Supplemental Material for: "Wirtinger Flow Meets Constant Modulus Algorithm: Revisiting Signal Recovery for Grant-Free Access"

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Abstract

This document presents the proofs for Lemmas 5, 6, 7 and 8 stated in the paper "Wirtinger Flow Meets Constant Modulus Algorithm: Revisiting Signal Recovery for Grant-Free Access", which in turn establish Theorem 2, demonstrating the local convergence guarantees for multiple source recovery via constant modulus algorithm with limited samples.

I. Definitions and Expectations

From the main text, we recall the expressions for the cost function, gradient and Hessian in the case of multiple source recovery (MSR). Using the overall system parameter space, the cost function for MSR is

$$g(\boldsymbol{q}) = \sum_{j=1}^{J} f(\boldsymbol{q}_j) + \gamma_0 \sum_{j_1 \neq j_2}^{J} \left| \boldsymbol{q}_{j_1} \boldsymbol{S} \boldsymbol{q}_{j_2} \right|^2 \tag{1}$$

where $\boldsymbol{q} = [\boldsymbol{q}_1^{\mathsf{T}} \dots \boldsymbol{q}_J^{\mathsf{T}}]^{\mathsf{T}}$ is the aggregation of the J demixers, f is the CMA cost function for single source recovery, and $\boldsymbol{S}$ is the sample covariance matrix of transmitted signals as defined in the main text. The gradient of g, using Wirtinger calculus, is given by:

$$\nabla_j g = \frac{1}{K} \sum_{k=1}^{K} \Big(|\boldsymbol{s}_k^{\mathsf{H}} \boldsymbol{q}_j|^2 - R_2 \Big) \boldsymbol{s}_k \boldsymbol{s}_k^{\mathsf{H}} \boldsymbol{q}_j + 2\gamma_0 \sum_{i \neq j}^{J} \boldsymbol{S} \boldsymbol{q}_i \boldsymbol{q}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{q}_j, \tag{2}$$

and the Wirtinger Hessian of the cost function g is

$$\nabla^2 g(\boldsymbol{q}) = \mathrm{Bdiag}\Big(\big\{ \nabla^2 f(\boldsymbol{q}_j) \big\}_{j=1}^{J} \Big) + 2\gamma_0 \begin{bmatrix} \boldsymbol{G}_1(\boldsymbol{q}) & \cdots & \boldsymbol{H}_{1J}(\boldsymbol{q}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{H}_{J1}(\boldsymbol{q}) & \cdots & \boldsymbol{G}_J(\boldsymbol{q}) \end{bmatrix} \tag{3}$$

where $\mathrm{Bdiag}()$ constructs a block diagonal matrix out of the matrices

$$\nabla^2 f(\boldsymbol{q}_j) = \begin{bmatrix} 2\boldsymbol{A}(\boldsymbol{q}_j) - R_2 \boldsymbol{S} & \boldsymbol{B}(\boldsymbol{q}_j) \\ \overline{\boldsymbol{B}(\boldsymbol{q}_j)} & 2\overline{\boldsymbol{A}(\boldsymbol{q}_j)} - R_2 \overline{\boldsymbol{S}} \end{bmatrix}. \tag{4}$$

$\boldsymbol{A}(\boldsymbol{q}_j)$, $\boldsymbol{B}(\boldsymbol{q}_j)$, and $\boldsymbol{S}$ have been defined in the main text, and

$$\boldsymbol{G}_j(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{C}_j(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{C}_j(\boldsymbol{q})} \end{bmatrix}, \quad \boldsymbol{C}_j(\boldsymbol{q}) = \sum_{i \neq j}^{J} \boldsymbol{S} \boldsymbol{q}_i \boldsymbol{q}_i^{\mathsf{H}} \boldsymbol{S}. \tag{5}$$

