Introductory lecture: examples of optimization problems, optimization methods

Optimization methods in machine learning

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Lecturer

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Grading

- Homeworks (70 %). 3 assignments for 2-3 weeks each. Deadlines are strict.
- Project (30 %). There are two types of projects in the course: 1) read 1-3 papers and implement methods from there, test their workability (check project progress every two weeks via github commits), 2) real project aimed at writing a scientific paper (call and discussion every week).
- A $\geq$ 80%, B $\geq$ 60 %, C $\geq$ 40%.

Recommended literature on the topic of the lecture

This lecture is mostly based on the book by Y.E. Nesterov

- Нестеров, Юрий Евгеньевич. «Введение в выпуклую оптимизацию» (2010). URL: `https://mipt.ru/dcam/upload/abb/nesterovfinal-arpgzk47dcy.pdf`
- Yurii Nesterov. «Introductory Lectures on Convex Optimization: A Basic Course»

ML approach

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- In practice, only a finite sample/final data set $\xi_1, \ldots, \xi_m$ is usually available, on which the problem of minimizing the empirical loss function is formulated:

$$\min_{x \in Q \subseteq \mathbb{R}^d} \left\{ \hat{f}(x) = \frac{1}{m} \sum_{i=1}^{m} f(x, \xi_i) = \frac{1}{m} \sum_{i=1}^{m} \ell(g(x, \xi_{i,data}), \xi_{i,label}) \right\}. \tag{1}$$

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- However, it is assumed that the sample came from some distribution $\mathcal{D}$ and one wants to get a good approximate solution to the problem of minimizing the expected loss function:

$$\min_{x \in Q \subseteq \mathbb{R}^d} \left\{ f(x) = \mathbb{E}_{\xi \sim \mathcal{D}}[f(x, \xi)] = \mathbb{E}_{\xi \sim \mathcal{D}}[\ell(g(x, \xi_{data}), \xi_{label})] \right\}. \tag{2}$$

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- In (1), we have data samples that can be treated any way we want, forgetting about the original distribution $\mathcal{D}$. In (2), there may be no samples. The data come online one point at a time. We need to process this data at the moment of arrival, there is no time to collect a large data set.

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ML approach

The classic results from

Shalev-Shwartz, S., Shamir, O., Srebro, N. and Sridharan, K., 2009, June. Stochastic Convex Optimization. In COLT.
https://ttic.uchicago.edu/~nati/Publications/nonlinearTR.pdf

Feldman, V. and Vondrak, J., 2019, June. High probability generalization bounds for uniformly stable algorithms with nearly optimal rate. In Conference on Learning Theory (pp. 1270-1279). PMLR.
http://proceedings.mlr.press/v99/feldman19a/feldman19a.pdf

give: if the functions $f(x, \xi)$ are convex and M-Lipschitz, Q has diameter D and $\hat{x}^* = \text{argmin}_{x \in Q} \hat{f}(x)$, then with probability at least $1 - \delta$

$$f(\hat{x}^*) - \min_{x \in Q} f(x) = O\left(\sqrt{\frac{M^2 D^2 n \ln(m) \ln(n/\delta)}{m}}\right).$$

Statistical approach: linear regression

Suppose that some variable y depends on the variables $a_2, a_3, \ldots, a_n$ in a linear manner:

$$y(a_2, \ldots, a_n) = x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n,$$

where the coefficients $x_1, \ldots, x_n$ are unknown to us. Suppose we want to find these coefficients by measuring the variable y at different values of $a_2, \ldots, a_n$. It would seem to be a problem, because it is enough to make n measurements to solve the system. **Question:** what problem?

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$$y_i = x_1 + a_2^i x_2 + a_3^i x_3 + \ldots + a_n^i x_n + \xi_i,$$

where $\xi_i \sim \mathcal{N}(0, \sigma^2)$.

Statistical approach: linear regression

- In other words, we assume that $y_i \sim \mathcal{N}(x_1 + a_2^i x_2 + a_3^i x_3 + \ldots + a_n^i x_n, \sigma^2)$, where the parameters $x = (x_1, \ldots, x_n)^\top$ are required to be found from a finite simple sample $\{y_i\}_{i=1}^m$ (we will assume that $y_1, \ldots, y_n$ are independent random variables). **Question:** How better to choose the parameters $x_1, \ldots, x_n$? And what does better mean?

