PART A

Ordinary Differential Equations (ODEs)

CHAPTER 2

Second-Order Linear ODEs

2.1 Homogeneous Linear ODEs of Second Order

A second-order ODE is called **linear** if it can be written

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

and **nonlinear** if it cannot be written in this form.

If $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

(2) $$y'' + p(x)y' + q(x)y = 0$$

and is called **homogeneous**. If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous**.

If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term.

An example of a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0,$$

written in standard form

$$y'' + \frac{1}{x} y' + y = 0.$$

An example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs.

Solutions are defined similarly as for first-order ODEs in Chap. 1.
A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''.

Homogeneous Linear ODEs: Superposition Principle

Linear ODEs have a rich solution structure. For the homogeneous equation the backbone of this structure is the *superposition principle* or *linearity principle,* which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants.
Of course, this is a great advantage of homogeneous linear ODEs.

Example 1 - Homogeneous Linear ODEs: Superposition of Solutions

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all x. We verify this by differentiation and substitution. We obtain $(\cos x)'' = -\cos x$; hence

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

Similarly for $y = \sin x$(verify!). We can go an important step further. We multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by, say, -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x)$$
$$= -4.7 \cos x + 2 \sin x + 4.7 \cos x - 2 \sin x = 0.$$

Theorem 1

Fundamental Theorem
for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

(2) $y'' + py' + qy = 0$

If $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ are solutions of (2) on I, then $\boldsymbol{y = c_1 y_1 + c_2 y_2}$ is a solution of (2).

Don't forget that this highly important theorem holds for *homogeneous linear* ODEs only, but ***does not hold*** for nonhomogeneous linear or nonlinear ODEs.

Initial Value Problem. Basis. General Solution

For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

(4) $$y(x_0) = K_0, \qquad y'(x_0) = K_1.$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

(5) $$y = c_1 y_1 + c_2 y_2$$

of the ODE;
here, y_1 and y_2 are suitable solutions of the ODE.
This results in a unique solution, passing through the point (x_0, K_0) with K_1 as the tangent direction (the slope) at that point.
That solution is called a **particular solution** of the ODE (2).

Example 4 – Solve the Initial Value Problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution. *Step 1. General solution.* The functions $\cos x$ and $\sin x$ are solutions of the ODE (by Example 1), and we take

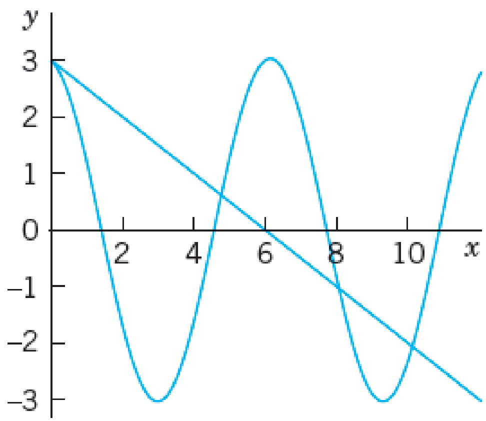

Fig. 29. Particular solution and initial tangent in Example 4

$$y = c_1 \cos x + c_2 \sin x.$$

This will turn out to be a general solution as defined below.

Step 2. Particular solution. We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, since $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3.0 \qquad \text{and} \qquad y'(0) = c_2 = -0.5.$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x.$$

Figure 29 shows that at $x = 0$ it has the value 3.0 and the slope -0.5, so that its tangent intersects the x-axis at $x = 3.0/0.5 = 6.0$. (The scales on the axes differ!)

Observation. Our choice of y_1 and y_2 was general enough to satisfy both initial conditions. Now let us take instead two proportional solutions $y_1 = \cos x$ and $y_2 = k \cos x$, so that $y_1/y_2 = 1/k = \text{const}$. Then we can write $y = c_1 y_1 + c_2 y_2$ in the form

$$y = c_1 \cos x + c_2(k \cos x) = C \cos x \qquad \text{where} \qquad C = c_1 + c_2 k.$$

No longer able to satisfy two initial conditions with only one arbitrary constant C. Consequently, in defining the concept of a general solution, we must **exclude proportionality**.

Definition

General Solution, Basis, Particular Solution

A **general solution** of an ODE (2) on an open interval I is a solution (5) in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called a **basis** (or a **fundamental system**) of solutions of (2) on I.

A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (5).

