Engineering Mathematics
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Homework #4 Solution

Due: *December 18, 2020*

1. Problem Set 6.1 – 5

Find the Laplace transform.

5. $e^{3t}\sinh t$

Sol:

$$e^{3t}\sinh t = e^{3t}\left(\frac{e^{t}-e^{-t}}{2}\right) = \frac{e^{4t}-e^{2t}}{2}$$

$$\mathcal{L}\left\{\frac{e^{4t}-e^{2t}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-4}-\frac{1}{s-2}\right) = \frac{1}{2}\frac{2}{(s-4)(s-2)} = \frac{1}{(s-4)(s-2)}$$

2. Problem Set 6.1 – 43

Find the inverse Laplace transform.

43. $\frac{6s+7}{2s^2+4s+10}$

Sol:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{6s+7}{2(s+1)^2+8}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+\frac{7}{2}}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s+1)+\frac{1}{2}}{(s+1)^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s+1)}{(s+1)^2+2^2} + \frac{\frac{1}{4}\times 2}{(s+1)^2+2^2}\right\} = 3e^{-t}\cos 2t + \frac{1}{4}e^{-t}\sin 2t$$

3. Problem Set 6.2 – 11

Solve the IVPs by the Laplace transform.

11. $y'' + 3y' + 2.25y = 9t^3 + 64,\ y(0) = 1,\ y'(0) = 31.5$

Sol:

Set $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y''+3y'+2.25y\} = 9\times\frac{3!}{s^4}+\frac{64}{s},$$

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2.25Y(s) = \frac{54}{s^4}+\frac{64}{s}$$

Insert $y(0)=1,\ y'(0)=31.5$

$$\left(s^2+3s+2.25\right)Y(s) = s+31.5+3+\frac{54}{s^4}+\frac{64}{s} = s+34.5+\frac{54}{s^4}+\frac{64}{s}$$

$$Y(s) = \frac{s+34.5}{\left(s^2+3s+2.25\right)} + \frac{54}{s^4\left(s^2+3s+2.25\right)} + \frac{64}{s\left(s^2+3s+2.25\right)}$$

$$= \frac{s+34.5}{\left(s^2+3s+2.25\right)} + \frac{32s^2-32s+24}{s^4} - \frac{32}{s^2+3s+2.25}$$

$$= \frac{s+2.5}{s^2+3s+2.25} + \frac{32s^2-32s+24}{s^4}$$

$$= \frac{(s+1.5)+1}{(s+1.5)^2} + \frac{32}{s^2} - \frac{32}{s^3} + \frac{24}{s^4}$$

$$= \frac{1}{s+1.5} + \frac{1}{(s+1.5)^2} + \frac{32}{s^2} - 16\times\frac{2}{s^3} + 4\times\frac{3!}{s^4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-1.5t} + te^{-1.5t} + 32t - 16t^2 + 4t^3$$

4. Problem Set 6.2 – 13

   Solve the shifted data IVPs by the Laplace transform.

   13. $y' - 6y = 0,\ y(-1) = 4$

   Sol:

   Set $t+1=\tau,\ \mathrm{t}=\tau-1, \mathrm{t}=-1 \to \tau=0, y(-1)\to \tilde{y}(0), y(t)\to\tilde{y}(\tau)$

   Let $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{\tilde{y}(\tau)\} = \tilde{Y}(\tilde{s})$

   $$\tilde{s}\tilde{Y}(\tilde{s}) - \tilde{y}(0) - 6\tilde{Y}(\tilde{s}) = 0 \qquad (\tilde{s}-6)\tilde{Y}(\tilde{s}) = 4$$

   $$\tilde{Y}(\tilde{s}) = \frac{4}{\tilde{s}-6}$$

   $$\tilde{y}(\tau) = \mathcal{L}^{-1}\{\tilde{Y}(\tilde{s})\} = 4e^{6\tau} \quad \text{and} \quad \tau = t+1, \qquad y(t) = 4e^{6(t+1)}$$

5. Problem Set 6.2 – 27

   Using Theorem 3, find $f(t)$ if $\mathcal{L}(F)$ equals:

