Supplemental Material for: "Wirtinger Flow Meets Constant Modulus Algorithm: Revisiting Signal Recovery for Grant-Free Access"

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Abstract

This document presents the proofs for Lemmas 5, 6, 7 and 8 stated in the paper "Wirtinger Flow Meets Constant Modulus Algorithm: Revisiting Signal Recovery for Grant-Free Access" [1], which in turn establish Theorem 2, demonstrating the local convergence guarantees for multiple source recovery via constant modulus algorithm with limited samples.

I. Definitions and Expectations

From the main text, we recall the expressions for the cost function, gradient and Hessian in the case of multiple source recovery (MSR). Using the overall system parameter space, the cost function for MSR is

$$g(\boldsymbol{q}) = \sum_{j=1}^{J} f(\boldsymbol{q}_j) + \gamma_0 \sum_{j=1}^{J}\sum_{i\neq j}^{J} \left|\boldsymbol{q}_i \boldsymbol{S}\boldsymbol{q}_j\right|^2 = \sum_{j=1}^{J} f(\boldsymbol{q}_j) + \gamma_0 r(\boldsymbol{q}) \tag{1}$$

where $\boldsymbol{q} = [\boldsymbol{q}_1^{\mathsf{T}} \dots \boldsymbol{q}_J^{\mathsf{T}}]^{\mathsf{T}}$ is the aggregation of the J demixers, f is the CMA cost function for single source recovery, and $\boldsymbol{S}$ is the sample covariance matrix of transmitted signals as defined in the main text. The gradient of g, using Wirtinger calculus, is given by:

$$\nabla_j g = \frac{1}{K}\sum_{k=1}^{K}\Big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{q}_j|^2 - R_2\Big)\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{q}_j + \gamma_0 \sum_{i\neq j}^{J} \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j = \nabla f(\boldsymbol{q}_j) + \gamma_0 \nabla_j r(\boldsymbol{q}), \tag{2}$$

and the Wirtinger Hessian of the cost function g is

$$\nabla^2 g(\boldsymbol{q}) = \mathrm{Bdiag}\Big(\big\{\nabla^2 f(\boldsymbol{q}_j)\big\}_{j=1}^{J}\Big) + \gamma_0 \begin{bmatrix} \boldsymbol{G}_1(\boldsymbol{q}) & \cdots & \boldsymbol{H}_{1J}(\boldsymbol{q}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{H}_{J1}(\boldsymbol{q}) & \cdots & \boldsymbol{G}_J(\boldsymbol{q}) \end{bmatrix} = \mathrm{Bdiag}\Big(\big\{\nabla^2 f(\boldsymbol{q}_j)\big\}_{j=1}^{J}\Big) + \gamma_0 \nabla^2 r(\boldsymbol{q}) \tag{3}$$

where $\mathrm{Bdiag}()$ constructs a block diagonal matrix out of the matrices

$$\nabla^2 f(\boldsymbol{q}_j) = \begin{bmatrix} 2\boldsymbol{A}(\boldsymbol{q}_j) - R_2\boldsymbol{S} & \boldsymbol{B}(\boldsymbol{q}_j) \\ \overline{\boldsymbol{B}(\boldsymbol{q}_j)} & 2\overline{\boldsymbol{A}(\boldsymbol{q}_j)} - R_2\overline{\boldsymbol{S}} \end{bmatrix}. \tag{4}$$

$\boldsymbol{A}(\boldsymbol{q}_j)$, $\boldsymbol{B}(\boldsymbol{q}_j)$, and $\boldsymbol{S}$ have been defined in the main text, and

$$\boldsymbol{G}_j(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{C}_j(\boldsymbol{q}) & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{C}_j(\boldsymbol{q})} \end{bmatrix}, \quad \boldsymbol{C}_j(\boldsymbol{q}) = \sum_{i\neq j}^{J} \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}. \tag{5}$$

Furthermore, we have

$$\boldsymbol{H}_{ji}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{E}_{ji}(\boldsymbol{q}) & \boldsymbol{F}_{ji}(\boldsymbol{q}) \\ \overline{\boldsymbol{F}_{ji}(\boldsymbol{q})} & \overline{\boldsymbol{E}_{ji}(\boldsymbol{q})} \end{bmatrix}, \quad \boldsymbol{E}_{ji}(\boldsymbol{q}) = \boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j\boldsymbol{S}, \quad \boldsymbol{F}_{ji}(\boldsymbol{q}) = \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_j^{\mathsf{T}}\boldsymbol{S}^{T}. \tag{6}$$

The expectation of matrices $\boldsymbol{A}(\boldsymbol{q})$ and $\boldsymbol{B}(\boldsymbol{q})$ are:

$$\begin{aligned} \mathbb{E}\{\boldsymbol{A}(\boldsymbol{q})\} &= m_2^2(\|\boldsymbol{q}\|^2\boldsymbol{I} + \boldsymbol{q}\boldsymbol{q}^{\mathsf{H}}) + \kappa\,\mathrm{ddiag}(\boldsymbol{q}\boldsymbol{q}^{\mathsf{H}}), \\ \mathbb{E}\{\boldsymbol{B}(\boldsymbol{q})\} &= 2m_2^2\boldsymbol{q}\boldsymbol{q}^{\mathsf{T}} + \kappa\,\mathrm{ddiag}(\boldsymbol{q}\boldsymbol{q}^{\mathsf{T}}), \end{aligned}$$

whereas matrices $\boldsymbol{C}_j(\boldsymbol{q})$, $\boldsymbol{E}_{ji}(\boldsymbol{q})$ and $\boldsymbol{F}_{ji}(\boldsymbol{q})$ of the MSR Hessian satisfy

$$\mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{q})\} = \sum_{i\neq j}^{J} \big(m_2^2 \boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}} + \frac{m_2^2}{K}\|\boldsymbol{q}_i\|^2\boldsymbol{I} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}})\big),$$
$$\mathbb{E}\{\boldsymbol{E}_{ji}(\boldsymbol{q})\} = m_2^2(\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{q}_j)\boldsymbol{I} + \frac{m_2^2}{K}\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{H}} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{H}}),$$
$$\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{q})\} = m_2^2\boldsymbol{q}_i\boldsymbol{q}_j^{\mathsf{T}} + \frac{m_2^2}{K}\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{T}} + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_j\boldsymbol{q}_i^{\mathsf{T}}).$$

The expectation of the j-th gradient of the regularizing term is

$$\mathbb{E}\{\nabla_j r(\boldsymbol{q})\} = \mathbb{E}\big\{\boldsymbol{C}_j(\boldsymbol{q})\big\}\boldsymbol{q}_j = \sum_{i\neq j}^{J}\big(m_2^2\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{q}_j + \frac{m_2^2}{K}\|\boldsymbol{q}_i\|^2\boldsymbol{q}_j + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}})\big)\boldsymbol{q}_j \tag{7}$$

