{i,:}x + b_i \quad &\Rightarrow \quad J_i = B_{i,:} \end{aligned}$$

Otherwise, $i \in I(x)$, H_i nondifferentiable at x

$$v_i^{\mathsf{T}} d_{(k)} \neq 0 : \quad \begin{array}{l} \text{if } v_i^{\mathsf{T}} d_{(k)} > 0 \Rightarrow B_{i,:}d_{(k)} > A_{i,:}d_{(k)} : J_i = A_{i,:} \\ \text{if } v_i^{\mathsf{T}} d_{(k)} < 0 \Rightarrow B_{i,:}d_{(k)} < A_{i,:}d_{(k)} : J_i = B_{i,:} \end{array}$$

$|I(x)| > 1$: which inequations can be verified simultaneously?

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$|I(x)| > 1$: which inequations can be verified <u>simultaneously</u>?

Vectors and hyperplanes - 3

$|I(x)| = m$ hyperplanes: space without them = ?
Connected sets := region; on $+$ or on $-$ side of every hyperplane
$\Leftrightarrow \pm v_i^\mathsf{T} d > 0$ has a solution d.

Fundamental question

$$\text{given } v_i := (B_{i,:} - A_{i,:})^\mathsf{T}$$
$$\text{find all } s = (s_1, \dots, s_m) \in \{\pm 1\}^m,$$
$$\text{s.t. } \exists\ d_s, \forall\ i \in [1:m], s_i v_i^\mathsf{T} d_s > 0$$

Brute force: 2^m linear feasibility pbs to solve... How can one improve?

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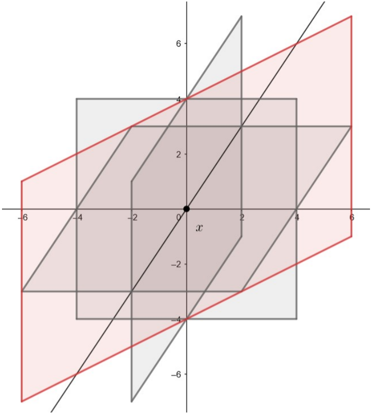
Illustration of systems of equations



$(e_1 + e_2)^\perp$; $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = +e_1 + e_2$; if $v_1^\mathsf{T} d < 0, v_2^\mathsf{T} d < 0$
$\Rightarrow v_4^\mathsf{T} d < 0$: $s = (-, -, \cdot, +)$ unfeasible

From literature

Algebraic approach: arrangement of hyperplanes

Our setting: degenerate case
A priori no known theoretical answer
Various examples: [Zas75], [AW81], [CS95]

Computational result, [BN82]

Computes more elements: objects of all intermediate dimensions.
Dimensional recursive sweeping by hyperplanes; hard to evaluate complexity / real time.

Plan

Framework

Given m vectors, $V = [v_1, \ldots, v_m] \in \mathbb{R}^{n\times m}$
Up to 2^m elements to compute, likely less.
Element $\equiv$ signs of a feasible system: how to determine the s's?
Some easy cases:

- n or $m \in \{1, 2\}$ (in fact, $\mathrm{rank}(V) = 1, 2$)
- $\mathrm{rank}(V) = m$ (2^m sign vectors)
- $\mathrm{rank}(V) = 2$: $2m$ elements

Full rank case: observed in [CX11] (Chen, Xiang)

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Method - adding vectors one at a time

Assuming all sign vectors are found for $(v_1, \ldots, v_{k-1})$: for $(v_1, \ldots, v_k)$?

With one more vector

- Given $(v_1, \ldots, v_{k-1})$; v_k ; $\mathcal{S}_{k-1} \subseteq \{\pm 1\}^{k-1}$

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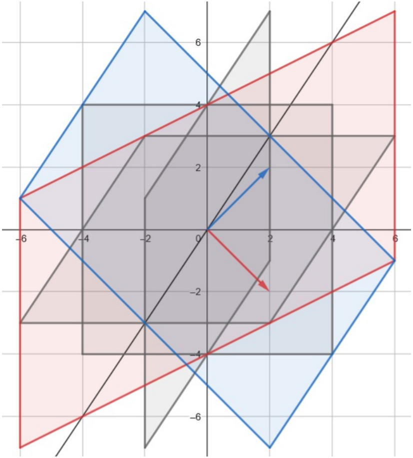
- $v_k^\mathsf{T} d_s^{k-1} \simeq 0$ both by perturbation

Illustration of the method



Grey: hyperplanes orthogonal to e_1, e_2, e_3, red: $(e_1 + e_2)^{\perp}$, blue: $(e_2 + e_3)^{\perp}$

Plan

Study of the iterative subproblem - 1

- vectors $v_1, \dots, v_{k-1}$ and associated signs $s = (s_1, \dots, s_{k-1})$
- d_s s.t. $\forall\ i \in [1 : k-1], s_i v_i^\mathsf{T} d_s > 0$
- $\mathrm{sign}(v_k^\mathsf{T} d_s) v_k^\mathsf{T} d_s$ already > 0
- feasible with $s_k = -\mathrm{sign}(v_k^\mathsf{T} d_s)$? [new d?]