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- In other words, we assume that $y_i \sim \mathcal{N}(x_1 + a^i_2 x_2 + a^i_3 x_3 + \ldots + a^i_n x_n, \sigma^2)$, where the parameters $x = (x_1, \ldots, x_n)^\top$ are required to be found from a finite simple sample $\{y_i\}_{i=1}^m$ (we will assume that $y_1, \ldots, y_n$ are independent random variables). **Question:** How better to choose the parameters $x_1, \ldots, x_n$? And what does better mean?
- We can, for example, consider the maximum likelihood estimator (for brevity we introduce the vector $a^i = (1, a^i_2, \ldots, a^i_n)^\top$):

$$
\begin{aligned}
\hat{x} &= \arg\max_{x \in \mathbb{R}^n} \prod_{i=1}^m \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \langle a^i, x\rangle)^2\right) \\
&= \arg\max_{x \in \mathbb{R}^n} \ln\left(\prod_{i=1}^m \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \langle a^i, x\rangle)^2\right)\right).
\end{aligned}
$$

Statistical approach: linear regression

- Since the logarithm of the product is the sum of logarithms and since additive and multiplicative constants do not change the point of minimum, we obtain:

$$\begin{aligned}\hat{x} &= \arg\max_{x\in\mathbb{R}^n}\left\{\text{Const} + \sum_{i=1}^{m} -\frac{1}{2\sigma^2}(y_i - \langle a^i, x\rangle)^2\right\}\\ &= \arg\min_{x\in\mathbb{R}^n}\sum_{i=1}^{m}(y_i - \langle a^i, x\rangle)^2\\ &= \arg\min_{x\in\mathbb{R}^n}\|\mathrm{A}x - y\|_2^2,\end{aligned}$$

where the matrix A is composed of rows $(a^i)^\top$.

Optimization problem

$$\min_{\substack{g_i(x)\&0,\\ i=1,\dots,m,\\ x\in Q}} f(x) \tag{3}$$

- $Q \subseteq \mathbb{R}^d$ — subset of d-dimensional space
- $f : Q \to \mathbb{R}$ — some function defined on the set Q
- Either $\leq$ or $=$ is taken as &
- $g_i(x) : Q \to \mathbb{R}$, $i = 1, \dots, m$ — constraint functions

Optimization tasks. First observations.

1. In general, optimization problems may not have a solution. For example, the problem $\min_{x \in \mathbb{R}} x$ has no solution.
2. Optimization problems often cannot be solved analytically.
3. Their complexity depends on the type of the target function f, the set Q and may depend on the dimensionality x.

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If an optimization problem has a solution, then in practice it is usually solved, generally speaking, approximately. For this purpose, special algorithms are used, which are called optimization methods.

Optimization methods I

- There is no point in looking for the best method to solve a particular problem. For example, the best method for solving the problem $\min_{x \in \mathbb{R}^d} \|x\|^2$ converges for 1 iteration: this method simply always gives the answer $x^* = 0$. Obviously, this method is not suitable for other problems.
- The efficiency of a method is determined for a class of problems, since numerical methods are usually developed for the *approximate* solution of a set of similar problems.
- The method is developed for a class of problems $\Longrightarrow$ method may not have complete information about the problem from the beginning. Instead, the method uses a model of the problem, e.g., the problem formulation, a description of the functional components, the set on which the optimization takes place, etc.

Optimization methods II

- It is assumed that the numerical method can accumulate specific information about the problem by means of some *oracle*. The oracle can be understood as some device that answers successive questions of the numerical method.

Examples of oracles

- **Zeroth order oracle** at the requested point x returns the value of the target function $f(x)$.
- **First order oracle** at the requested point returns the value of the function $f(x)$ and its gradient at that point $\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$.

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Input: initial point x^0 (0 – upper index), required accuracy of the problem solution $\varepsilon > 0$.

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4. Check the stopping criterion $\mathcal{T}_\varepsilon$. If the criterion is met, output the answer $\bar{x}$, otherwise put $k := k + 1$ and get back to the step 1.

Examples of iterative methods. Gradient descent

Let us consider the optimization problem

$$\min_{x \in \mathbb{R}^d} f(x), \tag{4}$$

where the function $f(x)$ is differentiable. Suppose that at any point we can calculate its gradient.

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Алгоритм 1 Gradient descent with constant step size

Input: step size $\gamma > 0$, initial point $x^0 \in \mathbb{R}^d$, number of iterations N

1: **for** $k = 0, 1, \ldots, N-1$ **do**

2: Compute $\nabla f(x^k)$

3: $x^{k+1} = x^k - \gamma \nabla f(x^k)$

4: **end for**

Output: x^N

Question: what are an oracle $\mathcal{O}$, a model I and a stopping criterion here?

