Convex sets and functions

Mathematical Optimization

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Definitions

Convex set

A set $S \subseteq \mathbb{R}^n$ is called *convex* if for any two points $x_1, x_2 \in S$ and any $\theta \in [0, 1]$, the point $\theta x_1 + (1 - \theta)x_2$ also lies in S.

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Affine set

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Q: What is the connection between them?

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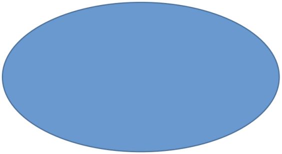
A: Every affine set is convex.

Examples



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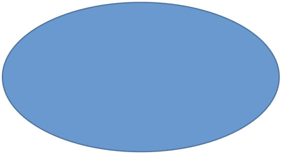

Half-plane

Let $a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$, then the half-plane $\{x \mid a^\top x \geq b\}$ is convex.

Examples

Hyperplane

Let $a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$, then the hyperplane $\{x \mid a^\top x = b\}$ is an affine set.

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Ball

Let $\|\cdot\|$ be the norm (any norm!) in $\mathbb{R}^n$, $r > 0$ and $c \in \mathbb{R}^n$. Then the ball $\overline{B}(c, r) = \{x \in \mathbb{R}^n \mid \|x - c\| \leq r\}$ is a convex set.

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$$\|\theta x_1 + (1-\theta)x_2 - c\| = \|\theta(x_1 - c) + (1-\theta)(x_2 - c)\| \leq$$
$$\leq \theta\|x_1 - c\| + (1-\theta)\|x_2 - c\| \leq \theta r + (1-\theta)r = r.$$

Counterexample?

More complex examples

Positively semi-defined matrices

$\mathcal{S}_+^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^\top z^T Xz \geq 0,\ z \in \mathbb{R}^n\}$, is a convex set.

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Visual one

Show that the set $M = \{x \in \mathbb{R}^2_{++} | \sqrt{x_1} + x_2 \geq 1\}$ is convex.

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$$\sqrt{\theta x_1 + (1-\theta) y_1} + \theta x_2 + (1-\theta) y_2 \geq$$
$$\geq \theta\sqrt{x_1} + (1-\theta)\sqrt{y_1} + \theta x_2 + (1-\theta) y_2 \geq \theta + (1-\theta) = 1.$$

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Let S be a convex set. Then $\text{int } S$ and $\text{cl } S$ (interior and closure) are also convex sets.

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Convex combination

A convex combination of points $x_1, \ldots, x_k$ is any point of the form $\theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_1 + \ldots + \theta_k = 1$ and $0 \leq \theta_i \leq 1$ for all i.

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$$\theta_1 x_1 + \ldots + \theta_k x_k = (1 - \theta_k)\left(\frac{\theta_1}{1-\theta_k} x_1 + \ldots + \frac{\theta_{k-1}}{1-\theta_k} x_{k-1}\right) + \theta_k x_k,$$

Convex Hull

Definition

The convex hull of a set S is the smallest convex set T containing S. That is, it is the intersection of all convex sets containing S. Usually the convex hull is denoted by conv$S$.

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Properties

Let $S, T \subseteq \mathbb{R}^n$. Then the following statements are true:

1. $S \subseteq \text{conv } S$
2. $S \subseteq T \to \text{conv } S \subseteq \text{conv } T$
3. S is convex if and only if $S = \text{conv } S$.

Convex Hull. Theorem

Theorem

The convex hull of the set S is equal to the set of all convex combinations of elements S, that is,
$\mathrm{conv} S = \cup_{k \in \mathbb{N}} \{\theta_1 x_1 + \ldots + \theta_k x_k | \theta_1 + \ldots + \theta_k = 1, 0 \le \theta_i \le 1, x_i \in S\}$.

Example of how to use this

Show that conv $\{xx^\top | x \in \mathbb{R}^n, \|x\|_2 = 1\} = \{A \in \mathcal{S}^n_+ | \mathrm{Tr}\,(A) = 1\}$.