A kind of subproblem to solve many times.
We work on improvements of the current version.
The metric: number of subproblems ("hard things").

Example of code's return

```
--------------------------------------------------------------------------
[0;34mConsidering the linearly dependent vector 2[0m
Direct and complementary sign vectors
   1:  111111      1  1111110   | 0000000  0000001  (P)
   2:  011111      1  0111110   | 1000000  1000001  (P)
   3:  101111      1  1011110   | 0100000  0100001  (Q)
   4:  001111      1  0011110   | 1100000  1100001  (Q)
   5:  110111      0  1101111   | 0010001  0010000  (Q)
   6:  010111      0            | 1010001           (Q)
   7:  100111      0  1001111   | 0110001  0110000  (Q)
   8:  000111      0  0001111   | 1110001  1110000  (Q)
   9:  111011      1  1110110   | 0001000  0001001  (Q)
  10:  011010      1  0110100   | 1001010  1001011  (P)
  11:  101011      1  1010110   | 0101000  0101001  (Q)
  12:  001010      1  0010100   | 1101010  1101011  (P)
  13:  110011      0  1100111   | 0011001  0011000  (Q)
  14:  010011      0            | 1011001           (Q)
  15:  100011      0  1000111   | 0111001  0111000  (Q)
  16:  000011      0            | 1111001           (Q)
  17:  111110      1            | 0000010           (Q)
  18:  011110      1  0111100   | 1000010  1000011  (P)
  19:  101110      1  1011100   | 0100010  0100011  (Q)
  20:  001110      1  0011100   | 1100010  1100011  (P)
  21:  010110      0            | 1010011           (Q)
  22:  100110      1  1001100   | 0110010  0110011  (Q)
  23:  000110      0  0001101   | 1110011  1110010  (Q)
  24:  111010      1  1110100   | 0001010  0001011  (Q)
  25:  011011      0  0110111   | 1001001  1001000  (Q)
  26:  101010      1  1010100   | 0101010  0101011  (Q)
  27:  001011      0  0010111   | 1101001  1101000  (Q)
  28:  110010      0  1100101   | 0011011  0011010  (Q)
  29:  010010      0  0100101   | 1011011  1011010  (Q)
  30:  100010      0  1000101   | 0111011  0111010  (Q)
  31:  000010      0  0000101   | 1111011  1111010  (Q)
```

First vector added, from $2^5 = 32$ to $62 = (2^5 - 2) * 2 + 2 = 2^{5+1} - 2$ [expected].

Theoretical bound

- General formula unknown
- Upper bound of efficiency?
- Default: next iteration can double, or "much less"
- Simple upper bound obtained: # QP
- Idea: step $k-1$, less than 2^{k-1} and $|\mathcal{S}_{k-1}| \leq |\mathcal{S}|$

$$\mathcal{B} = \min(2^m - 2^r, (m-r)|\mathcal{S}|), \quad |\mathcal{S}| = |\partial_B| \tag{5}$$

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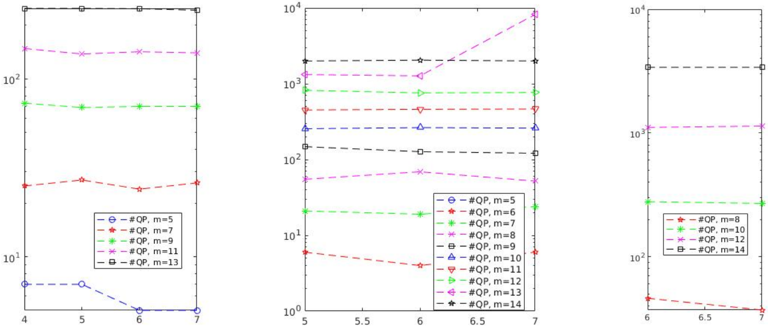
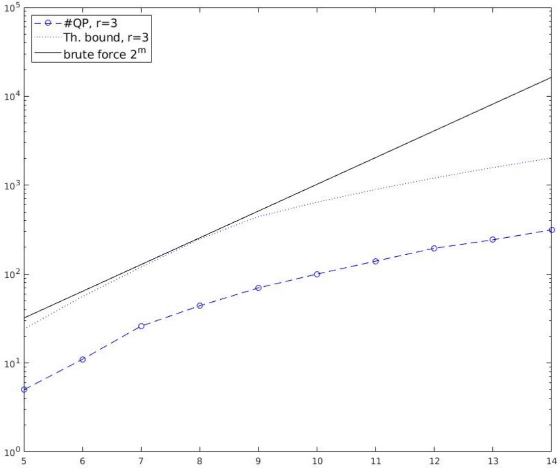
Numerical results - #QPs depending on dimension