Furthermore, we have

$$\boldsymbol{H}_{ji}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{E}_{ji}(\boldsymbol{q}) & \boldsymbol{F}_{ji}(\boldsymbol{q}) \\ \overline{\boldsymbol{F}_{ji}(\boldsymbol{q})} & \overline{\boldsymbol{E}_{ji}(\boldsymbol{q})} \end{bmatrix}, \quad \boldsymbol{E}_{ji}(\boldsymbol{q}) = \boldsymbol{q}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{q}_j \boldsymbol{S}, \quad \boldsymbol{F}_{ji}(\boldsymbol{q}) = \boldsymbol{S} \boldsymbol{q}_i \boldsymbol{q}_j^{\mathsf{T}} \boldsymbol{S}^{T}. \tag{6}$$

The expectation of matrices $\boldsymbol{A}(\boldsymbol{q})$ and $\boldsymbol{B}(\boldsymbol{q})$ are:

$$\mathbb{E}\{\boldsymbol{A}(\boldsymbol{q})\} = m_2^2 (\|\boldsymbol{q}\|^2 \boldsymbol{I} + \boldsymbol{q}\boldsymbol{q}^{\mathsf{H}}) + \kappa \,\mathrm{ddiag}(\boldsymbol{q}\boldsymbol{q}^{\mathsf{H}}),$$
$$\mathbb{E}\{\boldsymbol{B}(\boldsymbol{q})\} = 2m_2^2 \boldsymbol{q}\boldsymbol{q}^{\mathsf{T}} + \kappa \,\mathrm{ddiag}(\boldsymbol{q}\boldsymbol{q}^{\mathsf{T}}),$$

whereas matrices $\boldsymbol{C}_j(\boldsymbol{q})$, $\boldsymbol{E}_{ji}(\boldsymbol{q})$ and $\boldsymbol{F}_{ji}(\boldsymbol{q})$ of the MSR Hessian satisfy

$$\mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{q})\} = \sum_{i\neq j}^{J} \big(m_2^2 \boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}} + \frac{m_2^2}{K}\|\boldsymbol{q}_i\|^2\boldsymbol{I} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}})\big),$$

$$\mathbb{E}\{\boldsymbol{E}_{ji}(\boldsymbol{q})\} = m_2^2(\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{q}_j)\boldsymbol{I} + \frac{m_2^2}{K}\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{H}} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{H}}),$$

$$\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{q})\} = m_2^2\boldsymbol{q}_i\boldsymbol{q}_j^{\mathsf{T}} + \frac{m_2^2}{K}\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{T}} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{T}}).$$

Finally, we define corollaries that will be useful to prove Lemmas 6 and 7:

Corollary 13. *Suppose $K \geq C_2(\delta)L$. Then, with probability at least $1 - 12\mathrm{e}^{-c_2(\delta)K}$, for all $\boldsymbol{v} \in \mathbb{C}^L$ such that $\|\boldsymbol{v}\| = 1$, we have*

$$(J-1)\Big(m_2^2\frac{K+1}{K} + \delta\Big)\|\boldsymbol{v}\|^2 \geq \boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{v} \geq (J-1)\Big(\frac{m_2^2+\kappa}{K} - \delta\Big)\|\boldsymbol{v}\|^2,\ \forall i \in \{1,\dots,J\}.$$

Proof. From Lemma 5, we have that $-\delta\boldsymbol{I} \preceq \boldsymbol{C}_j(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{z})\} \preceq \delta\boldsymbol{I}$.With $\kappa < 0$ and dropping the summation in i, we obtain the result. ■

II. Proof of Lemma 5

Using the triangle inequality and Eq.(3), to prove the lemma we show that $\forall j \in \{1,\dots,J\}$ and $\forall i \neq j$,

$$\Big\|\nabla^2 f(\boldsymbol{z}_j) - \mathbb{E}\big\{\nabla^2 f(\boldsymbol{z}_j)\big\}\Big\| \leq \delta_f = \frac{\delta}{2J}, \tag{7}$$

$$\|\boldsymbol{C}_j(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{z})\}\| \leq \delta_C = \frac{\delta}{8\gamma_0 J}. \tag{8}$$

$$\|\boldsymbol{E}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{E}_{ji}(\boldsymbol{z})\}\| \leq \delta_E = \frac{\delta}{8\gamma_0 J(J-1)}, \tag{9}$$

$$\|\boldsymbol{F}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}\| \leq \delta_F = \frac{\delta}{8\gamma_0 J(J-1)}. \tag{10}$$

Eq.(7) corresponds to the concentration inequality of J Hessians of the single source recovery cost function. Thus, Lemma 1 states that Eq.(7) holds with probability at least $1 - 6\mathrm{e}^{-c_1(\delta_f)K}$ by choosing $K \geq C_1(\delta_f)L$.