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$$\boxed{\subset}\text{Tr}\left(xx^\top\right) = \text{Tr}\left(x^\top x\right) = \|x\|_2^2 = 1; \forall A = \sum_{i=1}^{n} \theta_i x_i x_i^\top, \text{Tr}\,(A) = 1$$

$$\boxed{\supset}\text{Tr}\left(S^\top \Lambda S\right) = \boxed{\text{Tr}\left(SS^\top \Lambda\right)} = \text{Tr}\,(\Lambda) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1.$$

List of basic operations

1. $\boxed{+}$ (linear comb): For S_1, S_2 convex sets and $c_1, c_2 \in \mathbb{R}$, $c_1 S_1 + c_2 S_2 = \{c_1 x_1 + c_2 x_2 | x_1 \in S_1, x_2 \in S_2\}$ — convex. **Minkowski sum** — $c_i = 1$.
2. $\boxed{L, L^{-1}}$: Consider **affine function** $L : \mathbb{R}^n \to \mathbb{R}^m$, $L(x) = Ax + b$ If $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ are convex, then the **image** $L(X)$ and **preimage** $L(Y)$ are convex.
3. $\boxed{\cap}$: Let $\{S_i\}_{i \in I}$ be convex sets $\Rightarrow \cap_{i \in I} S_i$ — convex.

Example. Polyhedron

$\{x \in \mathbb{R}^n | Ax \preceq b, Cx = d\}$, with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$, $d \in \mathbb{R}^k$

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4. $\boxed{\times}$: Let $\{S_i\}_{i=1}^k$ be convex sets $\Rightarrow \bigotimes_{i=1}^k S_i$ — convex.

More examples to play with

- $S = \{x \| \|Ax + b\| \leq c^\top x + d\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$

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- $\{a \in \mathbb{R}^k | p(0) = 1, |p(t)| \leq 1 \forall t : \alpha \leq t \leq \beta\}$, where $p(t) = a_1 + a_2 t + \ldots + a_k t^{k-1}$
- With given $S \subseteq \mathbb{R}^n$ и $z \in \mathbb{R}^n$, $M = \{x | \|x - z\| + \|x - y\| \leq 1 \forall y \in S\}$ is convex.

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- With given $S \subseteq \mathbb{R}^n$ и $z \in \mathbb{R}^n$, $M = \{x | \|x - z\| + \|x - y\| \leq 1 \forall y \in S\}$ is convex.
- (Hyperbolic cone) Let $P \in \mathcal{S}^n_+$ and $c \in \mathbb{R}^n$, then the set $K = \{x | x^\top P x \leq (c^\top x)^2, c^\top x \geq 0\}$ is convex.
- $C = \{x \in \mathbb{R}^n | x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succ B\}$, where $A_1, A_2, \ldots, A_n, B \in \mathbb{R}^{n \times n}$

Definition, criteria of convexity

From the examples, we found interesting sets - cones:

Definition. Cone

The set C is called *cone* if for any $c \in C$ and $\theta \geq 0$ the point θc also belongs to C.

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Examples?

Criteria of convex cone

Cone is convex if and only if for any $c_1, c_2 \in C$, $\theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 = 1$ it is fulfilled

$$\theta_1 c_1 + \theta_2 c_2 \in C.$$

The most important thing about cones

The separability theorem

1. Let S and T be disjoint nonempty convex sets in $\mathbb{R}^n$. Then $\exists a \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$: $\sup_{s \in S} a^\top s - b \leq \inf_{t \in T} a^\top t - b$.
2. (Strong/strict separability). Let S and T be disjoint nonempty convex sets in $\mathbb{R}^n$, with S being compact and T being closed. Then $\exists a \in \mathbb{R}^n \setminus \{0\}, b \in \mathbb{R} : \sup_{s \in S} a^\top s < b < \inf_{t \in T} a^\top t$.

Where did we see it before Optimal Control?:)

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Where did we see it before Optimal Control?:)
Consider the system of strict inequalities $Ax \prec b$, where $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$. It is undecidable if and only if $\exists \lambda \in \mathbb{R}^n \setminus \{0\}$: $\lambda^\top A = 0$, $\lambda \succeq 0$ and $\lambda^\top b \le 0$.