Examples of iterative methods. Newton's method

Let us consider the optimization problem

$$\min_{x \in \mathbb{R}^d} f(x), \tag{5}$$

where the function $f(x)$ is twice continuously differentiable. Suppose that at any point we can calculate its gradient and the matrix of second derivatives of $\nabla^2 f(x)$.

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Алгоритм 2 Newton's method

Input: initial point $x^0 \in \mathbb{R}^d$, number of iterations N

1: **for** $k = 0, 1, \ldots, N-1$ **do**
2: Compute $\nabla f(x^k)$ и $\nabla^2 f(x^k)$
3: $x^{k+1} = x^k - \left(\nabla^2 f(x^k)\right)^{-1} \nabla f(x^k)$
4: **end for**

Output: x^N

Complexity of optimization methods

- **Analytic Complexity** — the number of oracle calls required to solve the problem with ε accuracy.
- **Arithmetic Complexity** — the total number of calculations (including oracle work) required to solve the problem with ε accuracy.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Consider the problem:

$$\min_{x \in B_d} f(x) \tag{6}$$

- $B_d = \{x \in \mathbb{R}^d \mid 0 \leq x_i \leq 1, \quad i = 1, \ldots, d\}$
- The function $f(x)$ is M-Lipschitz on B_d with respect to the ℓ_∞-norm:

$$\forall x, y \ |f(x) - f(y)| \leq M\|x - y\|_\infty = M \max_{i=1,\ldots,d} |x_i - y_i|. \tag{7}$$

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Observation

The set B_d is bounded and closed, i.e., a compact, and the Lipschitzness of the function f implies its continuity, the the problem (6) has a solution, because the function continuous on the compact reaches its minimum and maximum values. Let $f_* = \min_{x \in B_d} f(x)$.

- **Class of methods.** For this problem, we consider zero-order methods.
- **Goal.** Find $\bar{x} \in B_d$: $f(\bar{x}) - f_* \leq \varepsilon$.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Let us consider one of the simplest ways of solving this problem — the method of uniform enumeration.

Алгоритм 3 Uniform search method

Input: целочисленный параметр перебора $p \geq 1$

1: Сформировать $(p+1)^d$ точек вида $x_{(i_1,\ldots,i_d)} = \left(\frac{i_1}{p}, \frac{i_2}{p}, \ldots, \frac{i_d}{p}\right)^\top$, где $(i_1, \ldots, i_d) \in \{0, 1, \ldots, p\}^n$

2: Среди точек $x_{(i_1,\ldots,i_d)}$ найти точку $\bar{x}$ с наименьшим значением целевой функции f.

Output: $\bar{x}, f(\bar{x})$

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Theorem 1 (Theorem 1.1.1 from Nesterov's book)

The algorithm 3 with parameter p returns a point $\bar{x}$ such that

$$f(\bar{x}) - f_* \leq \frac{M}{2p}, \tag{8}$$

whence it follows that the uniform search method needs in the worst case

$$\left(\left\lfloor \frac{M}{2\varepsilon} \right\rfloor + 2 \right)^d \tag{9}$$

calls of the oracle to guarantee $f(\bar{x}) - f_* \leq \varepsilon$.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

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Proof of Theorem 1

Let x_* — the solution of the problem (the point of minimum of the function f). Then in the constructed «grid» of points there is a point $x_{(i_1,\ldots,i_d)}$ such that $x := x_{(i_1,\ldots,i_d)} \leq x^* \leq x_{(i_1+1,\ldots,i_d+1)} =: y$, where the «$\leq$» sign is applied component by component.

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$$\tilde{x}_i = \begin{cases} y_i, & \text{if } x_i^* \geq \hat{x}_i, \\ x_i, & \text{otherwise.} \end{cases}$$

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Proof of Theorem 1 (continued)

Note that $\tilde{x}$ belongs to «grid» and $|\tilde{x}_i - x_i^*| \leq \frac{1}{2p}$, and hence $\|\tilde{x} - x^*\|_\infty \leq \frac{1}{2p}$.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

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$$f(\bar{x}) - f_* \leq f(\tilde{x}) - f_* \leq M\|\tilde{x} - x^*\|_\infty \leq \frac{M}{2p}.$$

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The above estimate is achieved by a uniform enumeration over $(p+1)^d$ calls to the oracle. Hence, to guarantee $f(\bar{x}) - f_* \leq \varepsilon$, we need to take $p = \left\lfloor \frac{M}{2\varepsilon} \right\rfloor + 1$, i.e. the method will make $\left(\left\lfloor \frac{M}{2\varepsilon} \right\rfloor + 2 \right)^d$ calls to the oracle.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

- Suppose $M = 2$, $d = 13$ and $\varepsilon = 0.01$, that is, the dimensionality of the problem is relatively small and the accuracy of solving the problem is not too high.