Let $\boldsymbol{c}_{i,k} = \boldsymbol{s}_k(\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i)$, which are independent for $k \in \{1,\dots,K\}$, and in particular, $\boldsymbol{c}_{i,k}$ is independent of $\boldsymbol{c}_{i,m}$ for $m \neq k$. Note that $\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i = \sqrt{m_2}\,\mathrm{e}^{i\varphi}\overline{s_{\ell_i}[k]}$, and therefore the vectors $\boldsymbol{c}_{i,k}$ have bounded, discrete elements over an exponentially large set, and as such they are subgaussian [1]. Moreover, the sum of these vectors over index i is also subgaussian. Thus, we have

$$\boldsymbol{C}_j(\boldsymbol{z}) = \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{i,m}^{\mathsf{H}} = \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{i,k}^{\mathsf{H}} + \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m\neq k}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{i,m}^{\mathsf{H}}, \tag{11}$$

and invoking Lemma 9 for each $i \neq j$ and results of concentration of quadratic forms [1, Chapter 6], Eq.(8) holds with probability at least $1 - 2\mathrm{e}^{-c_6(\delta_C)K}$ by choosing $K \geq C_6(\delta_C)L$.

In a similar fashion, note that

$$\boldsymbol{F}_{ji}(\boldsymbol{z}) = \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{j,m}^{\mathsf{T}} = \frac{1}{K^2}\sum_{k=1}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{j,k}^{\mathsf{T}} + \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m\neq k}^{K}\boldsymbol{c}_{i,k}\boldsymbol{c}_{j,m}^{\mathsf{T}}, \tag{12}$$

where we leverage the reasoning of the previous result, the fact that $\boldsymbol{c}_{i,k}$ is independent of $\boldsymbol{c}_{j,m}$ for $m \neq k$ and $i \neq j$, and the concentration of measure of $\boldsymbol{U}(\boldsymbol{z})$ in Lemma 1 of the main text (for the transposition instead of conjugate transpose). Hence, Eq.(10) holds with probability $1 - 2\mathrm{e}^{-c_7(\delta_F)K}$ by choosing $K \geq C_7(\delta_F)L$.

Now define $\boldsymbol{e}_{j,k,m} = (\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_j)\boldsymbol{s}_m$. The vectors $\boldsymbol{e}_{j,k,k}$ are independent for $k \in \{1,\dots,K\}$, and $\boldsymbol{e}_{j,k,m}$ is independent of $\boldsymbol{e}_{i,k,m}$ for $m \neq k$ and $i \neq j$. Following a similar reasoning as above, the $\boldsymbol{e}_{j,k,m}$ are subgaussian, and we obtain

$$\boldsymbol{E}_{ji}(\boldsymbol{z}) = \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K}\boldsymbol{e}_{j,k,m}\boldsymbol{e}_{i,k,m}^{\mathsf{H}} = \frac{1}{K^2}\sum_{k=1}^{K}\boldsymbol{e}_{j,k,k}\boldsymbol{e}_{i,k,k}^{\mathsf{H}} + \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m\neq k}^{K}\boldsymbol{e}_{j,k,m}\boldsymbol{e}_{i,k,m}^{\mathsf{H}}, \tag{13}$$

thus Eq.(9) holds with probability $1 - 2\mathrm{e}^{-c_8(\delta_E)K}$ by choosing $K \geq C_8(\delta_E)L$.