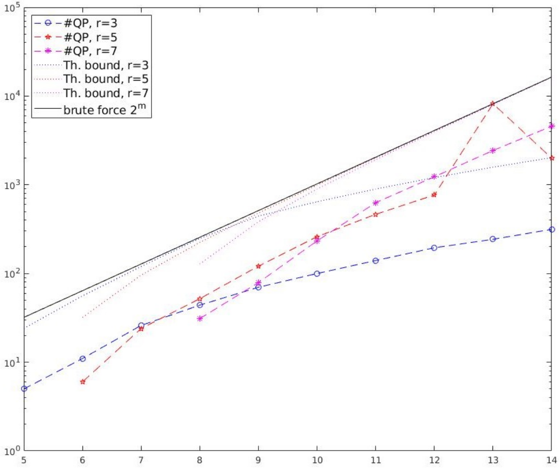
For $n \in [4, 5, 6, 7]$, number of QPs solved for various m's. From left to right: $r = 3$, $r = 5$, $r = 6$

Numerical results - 1



Number of problems solved for $n = 7$, $m \in [5 : 14], r = 3$

Numerical results - 2



Number of problems solved for $n = 7$, $m \in [5 : 14]$, $r = 3, 5, 7$

Conclusion

Results

- ∂_B computed efficiently
- In practice much less QPs the theoretical bound
- Improvements with IP / unconstrained

Open questions

- ε-issues
- complexity results
- larger n, m, r's
- algebraic insights?

Thank you for your attention! Any question?

Short bibliography I

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Study of the iterative subproblem - 1

- vectors $v_1, \dots, v_{k-1}$ and associated signs $s = (s_1, \dots, s_{k-1})$
- d_s s.t. $\forall\ i \in [1:k-1], s_i v_i^\mathsf{T} d_s > 0$
- $\mathrm{sign}(v_k^\mathsf{T} d_s) v_k^\mathsf{T} d_s$ already > 0
- feasible with $s_k = -\mathrm{sign}(v_k^\mathsf{T} d_s)$? [new d?]

Linear optimization formulation

$$\exists\ d, \text{s.t.} \begin{cases} s_k v_k^\mathsf{T} d > 0 \\ s_i v_i^\mathsf{T} d > 0, \quad i \in [1:k-1] \end{cases}$$
$$\Leftrightarrow \begin{cases} \inf & -s_k v_k^\mathsf{T} d \\ \text{s.t.} & s_i v_i^\mathsf{T} d > 0 \end{cases} \text{unbounded} \tag{6}$$

Study of the iterative subproblem - 2

Hypothesis for step $k-1$: $s_i v_i^\top d > 0 \rightarrow$ interior point

$$\varphi_\mu(d) = -s_k v_k^\top d - \mu \sum \log(s_i v_i^\top d)$$

$$\begin{cases} \inf \; -s_k v_k^\top d \\ s_i v_i^\top d \geq 0 \\ ||d|| \leq D \end{cases}$$

$$\inf \; \varphi_\mu$$

- interior point technique
- bound constraint
- finite solution
- sign?

- direct opt.
- constraints $\rightarrow$ log
- $d \rightarrow 0 / -\infty$
- sign of $-s_k v_k^\top d^j$?

Study of the iterative subproblem - 2

Hypothesis for step $k-1$: $s_i v_i^\top d > 0 \rightarrow$ interior point

$$\varphi_\mu(d) = -s_k v_k^\top d - \mu \sum \log(s_i v_i^\top d)$$

$$\begin{cases} \inf \ -s_k v_k^\top d \\ s_i v_i^\top d \geq 0 \\ ||d|| \leq D \end{cases}$$

$\inf \ \varphi_\mu$

- interior point technique
- bound constraint
- finite solution
- sign?

