# From LP to SDP. Simplex method

## Mathematical Optimization

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# General VS Standart

## General form

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \; & c^\top x, \\
\text{s.t.} \; & Ax = b, \\
& Gx \preceq h,
\end{aligned}
\tag{GF}
$$

$A \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{k \times n}$.

# General VS Standart

## General form

$$\min_{x\in\mathbb{R}^n} c^\top x, \quad \text{s.t. } Ax = b, \quad Gx \le h, \tag{GF}$$

$A \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{k\times n}$.

## Standart form

$$\min_{x\in\mathbb{R}^n} c^\top x, \quad \text{s.t. } Ax = b, \quad x \ge 0, \tag{SF}$$

$A \in \mathbb{R}^{m\times n}$.

General $\Rightarrow$ standart?

# Basic form

## Basic form

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^\top x, \\ &\text{s.t. } Ax \preceq b, \end{aligned} \tag{BF}$$

$c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

# Basic form

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\end{aligned}
\tag{BF}
$$

$c \in \mathbb{R}^n, b \in \mathbb{R}^m, \; A \in \mathbb{R}^{m \times n}.$

For what?

## Canonical form

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \; & c^\top x, \\
\text{s.t.} \;\; & Ax \preceq b, \\
& x \succeq 0
\end{aligned}
\tag{CF}
$$

# Duality of SF and BF

## Standart form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & c^\top x, \\ \text{s.t. } \ & Ax = b, \\ & x \geq 0, \end{aligned} \tag{SF}$$

# Duality of SF and BF

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&\text{s.t. } Ax = b, \\
&\quad\quad x \geq 0,
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\tag{SF}
$$

$$
\mathcal{L}(x, \nu, \lambda) = c^\top x + \nu^\top (Ax - b) + \lambda^\top (-x)
$$

# Duality of SF and BF

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$$g(\nu,\lambda) = \inf_x \mathcal{L}(x,\nu,\lambda) = \inf_x (c + A^\top \nu - \lambda)^\top x - \nu^\top b$$

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$$\min_{x \in \mathbb{R}^n} c^\top x, \quad \text{s.t. } Ax = b, \quad x \succeq 0, \qquad \text{(SF)}$$

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$$\max_{\nu, \lambda} -b^\top \nu, \quad \text{s.t. } c + A^\top \nu = \lambda, \quad \lambda \succeq 0.$$

# Duality of SF and BF

**Standart form**

$$\begin{aligned} \min_{x\in\mathbb{R}^n} c^\top x, \\ \text{s.t. } Ax = b, \\ x \geq 0, \end{aligned} \qquad \text{(SF)}$$

$$\mathcal{L}(x,\nu,\lambda) = c^\top x + \nu^\top (Ax - b) + \lambda^\top (-x)$$

$$g(\nu,\lambda) = \inf_x \mathcal{L}(x,\nu,\lambda) = \inf_x (c + A^\top \nu - \lambda)^\top x - \nu^\top b$$

$$\begin{aligned} \max_{\nu,\lambda} - b^\top \nu, \\ \text{s.t. } c + A^\top \nu = \lambda \\ \lambda \geq 0. \end{aligned}$$

**Basic form**

$$\begin{aligned} \max_{\nu} - b^\top \nu, \\ \text{s.t. } c + A^\top \nu \geq 0. \end{aligned}$$

# Strong duality theorem

## Theorem (Strong duality)

- Direct (SF) or dual (BF) task has a (finite) solution $\Rightarrow$ the other task has the same solution (**strong duality is performed**).
- Direct (SF) or dual (BF) task is unlimited $\Rightarrow$ the other task has no solution.