Finally, set $C_2(\delta) \geq \max\{C_5(\delta_f), C_6(\delta_C), C_7(\delta_F), C_8(\delta_E)\}$. By selecting $K \geq C_2(\delta)L$, Lemma 5 holds with probability at least $1 - 12\mathrm{e}^{-c_2(\delta)K}$, where we define $c_2(\delta) = \min\{c_5(\delta_f), c_6(\delta_C), c_7(\delta_F), c_8(\delta_E)\}$.

III. Proof of Lemma 6

Let $\boldsymbol{T}$ be a matrix that randomly rotates each of the combiners, that is,

$$\boldsymbol{T} = \mathrm{Bdiag}\Big(\big\{\mathrm{e}^{i\theta_j}\boldsymbol{I}_L\big\}_{j=1}^{J}\Big), \quad \theta_j \in [0, 2\pi] \ \forall j \in \{1, \ldots, J\}, \tag{14}$$

and let

$$\hat{\boldsymbol{T}} = \mathrm{Bdiag}\Big(\big\{\mathrm{e}^{i\phi_j(\boldsymbol{q})}\boldsymbol{I}_L\big\}_{j=1}^{J}\Big) \tag{15}$$

be the matrix that collects the optimal rotations of the combiners $\boldsymbol{q}_j$ with their respective ground truths $\boldsymbol{z}_j$. Let $\boldsymbol{q} \in E(\epsilon)$, and $\boldsymbol{h} = \boldsymbol{T}\boldsymbol{q} - \boldsymbol{z}$. Hence, $\|\boldsymbol{h}\| \leq \epsilon$, $\boldsymbol{h}_j = \mathrm{e}^{-i\phi(\boldsymbol{q}_j)}\boldsymbol{q}_j - \boldsymbol{z}_j$, and $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$, as $\boldsymbol{h}_j$ and $\boldsymbol{z}_j$ are geometrically aligned.

Recall that $\nabla f(\mathrm{e}^{i\theta}\boldsymbol{z}_j) \approx 0$ for any $\theta \in [0, 2\pi]$ and $\forall j \in \{1, \ldots, J\}$, and the same principle applies to the MSR cost function, i.e. $g(\boldsymbol{T}\boldsymbol{z}) \approx 0$ for any $\boldsymbol{T}$ as defined in Eq.(14), and in particular for $\hat{\boldsymbol{T}}$. Thus, the proof is equivalent to proving

$$\begin{aligned}
&\mathrm{Re}\Big(\big\langle \nabla_j g(\boldsymbol{q}) - \nabla_j g(\hat{\boldsymbol{T}}\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad = \mathrm{Re}\Big(\big\langle \nabla f(\boldsymbol{q}_j) - \nabla_j f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) + 2\gamma_0 \sum_{i\neq j}^{J} \mathrm{Re}\Big(\big\langle \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j, \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad \geq \frac{1}{\alpha}\|\boldsymbol{h}_j\|^2 + \Big(\frac{2m_2^2 - R_2 m_2 + \delta}{4}\Big)\|\boldsymbol{h}\|^2 + \frac{1}{10K}\sum_{k=1}^{K}\sum_{j=1}^{J}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j\big|^4
\end{aligned}$$

for all $\boldsymbol{h}$ satisfying $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}\| \leq \epsilon$. In the following, we set $\gamma_0 = 1/2$, and for simplicity, we now assume that $\|\boldsymbol{h}_j\| \leq \epsilon/\sqrt{J}$ for all $j \in \{1, \ldots, J\}$. Therefore, thanks to Lemma 2, we know that for $\alpha \geq 30$ and $\epsilon_j = (2B\sqrt{L})^{-1}$, we have

$$\mathrm{Re}\Big(\big\langle \nabla f(\boldsymbol{q}_j) - \nabla_j f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \geq \Big(\frac{1}{\alpha} + \frac{2m_2^2 - R_2 m_2 + \delta}{4}\Big)\|\boldsymbol{h}_j\|^2 + \frac{1}{10K}\sum_{k=1}^{K}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j\big|^4,$$

so it suffices to show that for all $\boldsymbol{h}$ satisfying $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}_j\| \leq \epsilon/\sqrt{J}$,