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- Computational power: 10^{11} arithmetic operations per second.
- Total time: at least 10^{15} seconds, which is over 30 million years.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

- **Question:** What did we get in Theorem 1? Is it a lower or an upper bound? What are lower or an upper bounds?
- The upper bounds are guarantees of the performance of a particular algorithm on a class of problems. In Theorem 1, we obtained upper bounds for our uniform search algorithm for Lipschitz functions.
- Lower bounds are the opposite of that. When obtaining lower bounds, we pick a particular bad function from the class and show that any algorithm on that function performs no better than the lower bound.

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Theorem 2 (Theorem 1.1.2 from Nesterov's book)

Let $\varepsilon < \frac{M}{2}$. Then the analytical complexity of the described class of problems, i.e., the analytical complexity of the method on the «worst» problem from this class is at least

$$\left(\left\lfloor \frac{M}{2\varepsilon} \right\rfloor\right)^d \quad \text{oracle calls.} \tag{10}$$

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

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Scheme of the proof of Theorem 2

Let $p = \left\lfloor \frac{M}{2\varepsilon} \right\rfloor$. Prove from the contrary: suppose there exists a method that solves the problem in $N < p^d$ calls to the oracle to solve the problem with ε accuracy (by function).

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Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Scheme of the proof of Theorem 2 (continued)

Let x^* — be the center of the «cube» B, i.e., $x^* = \hat{x} + \frac{1}{2p}e$.

Complexity of optimization problems. Class of problems of minimization of Lipschitz functions

Scheme of the proof of Theorem 2 (continued)

Let x^* — be the center of the «cube» B, i.e., $x^* = \hat{x} + \frac{1}{2p}e$. Consider the function $\bar{f}(x) = \min\{0, M\|x - x^*\|_\infty - \varepsilon\}$.

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Thus, in the our class of problems any method has rather pessimistic estimates for the convergence rate. The question arises: what properties should be required from the class of optimized functions to make the estimates more optimistic?

Convex and smooth functions

Good news:

- A rich and interesting theory
- There are efficient algorithms for finding approximate solutions.

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Good news:

- A rich and interesting theory
- There are efficient algorithms for finding approximate solutions.

Bad news:

- The class of convex and smooth problems is not very broad
- In practice we often have to face non-convex problems

Convex and smooth functions

- Nevertheless, sometimes convex smooth optimization methods also perform well in practice on nonconvex or nonsmooth problems that are locally convex or smooth.

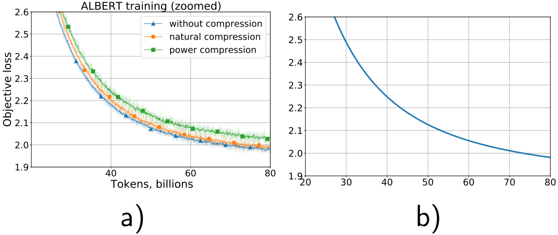

Figure: a) ALBERT (big NLP) model training: loss function, b) rate $1/k^2$ – theoretical rate of GD with momentum for smooth and convex problems

- The theory for convex and smooth problems allows us to compare methods and determine which of them is better at all/better suited for a certain class of problems.

Convex sets

Definition 1

A set $Q \subseteq \mathbb{R}^d$ is called convex if for any two points $x, y \in Q$ and for any number $\alpha \in [0, 1]$ the point $z = \alpha x + (1 - \alpha) y$ belongs to the set Q.

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This means that along with any two points, the set contains a segment connecting them.

Convex sets. Examples

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- $Q = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ — non-convex set. Let us show: take an arbitrary vector $x \in Q$. Тогда $-x \in Q$, since $\|x\| = \|-x\| = 1$. However, any nontrivial convex combination of x and $-x$ does not lie in Q: if $\alpha \in (0,1)$, then $\|\alpha x + (1-\alpha)(-x)\| = \|(2\alpha - 1)x\| = |2\alpha - 1| < 1$.