Note that at a CMA solution $\boldsymbol{z} = \big[\mathrm{e}^{i\theta_{\ell_1}}\boldsymbol{e}_{\ell_1}^{\mathsf{T}} \quad \ldots \mathrm{e}^{i\theta_{\ell_J}}\boldsymbol{e}_{\ell_J}^{\mathsf{T}}\big]^{\mathsf{T}}$, we have $\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{z}_j = 0$ for all $i \neq j$. Hence,

$$\mathbb{E}\{\nabla_j r(\boldsymbol{z})\} = \sum_{i\neq j}^{J}\Big(m_2^2\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{z}_j + \frac{m_2^2}{K}\|\boldsymbol{z}_i\|^2\boldsymbol{z}_j + \frac{\kappa}{K}\,\mathrm{ddiag}(\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{z}_j\Big) = \sum_{i\neq j}^{J}\Big(\frac{m_2^2}{K}\boldsymbol{z}_j + \frac{\kappa}{K}\boldsymbol{e}_i\boldsymbol{e}_i^{\mathsf{T}}\boldsymbol{z}_j\Big) = \frac{(J-1)m_2^2}{K}\boldsymbol{z}_j$$

which is non-zero, but decreases with the number of samples K and is aligned with the j-th component of the solution. Therefore, for large K, the CMA solution corresponds to an approximate stationary point in gradient-descent schemes, which often yields satisfactory numerical solutions. In particular, we have the following result that defines the CMA solution in multiple source recovery as a approximate stationary point.

Corollary 15. *Let $K \geq C_2(\delta)L$. Then, with probability at least $1 - 12\mathrm{e}^{-c_2(\delta)K}$,*

$$\Big\|\nabla_j g(\boldsymbol{z}) - \mathbb{E}\big\{\nabla_j g(\boldsymbol{z})\big\}\Big\| \leq (1 + R_2 + \gamma_0)\delta.$$

Proof. By definition and the triangular inequality,

$$\Big\|\nabla_j g(\boldsymbol{z}) - \mathbb{E}\big\{\nabla_j g(\boldsymbol{z})\big\}\Big\| = \max_{\boldsymbol{v}\in\mathbb{C}^L,\,\|\boldsymbol{v}\|=1}\Big|\boldsymbol{v}^{\mathsf{H}}\Big(\nabla_j g(\boldsymbol{z}) - \mathbb{E}\big\{\nabla_j g(\boldsymbol{z})\big\}\Big)\Big|$$
$$\leq \max_{\boldsymbol{v}\in\mathbb{C}^L,\,\|\boldsymbol{v}\|=1}\Big|\boldsymbol{v}^{\mathsf{H}}\Big(\nabla f(\boldsymbol{z}_j) - \mathbb{E}\big\{\nabla f(\boldsymbol{z}_j)\big\}\Big)\Big| + \gamma_0 \max_{\boldsymbol{v}\in\mathbb{C}^L,\,\|\boldsymbol{v}\|=1}\Big|\boldsymbol{v}^{\mathsf{H}}\Big(\nabla_j r(\boldsymbol{z}) - \mathbb{E}\big\{\nabla_j r(\boldsymbol{z})\big\}\Big)\Big|.$$

Note that $\nabla f(\boldsymbol{z}_j) = \boldsymbol{A}(\boldsymbol{z}_j)\boldsymbol{z}_j - R_2\boldsymbol{S}\boldsymbol{z}_j$ and $\nabla_j r(\boldsymbol{z}) = \boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{z}_j$. Invoking Lemma 5 and the triangular inequality, we have that for all $\boldsymbol{v} \in \mathbb{C}^L$ such that $\|\boldsymbol{v}\| = 1$,

$$\Big|\boldsymbol{v}^{\mathsf{H}}\Big(\nabla f(\boldsymbol{z}_j) - \mathbb{E}\big\{\nabla f(\boldsymbol{z}_j)\big\}\Big)\Big| \leq \Big(\big\|\boldsymbol{A}(\boldsymbol{z}_j) - \mathbb{E}\big\{\boldsymbol{A}(\boldsymbol{z}_j)\big\}\big\| + R_2\big\|\boldsymbol{S} - \mathbb{E}\big\{\boldsymbol{S}\big\}\big\|\Big)\|\boldsymbol{z}_j\|\|\boldsymbol{v}\| \leq (1 + R_2)\delta,$$
$$\Big|\boldsymbol{v}^{\mathsf{H}}\Big(\nabla_j r(\boldsymbol{z}) - \mathbb{E}\big\{\nabla_j r(\boldsymbol{z})\big\}\Big)\Big| \leq \Big\|\boldsymbol{C}_j(\boldsymbol{z}) - \mathbb{E}\big\{\boldsymbol{C}_j(\boldsymbol{z})\big\}\Big\|\|\boldsymbol{z}_j\|\|\boldsymbol{v}\| \leq \delta.$$

∎

Finally, we define corollaries that will be useful to prove Lemmas 6 and 7:

Corollary 16. *Let $K \geq C_2(\delta)L$. Then, with probability at least $1 - 12\mathrm{e}^{-c_2(\delta)K}$, for all $\boldsymbol{v} \in \mathbb{C}^L$ such that $\|\boldsymbol{v}\| = 1$, we have*

$$\Big(\frac{J-1}{K}m_2^2 - \delta\Big)\|\boldsymbol{v}\|^2 \leq \boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{v} \leq \Big(\frac{K+J-1}{K}m_2^2 + \frac{\kappa}{K} + \delta\Big)\|\boldsymbol{v}\|^2.$$

Proof. From Lemma 5, we have that $-\delta\boldsymbol{I} \preceq \boldsymbol{C}_j(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{z})\} \preceq \delta\boldsymbol{I}$. Hence, the lower bound is

$$\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{v} \geq \sum_{i\neq j}^{J}\big(m_2^2|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{z}_i|^2 + \frac{m_2^2}{K}\|\boldsymbol{z}_i\|^2\|\boldsymbol{v}\|^2 + \frac{\kappa}{K}\boldsymbol{v}^{\mathsf{H}}\,\mathrm{ddiag}(\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{v}\big) - \delta\|\boldsymbol{v}\|^2$$
$$\geq (J-1)\frac{m_2^2}{K}\|\boldsymbol{v}\|^2 + \sum_{i\neq j}^{J}\big(m_2^2|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{z}_i|^2 + \frac{\kappa}{K}\boldsymbol{v}^{\mathsf{H}}\,\mathrm{ddiag}(\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{v}\big) - \delta\|\boldsymbol{v}\|^2.$$

Note that $Km_2^2 + \kappa > 0$ for all $K > 1$ and all QAM modulations. Additionally, knowing that $\boldsymbol{z}_j = \mathrm{e}^{\theta_{\ell_j}}\boldsymbol{e}_{\ell_j}$ for all $j \in \{1, \ldots, J\}$, we have that

$$0 \leq \sum_{i\neq j}^{J}|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{z}_i|^2 = \sum_{i\neq j}^{J}\boldsymbol{v}^{\mathsf{H}}\,\mathrm{ddiag}(\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}})\boldsymbol{v} = \sum_{i\neq j}^{J}|v_{\ell_i}|^2 \leq \|\boldsymbol{v}\|^2.$$