$$\begin{aligned}
&\sum_{i\neq j}^{J} \mathrm{Re}\Big(\big\langle \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j, \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad = \boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j + \boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j + \mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\Big) + 2\sum_{i\neq j}\mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j + \boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\Big) \\
&\quad \geq \frac{2m_2^2 - R_2 m_2 + \delta}{8}\sum_{i\neq j}^{J}\|\boldsymbol{h}_i\|^2 + \frac{1}{20K}\sum_{k=1}^{K}\sum_{i\neq j}^{J}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i\big|^4, \quad \forall j \in \{1, \ldots, J\}.
\end{aligned} \tag{16}$$

Equivalently, we show that for all $\boldsymbol{h}_j$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}_j\| = 1$, and for all ξ with $0 \leq \xi \leq \epsilon/\sqrt{J}$, the following inequality holds

$$\begin{aligned}
&\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j + \xi^2\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j + \xi\mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\Big) + 2\sum_{i\neq j}\xi\mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\Big) + \mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\Big) \\
&\quad \geq (J-1)\Big(\frac{2m_2^2 - R_2 m_2 + \delta}{8}\Big) + \frac{\xi^2}{20K}\sum_{k=1}^{K}\sum_{i\neq j}^{J}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i\big|^4, \quad \forall j \in \{1, \ldots, J\}.
\end{aligned} \tag{17}$$

Invoking Corollary 12, we show that for all $\boldsymbol{h}_j$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}_j\| = 1$, and for all ξ with $0 \leq \xi \leq \epsilon/\sqrt{J}$,

$$\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K}\Big(\frac{5}{2}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2 + 3\xi\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2 + \frac{9}{10}\xi^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4 + \big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}|^2 - R_2\big)|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big) \\
&\quad \geq (J-1)\Big(\frac{2m_2^2 - R_2 m_2 + \delta}{8}\Big) + \frac{\xi^2}{20K}\sum_{k=1}^{K}\sum_{i\neq j}^{J}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i\big|^4, \quad \forall j \in \{1, \ldots, J\}.
\end{aligned} \tag{18}$$

For constant modulus signals, the last averaging term of the LHS of Eq.(18) is zero. For non-constant modulus QAM signals, the term is bounded by Corollaries 10 and 11:

$$\frac{1}{K}\sum_{k=1}^{K}\Big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}|^2 - R_2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big) \geq \big(m_2^2 - R_2 m_2 + \kappa - (1+R_2)\delta\big)\cdot\mathbf{1}[Q \neq 4].$$

Let

$$\boldsymbol{Y}(\boldsymbol{h},\xi)=\frac{1}{K}\sum_{k=1}^{K}\Big(\frac{5}{2}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2+3\xi\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2+\frac{9\xi^2}{10}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4\Big).$$

Since $(a-b)^2\geq\frac{a^2}{2}-b^2$, Cauchy-Schwarz inequality leads to

$$\begin{aligned}\boldsymbol{Y}(\boldsymbol{h},\xi)&\geq\left(\sqrt{\frac{5}{2K}\sum_{k=1}^{K}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2}-\sqrt{\frac{9\xi^2}{10K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4}\right)^2\\&\geq\frac{5}{4K}\sum_{k=1}^{K}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2-\frac{9\xi^2}{10K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4.\end{aligned}$$

By means of Corollary 10, with high probability we have

$$\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4\leq\max_k\|\boldsymbol{s}_k\|^2\Big(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big)\leq B^2L(m_2+\delta).$$

Using this result and Corollary 12, if $\|\boldsymbol{h}\|=1$, it holds with high probability that

$$\boldsymbol{Y}(\boldsymbol{h},\xi)\geq\frac{5m_2^2}{2}\mathrm{Re}(\boldsymbol{z}^{\mathsf{H}}\boldsymbol{h})^2+\frac{5}{8}(2m_2^2-R_2m_2-\delta)-\frac{9B^2L}{10}\xi^2(m_2+\delta)+\frac{5}{4}\kappa.$$

