Conjugate (dual) sets and functions

Mathematical Optimization

Georgiy Kormakov

CMC MSU

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Definitions

Conjugate set

Let $X \subseteq \mathbb{R}^n$ be an arbitrary nonempty set. Then the set

$$X^* = \{y \in \mathbb{R}^n \mid \langle y, x\rangle \geqslant -1 \quad \forall x \in X\}$$

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1. The sets X_1 and X_2 are called mutually conjugate (взаимносопряжёнными) if $X_1^* = X_2$ and $X_2^* = X_1$
2. A set X is called self-conjugate if $X^* = X$
3. Set $X^{**} = \{x \in \mathbb{R}^n \mid \langle y, x\rangle \geqslant -1 \quad \forall y \in X^*\}$ is called *the second conjugate* to X.

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- $X^* = (clX)^*$

Example on conjugate ball. Dual norm

Definition

Let $\|\cdot\|$ be the norm in direct space. Then the *Dual norm* is:

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Q: Do you remember any properties?

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Q: Do you remember any properties? The Hoelder inequality (or also CBS)

$$|\langle x, y \rangle| \leqslant \|x\|_p \|y\|_q$$

Let $B_{\|\cdot\|}(0, r) = \{x \in \mathbb{R}^n : \|x\| \leqslant r\}$ be a closed ball with center 0 and radius r according to the norm $\|\cdot\|$. Prove that $(B_{\|\cdot\|}(0, r))^* = B_{\|\cdot\|_*}(0, 1/r)$.

$$\boxed{\subset} \text{CBS for x} : -\langle x, y\rangle \leqslant \|x\|\|y\|_* \geqslant r\|y\|_*$$

$$\boxed{\supset} \sqcap \langle x, y\rangle = \left\langle -\frac{er}{\|e\|}, y\right\rangle = \frac{r}{\|e\|}\langle -e, y\rangle < -\frac{r}{\|e\|}\frac{1}{r} \leqslant -\frac{1}{\|e\|} \leqslant -1.?!$$

Definition

Proposition

Let K be a cone in $\mathbb{R}^n$. Then

$$K^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geqslant 0 \quad \forall x \in K\}$$

Q: What is the difference?

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Q: What is the difference?
$K^* = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geqslant -1 \quad \forall x \in K\}$ VS
$\tilde{K} = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geqslant 0 \quad \forall x \in K\}$

Geometrical interpretation

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- If $cl(K)$ is a pointed cone, then K^* has non-empty interior
- If K is a convex and closed cone, then $K^{**} = K$

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$$K^* = \{Y \in \mathbb{S}^n \mid \langle X, Y \rangle \geq 0 \quad \forall X \in \mathbb{S}^n_+\}.$$

$$\mathrm{Tr}\,(YX) = \mathrm{Tr}\left(\left(Y \sum_{i=1}^n \lambda_i q_i q_i^\top\right)\right) = \sum_{i=1}^n \lambda_i q_i^\top Y q_i \geq 0.$$

Semidefinite matrix in real space

What if

$$\hat{K} = \{Y \in \mathbb{R}^{n \times n} \mid \langle X, Y \rangle \geqslant 0 \quad \forall X \in \mathbb{S}^n_+\} \ ?$$

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For $K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leqslant t\}$ the dual cone is
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$$K^* = \{(u, v) \in \mathbb{R}^{n+1} \mid \langle x, u\rangle + tv \geqslant 0 \quad \forall x : \|x\| \leqslant t\}.$$

Definition

Dual function

Let $f : \mathbb{R}^n \to \mathbb{R}$. Function $f^* : \mathbb{R}^n \to R$ defined as

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}$$

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Example

Find the conjugate function to the linear function $f(x) = \langle a, x \rangle + b$, where $x \in \mathbb{R}^n$.

Properties

1. f^* is always a convex and closed function, since it is the supremum of affine functions (they are convex and closed).
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1. f^* is always a convex and closed function, since it is the supremum of affine functions (they are convex and closed).
2. (Fenhel-Moreau theorem) If $f(x)$ is a convex, closed, eigenfunction, then $f^{**} = f$.
3. For separable function $f(x, y) = f_1(x) + f_2(y)$ and convex f_1, f_2, $f^*(p, q) = f_1^*(p) + f_2^*(q)$

One more simple example

Example

Find the conjugate function to the exponential $f(x) = e^x$, where $x \in \mathbb{R}$.

More complex example

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$$\nabla_x(x^\top y - f(x)) = 0 \Rightarrow y_i = \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}, \quad i = 1, \dots, n.$$

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$$f^*(y) = \sum_{i=1}^{n} y_i(\log y_i + f(x)) - f(x) = \sum_{i=1}^{n} y_i \log y_i,$$

Dual function for norm

Example

Let $\| \cdot \|$ be the norm for $\mathbb{R}^n$, and $\| \cdot \|_*$ be the dual norm. Prove that the conjugate function for $f(x) = \|x\|$ is $f^*(y) = 0$ with the domain of definition $dom\, f^* = B_{\|\cdot\|_*}(0, 1)$.

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$\boxed{\|y\|_* > 1 :}$

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