The upper bound is obtained similarly. ∎

Corollary 17. *Let $K \geq C_2(\delta)L$. Then, with probability at least $1-12\mathrm{e}^{-c_2(\delta)K}$, we have*

$$\left|\boldsymbol{z}_j \boldsymbol{S} \boldsymbol{z}_i\right|^2 \leq \delta.$$

Proof. From Lemma 1, we have that $-\delta \boldsymbol{I} \preceq \boldsymbol{S} - m_2 \boldsymbol{I} \preceq \delta \boldsymbol{I}$. Then, we have that for all $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$,

$$\left|\boldsymbol{u}^{\mathsf{H}} \boldsymbol{S} \boldsymbol{v}\right|^2 \leq m_2 \left|\boldsymbol{u}^{\mathsf{H}} \boldsymbol{v}\right| + \left|\boldsymbol{u}^{\mathsf{H}}\left(\boldsymbol{S} - m_2 \boldsymbol{I}\right)\boldsymbol{v}\right|^2 \leq m_2 \left|\boldsymbol{u}^{\mathsf{H}} \boldsymbol{v}\right| + \left\|\boldsymbol{S} - m_2 \boldsymbol{I}\right\| \|\boldsymbol{u}\| \|\boldsymbol{v}\| \leq m_2 \left|\boldsymbol{u}^{\mathsf{H}} \boldsymbol{v}\right| + \delta \|\boldsymbol{u}\| \|\boldsymbol{v}\|.$$

Knowing that $\boldsymbol{z}_j = \mathrm{e}^{\theta_{\ell_j}} \boldsymbol{e}_{\ell_j}$ for all $j \in \{1, \ldots, J\}$, we have that $|\boldsymbol{z}_j^{\mathsf{H}} \boldsymbol{z}_i| = 0$ for all $i \neq j$, and we obtain the bound. ∎

Corollary 18. *Let $K \geq C_2(\delta)L$. Then, with probability at least $1-12\mathrm{e}^{-c_2(\delta)K}$, we have*

$$\left\|\boldsymbol{F}_{ji}(\boldsymbol{z})\right\| \leq m_2^2 + \delta.$$

Proof. From Lemma 5, we have that $-\delta \boldsymbol{I} \preceq \boldsymbol{F}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\} \preceq \delta \boldsymbol{I}$, or equivalently,

$$\left\|\boldsymbol{F}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}\right\| \leq \delta.$$

Furthermore, using [2, Corollary 8.6.2] for the largest singular value (i.e. the operator norm), we have

$$\left\|\boldsymbol{F}_{ji}(\boldsymbol{z})\right\| \leq \left\|\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}\right\| + \delta.$$

Knowing that $\boldsymbol{z}_j = \mathrm{e}^{\theta_{\ell_j}} \boldsymbol{e}_{\ell_j}$ for all $j \in \{1, \ldots, J\}$, we also have that

$$\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\} = m_2^2 \boldsymbol{z}_i \boldsymbol{z}_j^{\mathsf{T}} + \frac{m_2^2}{K} \boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{T}} + \frac{\kappa}{K} \operatorname{ddiag}\left(\boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{T}}\right) = m_2^2 \mathrm{e}^{i(\theta_{\ell_j}+\theta_{\ell_i})} \boldsymbol{e}_i \boldsymbol{e}_j^{\mathsf{T}} + \frac{m_2^2}{K} \mathrm{e}^{i(\theta_{\ell_j}+\theta_{\ell_i})} \boldsymbol{e}_j \boldsymbol{e}_i^{\mathsf{T}},$$

which means that $\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}$ has only two non-zero elements in positions (j, i) and (i, j) with $i \neq j$. Hence, its non-zero columns are independent, and its norm is the largest absolute value of the non-zero elements, i.e.

$$\left\|\mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}\right\| = \max\left\{\left|\mathrm{e}^{i(\theta_{\ell_j}+\theta_{\ell_i})} m_2^2\right|, \left|\mathrm{e}^{i(\theta_{\ell_j}+\theta_{\ell_i})} \frac{m_2^2}{K}\right|\right\} = m_2^2.$$ ∎

Corollary 19. *Let $K \geq C_2(\delta)L$. Then, with probability at least $1-12\mathrm{e}^{-c_2(\delta)K}$, for all $\boldsymbol{h} \in \mathbb{C}^{JL}$ such that its J components have unit norm, i.e., $\|\boldsymbol{h}_j\| = 1$, we have*

$$\sum_{j=1}^{J} \sum_{i \neq j}^{J} |\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i|^2 + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{h}_i \boldsymbol{z}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i \boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) \geq -\left(\frac{m_2^2}{K} + \delta\right) \left\|\boldsymbol{h}\right\|^2. \tag{8}$$

Proof. Recall $\nabla^2 r(\boldsymbol{z})$ as defined in Eq.(3). Now, notice that

$$\begin{aligned}
&\sum_{j=1}^{J} \sum_{i \neq j}^{J} |\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i|^2 + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{h}_i \boldsymbol{z}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i \boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) \\
&\quad = \sum_{j=1}^{J} \sum_{i < j}^{J} |\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i|^2 + |\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j|^2 + 2\operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{h}_i \boldsymbol{z}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) + 2\operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_i \boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{S} \boldsymbol{z}_j\right) \\
&\quad = \frac{1}{2} \tilde{\boldsymbol{h}}^{\mathsf{H}} \nabla^2 r(\boldsymbol{z}) \tilde{\boldsymbol{h}}
\end{aligned} \tag{9}$$

where in the second equality we collect pairs (i, j) and (j, i) in one summation, and $\tilde{\boldsymbol{h}}$ is the stacked version of all $\boldsymbol{h}_j$ and their complex conjugates, i.e. $\tilde{\boldsymbol{h}} = \begin{bmatrix} \boldsymbol{h}_1 & \overline{\boldsymbol{h}_1} & \cdots & \boldsymbol{h}_J & \overline{\boldsymbol{h}_J} \end{bmatrix}^{\mathsf{T}}$. From Lemma 5, we have that $-\delta \boldsymbol{I} \preceq \nabla^2 r(\boldsymbol{z}) - \mathbb{E}\{\nabla^2 r(\boldsymbol{z})\} \preceq \delta \boldsymbol{I}$. Hence, we have that