Hence, Lemma 6 holds under the following condition:

$$\begin{aligned}&\frac{5m_2^2}{2}\mathrm{Re}(\boldsymbol{z}^{\mathsf{H}}\boldsymbol{h})^2+\frac{5}{8}(2m_2^2-R_2m_2-\delta)-\frac{9B^2L}{10}\xi^2(m_2+\delta)+\frac{5}{4}\kappa+\big(m_2^2-R_2m_2+\kappa-(1+R_2)\delta\big)\cdot\mathbf{1}[Q\neq4]\\&\geq\frac{1}{\alpha}+\frac{2m_2^2-R_2m_2+\kappa}{2}+\frac{3m_2^2}{4}\mathrm{Re}(\boldsymbol{z}^{\mathsf{H}}\boldsymbol{h})^2.\end{aligned}\tag{19}$$

With $\epsilon=(2B\sqrt{L})^{-1}$ and $\delta\leq0.1$, Eq.(19) holds for $\alpha\geq30$.

IV. Proof of Lemma 7

For any $\boldsymbol{u}\in\mathbb{C}^L$ such that $\|\boldsymbol{u}\|=1$, let $\boldsymbol{v}=\mathrm{e}^{-i\phi(\boldsymbol{q}_j)}\boldsymbol{u}$. Recall that $\nabla f(\mathrm{e}^{i\theta}\boldsymbol{z}_j)\approx0$ and $\nabla_j g(\hat{\boldsymbol{T}}\boldsymbol{z})\approx0$ for any $\theta\in[0,2\pi]$ and $\forall j\in\{1,\ldots,J\}$, . By means of the triangle inequality, we prove that

$$\begin{aligned}\big|\boldsymbol{u}^{\mathsf{H}}\nabla_j g(\boldsymbol{q})\big|^2&=\Big|\boldsymbol{u}^{\mathsf{H}}\big(\nabla_j g(\boldsymbol{q})-\nabla_j g(\hat{\boldsymbol{T}}\boldsymbol{z})\big)\Big|^2\\&=\left|\boldsymbol{u}^{\mathsf{H}}\big(\nabla f(\boldsymbol{q}_j)-\nabla f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j)\big)+2\gamma_0\boldsymbol{u}^{\mathsf{H}}\Big(\sum_{i\neq j}^{J}\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\sum_{i\neq j}^{J}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j\Big)\right|^2\\&\leq\Big|\boldsymbol{u}^{\mathsf{H}}\big(\nabla f(\boldsymbol{q}_j)-\nabla f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j)\big)\Big|^2+4\gamma_0^2\left|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2\\&\leq\beta\bigg(\frac{2m_2^2-R_2m_2+\delta}{4}\|\boldsymbol{h}\|^2+\frac{1}{10K}\sum_{j=1}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^4\bigg).\end{aligned}\tag{20}$$

Now, we set $\gamma_0=1$, and for simplicity, we now assume that $\|\boldsymbol{h}_j\|\leq\epsilon/\sqrt{J}$ for all $j\in\{1,\ldots,J\}$. Hence, assuming $\beta\geq580$ and $\epsilon_j=(2B\sqrt{L})^{-1}$, Lemma 3 holds and it suffices to show that

$$\begin{aligned}\left|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2&=\left|\sum_{i\neq j}^{J}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}(\boldsymbol{h}_i+\boldsymbol{z}_i)(\boldsymbol{h}_i+\boldsymbol{z}_i)^{\mathsf{H}}\boldsymbol{S}(\boldsymbol{h}_j+\boldsymbol{z}_j)-\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\right|^2\\&=\left|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j+\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j+\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j+\sum_{i\neq j}^{J}\Big(2\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j+2\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2\\&\leq\left(\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|+\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|+\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|+2\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\big|+\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\big|\right)^2\\&\leq\beta\Big(\frac{2m_2^2-R_2m_2+\delta}{16}\sum_{i\neq j}^{J}\|\boldsymbol{h}_i\|^2+\frac{1}{40K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4\Big)\end{aligned}$$