- direct opt.
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Study of the iterative subproblem - 2

Hypothesis for step $k-1$: $s_i v_i^\top d > 0 \rightarrow$ interior point

$$\varphi_\mu(d) = -s_k v_k^\top d - \mu \sum \log(s_i v_i^\top d)$$

$$\begin{cases} \inf \; -s_k v_k^\top d \\ s_i v_i^\top d \geq 0 \\ ||d|| \leq D \end{cases} \qquad\qquad \inf \; \varphi_\mu$$

- interior point technique
- bound constraint
- finite solution
- sign?

- direct opt.
- constraints $\rightarrow$ log
- $d \rightarrow 0 / -\infty$
- sign of $-s_k v_k^\top d^j$?

Study of the iterative subproblem - 3

Bound constraint $\Rightarrow$ finite solution $\Rightarrow$ dual problem
Dual = projection $\leftrightarrow$ quadratic problem (QP)

Interior points

bounded domain: constraint $||d|| \leq D$ [homogeneity].
$\rightarrow$ recent framework of [Nes22] (Nesterov): complexity result

Example of code's return - 1

```
========================================================================
Bdiffmin_inc (Version 0.1, 2022-03-09)
^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
Current date (time): 2022-05-14 (8:20)

Vector dimension (n)  = 5
Number of vectors (m) = 9
Incremental method
Tolerances
. distance  = 1.0e-08
. QP solver = 1.0e-12
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
Vector dimension (n)  = 5
Number of vectors (m) = 9
Rank (r)              = 5
```

Beginning of a en execution

Example of code's return - 2

```
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
Vector dimension (n)  = 5
Number of vectors (m) = 9
Rank (r)              = 5
-------------------------------------------------------------------------
[0;34mConsidering the linearly independent vectors 1, 7, 4, 9, 3[0m
Direct and complementary sign vectors
   1:  11111 | 00000
   2:  01111 | 10000
   3:  10111 | 01000
   4:  00111 | 11000
   5:  11011 | 00100
   6:  01011 | 10100
   7:  10011 | 01100
   8:  00011 | 11100
   9:  11101 | 00010
  10:  01101 | 10010
  11:  10101 | 01010
  12:  00101 | 11010
  13:  11001 | 00110
  14:  01001 | 10110
  15:  10001 | 01110
  16:  00001 | 11110
```

Base case - $r = 5$ independent vectors, 2^5 sign vectors [simplicity: $-1 \to 0, +1 \to +1$]

Example of code's return - 3

```
-----------------------------------------------------------------------
[0;34mConsidering the linearly dependent vector 6[0m
Direct and complementary sign vectors
   1:  11111      1  111110    | 000000  000001  (P)
   2:  01111      1  011110    | 100000  100001  (P)
   3:  10111      1  101110    | 010000  010001  (P)
   4:  00111      1  001110    | 110000  110001  (P)
   5:  11011      1            | 001000          (Q)
   6:  01011      1  010110    | 101000  101001  (Q)
   7:  10011      1  100110    | 011000  011001  (Q)
   8:  00011      1  000110    | 111000  111001  (P)
   9:  11101      1  111010    | 000100  000101  (P)
  10:  01101      0  011011    | 100101  100100  (Q)
  11:  10101      1  101010    | 010100  010101  (P)
  12:  00101      0  001011    | 110101  110100  (Q)
  13:  11001      1  110010    | 001100  001101  (Q)
  14:  01001      1  010010    | 101100  101101  (P)
  15:  10001      1  100010    | 011100  011101  (P)
  16:  00001      1  000010    | 111100  111101  (P)
```

First dependent vector being processed.