$$\begin{aligned}
\frac{1}{2} \tilde{\boldsymbol{h}}^{\mathsf{H}} \nabla^2 r(\boldsymbol{z}) \tilde{\boldsymbol{h}} &\geq \frac{1}{2} \tilde{\boldsymbol{h}}^{\mathsf{H}} \mathbb{E}\{\nabla^2 r(\boldsymbol{z})\} \tilde{\boldsymbol{h}} - \frac{\delta}{2} \|\tilde{\boldsymbol{h}}\|^2 \\
&= \sum_{j=1}^{J} \left(\operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{z})\} \boldsymbol{h}_j\right) + \sum_{i \neq j}^{J} \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \mathbb{E}\{\boldsymbol{E}_{ji}(\boldsymbol{z})\} \boldsymbol{h}_i\right) + \sum_{i \neq j}^{J} \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\} \overline{\boldsymbol{h}_i}\right)\right) - \delta \|\boldsymbol{h}\|^2 \\
&= \sum_{j=1}^{J} \sum_{i \neq j}^{J} \left(\operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \left(m_2^2 \boldsymbol{z}_i \boldsymbol{z}_i^{\mathsf{H}} + \frac{m_2^2}{K} \|\boldsymbol{z}_i\|^2 \boldsymbol{I} + \frac{\kappa}{K} \operatorname{ddiag}(\boldsymbol{z}_i \boldsymbol{z}_i^{\mathsf{H}})\right) \boldsymbol{h}_j\right)\right. \\
&\quad + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \left(m_2^2 (\boldsymbol{z}_i^{\mathsf{H}} \boldsymbol{z}_j) \boldsymbol{I} + \frac{m_2^2}{K} \boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{H}} + \frac{\kappa}{K} \operatorname{ddiag}(\boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{H}})\right) \boldsymbol{h}_i\right) \\
&\quad \left. + \operatorname{Re}\left(\boldsymbol{h}_j^{\mathsf{H}} \left(m_2^2 \boldsymbol{z}_i \boldsymbol{z}_j^{\mathsf{T}} + \frac{m_2^2}{K} \boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{T}} + \frac{\kappa}{K} \operatorname{ddiag}(\boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{T}})\right) \overline{\boldsymbol{h}_i}\right)\right) - \delta \|\boldsymbol{h}\|^2.
\end{aligned}$$

Knowing that $\boldsymbol{z}_j = \mathrm{e}^{\theta_{\ell_j}} \boldsymbol{e}_{\ell_j}$ for all $j \in \{1, \ldots, J\}$, we have that $|\boldsymbol{h}_{j,i}| = |\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i|$, and $\mathrm{ddiag}(\boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{H}}) = \mathrm{ddiag}(\boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{H}}) = \boldsymbol{0}$ and $(\boldsymbol{z}_j^{\mathsf{H}} \boldsymbol{z}_i) = 0$ for all $i \neq j$. Hence,

$$\begin{aligned}
\frac{1}{2}\tilde{\boldsymbol{h}}^{\mathsf{H}} \nabla^2 r(\boldsymbol{z}) \tilde{\boldsymbol{h}} &\geq \sum_{j=1}^{J} \sum_{i \neq j}^{J} \Big(m_2^2 \big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + \frac{m_2^2}{K} \|\boldsymbol{z}_i\|^2 \big\|\boldsymbol{h}_j\big\|^2 + \frac{\kappa}{K} + \frac{m_2^2}{K} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{H}} \boldsymbol{h}_i\big) \\
&\qquad + m_2^2 \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i \boldsymbol{z}_j^{\mathsf{T}} \overline{\boldsymbol{h}_i}\big) + \frac{m_2^2}{K} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j \boldsymbol{z}_i^{\mathsf{T}} \overline{\boldsymbol{h}_i}\big) \Big) - \delta \|\boldsymbol{h}\|^2 \\
&= \sum_{j=1}^{J} \sum_{i \neq j}^{J} \Big(\Big(m_2^2 + \frac{\kappa}{K} \Big) \big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + \frac{m_2^2}{K} \big\|\boldsymbol{h}_j\big\|^2 + \frac{2m_2^2}{K} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) + m_2^2 \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i \boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_j\big) \Big) - \delta \|\boldsymbol{h}\|^2 \\
&= m_2^2 \sum_{j=1}^{J} \sum_{i<j}^{J} \Big(\big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + \big|\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_j\big|^2 + 2\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i \boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_j\big) \Big) \\
&\qquad + \frac{1}{K} \sum_{j=1}^{J} \sum_{i \neq j}^{J} \Big(\kappa \big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + m_2^2 \big\|\boldsymbol{h}_j\big\|^2 + 2m_2^2 \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) \Big) - \delta \|\boldsymbol{h}\|^2.
\end{aligned} \tag{10}$$

The first term in the RHS of Eq.(10) is a perfect square, and is bounded below by 0. Rewriting $\|\boldsymbol{h}_j\| = \sum_{a=1}^{L} |\boldsymbol{h}_{j,a}|^2$ in the second term in the RHS, we have

$$\begin{aligned}
\frac{1}{2}\tilde{\boldsymbol{h}}^{\mathsf{H}} \nabla^2 r(\boldsymbol{z}) \tilde{\boldsymbol{h}} &\geq \frac{1}{K} \sum_{j=1}^{J} \sum_{i \neq j}^{J} \Big((m_2^2 + \kappa) \big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + m_2^2 \sum_{a \neq i}^{L} |\boldsymbol{h}_{j,a}|^2 + 2m_2^2 \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) \Big) - \delta \|\boldsymbol{h}\|^2 \\
&= \frac{m_2^2 + \kappa}{K} \sum_{j=1}^{J} \sum_{i \neq j}^{J} \big|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_i\big|^2 + \frac{m_2^2}{K} \sum_{j=1}^{J} \sum_{i \neq j}^{J} \sum_{a \neq i,j}^{L} |\boldsymbol{h}_{j,a}|^2 + \frac{m_2^2}{K} \sum_{j=1}^{J} \sum_{i<j}^{J} \Big(|\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j|^2 + |\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i|^2 + 2\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) \Big) \\
&\qquad + \frac{2m_2^2}{K} \sum_{j=1}^{J} \sum_{i<j}^{J} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) - \delta \|\boldsymbol{h}\|^2.
\end{aligned} \tag{11}$$

Knowing that in square QAM modulations $|\kappa| \leq m_2^2$, we have that the first term of the RHS is bounded below by zero. Furthermore, the second and third terms in the RHS of Eq.(11) are perfect squares, and are also bounded below by zero. We now focus on bounding the remaining term

$$\frac{2m_2^2}{K} \sum_{j=1}^{J} \sum_{i<j}^{J} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big). \tag{12}$$

For each (j, i) pair, the summands can be negative. However, note that for $J > 2$, it is not possible that all summands are negative, i.e., for all pairs (j, i) with $i < j$: some summands *will* be positive, as they are composed of pairwise products. Moreover, every $\boldsymbol{h}_a^{\mathsf{H}} \boldsymbol{z}_a$ can have at most magnitude equal to 1 by construction, and we also have that $\|\boldsymbol{h}\| = J$. We then count the number of negative summands to obtain the worst-case scenario. There is a total of $J(J-1)/2$ summands, and the maximum number of negative summands is equal to all combinations of half of summands with one sign and the other with the opposite sign:

$$\binom{\lfloor J/2 \rfloor}{1} \binom{\lceil J/2 \rceil}{1} = \left\lfloor \frac{J}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil = \begin{cases} J^2/4 & J \text{ even}, \\ (J^2-1)/4 & J \text{ odd}. \end{cases} \tag{13}$$

Thus, the worst sum (with each $\boldsymbol{h}_a^{\mathsf{H}} \boldsymbol{z}_a$ having a magnitude of 1) corresponds to the maximum amount of negative summands with a negative sign, plus the remaining positive summands, i.e.,

$$-\left\lfloor \frac{J}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil + \left(\frac{J(J-1)}{2} - \left\lfloor \frac{J}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil \right) \geq -2 \max \left\{ \frac{J^2}{4}, \frac{J^2-1}{4} \right\} + \frac{J^2}{2} - \frac{J}{2} = -\frac{J}{2}. \tag{14}$$