   27. $\frac{s+8}{s^4+4s^2}$

   Sol:

   $$F(s) = \frac{s+8}{s^4+4s^2} = \frac{s+8}{s^2\left(s^2+4\right)}$$

   and $\frac{s+8}{s^2+4} = \frac{s}{s^2+4} + \frac{8}{s^2+4} = \frac{s}{s^2+2^2} + \frac{4\cdot 2}{s^2+2^2}$

$$\mathcal{L}^{-1}\left\{\frac{s+8}{s^2+4}\right\}=\cos 2t+4\sin 2t$$

Then $\mathcal{L}^{-1}\left\{\dfrac{s+8}{s\left(s^2+4\right)}\right\}=\int_{\tau=0}^{t}(\cos 2\tau+4\sin 2\tau)d\tau=\left(\frac{1}{2}\sin 2\tau-2\cos 2\tau\right)_{\tau=0}^{t}$

$$=\frac{1}{2}(\sin 2t-\sin 0)-2(\cos 2t-\cos 0)=\frac{1}{2}\sin 2t-2\cos t+2$$

$$\mathcal{L}^{-1}\left\{\frac{s+8}{s^2\left(s^2+4\right)}\right\}=\int_{\tau=0}^{t}\left(\frac{1}{2}\sin 2\tau-2\cos\tau+2\right)d\tau$$

$$=\left(-\frac{1}{4}\cos 2\tau-\sin 2\tau+2\tau\right)_{\tau=0}^{t}$$

$$=-\frac{1}{4}\cos 2t+\frac{1}{4}-\sin 2t+2t$$

6. Problem Set 6.3 – 19

Using the Laplace transform and showing the details, solve

19. $y''-6y'+8y=e^{-t}-e^{-4t},\ y(0)=1,\ y'(0)=4$

<u>Sol:</u>

$$\mathcal{L}\{y''-6y'+8y\}=\mathcal{L}\left\{e^{-t}-e^{-4t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-6[sY(s)-y(0)]+8Y(s)=\frac{1}{s+1}-\frac{1}{s+4}$$

Insert $y(0)=1, y'(0)=4$ into

$$s^2Y(s)-s-4-6sY(s)+6+8Y(s)=\frac{1}{s+1}-\frac{1}{s+4}$$

$$\left(s^2-6s+8\right)Y(s)=s-2+\frac{1}{s+1}-\frac{1}{s+4}$$

$$(s-2)(s-4)Y(s)=s-2+\frac{1}{s+1}-\frac{1}{s+4}$$

$$Y(s)=\frac{1}{s-4}+\frac{1}{(s-2)(s-4)(s+1)}-\frac{1}{(s-2)(s-4)(s+4)}$$

$$=\frac{1}{s-4}-\frac{\frac{1}{6}}{(s-2)}+\frac{\frac{1}{10}}{(s-4)}+\frac{\frac{1}{15}}{(s+1)}+\frac{\frac{1}{12}}{(s-2)}-\frac{\frac{1}{16}}{(s-4)}-\frac{\frac{1}{48}}{(s+4)}$$

$$=\frac{\frac{83}{80}}{s-4}-\frac{\frac{1}{12}}{(s-2)}+\frac{\frac{1}{15}}{(s+1)}-\frac{\frac{1}{48}}{(s+4)}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\tfrac{83}{80}e^{4t}-\tfrac{1}{12}e^{2t}+\tfrac{1}{15}e^{-t}-\tfrac{1}{48}e^{-4t}$$

7. Problem Set 6.3 – 23

Using the Laplace transform and showing the details, solve

23. $y''+y'-2y=3\sin t-\cos t,\ 0<t<2\pi$, and $3\sin 2t-\cos 2t,\ t>2\pi;\ y(0)=1,\ y'(0)=-1$