holds for all $\boldsymbol{h}$ and $\boldsymbol{v}$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j)=0$, $\|\boldsymbol{h}_j\|\le\epsilon/\sqrt{J}$, and $\|\boldsymbol{v}\|=1$. Equivalently, we prove that for all $\boldsymbol{h}_j$ and $\boldsymbol{v}$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j)=0$, $\|\boldsymbol{h}_j\|=\|\boldsymbol{v}\|=1$ and for all ξ with $0\le\xi\le\epsilon/\sqrt{J}$, the following inequality holds

$$\begin{aligned}&\bigg(\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|+\xi\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|+\xi^2\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|+2\sum_{i\neq j}^{J}\xi\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\big|+\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\big|\bigg)^2\\&\quad\le\beta\bigg((J-1)\frac{2m_2^2-R_2m_2+\delta}{16}+\frac{\xi^2}{40K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4\bigg).\end{aligned}\tag{21}$$

Knowing that $(a+b+c+d+e)^2\le5(a^2+b^2+c^2+d^2+e^2)$,

$$\begin{aligned}&\bigg(\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|+\xi\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|+\xi^2\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|+2\sum_{i\neq j}^{J}\xi\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\big|+\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\big|\bigg)^2\\&\quad\le5\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|^2+5\xi^2\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|^2+5\xi^4\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|^2+20\xi^2\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2\\&\qquad+20\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2\\&\quad=5I_1+5\xi^2I_2+5\xi^4I_3+20\xi^2I_4+20I_5.\end{aligned}$$

We now bound the terms on the right-hand side. By means of Corollary 13,

$$I_1=\big|\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{v}\big|\le(J-1)^2\Big(m_2^2\frac{K+1}{K}+\delta\Big)^2\le(J-1)^2(2m_2^2+\delta)^2.$$

Additionally, knowing that $\max_k\|\boldsymbol{s}_k\|=B\sqrt{L}\ge1$ for all $k\in\{1,\dots,K\}$, and by means of Corollaries 10 and 11 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}I_2&=\bigg|\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{s}_m\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{z}_j\bigg|^2\le\bigg(\frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{z}_j|^2\bigg)\bigg(\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K}|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_i|^2\bigg)\\&=\bigg(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_j|^2+\frac{1}{K^2}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2\sum_{m\neq k}^{K}|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{z}_j|^2\bigg)\bigg(\frac{1}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^2\sum_{m\neq k}^{K}|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_i|^2\bigg)\\&\le\bigg(2m_2^2+\delta+\frac{K-1}{K}(m_2+\delta)^2\bigg)\bigg(\frac{1}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+(J-1)\frac{K-1}{K}(m_2+\delta)^2\bigg)\\&\le\frac{2m_2^2+\delta+(m_2+\delta)^2}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+(J-1)B^2L(m_2^2+\delta)^2\big(2m_2^2+\delta+(m_2+\delta)^2\big).\end{aligned}$$

Similarly,

$$\begin{aligned}I_3&=\bigg|\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{s}_m\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_j\bigg|^2\le\bigg(\frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_j|^2\bigg)\bigg(\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K}|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_i|^2\bigg)\\&=\bigg(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^2+\frac{1}{K^2}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2\sum_{m\neq k}^{K}|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_j|^2\bigg)\bigg(\frac{1}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+\frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^2\sum_{m\neq k}^{K}|\boldsymbol{s}_m^{\mathsf{H}}\boldsymbol{h}_i|^2\bigg)\\&\le\bigg(B^2L(m_2+\delta)+\frac{K-1}{K}(m_2+\delta)^2\bigg)\bigg(\frac{1}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+(J-1)\frac{K-1}{K}(m_2+\delta)^2\bigg)\\&\le\frac{B^2L(m_2^2+\delta)\big(1+m_2^2+\delta\big)}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4+(J-1)B^4L^2(m_2+\delta)^3\big(1+m_2+\delta\big).\end{aligned}$$