Replacing in (12), we have

$$\frac{2m_2^2}{K} \sum_{j=1}^{J} \sum_{i<j}^{J} \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}} \boldsymbol{z}_j\big) \mathrm{Re}\big(\boldsymbol{h}_i^{\mathsf{H}} \boldsymbol{z}_i\big) \geq -\frac{m_2^2}{K} J = -\frac{m_2^2}{K} \|\boldsymbol{h}\|^2, \tag{15}$$

which completes the proof. ∎

II. Proof of Lemma 5

Using the triangle inequality and Eq.(3), to prove the lemma we show that $\forall j \in \{1, \ldots, J\}$ and $\forall i \neq j$,

$$\left\| \nabla^2 f(\boldsymbol{z}_j) - \mathbb{E}\{\nabla^2 f(\boldsymbol{z}_j)\} \right\| \leq \delta_f = \frac{\delta}{2J}, \tag{16}$$

$$\|\boldsymbol{C}_j(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{C}_j(\boldsymbol{z})\}\| \leq \delta_C = \frac{\delta}{8\gamma_0 J}. \tag{17}$$

$$\|\boldsymbol{E}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{E}_{ji}(\boldsymbol{z})\}\| \leq \delta_E = \frac{\delta}{8\gamma_0 J(J-1)}, \tag{18}$$

$$\|\boldsymbol{F}_{ji}(\boldsymbol{z}) - \mathbb{E}\{\boldsymbol{F}_{ji}(\boldsymbol{z})\}\| \leq \delta_F = \frac{\delta}{8\gamma_0 J(J-1)}. \tag{19}$$

Eq.(16) corresponds to the concentration inequality of J Hessians of the single source recovery cost function. Thus, Lemma 1 states that Eq.(16) holds with probability at least $1 - 6\mathrm{e}^{-c_1(\delta_f)K}$ by choosing $K \geq C_1(\delta_f)L$.

Let $\boldsymbol{c}_{i,k} = \boldsymbol{s}_k(\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i)$, which are independent for $k \in \{1, \ldots, K\}$, and in particular, $\boldsymbol{c}_{i,k}$ is independent of $\boldsymbol{c}_{i,m}$ for $m \neq k$. Note that $\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_i = \sqrt{m_2}\,\mathrm{e}^{i\varphi}\overline{s_{\ell_i}[k]}$, and therefore the vectors $\boldsymbol{c}_{i,k}$ have bounded, discrete elements over an exponentially large set, and as such they are subgaussian [3]. Moreover, the sum of these vectors over index i is also subgaussian. Thus, we have

$$\boldsymbol{C}_j(\boldsymbol{z}) = \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m=1}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{i,m}^{\mathsf{H}} = \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{i,k}^{\mathsf{H}} + \frac{1}{K^2}\sum_{i\neq j}^{J}\sum_{k=1}^{K}\sum_{m\neq k}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{i,m}^{\mathsf{H}}, \tag{20}$$

and invoking Lemma 9 for each $i \neq j$ and results of concentration of quadratic forms [3, Chapter 6], Eq.(17) holds with probability at least $1 - 2\mathrm{e}^{-c_6(\delta_C)K}$ by choosing $K \geq C_6(\delta_C)L$.

In a similar fashion, note that

$$\boldsymbol{F}_{ji}(\boldsymbol{z}) = \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{j,m}^{\mathsf{T}} = \frac{1}{K^2}\sum_{k=1}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{j,k}^{\mathsf{T}} + \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m\neq k}^{K} \boldsymbol{c}_{i,k}\boldsymbol{c}_{j,m}^{\mathsf{T}}, \tag{21}$$

where we leverage the reasoning of the previous result, the fact that $\boldsymbol{c}_{i,k}$ is independent of $\boldsymbol{c}_{j,m}$ for $m \neq k$ and $i \neq j$, and the concentration of measure of $\boldsymbol{U}(\boldsymbol{z})$ in Lemma 1 of the main text (for the transposition instead of conjugate transpose). Hence, Eq.(19) holds with probability $1 - 2\mathrm{e}^{-c_7(\delta_F)K}$ by choosing $K \geq C_7(\delta_F)L$.

Now define $\boldsymbol{e}_{j,k,m} = (\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}_j)\boldsymbol{s}_m$. The vectors $\boldsymbol{e}_{j,k,k}$ are independent for $k \in \{1, \ldots, K\}$, and $\boldsymbol{e}_{j,k,m}$ is independent of $\boldsymbol{e}_{i,k,m}$ for $m \neq k$ and $i \neq j$. Following a similar reasoning as above, the $\boldsymbol{e}_{j,k,m}$ are subgaussian, and we obtain

$$\boldsymbol{E}_{ji}(\boldsymbol{z}) = \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m=1}^{K} \boldsymbol{e}_{j,k,m}\boldsymbol{e}_{i,k,m}^{\mathsf{H}} = \frac{1}{K^2}\sum_{k=1}^{K} \boldsymbol{e}_{j,k,k}\boldsymbol{e}_{i,k,k}^{\mathsf{H}} + \frac{1}{K^2}\sum_{k=1}^{K}\sum_{m\neq k}^{K} \boldsymbol{e}_{j,k,m}\boldsymbol{e}_{i,k,m}^{\mathsf{H}}, \tag{22}$$

thus Eq.(18) holds with probability $1 - 2\mathrm{e}^{-c_8(\delta_E)K}$ by choosing $K \geq C_8(\delta_E)L$.

Finally, set $C_2(\delta) \geq \max\{C_1(\delta_f), C_6(\delta_C), C_7(\delta_F), C_8(\delta_E)\}$. By selecting $K \geq C_2(\delta)L$, Lemma 5 holds with probability at least $1 - 12\mathrm{e}^{-c_2(\delta)K}$, where we define $c_2(\delta) = \min\{c_5(\delta_f), c_6(\delta_C), c_7(\delta_F), c_8(\delta_E)\}$.

