Engineering Mathematics
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Homework #5 Solution

Due: *December 25, 2020*

1. Problem Set 6.5 – 3

   Find the convolution.

   3. $e^{-t} * e^{t}$

   Sol:

$$e^{-t} * e^{t} = \int_{\tau=0}^{t} e^{-\tau} \cdot e^{t-\tau} d\tau = \int_{\tau=0}^{t} e^{t} e^{-2\tau} d\tau$$

$$= e^{t}\left(-\frac{1}{2}e^{-2\tau}\right)_{\tau=0}^{t} = e^{t}\left(\frac{-e^{-2t}+1}{2}\right) = \frac{e^{t}-e^{-t}}{2} = \sinh t$$

2. Problem Set 6.5 – 9

   Solve by the Laplace transform, showing details:

   9. $y(t) + \int_0^t y(\tau)\, d\tau = 2$

   Sol:

$$\mathcal{L}\left\{y(t) + \int_{\tau=0}^{t} y(\tau) d\tau\right\} = \frac{2}{s}$$

$$Y(s) + Y(s)\frac{1}{s} = \frac{2}{s}$$

$$Y(s)\left(1+\frac{1}{s}\right) = \frac{2}{s} \qquad Y(s)\left(\frac{s+1}{s}\right) = \frac{2}{s}$$

$$Y(s) = \frac{2}{s+1}$$

$$y(t) = 2e^{-t}$$

3. Problem Set 6.5 – 11

   Solve by the Laplace transform, showing details:

   11. $y(t) - \int_0^t (t-\tau) y(\tau)\, d\tau = 1$

   Sol:

$$\mathcal{L}\left\{y(t) - \int_{\tau=0}^{t} (t-\tau) y(\tau) d\tau\right\} = \frac{1}{s}$$

$$Y(s) - Y(s)\frac{1}{s^2} = \frac{1}{s}$$

$$Y(s)\left(1-\frac{1}{s^2}\right) = \frac{1}{s} \qquad Y(s)\left(\frac{s^2-1}{s^2}\right) = \frac{1}{s}$$

$$Y(s)=\frac{s}{s^2-1}=\frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)$$

$$y(t)=\frac{1}{2}\left(e^t+e^{-t}\right)=\cosh t$$

4. Problem Set 6.5 – 13

   Solve by the Laplace transform, showing details:

   13. $y(t)+2e^t\int_0^t y(\tau)e^{-\tau}\,d\tau=te^t$

   Sol:

$$\mathcal{L}\left\{y(t)+2e^t\int_{\tau=0}^t y(\tau)e^{-\tau}d\tau\right\}=\frac{1}{(s-1)^2} \qquad \mathcal{L}\left\{y(t)+2e^te^{-t}\int_{\tau=0}^t y(\tau)e^{(t-\tau)}d\tau\right\}=\frac{1}{(s-1)^2}$$

$$Y(s)+2Y(s)\left(\frac{1}{s-1}\right)=\frac{1}{(s-1)^2}$$

$$Y(s)\left(1+\frac{2}{s-1}\right)=\frac{1}{(s-1)^2} \qquad Y(s)\left(\frac{s+1}{s-1}\right)=\frac{1}{(s-1)^2}$$

$$Y(s)=\frac{1}{(s-1)(s+1)}=\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

$$y(t)=\frac{1}{2}\left(e^t-e^{-t}\right)=\sinh t$$

5. Problem Set 6.7 – 3

   Using the Laplace transform and showing the details of your work, solve the IVP:

   3. $y_1'-2y_1+3y_2=0,\ y_2'-y_1+2y_2=0,\ y_1(0)=1,\ y_2(0)=0$

   Sol:

Writing $\mathcal{L}\{y_1\}=Y_1(s),\ \ \mathcal{L}\{y_2\}=Y_2(s)$

$sY_1(s)-y_1(0)-2Y_1(s)+3Y_2(s)=0$
$sY_2(s)-y_2(0)-Y_1(s)+2Y_2(s)=0$ Replace $y_1(0)=1, y_2(0)=0$
$(s-2)Y_1(s)+3Y_2(s)=1$ ①
$-Y_1(s)+(s+2)Y_2(s)=0$ ②

①×(s+2)－② × 3 $\left(s^2-4+3\right)Y_1(s)=s+2$

$Y_1(s)=\frac{s+2}{s^2-1}$ Then from② $Y_2(s)=\frac{1}{s+2}Y_1(s)=\frac{1}{s^2-1}$

Furthermore $Y_1(s)=\frac{s+2}{s^2-1}=\frac{\frac{3}{2}}{s-1}-\frac{\frac{1}{2}}{s+1}$

$$Y_2(s)=\frac{1}{s^2-1}=\frac{\frac{1}{2}}{s-1}-\frac{\frac{1}{2}}{s+1}$$

$$y_1(t)=\frac{3}{2}e^t-\frac{1}{2}e^{-t}=\frac{1}{2}e^t+\frac{1}{2}e^{-t}+2\left(\frac{1}{2}e^t-\frac{1}{2}e^{-t}\right)=\cosh t+2\sinh t$$

$$y_2(t)=\frac{1}{2}e^t-\frac{1}{2}e^{-t}=\sinh t$$

6. Problem Set 6.7 – 5
   Using the Laplace transform and showing the details of your work, solve the IVP:
   5. $y_1'=y_2+2-u(t-1),\ y_2'=-y_1+1-u(t-1),\ y_1(0)=1,\ y_2(0)=0$
   <u>Sol:</u>

