Maximum Principle Proof by Fritz John

From: F. John. "Partial Differential Equations". New York: Springer-Verlag. 1982.

Let ω denote an open bounded set of $\Re^n$. For a fixed $T > 0$ we form the cylinder Ω in $\Re^{n+1}$ with base ω and height T:

$$\Omega = \{(x,t)|x \in \omega, 0 < t < T\} \tag{1}$$

The boundary $\partial\Omega$ consists of two dijoint portions, a "lower" boundar $\partial'\Omega$, and an "upper" one $partial''\Omega$ (see Figure1):

$$\partial'\Omega = \{(x,t)|eigherx \in \partial\omega, 0 \leq t \leq Torx \in \omega, t = 0\} \tag{2}$$

$$\partial''\Omega = \{(x,t)|x \in \omega, t = T\}. \tag{3}$$

As in the second-order elliptic case the maximum of a solution of the heat equation in Ω is taken on $\partial\Omega$; but a more subtle distinction between the forwards and backwards t-directions makes itself felt:

Theorem 1. *Let u be continuous in $\bar{\Omega}$ and $u_t, u_{x_i x_k}$ exist and be continuous in Ω and satisfy $u_t - \triangle u \leq 0$. Then*

$$\max_{\bar{\Omega}} u = \max_{\partial'\Omega} u \tag{4}$$

Proof. Let at first $u_t - \triangle u < 0$ in Ω. Let Ω_ϵ for $0 < \epsilon < T$ denote the set

$$\Omega_\epsilon = \{(x,t)|x \in \omega, 0 < t < T - \epsilon\}.$$

Since $u \in C^0(\bar{\Omega}_\epsilon)$ there exists a point $(x,t) \in \bar{\Omega}_\epsilon$ with

$$u(x,t) = \max_{\bar{\Omega}_\epsilon} u.$$

If here $(x,t) \in \Omega_\epsilon$ the necessary relations $u_t = 0, \triangle \leq 0$ would contradict $u_t - \triangle u < 0$. If $(x,t) \in \partial''\Omega_\epsilon$ we would have

$$u_t \geq 0, \triangle u \leq 0$$

leading to the same contradiction. Thus $(x,t) \in \partial'\Omega_\epsilon$, and

$$\max_{\bar{\Omega}_\epsilon} u = \max_{\partial'\Omega_\epsilon} u \leq \max_{\partial'\Omega} u.$$

Since every point of $\bar{\Omega}$ with $t < T$ belongs to some $\bar{\Omega}_\epsilon$ and u is continuous in $\bar{\Omega}$, (4) follows. Let next $u_t - \triangle u \leq 0$ in Ω. Introduce

$$v(x,t) = u(x,t) - kt$$

with a constant positive k. Then $v_t - \triangle v = u_t - \triangle u - k < 0$ and

$$\max_{\bar{\Omega}} u = max_{\bar{\Omega}}(v + kt) \leq \max_{\bar{\Omega}} v + kT = \max_{\partial'\Omega} v + kT \leq \max_{\partial'\Omega} u + kT.$$

For $k \to 0$ we obtain (4).

$\square$

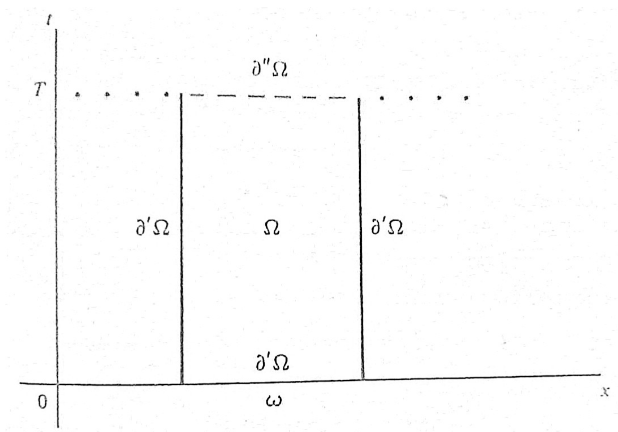

Figure 1: Illustration of the proof space, from book.