III. Generalized Regularity Condition for WF-CMA (MSR case)

Let $\nabla G(\boldsymbol{q}) = \begin{bmatrix}\nabla_1 g(\boldsymbol{q})^{\mathsf{T}} & \cdots & \nabla_J g(\boldsymbol{q})^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}}$ be the total gradient of the MSR function, stacking the gradients with respect to each CMA solution. Let $\boldsymbol{q} \in E(\epsilon)$, and let $\boldsymbol{T} = \mathrm{diag}(\mathrm{e}^{i\phi_1(\boldsymbol{q})}, \ldots, \mathrm{e}^{i\phi_J(\boldsymbol{q})}) \otimes \boldsymbol{I}_L$ be the matrix that collects the optimal rotations of the combiners $\boldsymbol{q}_j$ with their respective ground truths $\boldsymbol{z}_j$. Hence, the regularity condition that we aim to prove with Lemmas 6 and 7 is then

$$\begin{aligned}
\sum_{j=1}^{J} \mathrm{Re}\big(\langle \nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j\rangle\big) &= \mathrm{Re}\big(\langle \nabla G(\boldsymbol{q}) - \nabla G(\boldsymbol{T}\boldsymbol{z}), \boldsymbol{q} - \boldsymbol{T}\boldsymbol{z}\rangle\big) \\
&\geq \frac{1}{\alpha}\mathrm{dist}^2(\boldsymbol{q}, \boldsymbol{z}) + \frac{1}{\beta}\left\|\nabla G(\boldsymbol{q}) - \nabla G(\boldsymbol{T}\boldsymbol{z})\right\|^2 \\
&= \frac{1}{\alpha}\sum_{j=1}^{J}\mathrm{dist}^2(\boldsymbol{q}_j, \boldsymbol{z}_j) + \frac{1}{\beta}\sum_{j=1}^{J}\left\|\nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\right\|^2.
\end{aligned}$$

IV. Proof of Lemma 6

Let $\boldsymbol{q} \in E(\epsilon)$, $\boldsymbol{T}$ as defined in Section III, and $\boldsymbol{h} = \boldsymbol{T}^{\mathsf{H}}\boldsymbol{q} - \boldsymbol{z}$. Hence, $\|\boldsymbol{h}\| \leq \epsilon$, $\boldsymbol{h}_j = \mathrm{e}^{-i\phi(\boldsymbol{q}_j)}\boldsymbol{q}_j - \boldsymbol{z}_j$, and $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$, as $\boldsymbol{h}_j$ and $\boldsymbol{z}_j$ are geometrically aligned for all $j \in \{1, \dots, J\}$. To prove Lemma 6, we prove that

$$
\begin{aligned}
&\sum_{j=1}^{J} \mathrm{Re}\Big(\big\langle \nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad = \sum_{j=1}^{J} \mathrm{Re}\Big(\big\langle \nabla f(\boldsymbol{q}_j) - \nabla f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) + \gamma_0 \sum_{j=1}^{J}\sum_{i\neq j}^{J} \mathrm{Re}\Big(\big\langle \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j, \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad \geq \sum_{j=1}^{J} \Big(\frac{1}{\alpha} + \frac{2m_2^2 - R_2 m_2 - \delta}{19}\Big)\|\boldsymbol{h}_j\|^2 + \frac{1}{20K}\sum_{j=1}^{J}\sum_{k=1}^{K}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j\big|^4 + \gamma_0^2 \sum_{j=1}^{J}\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2
\end{aligned} \tag{23}
$$

for all $\boldsymbol{h}$ satisfying $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}\| \leq \epsilon$. In the following, we set $\gamma_0 = 1$, and for simplicity, we now assume that $\|\boldsymbol{h}_j\| \leq \epsilon/\sqrt{J}$ for all $j \in \{1, \dots, J\}$. In Lemma 2 we establish that the inner product of the CMA portion of the gradient, i.e., the terms with ∇f, are bounded below by the two first terms of the right-hand side of Eq.(23). Therefore, we focus on the regularizing term, which is

$$
\begin{aligned}
&\sum_{j=1}^{J}\sum_{i\neq j}^{J} \mathrm{Re}\Big(\big\langle \boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j, \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad = \sum_{j=1}^{J}\sum_{i\neq j}^{J} \bigg(\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + \big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big) + \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big) \\
&\qquad\qquad + \mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + \mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big)\bigg) \\
&\quad = \sum_{j=1}^{J}\sum_{i< j}^{J} \bigg(\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + \big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|^2 + 2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big) \\
&\qquad\qquad + 2\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + 2\mathrm{Re}\Big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big)\bigg) \\
&\quad \leq \sum_{j=1}^{J}\sum_{i< j}^{J} \bigg(2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big)\bigg) - \Big(\frac{m_2^2}{K} + \delta\Big)\|\boldsymbol{h}\|^2,
\end{aligned}
$$

where the second equality comes from realizing that the terms of pairs (i, j) and (j, i) are related through complex conjugation, and the last inequality comes from Corollary 19. Subs in Eq.(23), and invoking Lemma 2, we have that

$$
\begin{aligned}
&\sum_{j=1}^{J} \mathrm{Re}\Big(\big\langle \nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z}), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) + \gamma_0 \sum_{j=1}^{J} \mathrm{Re}\Big(\big\langle \nabla f(\boldsymbol{q}_j) - \nabla f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j), \boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j \big\rangle\Big) \\
&\quad \geq \frac{1}{K}\sum_{k=1}^{K}\Big(2\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2 + 3\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2 + |\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4 + \big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}|^2 - R_2\big)|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big) \\
&\qquad + \sum_{j=1}^{J}\sum_{i< j}^{J} \bigg(2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2 + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + 3\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big)\bigg) - \Big(\frac{m_2^2}{K} + \delta\Big)\|\boldsymbol{h}\|^2 \\
&\quad \geq \sum_{j=1}^{J} \Big(\frac{1}{\alpha} + \frac{2m_2^2 - R_2 m_2 - \delta}{19}\Big)\|\boldsymbol{h}_j\|^2 + \frac{1}{20K}\sum_{j=1}^{J}\sum_{k=1}^{K}\big|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j\big|^4 + \sum_{j=1}^{J}\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2.
\end{aligned} \tag{24}
$$

Equivalently, we prove that for all $\boldsymbol{h}_j$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}_j\| = 1$, and for all ξ with $0 \leq \xi \leq \epsilon/\sqrt{J}$,

$$
\begin{aligned}
&\frac{1}{K}\sum_{j=1}^{J}\sum_{k=1}^{K}\Big(2\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2 + 3\xi\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2 + \frac{19\xi^2}{20}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4 + \big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}|^2 - R_2\big)|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big) \\
&\qquad + \sum_{j=1}^{J}\sum_{i< j}^{J} \bigg(3\xi\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) + 3\xi\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big)\bigg) - J\Big(\frac{m_2^2}{K} + \delta\Big) \\
&\quad \geq J\Big(\frac{1}{\alpha} + \frac{2m_2^2 - R_2 m_2 - \delta}{19}\Big).
\end{aligned} \tag{25}
$$

Invoking Lemma 1 and Corollary 10, we have that

$$
\begin{aligned}
\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big) &\geq -\big|\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big| \geq -(m_2+\delta)^2,\\
\mathrm{Re}\big(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big) &\geq -\big|\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big| \geq -(m_2+\delta)^2,
\end{aligned} \tag{26}
$$

and replacing in Eq.(25), we obtain

$$
\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K}\Big(2\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})^2 + 3\xi\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z})|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2 + \frac{19\xi^2}{20}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^4 + \big(|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{z}|^2 - R_2\big)|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}|^2\Big)\\
&\qquad - 3(J-1)\xi(m_2+\delta)^2 - \frac{m_2^2}{K} - \delta\\
&\geq \Big(\frac{1}{\alpha} + \frac{2m_2^2 - R_2m_2 - \delta}{19}\Big)
\end{aligned} \tag{27}
$$