# Fractional LP

## FLP

$$\min_{x \in \mathbb{R}^n} \left[ f(x) := \frac{c^\top x + d}{e^\top x + f} \right], \; dom\ f = \left\{ x | e^T x + f > 0 \right\}.$$
$$\text{s.t. } \; Ax = b,$$
$$Gx \preceq h,$$

# Economic activity

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$$
\begin{aligned}
\max_{\{s(t)\}_{t=0}^{T} \subseteq \mathbb{R}^m, \{x(t)\}_{t=1}^{T} \subseteq \mathbb{R}^n} & \sum_{t=0}^{T} c^\top \gamma^t s(t), \\
\text{s.t. } & s(t) = Ax(t) - Bx(t+1), t = \overline{0, T-1}, \\
& s(T) = Ax(T), \\
& s(t) \geqslant 0, t = \overline{0, T}, \\
& x(t) \geqslant 0, t = \overline{1, T}.
\end{aligned}
$$

# Simplex method. Idea

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## Basic form

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^\top x, \\ &\text{s.t. } Ax \preceq b, \end{aligned} \tag{BF}$$

## Angular point

An *angular point* is a point from a feasible set lying on the boundary of $n$ linearly independent (l.i.) constraints.

# Simplex method. Idea

## Basic form

$$\min_{x \in \mathbb{R}^n} c^\top x, \quad \text{s.t. } Ax \preceq b, \tag{BF}$$

## Angular point

An *angular point* is a point from a feasible set lying on the boundary of $n$ linearly independent (l.i.) constraints.

## Basis

*The basis* of $B$ is a set of indices of $n$ l.i. vectors (constraints) from the matrix $A$ that define an angular point.

# Idea. Preparing an algorithm

## Feasible basis

A basis $B$ is called the *feasible basis* if the resulting angular point $x_B$ lies in a feasible set, i.e. $Ax_B \leqslant b$.

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A basis $B$ is called the *feasible basis* if the resulting angular point $x_B$ lies in a feasible set, i.e. $Ax_B \leqslant b$.

## Optimal basis

The basis $B$ is called the *optimal basis* if the resulting angular point is a solution to the LP problem, i.e. $c^\top x_B \leqslant c^\top x, \quad \forall x \in S$, where $S$ is a feasible set.

# Optimality condition

## Lemma

For a feasible basis $B$, we can decompose the vector $c$ according to this basis, and also find the scalar coefficients $\lambda_B$:

$$c^\top = \lambda_B^\top A_B \quad \Leftrightarrow \quad \lambda_B^\top = c^\top A_B^{-1},$$

Then the $B$ basis is optimal if $\lambda_B \leq 0$.

# Step

# Algorithm

1. Select a feasible basis $B_k \Rightarrow x_k = A_{B_k}^{-1} b_{B_k}$.
2. Decompose the vector $c$ into the selected basis $B_k$: $c = A_{B_k}^\top \lambda_{B_k}$.
3. Check the optimality of the basis.
   - If $\lambda_{B_k} \preceq 0$ (where $\lambda_{B_k} = A_{B_k}^{-\top} c$), then the algorithm terminates, $x_k$ is the solution
   - Else, we change the basis
4. Replace the basis: $x_{k+1} = x_k + \mu_k d_k$. Go back to Step 2.

# Quadratic Programming. Example

**QP**

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \ & \frac{1}{2} x^\top A x + b^\top x \\
\text{s.t.} \ & Cx = b, \\
& Gx \leqslant h,
\end{aligned}
$$

where
$A \in \mathbb{S}^n_+$, $C \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{k \times n}$.

# More examples

## Sphere points restoration

Consider $x_i = \overline{x}_i + v$, where $\overline{x}_i \in S^n_r(x_c)$, $v \sim \mathcal{N}(0, \varepsilon^2)$.

$$\min_{x_c \in \mathbb{R}^n, r \in \mathbb{R}_+} \sum_{i=1}^{k} \left(\|x_i - x_c\|^2 - r^2\right)^2.$$

# Sphere restoration. Optimallity

$$A = \begin{pmatrix} 2x_1^\top & (-1) \\ \cdots & \cdots \\ 2x_n^\top & (-1) \end{pmatrix}; b = \begin{bmatrix} \|x_1\|^2 \\ \cdots \\ \|x_n\|^2 \end{bmatrix}$$

# Quadratic Constraints Quadratic Programming

## QCQP

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} \frac{1}{2} x^\top A_0 x + b_0^\top x + c_0 \\
&\text{s.t. } \frac{1}{2} x^\top A_i x + b_i^\top x + c_i \leqslant 0, i = \overline{1, n} \\
&\qquad \frac{1}{2} x^\top A_j x + b_j^\top x + c_j = 0, j = \overline{n+1, N}
\end{aligned}
$$

where $A_i \in \mathbb{S}^n, i = \overline{1, N}$,

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where $A_i \in \mathbb{S}^n, i = \overline{1, N}$,