Example of code's return - 4

```
---------------------------------------------------------------------------
[0;34mConsidering the linearly dependent vector 2[0m
Direct and complementary sign vectors
   1:  111111    1  1111110   | 0000000  0000001  (P)
   2:  011111    1  0111110   | 1000000  1000001  (P)
   3:  101111    1  1011110   | 0100000  0100001  (Q)
   4:  001111    1  0011110   | 1100000  1100001  (Q)
   5:  110111    0  1101111   | 0010001  0010000  (Q)
   6:  010111    0            | 1010001           (Q)
   7:  100111    0  1001111   | 0110001  0110000  (Q)
   8:  000111    0  0001111   | 1110001  1110000  (Q)
   9:  111011    1  1110110   | 0001000  0001001  (Q)
  10:  011010    1  0110100   | 1001010  1001011  (P)
  11:  101011    1  1010110   | 0101000  0101001  (Q)
  12:  001010    1  0010100   | 1101010  1101011  (P)
  13:  110011    0  1100111   | 0011001  0011000  (Q)
  14:  010011    0            | 1011001           (Q)
  15:  100011    0  1000111   | 0111001  0111000  (Q)
  16:  000011    0            | 1111001           (Q)
  17:  111110    1            | 0000010           (Q)
  18:  011110    1  0111100   | 1000010  1000011  (P)
  19:  101110    1  1011100   | 0100010  0100011  (Q)
  20:  001110    1  0011100   | 1100010  1100011  (P)
  21:  010110    0            | 1010011           (Q)
  22:  100110    1  1001100   | 0110010  0110011  (Q)
  23:  000110    0  0001101   | 1110011  1110010  (Q)
  24:  111010    1  1110100   | 0001010  0001011  (Q)
  25:  011011    0  0110111   | 1001001  1001000  (Q)
  26:  101010    1  1010100   | 0101010  0101011  (Q)
  27:  001011    0  0010111   | 1101001  1101000  (Q)
  28:  110010    0  1100101   | 0011011  0011010  (Q)
  29:  010010    0  0100101   | 1011011  1011010  (Q)
  30:  100010    0  1000101   | 0111011  0111010  (Q)
  31:  000010    0  0000101   | 1111011  1111010  (Q)
```

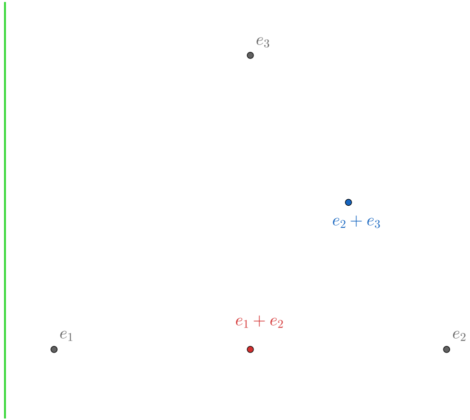
First vector added, from $2^5 = 32$ to $62 = (2^5 - 2) * 2 + 2 = 2^{5+1} - 2$ [expected].

Other illustration

Equivalent problems/representations

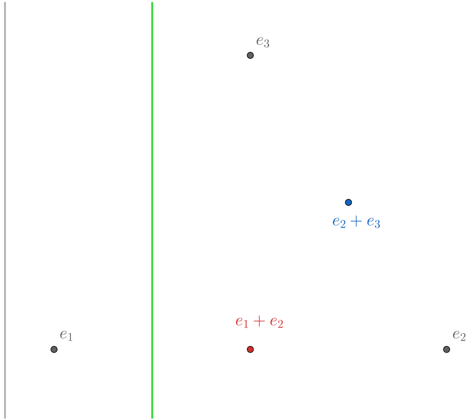
- (Fully) degenerate arrangement of hyperplanes
- How many pairs of convex subsets a set of points generate?
- Systems of inequations [chosen]

Illustration



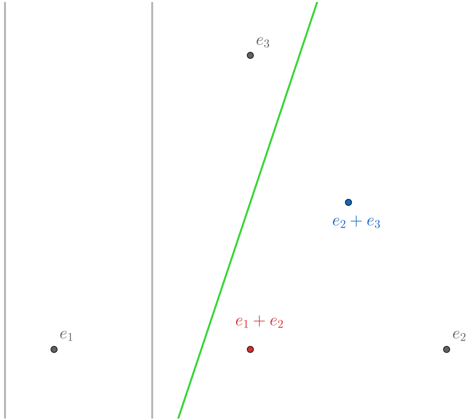
Previous problem under the convex separation form - 1

Illustration



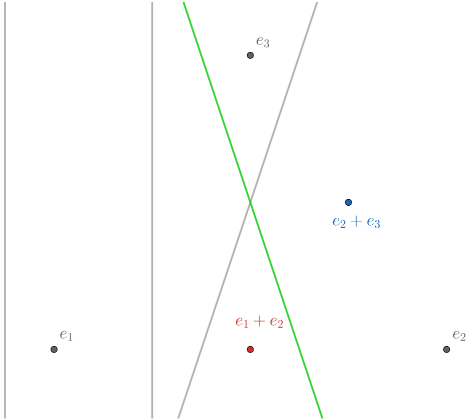
Previous problem under the convex separation form - 2