Now, following the procedure in Lemma 2, to bound the sum in the LHS, we obtain that Lemma 6 holds under the following condition:

$$
\begin{aligned}
&\frac{135m_2^2 + 90\kappa}{76}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{z})^2 + \frac{45}{76}(m_2^2 - \delta) - \frac{19B^2L}{20}\xi^2(m_2+\delta)\\
&+ \big(m_2^2 - R_2m_2 - (1+R_2)\delta + (m_2^2+\kappa)|\boldsymbol{h}^{\mathsf{H}}\boldsymbol{z}|^2\big)\cdot\mathbf{1}[Q\neq 4] - 3(J-1)\xi(m_2+\delta)^2 - \frac{m_2^2}{K} - \delta\\
&\geq \frac{1}{\alpha} + \frac{11m_2^2 - 2R_2m_2 + 5\delta}{19} + \frac{21m_2^2 + 14\kappa}{38}\mathrm{Re}(\boldsymbol{h}^{\mathsf{H}}\boldsymbol{z})^2.
\end{aligned} \tag{28}
$$

With $J \geq 2$, $\epsilon = (10JB\sqrt{LJ})^{-1}$ and $\delta \leq 0.001$, Eq.(28) holds for

$$
\begin{aligned}
&\alpha \geq 4 &&\text{for} \quad Q = 4,\\
&\alpha \geq 227 &&\text{for} \quad Q \neq 4.
\end{aligned}
$$

V. Proof of Lemma 7

Let $\boldsymbol{q} \in E(\epsilon)$ and $\boldsymbol{T}$ as defined in Section III. Let $\boldsymbol{h} = \boldsymbol{T}^{\mathsf{H}}\boldsymbol{q} - \boldsymbol{z}$. Hence, $\|\boldsymbol{h}\| \leq \epsilon$, $\boldsymbol{h}_j = \mathrm{e}^{-i\phi(\boldsymbol{q}_j)}\boldsymbol{q}_j - \boldsymbol{z}_j$ and $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$. We aim to prove that

$$\begin{aligned}&\big\|\nabla G(\boldsymbol{q}) - \nabla G(\boldsymbol{T}\boldsymbol{z})\big\|^2\\ &\qquad \leq \beta\Big(\frac{2m_2^2 - R_2 m_2 - \delta}{19}\big\|\boldsymbol{h}\big\|^2 + \frac{1}{20K}\sum_{j=1}^{J}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^4 + \gamma_0^2\sum_{j=1}^{J}\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\Big),\end{aligned} \tag{29}$$

We first notice that

$$\big\|\nabla G(\boldsymbol{q}) - \nabla G(\boldsymbol{T}\boldsymbol{z})\big\|^2 = \sum_{j=1}^{J}\big\|\nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\big\|^2, \tag{30}$$

where

$$\big\|\nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\big\|^2 = \max_{\boldsymbol{u}\in\mathbb{C}^L,\,\|\boldsymbol{u}\|=1}\Big|\boldsymbol{u}^{\mathsf{H}}\big(\nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\big)\Big|^2, \tag{31}$$

and therefore we bound each of the J gradients. By means of the triangle inequality, we have that

$$\begin{aligned}&\Big|\boldsymbol{u}^{\mathsf{H}}\big(\nabla_j g(\boldsymbol{q}) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\big)\Big|^2\\ &= \Big|\boldsymbol{u}^{\mathsf{H}}\nabla f(\boldsymbol{q}_j) + 2\gamma_0\boldsymbol{u}^{\mathsf{H}}\Big(\sum_{i\neq j}^{J}\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j\Big) - \boldsymbol{u}^{\mathsf{H}}\nabla f(\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{z}_j) - \gamma_0\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{u}^{\mathsf{H}}\Big(\sum_{i\neq j}^{J}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2\\ &\leq 2\Big|\boldsymbol{u}^{\mathsf{H}}\nabla f(\boldsymbol{q}_j) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla f(\boldsymbol{z}_j)\Big|^2 + 2\gamma_0^2\Big|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2.\end{aligned} \tag{32}$$

Let $D = 3 + \mathbf{1}[Q \neq 4]$. From the proof of Lemma 3, we know for all $\boldsymbol{h}_j$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j) = 0$ and $\|\boldsymbol{h}_j\| = 1$, and for all ξ with $0 \leq \xi \leq \epsilon/sqrtJ$, that

$$\begin{aligned}\Big|\boldsymbol{u}^{\mathsf{H}}\nabla f(\boldsymbol{q}_j) - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla f(\boldsymbol{z}_j)\Big|^2 &\leq 4D\xi^2 I_1 + 9D\xi^4 I_2 + D\xi^6 I_3 + 4\xi^2 I_4\cdot\mathbf{1}[Q\neq 4]\\ &\leq 4\xi^2 D(2m_2^2 + \kappa + \delta)^2 + \xi^2\big(m_2^2 + \kappa + (1+R_2)\delta\big)^2\cdot\mathbf{1}[Q\neq 4]\\ &\quad + \frac{9D\xi^4(2m_2^2+\kappa+\delta) + DB^2L\xi^6(m_2+\delta)}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^4,\end{aligned} \tag{33}$$

thus we only need to bound the second term in Eq.(32). For any $\boldsymbol{u} \in \mathbb{C}^L$ such that $\|\boldsymbol{u}\| = 1$, let $\boldsymbol{v} = \mathrm{e}^{-i\phi(\boldsymbol{q}_j)}\boldsymbol{u}$. Thus,

$$\begin{aligned}&\Big|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j - \mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2\\ &\quad= \Big|\sum_{i\neq j}^{J}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}(\boldsymbol{h}_i+\boldsymbol{z}_i)(\boldsymbol{h}_i+\boldsymbol{z}_i)^{\mathsf{H}}\boldsymbol{S}(\boldsymbol{h}_j+\boldsymbol{z}_j) - \boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big|^2\\ &\quad= \Big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j + \boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j + \boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j + \sum_{i\neq j}^{J}\Big(\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j + \boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\\ &\qquad\qquad + \boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j + \boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2\\ &\quad\leq \Bigg(\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big| + \big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big| + \big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big| + \sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big| + \sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\\ &\qquad\qquad + \sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big| + \sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Bigg)^2.\end{aligned}$$

Equivalently, for all $\boldsymbol{h}_j$ and $\boldsymbol{v}$ such that $\mathrm{Im}(\boldsymbol{h}_j^{\mathsf{H}}\boldsymbol{z}_j)=0$, $\|\boldsymbol{h}_j\|=\|\boldsymbol{v}\|=1$ and for all ξ with $0\leq\xi\leq\epsilon/\sqrt{J}$,

$$\begin{aligned}&\Big|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2\\&\quad\leq\Bigg(\xi\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|+\xi^2\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|+\xi^3\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|+\xi^2\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|+\xi^2\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\\&\qquad+\xi\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|+\xi\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Bigg)^2.\end{aligned}$$