## Convex QCQP

1. $A_i \succeq 0, i = \overline{0, n}$
2. $A_j = 0, i = \overline{n+1, N}$

# Second Order Conic Programming

## SOCP

$$\min_{x \in \mathbb{R}^n} c^\top x$$

$$\text{s.t. } Ax = b$$

$$\|G_i x - h_i\|_2 \leqslant e_i^\top x + f_i, i = \overline{1, M}$$

$$(or \ \|G_i x - h_i\|_2 \leqslant t).$$

where $A \in \mathbb{R}^{m \times n}$, $G_i \in \mathbb{R}^{k_i \times n}$, $i = \overline{1, M}$.

# SOCP examples

# Robust LinReg

## Problem statement

$$\min_{x \in \mathbb{R}^n} \left[ f(x) = \sup_{(A,b) \in \mathcal{A}} \|Ax - b\|_2 \right],$$

where $\mathcal{A} =$
$$\left\{ (A, b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \middle| \left\| \begin{pmatrix} A - A_0 \\ b - b_0 \end{pmatrix} \right\|_F \leqslant \alpha \right\}.$$

# Robust LinReg SOCP

# Robust LinReg SOCP

$$\min_{x \in \mathbb{R}^n} \|A_0 x - b_0\| + \alpha \left\| \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|_2,$$

# Semi-Defined Programming

## SDP

$$
\begin{aligned}
\min_{X \in \mathbb{S}^n} \ & \operatorname{tr}(CX) \\
\text{s.t.} \ & \operatorname{tr}(A_i X) = b_i, \, i = \overline{1, m} \\
& X \succeq 0.
\end{aligned}
$$

# Spectral Radius Minimisation

## Statement

Consider

$A(x) := A_0 + \sum_{i=1}^{n} A_i x_i$, with

$A_j \in \mathbb{R}^{m \times n}, j = \overline{1, n}.$

$$\min_{x \in \mathbb{R}^n} \|A(x)\|_2.$$

# Schur complement

$$\begin{pmatrix} sI_n & A(x)^\top \\ A(x) & sI_m \end{pmatrix} \succeq 0,$$

## Shur Complement

For a block matrix $M \in \mathbb{R}^{(p+q)\times(p+q)}$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the Shur complement is

$$(M/D) = A - BD^{-1}C \in \mathbb{R}^{p\times p}$$

# Final SRM statement

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$$\begin{aligned} \min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \quad & t \\ \text{s.t.} \quad & \begin{pmatrix} sI_n & A(x)^\top \\ A(x) & sI_m \end{pmatrix} \succeq 0. \end{aligned}$$

# Non-Convex QCQP to SDP

## QCQP

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \; & \frac{1}{2} x^\top A_0 x + b_0^\top x + c_0 \\
\text{s.t.} \; & \frac{1}{2} x^\top A_i x + b_i^\top x + c_i \leqslant 0, \\
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& \frac{1}{2} x^\top A_j x + b_j^\top x + c_j = 0
\end{aligned}
$$

## Equivalent

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n, X \in \mathbb{S}^n} \ & \frac{1}{2} \text{tr}\left(A_i X\right) + b_0^\top x + c_0, \\
\text{s.t.} \ & \frac{1}{2} \text{tr}\left(A_i X\right) + b_i^\top x + c_i \leqslant 0 \\
& \frac{1}{2} \text{tr}\left(A_j X\right) + b_j^\top x + c_j = 0, \\
& X = x x^\top.
\end{aligned}
$$

# The Greatest Shur again...

## Equivalent

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \ & \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_0^\top x + c_0, \\
\text{s.t.} \ & \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_i^\top x + c_i \leqslant 0 \\
& \frac{1}{2}\mathrm{tr}\left(A_j X\right) + b_j^\top x + c_j = 0, \\
& X = xx^\top.
\end{aligned}
$$