Illustration



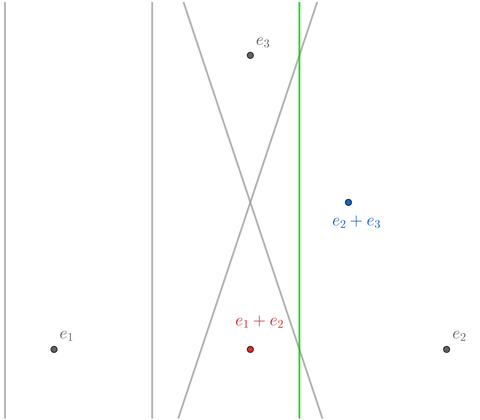
Previous problem under the convex separation form - 3

Illustration



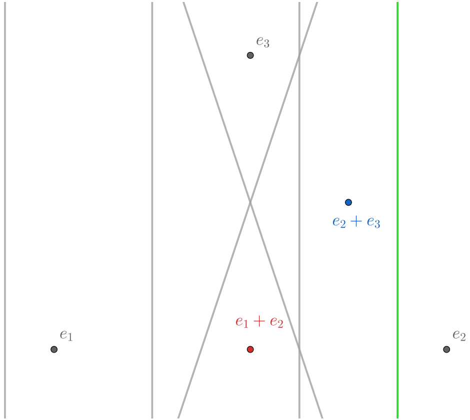
Previous problem under the convex separation form - 4

Illustration



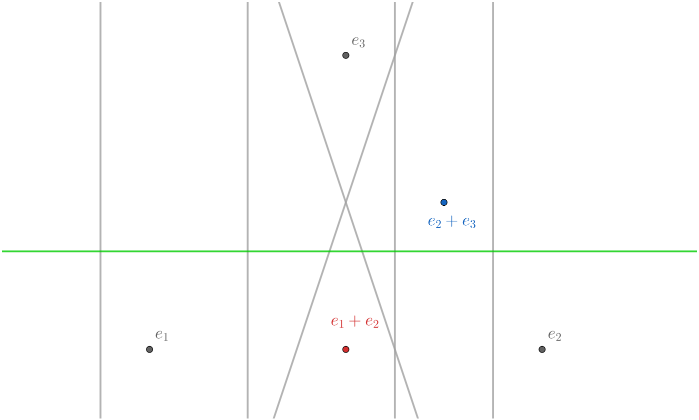
Previous problem under the convex separation form - 5

Illustration



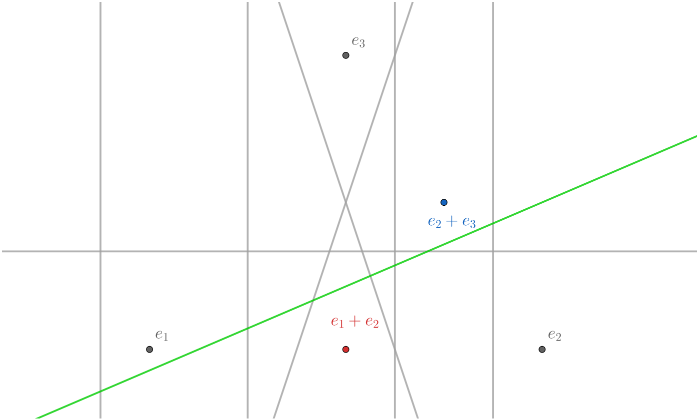
Previous problem under the convex separation form - 6

Illustration



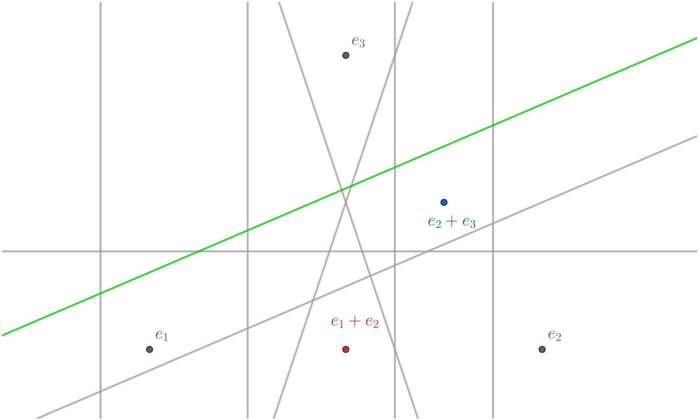
Previous problem under the convex separation form - 7

Illustration



Previous problem under the convex separation form - 8

Illustration



Previous problem under the convex separation form - 9