# 1.3 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

**(1)** $$g(y)\, y' = f(x)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to $x$, obtaining

(2) $$\int g(y)\, y'\, dx = \int f(x)\, dx + c.$$

On the left we can switch to $y$ as the variable of integration. By calculus, $y'\,dx = dy$ so that

**(3)** $$\int g(y)\, dy = \int f(x)\, dx + c.$$

If $f$ and $g$ are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: $x$ appears only on the right and $y$ only on the left.

# EXAMPLE 1

## Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

*It is very important to introduce the constant of integration immediately when the integration is performed.* If we wrote $\arctan y = x$, then $y = \tan x$, and *then* introduced $c$, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$). Verify this.

## Differentiation Table

$$(cu)' = cu' \qquad (c \text{ constant})$$

$$(u + v)' = u' + v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \qquad \text{(Chain rule)}$$

$$(x^n)' = nx^{n-1}$$

$$(e^x)' = e^x$$

$$(e^{ax})' = ae^{ax}$$

$$(a^x)' = a^x \ln a$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{\log_a e}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\text{arccot } x)' = -\frac{1}{1 + x^2}$$

## EXAMPLE 5

**Mixing Problem**

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time $t$.

## EXAMPLE 5 (continued)

***Solution.***

***Step 1. Setting up a model.***

Let $y(t)$ denote the amount of salt in the tank at time $t$. Its time rate of change is

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{Balance law.}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine.

This is $10/1000 = 0.01 (= 1\%)$ of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01\, y(t)$ .

Thus the model is the ODE

$$(4) \qquad y' = 50 - 0.01y = -0.01(y - 5000).$$

## EXAMPLE 5 (continued)

***Step 2. Solution of the model.***

The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y-5000} = -0.01\,dt, \qquad \ln|y - 5000| = -0.01t + c^*, \; y - 5000 \;\; ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution.

Substituting $y = 100$ and $t = 0$ in the last equation gives

$$100 - 5000 = ce^0 = c.$$

Hence $c = -4900$. Hence the amount of salt in the tank at time $t$ is

$$y(t) = 5000 - 4900e^{-0.01t} \tag{5}$$

This function shows an exponential approach to the limit 5000 lb; see Fig. 11. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

# Integration Table

$$\int uv'\,dx = uv - \int u'v\,dx \text{ (by parts)}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{arcsinh}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{arccosh}\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

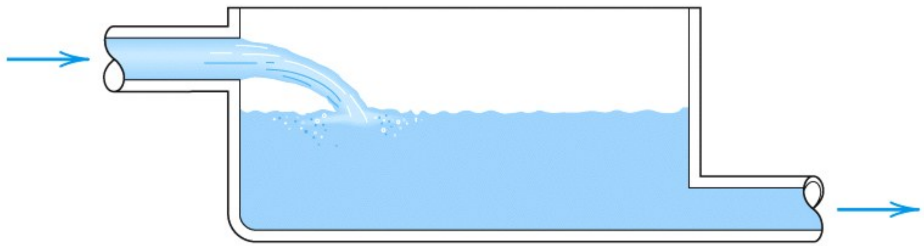
$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

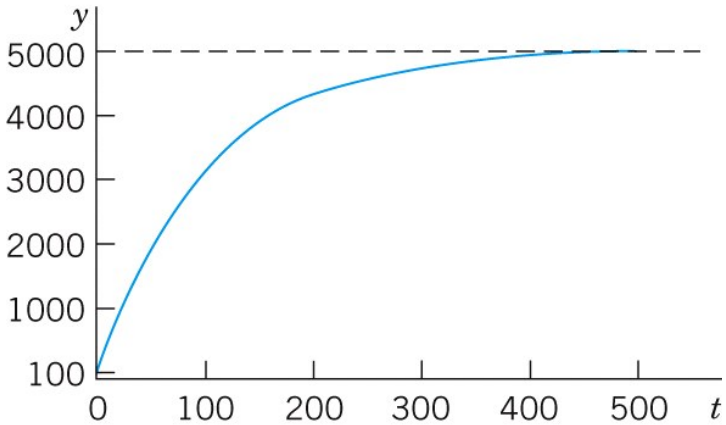
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$

## EXAMPLE 5 (continued)

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 35) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly.



Tank




Salt content $y(t)$

# Extended Method: Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for $y$ a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

**(8)** $$y' = f\left(\frac{y}{x}\right).$$

Here, $f$ is any (differentiable) function of $y/x$ such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE*, a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

# Extended Method: Reduction to Separable Form (continued)

The form of such an ODE suggests that we set $y/x = u$; thus,

(9) $y = ux$ and by product differentiation $y' = u'x + u$.