By inkoving the same tools, we also have

$$\begin{aligned}I_4&=\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2\le\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\le\sum_{i\neq j}^{J}\bigg(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^2\bigg)\bigg(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^2\bigg)\\&\le(J-1)B^2L(m_2+\delta)(2m_2^2+\delta),\end{aligned}$$

and similarly,

$$I_5 \leq \Big(\sum_{i\neq j}^{J} \big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\mathrm{Re}(\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2 \leq \sum_{i\neq j}^{J} \big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|^2 \leq \sum_{i\neq j}^{J}\left(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{s}_k|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^2\right)\left(\frac{1}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i|^2|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_j|^2\right)$$
$$\leq (J-1)B^2L(m_2+\delta)(2m_2^2+\delta).$$

Therefore, we obtain

$$\begin{aligned}
&\left|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2 \\
&\quad\leq 5(J-1)^2(2m_2^2+\delta)^2 \\
&\qquad + 5\xi^2\frac{\big(2m_2^2+\delta+(m_2^2+\delta)^2\big)}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4 + 5\xi^2(J-1)B^2L(m_2^2+\delta)^2\big(2m_2^2+\delta+(m_2+\delta)^2\big) \\
&\qquad + 5\xi^4\frac{B^2L(m_2+\delta)\big(1+m_2^2+\delta\big)}{K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4 + 5\xi^4(J-1)B^4L^2(m_2+\delta)^3\big(1+m_2+\delta\big) \\
&\qquad + 20\xi^2(B^2L)(J-1)(m_2+\delta)(2m_2^2+\delta) + 20B^2L(J-1)(m_2+\delta)(2m_2^2+\delta) \\
&\quad\leq \beta\left((J-1)\frac{2m_2^2-R_2m_2+\delta}{16}+\frac{\xi^2}{40K}\sum_{i\neq j}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_i|^4\right).
\end{aligned}$$

Hence, Lemma 7 holds under the conditions of Lemma 3 plus the following condition:

$$\begin{aligned}
\beta \geq \max\Big\{&\frac{80}{2m_2^2-R_2m_2+\delta}\Big((J-1)(2m_2^2+\delta)^2+\frac{\epsilon^2B^2L}{J}(m_2^2+\delta)^2\big(2m_2^2+\delta+(m_2+\delta)^2\big)+\frac{\epsilon^4B^4L^2}{J^2}(m_2+\delta)^3\big(1+m_2+\delta\big)\Big) \\
&+\frac{320}{2m_2^2-R_2m_2+\delta}\Big(\frac{\epsilon^2B^2L}{J}+B^2L\Big)(J-1)(m_2+\delta)(2m_2^2+\delta), \\
&200\big(2m_2^2+\delta+(m_2^2+\delta)^2\big)+200\frac{\epsilon^2B^2L}{J}(m_2+\delta)\big(1+m_2^2+\delta\big)\Big\}.
\end{aligned} \tag{22}$$

With $\epsilon = \sqrt{J}(2B\sqrt{L})^{-1}$ and $\delta \leq 0.1$, Eq.(22) holds for $\beta \geq 782$.

V. Proof of Lemma 8

We want to show that

$$\mathrm{dist}^2(\boldsymbol{q}^{t+1},\boldsymbol{z}) = \sum_{j=1}^{J}\min_{\phi\in[0,2\pi]}\|\boldsymbol{q}_j^{t+1}-\mathrm{e}^{i\phi}\boldsymbol{z}_j\|^2 \leq \sum_{j=1}^{J}\Big(1-\frac{2\mu}{\alpha}\Big)\min_{\phi\in[0,2\pi]}\|\boldsymbol{q}_j^{t}-\mathrm{e}^{i\phi}\boldsymbol{z}_j\|^2 = \Big(1-\frac{2\mu}{\alpha}\Big)\mathrm{dist}^2(\boldsymbol{q}^{t},\boldsymbol{z}). \tag{23}$$

Hence, we only need to prove the concentration inequality for each of the J combiners independently. However, we have shown that each combiner satisfies Lemmas 6 and 7, and therefore, each combiner satisfies its own inequality by following the procedure to prove Lemma 4, which in turn follows [2, Lemma 7.10].

References

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