<u>Sol:</u>

Let $r(t)=\begin{cases}3\sin t-\cos t, & 0<t<2\pi\\ 3\sin 2t-\cos 2t, & t>2\pi\end{cases}$

$$r(t)=(3\sin t-\cos t)[u(t)-u(t-2\pi)]+(3\sin 2t-\cos 2t)u(t-2\pi)$$
$$=(3\sin t-\cos t)u(t)-(3\sin t-\cos t)u(t-2\pi)+(3\sin 2t-\cos 2t)u(t-2\pi)$$
$$=(3\sin t-\cos t)u(t)-[3\sin(t-2\pi)-\cos(t-2\pi)]u(t-2\pi)+[3\sin 2(t-2\pi)-\cos 2(t-2\pi)]u(t-2\pi)$$

$$s^2Y-sy(0)-y'(0)+sY-y(0)-2Y$$
$$=\frac{3}{s^2+1}-\frac{s}{s^2+1}-(\frac{3}{s^2+1}-\frac{s}{s^2+1})e^{-2\pi s}+(\frac{6}{s^2+4}-\frac{s}{s^2+4})e^{-2\pi s}$$

$$(s^2+s-2)Y=-1+\frac{3}{s^2+1}-\frac{s}{s^2+1}-(\frac{3}{s^2+1}-\frac{s}{s^2+1})e^{-2\pi s}+(\frac{6}{s^2+4}-\frac{s}{s^2+4})e^{-2\pi s}$$

$$Y=-\frac{1}{(s-1)(s+2)}+\frac{3-s}{(s-1)(s+2)(s^2+1)}-[\frac{3-s}{(s-1)(s+2)(s^2+1)}]e^{-2\pi s}+[\frac{6-s}{(s-1)(s+2)(s^2+4)}]e^{-2\pi s}$$

$$Y=\frac{\frac{1}{3}}{s+2}-\frac{\frac{1}{3}}{s-1}+\left[\frac{\frac{1}{3}}{S-1}+\frac{-\frac{1}{3}}{S+2}+\frac{-1}{S^2+1}\right]-\left[\frac{\frac{1}{3}}{S-1}+\frac{-\frac{1}{3}}{S+2}+\frac{-1}{S^2+1}\right]e^{-2\pi s}+\left[\frac{\frac{1}{3}}{S-1}+\frac{-\frac{1}{3}}{S+2}+\frac{-1}{S^2+4}\right]e^{-2\pi s}$$

$$Y=-\frac{1}{s^2+1}+\left[\frac{1}{s^2+1}-\frac{1}{s^2+4}\right]e^{-2\pi s}$$

$$y(t)=-\sin t+\sin(t-2\pi)u(t-2\pi)-\tfrac{1}{2}\sin 2(t-2\pi)u(t-2\pi)$$

$$=-\sin t+\sin t\,u(t-2\pi)-\tfrac{1}{2}\sin 2t\,u(t-2\pi)$$

$$=-\sin t+[\sin t-\tfrac{1}{2}\sin 2t]\,u(t-2\pi)$$

$(i)\ 0<t<2\pi$

$y(t)=-\sin t$

$(ii)\ t>2\pi$

$$y(t)=-\sin t+\sin t-\tfrac{1}{2}\sin 2t=-\tfrac{1}{2}\sin 2t$$

8. Problem Set 6.4 – 5

Find the solution of the IVP. Show the details.

5. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi),\ y(0)=0,\ y'(0)=1$

Sol:

Make Laplace transformation on both sides of the equation

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s};\quad \text{y(0) = 0, y'(0) = 1} \quad \text{are then inserted}$$

$$\left(s^2+4\right)Y(s) = 1 + e^{-\pi s} - e^{-2\pi s} \qquad Y(s) = \frac{1}{s^2+4} + \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2(t-\pi)u(t-\pi) - \frac{1}{2}\sin 2(t-2\pi)u(t-2\pi)$$

$$= \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2tu(t-\pi) - \frac{1}{2}\sin 2tu(t-2\pi)$$

$$= \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2t[u(t-\pi) - u(t-2\pi)]$$

9. Problem Set 6.4 – 9

Find the solution of the IVP. Show the details.