Convex functions

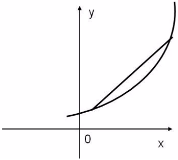
Definition 2

A function $f(x)$ defined on a **convex** set $Q \subseteq \mathbb{R}^d$ is called **convex** if for any two points $x, y \in Q$ and for any number $\alpha \in [0, 1]$ the following inequality holds:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{11}$$

(if we have sign $<$ instead of $\leq$ for all $x \neq y$, $\alpha \in (0, 1)$, then the function is **strictly convex**)

In the one-dimensional case, this means that between any two points $x, y \in Q$, the graph of the function f passes not above the segment connecting $f(x)$ and $f(y)$.



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  where K.-B. means that the transition is fair by virtue of the Cauchy-Bunyakovsky-Shwartz inequality.
- $f(x) = \langle a, x\rangle - \|x\|^2$ — non-convex function.

Strongly convex functions

Definition 3

A function $f(x)$ defined on a convex set $Q \subseteq \mathbb{R}^d$ is called μ strongly convex if for any two points $x, y \in Q$ and for any number $\alpha \in [0,1]$ the following inequality holds:

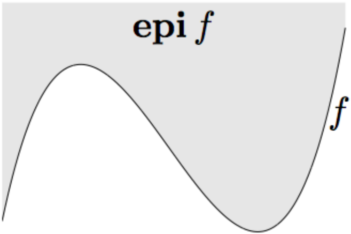
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) - \alpha(1-\alpha)\frac{\mu}{2}\|x-y\|_2^2. \quad (12)$$

Properties of convex functions I

1. Convex functions are continuous at all interior points of the domain of definition.
2. A strongly convex function is obviously strictly convex. The converse is not true.
3. A function is convex if and only if its epigraph is a convex set, where by the epigraph of a function f defined on the set $Q \subseteq \mathbb{R}^d$ we mean the following set:

$$\mathrm{epi} f = \{(x, t) \mid x \in Q,\ t \in \mathbb{R},\ t \geq f(x)\} \subseteq \mathbb{R}^{d+1}$$

Properties of convex functions II



4. If $f(x)$ is a convex function, then the set of

$$\mathrm{C}_\gamma = \{x \in Q \subseteq \mathbb{R}^d \mid f(x) \leq \gamma,\ \gamma = \text{const}\}$$

is convex.

Properties of convex functions III

5. For a convex function $f(x)$ the following is true **Jensen's inequality**

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right) \leq \sum_{i=1}^{n}\lambda_i f(x_i)$$

for all $\lambda_i \geq 0$ и $\sum\limits_{i=1}^{n}\lambda_i = 1$.
Jensen's inequality generalizes to the case of a convex combination of an infinite (countable or uncountable) number of points.

6. If $p(x) \geq 0,\ \int_{\mathrm{S}} p(x)dx = 1,\ \mathrm{S} \subseteq \mathrm{dom} f$

$$f\left(\int\limits_{\mathrm{S}} p(x)xdx\right) \leq \int\limits_{\mathrm{S}} f(x)p(x)dx.$$

Properties of convex functions IV

7. If x is a random variable and $x \in \mathrm{dom} f$ with probability 1, then

$$f(\mathbb{E}x) \leq \mathbb{E}[f(x)],$$

assuming the mathematical expectation exists.

Operations that preserve convexity I

1. Positive weighted sum

Let $f_1, ..., f_m$ be convex functions on the convex set G; $\lambda_1, ..., \lambda_m$ are non-negative numbers. Then the function

$$f(x) = \sum_{i=1}^{m} \lambda_i f_i(x) \quad \text{– convex on } \mathrm{G}.$$

Proof. Let us prove by definition. Let $x, y \in \mathrm{G}$, $\alpha \in [0, 1]$:

$$f(\alpha x + (1-\alpha)y) = \sum_{i=1}^{m} \lambda_i f_i(\alpha x + (1-\alpha)y) \leq \sum_{i=1}^{m} \lambda_i \left[\alpha f_i(x) + (1-\alpha) f_i(y)\right]$$

$$= \alpha \sum_{i=1}^{m} \lambda_i f_i(x) + (1-\alpha) \sum_{i=1}^{m} \lambda_i f_i(y) = \alpha f(x) + (1-\alpha) f(y)$$

