11   (dual) One operator is the dual of another operator if it negates the result when applied to the negated operands. The zero-operand operators $\top$ and $\bot$ are each other's duals. If $op_0 \neg a = \neg op_1 a$ then op_0 and op_1 are duals. If $\neg a \, op_0 \, \neg b = \neg(a \, op_1 \, b)$ then op_0 and op_1 are duals. And so on for more operands.

(a)   Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

§   To answer this question, I'll use the symbols I introduce in part (f). The pair of duals is: $\overline{\top}$ (always $\top$) and $\underline{\bot}$ (always $\bot$). The two self-duals are: $\mathtt{I}$ (identity) and $\pm$ (negation).

(b)   Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

§   To answer this question, I'll use the symbols I will introduce in part (f). The six dual pairs are: $\overline{\triangle}\,\underline{\triangledown}$, $\vee\,\wedge$, $\geqslant\,\geqslant$, $\leqslant\,\leqslant$, $\underline{\triangle}\,\overline{\triangledown}$, $\triangle\,\triangledown$. The four self-duals are: $<$, $>$, $\triangleright$, $\triangleleft$.

(c)   What is the dual of the three-operand operator **if then else fi** ? Express it using only the operator **if then else fi** .

§   Its truth table is

$\top\top\top$	$\top\top\bot$	$\top\bot\top$	$\top\bot\bot$	$\bot\top\top$	$\bot\top\bot$	$\bot\bot\top$	$\bot\bot\bot$
$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$

The dual of **if** a **then** b **else** c **fi** is equivalent to **if** a **then** c **else** b **fi** .

(d)   The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

§   I will show that for every expression P without variables, (dual of P) $= \neg P$. I do so by induction on the structure of expression P. The two binary values give us two base cases.

         (dual of $\top$)   use the dual-forming rules

$=$ $\bot$

$=$ $\neg\top$

         (dual of $\bot$)   use the dual-forming rules

$=$ $\top$

$=$ $\neg\bot$

There is an induction step for each of the binary operators. Suppose (this is an induction hypothesis) that (dual of P) $= \neg P$. Then

         (dual of $\neg P$)   use the dual-forming rules

$=$ $\neg$(dual of P)   use the induction hypothesis

$=$ $\neg\neg P$

Suppose that (dual of P) $= \neg P$ and (dual of Q) $= \neg Q$. Then

         (dual of $P \wedge Q$)   use the dual-forming rules

$=$ (dual of P) $\vee$ (dual of Q)   use the induction hypotheses

$=$ $\neg P \vee \neg Q$ use duality law

$=$ $\neg(P \wedge Q)$

And similarly for all other operators.

(e)   Let P be a binary expression without variables. From part (d) we know that every binary expression without variables of the form

         (dual of P) $= \neg P$

is a theorem. Therefore, to find the dual of a binary expression with variables, we must

replace each operator by its dual and negate each variable. For example, if a and b are binary variables, then the dual of $a \wedge b$ is $\neg a \vee \neg b$. And since

$$\text{(dual of } a \wedge b \text{)} = \neg(a \wedge b)$$

we have one of the Duality Laws:

$$\neg a \vee \neg b = \neg(a \wedge b)$$

The other of the Duality Laws is obtained by equating the dual and negation of $a \vee b$. Obtain five laws that do not appear in this book by equating a dual with a negation.

§ From $a=b$ we get $\neg a \not= \neg b = \neg(a=b)$

From **if** a **then** b **else** c **fi** we get **if** $\neg a$ **then** $\neg c$ **else** $\neg b$ **fi** $= \neg$**if** a **then** b **else** c **fi**

From $a=b \wedge c$ we get $\neg a \not= \neg b \vee \neg c = \neg(a=b \wedge c)$

From $a=b \vee c$ we get $\neg a \not= \neg b \wedge \neg c = \neg(a=b \vee c)$

From $a = (b \wedge c)$ we get $\neg a \not= (\neg b \vee \neg c) = \neg(a = (b \wedge c))$

(f)   Dual operators have truth tables that are each other's vertical mirror reflections. For example, the truth table for $\wedge$ (below left) is the vertical mirror reflection of the truth table for $\vee$ (below right).

$$\wedge: \quad \begin{array}{cc|c} T & T & T \\ T & \bot & \bot \\ \bot & T & \bot \\ \bot & \bot & \bot \end{array} \qquad\qquad \vee: \quad \begin{array}{cc|c} T & T & T \\ T & \bot & T \\ \bot & T & T \\ \bot & \bot & \bot \end{array}$$

Design symbols (you may redesign existing symbols where necessary) for the 4 one-operand and 16 two-operand binary operators according to the following criteria.

(i) Dual operators should have symbols that are vertical mirror reflections (like $\wedge$ and $\vee$). This implies that self-dual operators have vertically symmetric symbols, and all others have vertically asymmetric symbols.

(ii) If $a \; op_0 \; b = b \; op_1 \; a$ then op_0 and op_1 should have symbols that are horizontal mirror reflections (like $\Rightarrow$ and $\Leftarrow$). This implies that symmetric operators have horizontally symmetric symbols, and all others have horizontally asymmetric symbols.

§

old		¬		
new	⊤	I	≠	⊥

	⊤	I	≠	⊥
T	T	T	⊥	⊥
⊥	T	⊥	T	⊥

old		∨	⇐		⇒		=	∧		≠						
new	Ā	∨	⩾	<	⩽	>	≜	∧	ᐃ	⩒	▷	⩾	◁	⩽	▽	ᐁ

	Ā	∨	⩾	<	⩽	>	≜	∧	ᐃ	⩒	▷	⩾	◁	⩽	▽	ᐁ
T T	T	T	T	T	T	T	T	T	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
T ⊥	T	T	T	T	⊥	⊥	⊥	⊥	T	T	T	T	⊥	⊥	⊥	⊥
⊥ T	T	T	⊥	⊥	T	T	⊥	⊥	T	T	⊥	⊥	T	T	⊥	⊥
⊥ ⊥	T	⊥	T	⊥	T	⊥	T	⊥	T	⊥	T	⊥	T	⊥	T	⊥