Basic definitions

Jensen inequality definition

Let $Q \subset U \subset \mathbb{R}^n$, and Q is *nonempty and convex*. The function $f : Q \to \mathbb{R}$ is called **convex** if for any $x, y \in Q$ and any $0 \leq \alpha \leq 1$ is satisfied

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Is the norm $f(x) = \|x\|$ **convex at least?**

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Example of convex & concave function? $f(x) = Ax + b$
Vice versa? $f(x) = x^3$
Is the norm $f(x) = \|x\|$ **convex at least?** Yes!

$$f(\alpha x+(1-\alpha)y) = ||\alpha x+(1-\alpha)y|| \leq \alpha||x||+(1-\alpha)||y|| = \alpha f(x)+(1-\alpha)f(y)$$

Practically important theorem

Convexity on set X

The function $f(x)$ is convex on a convex set $X \subseteq \mathbb{R}^n$ if and only if the function of one argument $g(\alpha) = f(x + \alpha v)$ is convex by α on the set

$$\mathcal{A}_{x,v} = \{\alpha \in \mathbb{R} \,|\, x + \alpha v \in X\}$$

for any $x \in X$ and any nonzero $v \in \mathbb{R}^n$

$\boxed{\Rightarrow} \alpha_\lambda = \lambda\alpha_1 + (1 - \lambda)\alpha_2 \in \mathcal{A}_{x,v}$. And also, because f is convex:

$$g(\alpha_\lambda) = f(x + \alpha_\lambda v) = f(\lambda(x + \alpha_1 v) + (1 - \lambda)(x + \alpha_2 v)) \leq$$
$$\leq \lambda f(x + \alpha_1 v) + (1 - \lambda)(x + \alpha_2 v) = \lambda g(\alpha_1) + (1 - \lambda)g(\alpha_2)$$

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$$\leq \lambda f(x + \alpha_1 v) + (1-\lambda)(x + \alpha_2 v) = \lambda g(\alpha_1) + (1-\lambda) g(\alpha_2)$$

$$\boxed{\Leftarrow} f(\lambda x_1 + (1-\lambda)x_2) = f(x_1 + (1-\lambda)(x_2 - x_1)) = f(x + 0 \cdot v + (1-\lambda)v)$$

$$= f(x + (\lambda \cdot 0 + (1-\lambda)\cdot 1)v) = g(\lambda \cdot 0 + (1-\lambda)\cdot 1) \leq$$

$$\leq \lambda g(0) + (1-\lambda) g(1) = \lambda f(x_1) + (1-\lambda) f(x_2).$$

Expansion to the entire space $\mathbb{R}^n$

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What's stopping us?

Expansion to the entire space $\mathbb{R}^n$

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Let $Q \subset U \subset \mathbb{R}^n$, and Q is *nonempty and convex*. The function $f : Q \to \mathbb{R}$ is called **convex** if for any $x, y \in Q$ and any $0 \leq \alpha \leq 1$ is satisfied

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

What's stopping us? f is not defined outside the convex Q.
How to solve this problem?

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How to solve this problem? Define these values of functions as $+\infty$

Effective area (Domain)

For function $f : U \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, the **effective area** is $dom\, f = \{x \in U : |f(x)| < +\infty\}$

Extended definition

Convex and concave extended-valued functions)

Let U be a real vector space. Let the function $f : U \to \mathbb{R} \cup \{+\infty\}$ is set over the entire space and takes extended real values . A function f is called convex if for any $x, y \in U$ and any $0 < \alpha < 1$

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Example. Indicator

$$\delta_Q(x) : U \to \mathbb{R} \cup \{+\infty\}, \, \delta_Q(x) = \begin{cases} 0, x \in Q \\ +\infty, x \notin Q \end{cases}$$

Differential definitions

First order

Let $dom\, f$ be an open set and the function f is differentiable everywhere on $dom\, f$. The function f is convex if and only if $dom\, f$ is a convex set and

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \tag{2}$$

for all $x, y \in dom\, f$

Concave? Geometric interpretation?

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Differential optimality condition for a convex function

Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a convex function, $dom\, f$ is an open set, and let $x^* \in dom\, f$. Then x^* is the **global minimum** of the function f if and only if $\nabla f(x^*) = 0$.

Differential definitions

2nd order convexity criterion

Let *dom f* be an open set and the function f is twice differentiable on *dom f*. The function f is convex if and only if *dom f* is a convex set and

$$\nabla^2 f(x) \succeq 0$$

for all $x \in dom\, f$.