Furthermore, as usual, y_1 and y_2 are called *proportional* on I if for all x on I,

$$(6) \qquad \text{(a) } y_1 = ky_2 \qquad \text{or} \qquad \text{(b) } y_2 = ly_1$$

where k and l are numbers, zero or not. (Note that (a) implies (b) if and only if $k \neq 0$).

Two functions y_1 and y_2 are called **linearly independent** on an interval I where they are defined if

(7) $k_1y_1(x) + k_2y_2(x) = 0$ everywhere on I implies $k_1 = 0$ and $k_2 = 0$.

And y_1 and y_2 are called **linearly dependent** on I if (7) also holds for some constants k_1, k_2 not both zero.
Then, if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1} y_2 \quad \text{or} \quad y_2 = -\frac{k_1}{k_2} y_1.$$

In contrast, in the case of linear *independence* these functions are not proportional because then we cannot divide in (7). This gives the following definition. (See next slide.)

Definition

Basis (Reformulated)

A **basis** of solutions of (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

Example 6 - Basis, General Solution, Particular Solution

Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y'' - y = 0, \qquad y(0) = 6, \qquad y'(0) = -2.$$

Solution. $(e^x)'' - e^x = 0$ and $(e^{-x})'' - e^{-x} = 0$ show that e^x and e^{-x} are solutions. They are not proportional, $e^x/e^{-x} = e^{2x} \neq$ const. Hence e^x, e^{-x} form a basis for all x. We now write down the corresponding general solution and its derivative and equate their values at 0 to the given initial conditions,

$$y = c_1e^x + c_2e^{-x}, \qquad y' = c_1e^x - c_2e^{-x}, \qquad y(0) = c_1 + c_2 = 6, \qquad y'(0) = c_1 - c_2 = -2.$$

By addition and subtraction, $c_1 = 2$, $c_2 = 4$, so that the *answer* is $y = 2e^x + 4e^{-x}$. This is the particular solution satisfying the two initial conditions. ■

Find a Basis If One Solution Is Known. Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way.

Then a second linearly independent solution can be obtained by solving a first-order ODE.

This is called the method of **reduction of order**.

We first show how this method works in an example and then in general.

EXAMPLE 7

Reduction of Order If a Solution Is Known. Basis

Find a basis of solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

***Solution.* (See next slide.)**

EXAMPLE 7 (continued)

Solution.

Inspection shows that $y_1 = x$ is a solution because $y'_1 = 1$ and $y''_1 = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u, \quad y'' = u''x + 2u'$$

into the ODE. This gives

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

EXAMPLE 7 (continued)

Solution. *(continued 1)*

This ODE is of first order in $v = u'$,
namely, $(x^2 - x)v' + (x - 2)v = 0$.
Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}\,dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx,$$

$$\ln|v| = \ln|x-1| - 2\ \ln|x| = \ln\frac{|x-1|}{x^2}.$$

(continued)

EXAMPLE 7 (continued)

Solution. (*continued 2*)

We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x}$$

hence $y_2 = ux = x\ln|x| + 1.$

Since $y_1 = x$ and $y_2 = x \ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive x.

Reduction of order to a homogeneous linear ODE

$$(2) \qquad y'' + p(x)y' + q(x)y = 0.$$ ⇐ ODE in standard form

Assume a solution y_1 to be known and want to find a basis. For this we need a second linearly independent solution y_2. To get y_2, we substitute

$$y = y_2 = uy_1, \qquad y' = y_2' = u'y_1 + uy_1', \qquad y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into (2). This gives

$$(8) \qquad u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0.$$

Collecting terms in u'', u', and u, we have

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0.$$

Now comes the main point. Since y_1 is a solution of (2), the expression in the last parentheses is zero. Hence u is gone, and we are left with an ODE in u' and u''. We divide this remaining ODE by y_1 and set $u' = U$, $u'' = U'$,

$$u'' + u' \frac{2y_1' + py_1}{y_1} = 0, \qquad \text{thus} \qquad U' + \left(\frac{2y_1'}{y_1} + p\right) U = 0.$$

This is the desired first-order ODE, the reduced ODE. Separation of variables and integration gives

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right) dx \qquad \text{and} \qquad \ln |U| = -2 \ln |y_1| - \int p \, dx.$$

By taking exponents we finally obtain

$$U = \frac{1}{y_1^2} e^{-\int p \, dx}. \tag{9}$$

Here $U = u'$, so that $u = \int U \, dx$. Hence the desired second solution is

$$y_2 = y_1 u = y_1 \int U \, dx.$$

The quotient $y_2/y_1 = u = \int U \, dx$ cannot be constant (since $U > 0$), so that y_1 and y_2 form a basis of solutions.