9. $y'' + 2y' + 2y = [1 - u(t-2)]e^t - e^2\delta(t-2),\ y(0)=0,\ y'(0)=1$

Sol:

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = \frac{1}{s-1} - \frac{e^2e^{-2s}}{s-1} - e^2e^{-2s} \quad \text{and } y(0)=0,\ y'(0)=1$$

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = \frac{1}{s-1} - \frac{e^2e^{-2s}}{s-1} - e^2e^{-2s}$$

$$\left(s^2+2s+2\right)Y(s) = 1 + \frac{1}{s-1} - \frac{e^2e^{-2s}}{s-1} - e^2e^{-2s}$$

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{(s-1)\left(s^2+2s+2\right)} - \frac{e^2e^{-2s}}{(s-1)\left(s^2+2s+2\right)} - \frac{e^2e^{-2s}}{s^2+2s+2}$$

$$= \frac{1}{s^2+2s+2} + \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}s+\frac{3}{5}}{s^2+2s+2} - \left(\frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}s+\frac{3}{5}}{s^2+2s+2}\right)e^2e^{-2s} - \frac{e^2e^{-2s}}{s^2+2s+2}$$

$$= \frac{1}{(s+1)^2+1} + \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}s+\frac{3}{5}}{(s+1)^2+1} - \left(\frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}s+\frac{3}{5}}{(s+1)^2+1}\right)e^2e^{-2s} - \frac{e^2e^{-2s}}{(s+1)^2+1}$$

$$= \frac{1}{(s+1)^2+1} + \frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}(s+1)+\frac{2}{5}}{(s+1)^2+1} - \left[\frac{\frac{1}{5}}{s-1} - \frac{\frac{1}{5}(s+1)+\frac{2}{5}}{(s+1)^2+1}\right]e^2e^{-2s} - \frac{e^2e^{-2s}}{(s+1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= e^{-t}\sin t + \tfrac{1}{5}e^{t} - \tfrac{1}{5}e^{-t}\cos t + \tfrac{2}{5}e^{-t}\sin t - \tfrac{1}{5}e^{(t-2)}e^{2}u(t-2)$$

$$+ \tfrac{1}{5}e^{-(t-2)}e^{2}\cos(t-2)u(t-2) + \tfrac{2}{5}e^{-(t-2)}e^{2}\sin(t-2)u(t-2) - e^{-(t-2)}e^{2}\sin(t-2)u(t-2)$$

$$= \tfrac{3}{5}e^{-t}\sin t + \tfrac{1}{5}e^{t} - \tfrac{1}{5}e^{-t}\cos t - \tfrac{1}{5}e^{t}u(t-2) + \tfrac{1}{5}e^{-(t-2)}e^{2}\cos(t-2)u(t-2) - \tfrac{3}{5}e^{-(t-2)}e^{2}\sin(t-2)u(t-2)$$

$$= \tfrac{3}{5}e^{-t}\sin t + \tfrac{1}{5}e^{t} - \tfrac{1}{5}e^{-t}\cos t - \tfrac{1}{5}e^{t}u(t-2) + \left[\tfrac{1}{5}\cos(t-2) - \tfrac{3}{5}\sin(t-2)\right]e^{-(t-4)}u(t-2)$$

10. Problem Set 6.4 – 11

Find the solution of the IVP. Show the details.

11. $y'' + 3y' + 2y = u(t-1) + \delta(t-2),\ y(0) = 0,\ y'(0) = 1$

Sol:

$$s^{2}Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{e^{-s}}{s} + e^{-2s} \quad \text{and} \quad y(0)=0,\ y'(0)=1$$

$$s^{2}Y(s) - 1 + 3sY(s) + 2Y(s) = \frac{e^{-s}}{s} + e^{-2s}$$

$$\left(s^{2} + 3s + 2\right)Y(s) = 1 + \frac{e^{-s}}{s} + e^{-2s}$$

$$s^{2} + 3s + 2 = (s+1)(s+2)$$

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{e^{-s}}{s(s+1)(s+2)} + \frac{e^{-2s}}{(s+1)(s+2)}$$

$$= \frac{1}{s+1} - \frac{1}{s+2} + \left(\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right)e^{-s} + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-2s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= e^{-t} - e^{-2t} + \left[\tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}\right]u(t-1) + \left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2)$$