Writing $\mathcal{L}\{y_1\}=Y_1(s),\ \ \mathcal{L}\{y_2\}=Y_2(s)$

$$sY_1(s)-y_1(0)=Y_2(s)+\frac{2}{s}-\frac{e^{-s}}{s}$$

$$sY_2(s)-y_2(0)=-Y_1(s)+\frac{1}{s}-\frac{e^{-s}}{s}\quad \text{Replace}\quad y_1(0)=1,\ y_2(0)=0$$

$$sY_1(s)-Y_2(s)=1+\frac{2}{s}-\frac{e^{-s}}{s}\qquad ①$$

$$Y_1(s)+sY_2(s)=\frac{1}{s}-\frac{e^{-s}}{s}\qquad ②$$

①× s + ②

$$\left(s^2+1\right)Y_1(s)=s+2-e^{-s}+\frac{1}{s}-\frac{e^{-s}}{s}=2+\frac{s^2+1}{s}-\left(\frac{s+1}{s}\right)e^{-s}$$

$$Y_1(s)=\frac{2}{s^2+1}+\frac{1}{s}-\frac{s+1}{s\left(s^2+1\right)}e^{-s}$$

$$=\frac{2}{s^2+1}+\frac{1}{s}-\left(\frac{1}{s}-\frac{s-1}{s^2+1}\right)e^{-s}=\frac{2}{s^2+1}+\frac{1}{s}-\left(\frac{1}{s}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right)e^{-s}$$

① − ② × s

$$\left(-1-s^2\right)Y_2(s)=1+\frac{2}{s}-\frac{e^{-s}}{s}-1+e^{-s}=\frac{2}{s}-\frac{e^{-s}}{s}+e^{-s}$$

$$Y_2(s)=-\frac{2}{s\left(s^2+1\right)}+\left[-\frac{1}{s^2+1}+\frac{1}{s\left(s^2+1\right)}\right]e^{-s}=-\frac{2}{s}+\frac{2s}{s^2+1}+\left(-\frac{1}{s^2+1}+\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-s}$$

$$y_1(t)=1+2\sin t-[1-\cos(t-1)+\sin(t-1)]u(t-1)$$
$$y_2(t)=-2+2\cos t+[1-\sin(t-1)-\cos(t-1)]u(t-1)$$

7. Problem Set 6.7 – 11

Using the Laplace transform and showing the details of your work, solve the IVP:

11. $y_1'' = y_1 + 3y_2,\ y_2'' = 4y_1 - 4e^t,\ y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = 1,\ y_2'(0) = 2$

<u>Sol:</u>

Writing $\mathcal{L}\{y_1\} = Y_1(s),\ \ \mathcal{L}\{y_2\} = Y_2(s)$

$$s^2 Y_1(s) - s y_1(0) - y_1'(0) = Y_1(s) + 3Y_2(s)$$

$$s^2 Y_2(s) - s y_2(0) - y_2'(0) = 4Y_1(s) - \frac{4}{s-1}$$

Replace $y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = 1,\ y_2'(0) = 2$

$$\left(s^2 - 1\right)Y_1(s) - 3Y_2(s) = 2s + 3 \qquad ①$$

$$-4Y_1(s) + s^2 Y_2(s) = s + 2 - \frac{4}{s-1} \qquad ②$$

$① \times s^2 + ② \times 3$

$$\left[s^2\left(s^2 - 1\right) - 12\right]Y_1(s) = (2s+3)s^2 + 3s + 6 - \frac{12}{s-1} = 2s^3 + 3s^2 + 3s + 6 - \frac{12}{s-1}$$

$$Y_1(s) = \frac{2s^3 + 3s^2 + 3s + 6}{s^4 - s^2 - 12} - \frac{12}{\left(s^4 - s^2 - 12\right)(s-1)}$$

$$= \frac{2s^3 + 3s^2 + 3s + 6}{\left(s^2 - 4\right)\left(s^2 + 3\right)} - \frac{12}{\left(s^2 - 4\right)\left(s^2 + 3\right)(s-1)}$$

$$= \frac{\frac{11}{7}s + \frac{18}{7}}{s^2 - 4} + \frac{\frac{3}{7}s + \frac{3}{7}}{s^2 + 3} - \left(\frac{\frac{4}{7}s + \frac{4}{7}}{s^2 - 4} + \frac{\frac{3}{7}s + \frac{3}{7}}{s^2 + 3} - \frac{1}{s-1}\right)$$