Knowing that $\big(\sum_{i=1}^{n}a_i\big)^2\leq n\sum_{i=1}^{n}a_i^2$,

$$\begin{aligned}&\Big|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\Big|^2\\&\quad\leq 7\xi^2\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|^2+7\xi^4\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{z}_j\big|^2+7\xi^6\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{h})\boldsymbol{h}_j\big|^2+7\xi^4\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2\\&\qquad+7\xi^4\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2+7\xi^2\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2+7\xi^2\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2\\&\quad=7\xi^2I_5+7\xi^4I_6+7\xi^6I_7+7\xi^4I_8+7\xi^4I_9+7\xi^2I_{10}+7\xi^2I_{11}.\end{aligned}$$

We now bound the terms on the right-hand side. By means of Corollary 16 and large enough K,

$$I_5=\big|\boldsymbol{v}_j^{\mathsf{H}}\boldsymbol{C}_j(\boldsymbol{z})\boldsymbol{h}_j\big|^2\leq\Big(\big\|\boldsymbol{C}_j(\boldsymbol{z})\big\|\big\|\boldsymbol{h}_j\big\|\|\boldsymbol{v}\|\Big)^2\leq\Big(\frac{K+J-1}{K}m_2^2+\frac{\kappa}{K}+\delta\Big)^2.$$

Additionally, knowing that $\max_k\|\boldsymbol{s}_k\|=B\sqrt{L}\geq 1$ for all $k\in\{1,\ldots,K\}$, and by means of Corollaries 10 and 11 and the Cauchy-Schwarz inequality, we have

$$I_6=\Big|\sum_{i\neq j}^{J}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big|^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|^2\leq(J-1)(m_2+\delta)^4.\tag{34}$$

We also have that

$$I_7=\Big|\sum_{i\neq j}^{J}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\Big|^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\leq\|\boldsymbol{S}\|^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\leq(m_2+\delta)^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2.$$

By inkoving the same tools, we also have

$$\begin{aligned}I_8&=\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\leq\sum_{i\neq j}^{J}\big\|\boldsymbol{S}\big\|^4\leq(J-1)(m_2+\delta)^4,\\I_9&=\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|\Big)^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\big|^2\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\leq\|\boldsymbol{S}\|^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\leq(m_2+\delta)^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality and Corollary 17,

$$I_{10}\leq\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_i\big|^2\big|\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|^2\leq(J-1)(m_2+\delta)^2\delta^2.$$

Finally, via Cauchy-Schwarz inequality and Corollary 18,

$$I_{11}\leq\Big(\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\big|\Big)^2\leq\sum_{i\neq j}^{J}\big|\boldsymbol{v}^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_j^{\mathsf{T}}\boldsymbol{S}^{\mathsf{T}}\overline{\boldsymbol{h}_i}\big|^2\leq\sum_{i\neq j}^{J}\big\|\boldsymbol{F}_{ji}(\boldsymbol{z})\big\|^2\ \leq(J-1)(m_2+\delta)^2.$$

Therefore, we obtain

$$\begin{aligned}
&\left|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2\\
&\quad\le 7\xi^2\Big(\frac{K+J-1}{K}m_2^2+\frac{\kappa}{K}+\delta\Big)^2+7\xi^4(J-1)(m_2+\delta)^4\\
&\qquad+7\xi^6(m_2+\delta)^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2+7\xi^4(J-1)(m_2+\delta)^4\\
&\qquad+7\xi^4(m_2+\delta)^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2+7\xi^2(J-1)(m_2+\delta)^2\delta^2+7\xi^2(J-1)(m_2+\delta)^2. \qquad (35)
\end{aligned}$$

By substituting Eqs.(35) and (33) into Eq.(32), we have that

$$\begin{aligned}
&\frac{1}{\beta}\Big|\boldsymbol{u}^{\mathsf{H}}\big(\nabla_j g(\boldsymbol{q})-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla_j g(\boldsymbol{z})\big)\Big|^2\\
&\quad\le 2\Big|\boldsymbol{u}^{\mathsf{H}}\nabla f(\boldsymbol{q}_j)-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\nabla f(\boldsymbol{z}_j)\Big|^2+2\gamma_0^2\left|\sum_{i\neq j}^{J}\boldsymbol{u}^{\mathsf{H}}\Big(\boldsymbol{S}\boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{q}_j-\mathrm{e}^{i\phi(\boldsymbol{q}_j)}\boldsymbol{S}\boldsymbol{z}_i\boldsymbol{z}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{z}_j\Big)\right|^2.\\
&\quad 2\bigg(4D(2m_2^2+\kappa+\delta)^2+\big(m_2^2+\kappa+(1+R_2)\delta\big)^2\cdot\mathbf{1}[Q\neq 4]+\frac{9D\xi^2(2m_2^2+\kappa+\delta)+DB^2L\xi^4(m_2+\delta)}{K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^4\bigg)\\
&\qquad+14\gamma_0^2\xi^2\bigg(\Big(\frac{K+J-1}{K}m_2^2+\frac{\kappa}{K}+\delta\Big)^2+2\xi^2(J-1)(m_2+\delta)^4\\
&\qquad+\xi^2(1+\xi^2)(m_2+\delta)^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2+(J-1)(m_2+\delta)^2(1+\delta^2)\bigg)\\
&\quad\le\beta\bigg(\frac{2m_2^2-R_2m_2-\delta}{19}+\frac{\xi^2}{20K}\sum_{k=1}^{K}|\boldsymbol{s}_k^{\mathsf{H}}\boldsymbol{h}_j|^4+\gamma_0^2r_1+\gamma_0^2r_2\xi^2\sum_{i\neq j}^{J}\big|\boldsymbol{h}_i^{\mathsf{H}}\boldsymbol{S}\boldsymbol{h}_j\big|^2\bigg).
\end{aligned}$$

Hence, Lemma 7 holds under the following condition:

$$\begin{aligned}
\beta\ge\max\bigg\{&\frac{152D(2m_2^2+\kappa+\delta)^2}{2m_2^2-R_2m_2-\delta}+\frac{38\big(m_2^2+\kappa+(1+R_2)\delta\big)^2}{2m_2^2-R_2m_2-\delta}\cdot\mathbf{1}[Q\neq 4]\\
&+\frac{266\gamma_0^2}{2m_2^2-R_2m_2-\delta}\bigg(\Big(\frac{K+J-1}{K}m_2^2+\frac{\kappa}{K}+\delta\Big)^2+2\epsilon^2(J-1)(m_2+\delta)^4+(J-1)(m_2+\delta)^2(1+\delta^2)\bigg),\\
&360D(2m_2^2+\kappa+\delta)+40DB^2L\epsilon^2(m_2+\delta),\ 14\gamma_0^2(1+\epsilon^2)(m_2+\delta)^2\bigg\}. \qquad (36)
\end{aligned}$$

With $\epsilon=(10B\sqrt{JL})^{-1}$, $\gamma_0=1$ and $\delta\le 0.001$, Eq.(36) holds for

$$\begin{aligned}
\beta&\ge 730+267(J-1) &&\text{for}\quad Q=4,\\
\beta&\ge 1964+394(J-1) &&\text{for}\quad Q\neq 4.
\end{aligned}$$

VI. Proof of Lemma 8

Thanks to Lemma 6 and 7, this proof is a straightforward adaptation of [4, Lemma 7.10].

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