

Maximum Principle Proof by Fritz John

Notes of proof of maximum principle for a parabolic PDE, from: F. John. "Partial Differential Equations". New York: Springer-Verlag. 1982. pp. 215-216.

Let ω denote an open bounded set of \mathbb{R}^n . For a fixed $T > 0$ we form the cylinder Ω in \mathbb{R}^{n+1} with base ω and height T :

$$\Omega = \{(x, t) | x \in \omega, 0 < t < T\} \quad (1)$$

The boundary $\partial\Omega$ consists of two disjoint portions, a "lower" boundar $\partial'\Omega$, and an "upper" one $\partial''\Omega$ (see Figure1):

$$\partial'\Omega = \{(x, t) | \text{eigher } x \in \partial\omega, 0 \leq t \leq T \text{ or } x \in \omega, t = 0\} \quad (2)$$

$$\partial''\Omega = \{(x, t) | x \in \omega, t = T\}. \quad (3)$$

As in the second-order elliptic case the maximum of a solution of the heat equation in Ω is taken on $\partial\Omega$; but a more subtle distinction between the forwards and backwards t-directions makes itself felt:

Theorem 1. *Let u be continuous in $\bar{\Omega}$ and $u_t, u_{x_i x_k}$ exist and be continuous in Ω and satisfy $u_t - \Delta u \leq 0$. Then*

$$\max_{\Omega} u = \max_{\partial'\Omega} u \quad (4)$$

Proof. Let at first $u_t - \Delta u < 0$ in Ω . Let Ω_ϵ for $0 < \epsilon < T$ denote the set

$$\Omega_\epsilon = \{(x, t) | x \in \omega, 0 < t < T - \epsilon\}.$$

Since $u \in C^0(\bar{\Omega}_\epsilon)$ there exists a point $(x, t) \in \bar{\Omega}_\epsilon$ with

$$u(x, t) = \max_{\bar{\Omega}_\epsilon} u.$$

If here $(x, t) \in \Omega_\epsilon$ the necessary relations $u_t = 0, \Delta u \leq 0$ would contradict $u_t - \Delta u < 0$. If $(x, t) \in \partial''\Omega_\epsilon$ we would have

$$u_t \geq 0, \Delta u \leq 0$$

leading to the same contradiction. Thus $(x, t) \in \partial'\Omega_\epsilon$, and

$$\max_{\bar{\Omega}_\epsilon} u = \max_{\partial'\Omega_\epsilon} u \leq \max_{\partial'\Omega} u.$$

Since every point of $\bar{\Omega}$ with $t < T$ belongs to some $\bar{\Omega}_\epsilon$ and u is continuous in $\bar{\Omega}$, (4) follows. Let next $u_t - \Delta u \leq 0$ in Ω . Introduce

$$v(x, t) = u(x, t) - kt$$

with a constant positive k . Then $v_t - \Delta v = u_t - \Delta u - k < 0$ and

$$\max_{\bar{\Omega}} u = \max_{\bar{\Omega}} (v + kt) \leq \max_{\bar{\Omega}} v + kT = \max_{\partial'\Omega} v + kT \leq \max_{\partial'\Omega} u + kT.$$

For $k \rightarrow 0$ we obtain (4).

□

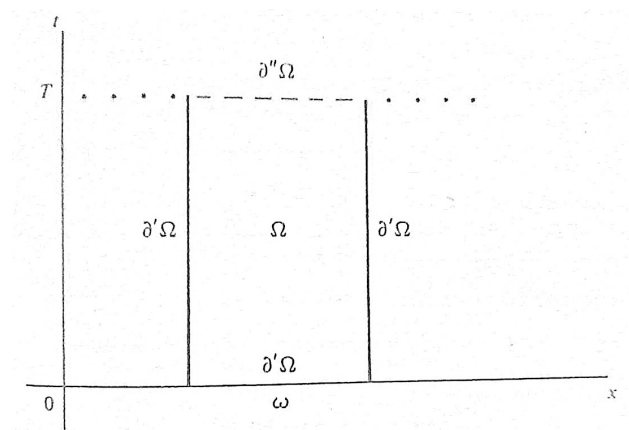


Figure 1: Illustration of the proof space, from book.