$$= \frac{s+2}{s^2 - 4} + \frac{1}{s-1} = \frac{1}{s-2} + \frac{1}{s-1}$$

$① \times 4 + ② \times \left(s^2 - 1\right)$ $\qquad \left[-12 + s^2\left(s^2 - 1\right)\right]Y_2(s) = 8s + 12 + \left(s^2 - 1\right)\left(s + 2 - \frac{4}{s-1}\right)$

$$\left(s^4 - s^2 - 12\right)Y_2(s) = 8s + 12 + s^3 - s + 2s^2 - 2 - \frac{4\left(s^2 - 1\right)}{s-1}$$

$$\left(s^2 - 4\right)\left(s^2 + 3\right)Y_2(s) = 8s + 12 + s^3 - s + 2s^2 - 2 - 4(s+1)$$

$$\left(s^2-4\right)\left(s^2+3\right)Y_2(s)=s^3+2s^2+3s+6$$

$$Y_2(s)=\frac{s^3+2s^2+3s+6}{\left(s^2-4\right)\left(s^2+3\right)}=\frac{(s+2)\left(s^2+3\right)}{\left(s^2-4\right)\left(s^2+3\right)}=\frac{1}{s-2}$$

$$y_1(t)=e^{2t}+e^t$$

$$y_2(t)=e^{2t}$$

8. Problem Set 6.7 – 15
   Using the Laplace transform and showing the details of your work, solve the IVP:
   15. $-y_1'+y_2'=2\cosh t,\ y_2'-y_3'=e^{-t},\ y_3'+y_1'=2e^{-t}\ \ y_1(0)=1,\ y_2(0)=0,$ $y_3(0)=1$

   <u>Sol:</u>

Writing $\mathcal{L}\{y_1\}=Y_1(s),\ \ \mathcal{L}\{y_2\}=Y_2(s),\ \ \mathcal{L}\{y_3\}=Y_3(s)$

$$\mathcal{L}\{\cosh t\}=\frac{s}{s^2-1}\qquad \mathcal{L}\left\{e^{-t}\right\}=\frac{1}{s+1}$$

$$-sY_1(s)+y_1(0)+sY_2(s)-y_2(0)=\frac{2s}{s^2-1}$$

$$sY_2(s)-y_2(0)-sY_3(s)+y_3(0)=\frac{1}{s+1}$$

$$sY_1(s)-y_1(0)+sY_3(s)-y_3(0)=\frac{2}{s+1}\qquad \text{Replace}\quad y_1(0)=0,\ \ y_2(0)=0, y_3(0)=1$$

$$-sY_1(s)+sY_2(s)=\frac{2s}{s^2-1}\qquad ①$$

$$sY_2(s)-sY_3(s)=-1+\frac{1}{s+1}\qquad ②$$

$$sY_1(s)+sY_3(s)=1+\frac{2}{s+1}\qquad ③$$

$$①-②\qquad -sY_1(s)+sY_3(s)=\frac{2s}{s^2-1}+1-\frac{1}{s+1}\qquad ④$$

$$③+④\quad 2sY_3(s)=1+\frac{2}{s+1}+\frac{2s}{s^2-1}+1-\frac{1}{s+1}=2+\frac{1}{s+1}+\frac{2s}{s^2-1}$$

$$Y_3(s)=\frac{1}{s}+\frac{1}{2s(s+1)}+\frac{1}{s^2-1}=\frac{1}{s}+\frac{\frac{1}{2}}{s}-\frac{\frac{1}{2}}{s+1}+\frac{\frac{1}{2}}{s-1}-\frac{\frac{1}{2}}{s+1}$$

$=\frac{\frac{3}{2}}{s}-\frac{1}{s+1}+\frac{\frac{1}{2}}{s-1}$ Then substituted into ② we get

$Y_2(s)=\frac{1}{2s}+\frac{\frac{1}{2}}{s-1}+\frac{-s+1}{s(s+1)}=\frac{1}{2s}+\frac{\frac{1}{2}}{s-1}+\frac{1}{s}-\frac{2}{s+1}=\frac{\frac{3}{2}}{s}+\frac{\frac{1}{2}}{s-1}-\frac{2}{s+1}$ And substituted into ①

$$sY_1(s)=sY_2(s)-\frac{2s}{s^2-1}=\frac{3}{2}+\frac{\frac{1}{2}s}{s-1}-\frac{2s}{s+1}-\frac{2s}{s^2-1}$$

$$Y_1(s)=\frac{3}{2s}+\frac{\frac{1}{2}}{s-1}-\frac{2}{s+1}-\frac{2}{s^2-1}=\frac{3}{2s}+\frac{\frac{1}{2}}{s-1}-\frac{2}{s+1}-\frac{1}{s-1}+\frac{1}{s+1}=\frac{3}{2s}-\frac{\frac{1}{2}}{s-1}-\frac{1}{s+1}$$

$$y_1(t)=\frac{3}{2}-\frac{1}{2}e^t-e^{-t}$$

$$y_2(t)=\frac{3}{2}+\frac{1}{2}e^t-2e^{-t}$$

$$y_3(t)=\frac{3}{2}-e^{-t}+\frac{1}{2}e^t$$