Power Spectral Density

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The power spectral density is the norm of the Fourier transform. In a way, it measures energy content at each frequency of response.

It is also the averaged(?) Fourier transform of the autocorrelation.

Norm of Fourier Transform

The power spectral density, $P(f)$, of a signal $y(t)$, is

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |Y_T(f)|^2 \, df$$

Where the Fourier transform $Y_T(f)$ of $y_T(t)$ is defined as:

$$Y_T(f) = \mathcal{F}\{y_T(t)\} = \int_{-\infty}^{\infty} e^{-i2\pi f t} y_T(t) dt$$

where $f \in \mathbb{R}_+$ is frequency in Hertz, and $t \in \mathbb{R}_+$ is time in seconds.

question.

$$y_T(t) = \begin{bmatrix} y(t) \\ y(t+\Delta t) \\ y(t+2\Delta t) \\ \vdots \\ y(t+(T-1)\Delta t) \end{bmatrix} \in \mathbb{R}^T \quad \textbf{??}$$

Fourier Transform of Autocorrelation

Autocorrelation, discrete:

$$R_{yy} \approx \frac{1}{N} \mathbf{Y}_p(k) \mathbf{Y}_p^*(k), \quad N \gg 0$$

$$\begin{aligned} \mathbf{Y}_p(k) &= \begin{bmatrix} \mathbf{y}_p(k) & \mathbf{y}_p(k+1) & \cdots & \mathbf{y}_p(k+N-1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{y}(k) & \mathbf{y}(k+1) & \cdots & \mathbf{y}(k+N-1) \\ \mathbf{y}(k+1) & \mathbf{y}(k+2) & \cdots & \mathbf{y}(k+N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}(k+p-1) & \mathbf{y}(k+p) & \cdots & \mathbf{y}(k+N+p-2) \end{bmatrix} \end{aligned}$$

Autocorrelation, continuous:

$$\begin{aligned} R_{yy}(\tau) &= \mathbb{E}\left\{\mathbf{y}(t+\tau)\mathbf{y}^*(t)\right\} \\ &= \int_{-\infty}^{\infty} \mathbf{y}(t+\tau)\mathbf{y}^*(t) f_y(t) dt \\ &= \int_{-\infty}^{\infty} \mathbf{y}(t+\tau)\mathbf{y}^*(t) dt \quad \mathbf{??} \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \mathbf{y}_T^*(t-\tau)\mathbf{y}_T(t) dt \quad \mathbf{??} \end{aligned}$$

Fourier transform of this autocorrelation:

$$\begin{aligned} \mathcal{F}\left\{R_{yy}(\tau)\right\} &= \mathbf{R}_{yy}(f) \\ &= \int_{-\infty}^{\infty} e^{-i2\pi f t} R_{yy}(\tau) d\tau \\ &= \int_{-\infty}^{\infty} e^{-i2\pi f t} \left[\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \mathbf{y}_T^*(t-\tau)\mathbf{y}_T(t) dt\right](\tau) d\tau \\ &= \lim_{N\to\infty} \frac{(\Delta t)^2}{T} \left| \sum_{n=-N}^{N} y_n e^{-i2\pi f n \Delta t} \right|^2 \\ &\approx \lim_{T\to\infty} \frac{1}{T} \left|Y_T(f)\right|^2 \end{aligned}$$

Averaged...**??** Fourier transform of this autocorrelation:

$$P = \int_{-\infty}^{\infty} \mathbf{R}_{yy}(f) df \quad = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} \left|Y_T(f)\right|^2 df$$