Simplificial Complexes

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1 Motivation

2 Definitions

Définition 2.1 (Simplicies):
Simplciies can be seen as "triangles" in higher dimensions.
A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. A k-simplex is the convex hull of k+1 affinely independent points in $\mathbb{R}^n$.

Remarque 2.1: The boundary of a k-simplex is the union of its (k-1)-dimensional faces.

A simplicial complex is a set of simplicies glued by their faces.

Définition 2.2 (Simplicial Complex): Let K be a set of simplicies in $\mathbb{R}^n$. K is a simplicial complex if:

1. Every face of a simplex in K is also in K.
2. The intersection of any two simplicies in K is a face of each of them.

Exemple 2.1:

For example can we construct a circle from a simplicial complex?
We can just take 3 three points and connect them with 1-simplicies to form a triangle And this is a 2-simplex. Then we can considere this as homeomorphic to a circle.
We can do the same with a 2-simplex to form a disk.
Let's considerate another definition of a simplicial complex.

Définition 2.3 (Abstract simplicial complexes): An abstract simplicial complex is a non-empty family of sets (called simplicies) closed under the operation of taking subsets, i.e. if A is a set in the family, and B is a non empty subset of A, then B is also in the family.
A family of sets X is an abstract simplicial complex if and only if: $Y_1 \in X$ and $Y_2 \subset Y1$ and $Y_2 \neq \emptyset$ implies $Y_1 \cap Y_2 \in X$

More intuitively we can think that we take the simplicial complexes and we decompose it into 0-simplicies, 1-simplicies, 2-simplicies, etc. And we can think of the simplicial complex as a set of vertices, edges, triangles, etc.

3 Application to data

Let's say we have a set of points in $\mathbb{R}^n$, and real number $\alpha \geq 0$.

1. The **Vietoris-Rips complex** $VR_\alpha(X)$ is the simplicial complex whose k-simplicies $[x_1, \ldots, x_k]$ such that $d_X(x_i, x_j) \leq \alpha$ for all i, j. So when the balls of radius $\frac{\alpha}{2}$ around the points in X are not disjoint we can connect them with a 1-simplex. And when the balls of radius $\frac{\alpha}{2}$ around the points in X are not disjoint we can connect them with a 2-simplex. And so on.

2. The **Cech complex** $Cech_\alpha(X)$ is the simplicial complex such that the k+1 closed balls of radius α around the points in X have a non-empty intersection.