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- $f(x) = ax_1^2 + bx_1x_2 + cx_2^2$
- $A \in \mathbb{S}^n, b \in \mathbb{R}^n,\ f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle.$
- $f(X) = \ln det(X),\ X \in S^n_{++}$

Epigraph, connection with convex set

Definition

Let U be a real vector space, and Q be a nonempty set in U. The supergraph (or epigraph) of the function $f : Q \to \mathbb{R}$ is the set

$$epi\, f = \{(x, t) \in Q \times \mathbb{R} : f(x) \leq t\}.$$

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Theorem about connection with convex set

The function $f(x)$ defined on a convex, nonempty set Q is convex on Q if and only if *epi f* is a convex set.

$$\boxed{\Rightarrow} f([x_1, x_2]) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \leq \lambda t_1 + (1-\lambda) t_2 = [t_1, t_2]$$

$$\boxed{\Leftarrow} [[x_1, x_2]], [t_1, t_2]] = \lambda [x_1, t_1] + (1-\lambda)[x_2, t_2] \in epi\, f$$

Jensen inequality

Theorem

Let $f(x)$ be a convex function on a convex set $X \subseteq \mathbb{R}^n$. Let also $x_1, \dots, x_k$ be points belonging to the set X and the coefficients $\alpha_1, \dots, \alpha_k$ are such that $\alpha_i \geq 0$ and $\sum\limits_{i=1}^{k} \alpha_i = 1$. Then the following inequality is true:

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum_{i=1}^{k} \alpha_i f(x_i), \tag{3}$$

moreover, equality is achieved if and only if the function f is affine or when all points x_i coincide.

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Generally, $f(\mathbb{E}x) \leq \mathbb{E}f(x)$

Jensen inequality. -ln x

$$\sqrt{ab} \leq (a+b)/2,$$

where $a, b \geq 0$

Basic list

1. $\boxed{+ \textit{ of epi}}$ $f_1, \ldots f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, $c_1, \ldots, c_n \in \mathbb{R}_+$. Then $f(x) = \sum\limits_{i=1}^{k} c_i f_i(x)$ is convex.
2. $\boxed{L \textit{ on epi}}$ Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$. Then $g(x) = f(Ax + b)$ is convex function with domain $dom\, g = \{x \mid Ax + b \in dom\, f\}$
   What is about intersection?

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   What is about intersection?
3. $\boxed{\cap \text{ of epi}}$ For two convex functions $f_1(x), f_1(x)$, defined respectively on the sets X_1, X_2, intersecting, we come to a convex set, which is the epigraph of the function $f(x) = max\, \{f_1(x), f_2(x)\}$. defined on $X = X_1 \cap X_2$.
4. $\boxed{\cap \text{ of epi}}$ (Danskin's theorem). Consider $g(x, y)$ is convex function by argument x for any $y \in Y$, then $f(x) = \sup_{y \in Y} g(x, y)$ is convex.

Monotonic superposition and derivable operations

6. Consider $f : \mathbb{R}^n \to \mathbb{R}$ — convex function, $h : \mathbb{R} \to \mathbb{R}$ — convex, non-decreasing function. Then $g(x) = h(f(x))$ — convex for $x \in \{x \in dom\, f | f(x) \in dom\, h\}$
7. If $g(x, y)$ is convex on $X \times Y$, Y is convex, $f(x) \neq -\infty$, then $f(x) = \inf_{y \in Y} g(x, y)$ is convex.
8. (From monotonic functions) If f is convex, then $g(x, t) = tf(\frac{x}{t})$ is convex

Examples

- Let $a, b \in \mathbb{R}^k$, $c \in \mathbb{R}^k_+$. $f(x) = \sum_{i=1}^{k} c_i \exp\left(\langle a_i, x\rangle + b_i\right)$

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- (Piecewise linear function) $f(x) = max\left\{a_1^\top x + b_1, ..., a_m^\top x + b_m\right\}$, where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$

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- Let's denote $x_{[i]}$ the i-th maximum coordinate of the vector $x \in \mathbb{R}^n$. Let's write the coordinates of the vector x in descending order: $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$. Then the function $f(x) = \sum_{i=1}^{r} x_{[i]}$, is a convex function.

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- Let $C \subseteq \mathbb{R}^n$, $||\cdot||$ be an arbitrary norm. Then the distance from the point x to the farthest point of the set C is written as follows:

$$f(x) = \sup_{y \in C} ||x - y||.$$

A bit more...

- Let M be an arbitrary (not necessarily convex) nonempty set. Then the support function of this set is

$$S_M(y) = \sup_{x \in M} \langle y, x \rangle$$

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- Let $g(x)$ be convex by $\mathbb{R}^n$. Then the function

$$f(x) = e^{g(x)}$$

is convex by $\mathbb{R}^n$.

- Let $||\cdot||$ be an arbitrary norm. Then the function

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is convex by $\mathbb{R}^n$ for ... $p \geq 1$.