SDP + Cones

Mathematical Optimization

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Second Order Conic Programming

SOCP

$$\min_{x \in \mathbb{R}^n} c^\top x$$

$$\text{s.t.} \quad Ax = b$$

$$\|G_i x - h_i\|_2 \leqslant e_i^\top x + f_i, \, i = \overline{1, M}$$

$$(or \ \|G_i x - h_i\|_2 \leqslant t).$$

where $A \in \mathbb{R}^{m \times n}$, $G_i \in \mathbb{R}^{k_i \times n}$, $i = \overline{1, M}$.

Semi-Defined Programming

SDP

$$
\begin{aligned}
\min_{X \in \mathbb{S}^n} \ & \operatorname{tr}(CX) \\
\text{s.t.} \ & \operatorname{tr}(A_i X) = b_i, \ i = \overline{1, m} \\
& X \geq 0.
\end{aligned}
$$

Spectral Radius Minimisation

Statement

Consider

$$A(x) := A_0 + \sum_{i=1}^{n} A_i \cancel{x_i}, \text{ with}$$

$$A_j \in \mathbb{R}^{m\times n}, j = \overline{1, \cancel{n}}$$

$$\min_{x\in\mathbb{R}^n} \|A(x)\|_2.$$

$\mathbb{R}^{m\times n}$

$$\min s$$

$$\|A(x)\|_2 \le s$$

$$\sqrt{\lambda_{max}(A^T A)} \le s^2$$

$$s \succeq 0$$

$$\begin{cases} s^2 I - A^T A \succeq 0 \\ s \succeq 0 \end{cases}$$

Schur complement

SOCP $\subset$ SDP

$s^2 I - A^T A \succeq 0$

$\frac{1}{s}$

$s \geq 0$

$$\begin{pmatrix} sI_n & A(x)^\top \\ A(x) & sI_m \end{pmatrix} \succeq 0,$$

$$\begin{pmatrix} I & A^T \\ A & s^2 I \end{pmatrix}$$

Shur Complement

For a block matrix $M \in \mathbb{R}^{(p+q)\times(p+q)}$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\cancel{AD - BC \succeq 0}$

the Shur complement is

$$(M/D) = A - BD^{-1}C \in \mathbb{R}^{p\times p}$$

$$M = \begin{pmatrix} A & C^T \\ C & B \end{pmatrix}$$

$$M \succeq 0 \iff \begin{cases} A \succeq 0 \\ B - C^T A^{-1} C \succeq 0 \end{cases}$$

Final SRM statement

$\min S$

$\cancel{S \succeq 0}$

$M \succeq 0$

Final SRM statement

$$\min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \quad t$$
$$\text{s.t.} \quad \begin{pmatrix} sI_n & A(x)^\top \\ A(x) & sI_m \end{pmatrix} \succeq 0.$$

Non-Convex QCQP to SDP

$\mathrm{Tr}(A_i X X^T)$

QCQP

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \ & \frac{1}{2}x^\top A_0 x + b_0^\top x + c_0 \\
\text{s.t.} \ & \frac{1}{2}x^\top A_i x + b_i^\top x + c_i \leqslant 0, \\
& \frac{1}{2}x^\top A_j x + b_j^\top x + c_j = 0
\end{aligned}
$$

Non-Convex QCQP to SDP

QCQP

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \ & \frac{1}{2}x^\top A_0 x + b_0^\top x + c_0 \\
\text{s.t.} \ & \frac{1}{2}x^\top A_i x + b_i^\top x + c_i \leqslant 0, \\
& \frac{1}{2}x^\top A_j x + b_j^\top x + c_j = 0
\end{aligned}
$$

Equivalent

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \ & \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_0^\top x + c_0, \\
\text{s.t.} \ & \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_i^\top x + c_i \leqslant 0 \\
& \frac{1}{2}\mathrm{tr}\left(A_j X\right) + b_j^\top x + c_j = 0, \\
& X = xx^\top.
\end{aligned}
$$

The Greatest Shur again...

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Equivalent

$$\min_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_0^\top x + c_0,$$

$$\text{s.t.} \quad \frac{1}{2}\mathrm{tr}\left(A_i X\right) + b_i^\top x + c_i \leqslant 0$$

$$\frac{1}{2}\mathrm{tr}\left(A_j X\right) + b_j^\top x + c_j = 0,$$

$$X = xx^\top.$$

$X - x I x^T \succeq 0$

$$\begin{pmatrix} I & x^T \\ x & X \end{pmatrix} = M$$

$I \succ 0$

$X - x I x^T \succeq 0$

$x \in \{-1, 1\} \Rightarrow x^2 - 1 = 0$

$\{0, 1\} \Rightarrow x^2 - x = 0$

Exponential Cone

$x \in K$

K_{exp}

$$K_{exp} = \left\{ (x_1, x_2, x_3) \middle| \; x_1 \geqslant x_2 e^{\frac{x_3}{x_2}}, \; x_2 > 0 \right\}$$

$\min t$

y

$\log\left(\sum e^{x_i}\right) \leq t$

$\log(e^t)$

$\sum e^{x_i - t} \leq 1$

u

e^{-t}

$(u, 1, x_i - t) \in K_{exp}$

Exponential Cone. Application

$\log(1+e^x) \le t$

$e^x + 1 \le e^t$

$\underbrace{e^{x-t}}_{v} + \underbrace{e^{-t}}_{u} \le 1$

$v + u \le 1$

$(u, 1, x-t) \in K_{exp}$

$(v, 1, -t) \in K_{exp}$

$W(x)e^{W(x)} = x$

$W(x) \ge t$

$x \ge te^t$

$x \ge te^{t^2}$

Power Cone

$\mathcal{P}_n^{\alpha,1-\alpha}$

$$\mathcal{P}_n^{\alpha,1-\alpha} = \left\{ x \in \mathbb{R}^n \mid x_1, x_2 \geq 0,\ x_1^{\alpha} x_2^{1-\alpha} \geq \sqrt{\sum_{i=3}^{n} x_i},\ \alpha \in (0,1) \right\}$$

$$\min c^T x - \sum \delta_i |x_i|^{\beta}, \quad \beta > 1$$

$$(t_i, 1, x_i) \quad |x_i|^{\beta} \leq t_i \quad 1/\beta$$

$$|x_i| \leq t_i^{1/\beta}$$

Quasiconvex Programming

QCP

$$\max_{x_i^+, x_i} \min_{i=\overline{1,n}} \frac{x_i^+}{x_i} = t \Rightarrow t x_i \geqslant x_i^+$$

$$s.t.\ x_i^+, x_i \geqslant 0$$

$$1^\top x_i = 1$$

$$B x_i^+ \leqslant A x_i$$