Operations that preserve convexity II

2. Maximum of convex functions

Let $f_1, ..., f_m$ be convex functions on a convex set G. Then the function

$$f(x) = \max_{i=\overline{1,m}} f_i(x) \quad \text{– convex on } \mathrm{G}.$$

Proof. Let us prove by definition. Let $x, y \in \mathrm{G}$, $\alpha \in [0,1]$:

$$f(\alpha x + (1-\alpha)y) = \max\{\ f_1(\alpha x + (1-\alpha)y),\ f_2(\alpha x + (1-\alpha)y)\ \} \leq$$

$$\max\{\ \alpha f_1(x) + (1-\alpha) f_1(y),\ \alpha f_2(x) + (1-\alpha) f_2(y)\ \} \leq$$

$$\alpha \max\{\ f_1(x), f_2(x)\ \} + (1-\alpha)\max\{\ f_1(y), f_2(y)\ \} = \alpha f(x) + (1-\alpha) f(y)$$

Operations that preserve convexity III

3. Pointwise supremum.

If a function of two arguments $g(x, y)$ is convex on $x \in \mathbb{R}^n$ for any $y \in \mathrm{Y} \subseteq \mathbb{R}^m$, then the function

$$f(x) = \sup_{y \in \mathrm{Y}} g(x, y) \quad \text{– convex on } x.$$

Operations that preserve convexity IV

Example

Maximum eigenvalue of a symmetric matrix

$$f(X) = \lambda_{\max}(X), \ \mathrm{dom} f = \mathbb{S}^m \quad \text{– convex.}$$

Proof.

$f(X)$ can be represented as

$$f(X) = \sup \left\{ y^\top X y \mid \|y\|_2 = 1 \right\},$$

i.e. as a pointwise supremum of the family of linear functions from X. □

Operations that preserve convexity V

4. Affine argument substitution

- Let $\varphi(x)$ be a convex function on a convex set $\mathrm{G} \subseteq \mathbb{R}^m$.
- Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\mathrm{X} = \{x \in \mathbb{R}^n \mid Ax + b \in \mathrm{G}\}$, $\mathrm{X} \neq \varnothing$.
- Then $f(x) = \varphi(Ax + b)$ - convex function on X.

Proof.

We prove by definition, let $x, y \in \mathrm{X}$, $\alpha \in [0, 1]$.

$$f(\alpha x + (1-\alpha)y) = \varphi\left(A(\alpha x + (1-\alpha)y) + b\right)$$

$$\varphi\left(\alpha(Ax + b) + (1-\alpha)(Ay + b)\right) \leq \alpha\varphi(Ax + b) + (1-\alpha)\varphi(Ay + b)$$
$$= \alpha f(x) + (1-\alpha) f(y)$$

$\square$

Operations that preserve convexity VI

5. Superposition of convex functions

- Let $h(x)$ be a convex function, $P(y)$ be convex and non-decreasing. Then $f(x) = P(h(x))$ is a convex function.

Proof.

We prove by definition, let $x, y \in \mathrm{X}$, $\alpha \in [0,1]$.

$$f(\alpha x + (1-\alpha)y) = P(h(\alpha x + (1-\alpha)y)) \leq P(\alpha h(x) + (1-\alpha)h(y))$$

$$\leq \alpha P(h(x)) + (1-\alpha)P(h(y)) \leq \alpha f(x) + (1-\alpha)f(y)$$

$\square$

Operations that preserve convexity VII

Proof

Is the function

$$f(x) = \exp\left(\sum_{i=1}^{m} |a_i^\top x - b_i|\right)$$

convex?

- $g_i(x) = |a_i^\top x - b_i|$ - is convex, since the absolute value $|\cdot|$ is a convex function and the function $a_i^\top x - b_i$ is linear, that is, an affine substitution of the argument
- $h(x) = \sum_{i=1}^{m} g_i(x)$ - is convex as the sum of convex functions
- $f(x) = P(h(x))$ - is convex as a superposition of convex, non-decreasing and convex functions

Smooth optimization problems

Definition 4

A differentiable function $f(x)$ defined on the set $Q \subseteq \mathbb{R}^d$ is called L-smooth if the following inequality holds for any two points $x, y \in Q$:

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2. \tag{13}$$

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Examples

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- $f(x) = \|x\|_2^3$

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- $f(x) = \|x\|_2^3$ is not L-smooth on $\mathbb{R}^d$ for any L.
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- $f(x) = \|x\|_2^3$ is not L-smooth on $\mathbb{R}^d$ for any L.
- $f(x) = x^3$ — 12-is a smooth function on the segment $[1, 2]$, which follows from Lagrange's mean and boundedness theorem $f''(x) = 6x$ on the segment $[1, 2]$.