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that if $f(u) - u \neq 0$, this can be separated:

(10) $$\boxed{\frac{du}{f(u)-u} = \frac{dx}{x}.}$$

# 1.4 Exact ODEs. Integrating Factors

We recall from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

From this it follows that if $u(x, y) = c = const$, $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2},$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'dx$ as in Sec. 1.3)

**(1)** $$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential

(2) $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

**(3)** $$u(x, y) = c.$$

This is called an **implicit solution**, in contrast to a solution $y = h(x)$ as defined in Sec. 1.1, which is also called an *explicit solution,* for distinction. Sometimes an implicit solution can be converted to explicit form. (Do this for $x^2 + y^2 = 1$.) If this is not possible, your CAS may graph a figure of the **contour lines** (3) of the function $u(x, y)$ and help you in understanding the solution.

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

(4) (a) $$\frac{\partial u}{\partial x} = M$$ (b) $$\frac{\partial u}{\partial y} = N.$$

From this we can derive a formula for checking whether (1) is exact or not, as follows.

Let $M$ and $N$ be continuous and have continuous first partial derivatives in a region in the $xy$-plane whose boundary is a closed curve without self-intersections. Then by partial differentiation of (4) (see App. 3.2 for notation),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Thus

$$\textbf{(5)} \qquad \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation.
If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to $x$

**(6)** $$u = \int M\, dx + k(y);$$

in this integration, $y$ is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration.
To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get $dk/dy$, and integrate $dk/dy$ to get $k$.

# EXAMPLE 1

## An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

**Solution.** *Step 1. Test for exactness.* Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

# EXAMPLE 1 (continued)

***Step 2. Implicit general solution.*** From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to $y$ and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

***Step 3. Checking an implicit solution.*** We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \cos(x + y)\,dx + (\cos(x + y) + 3y^2 + 2y)\,dy = 0. \tag{9}$$

This completes the check. ■

# EXAMPLE 2

**An Initial Value Problem**

Solve the initial value problem

$$(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0, \qquad y(1) = 2. \tag{10}$$

***Solution.*** You may verify that the given ODE is exact. We find $u$. For a change, let us use (6*),

$$u = -\int \sin y \cosh x\,dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = M = \cos y \sinh x + 1$. Hence $dl/dx = 1$. By integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$. Hence the answer is $\cos y \cosh x + x = 0.358$. Figure 17 shows the particular solutions for $c = 0, 0.358$ (thicker curve), 1, 2, 3. Check that the answer satisfies the ODE. (Proceed as in Example 1.) Also check that the initial condition is satisfied.

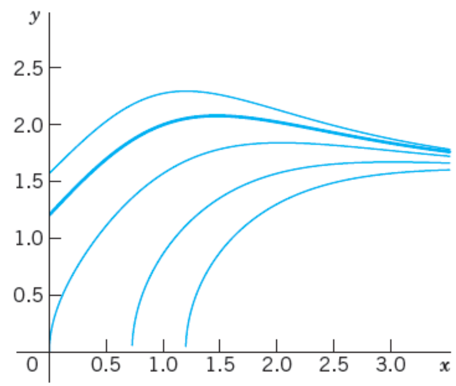



Fig. 17. Particular solutions in Example 2

# EXAMPLE 3

**WARNING! Breakdown in the Case of Nonexactness**

The equation $-y\,dx + x\,dy = 0$ is not exact because $M = -y$ and $N = x$, so that in (5), $\partial M/\partial y = -1$ but $\partial N/\partial x = 1$. Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + k(y) = -xy + k(y), \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on $y$. Try (6*); it will also fail. Solve the equation by another method that we have discussed.

# Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\,dx + x\,dy = 0$. It is not exact. However, if we multiply it by $1/x^2$, we get an exact equation [check exactness by (5)!],

$$\frac{-y\,dx + x\,dy}{x^2} = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right) = 0. \tag{11}$$

Integration of (11) then gives the general solution $y/x = c = \text{const}$.

We multiply a given nonexact equation,

**(12)** $$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

by a function $F$ that, in general, will be a function of both $x$ and $y$. The result was an equation

(13) $$FP\,dx + FQ\,dy = 0$$

that is exact, so we can solve it as just discussed. Such a function is then called an **integrating factor** of (12).

# How to Find Integrating Factors

For $M\,dx + N\,dy = 0$ the exactness condition (5) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FP\,dx + FQ\,dy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \tag{15}$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_y P + F P_y = F_x Q + F Q_x.$$

# How to Find Integrating Factors (continued)

Let $F = F(x)$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (15) becomes

$$FP_y = F'Q + FQ_x.$$

Dividing by $FQ$ and reshuffling terms, we have

(16) $$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

# Theorem 1

**Integrating Factor $F(x)$**

*If* (12) *is such that the right side R of* (16) *depends only on x, then* (12) *has an integrating factor F = F(x), which is obtained by integrating* (16) *and taking exponents on both sides.*

**(17)** $$F(x) = \exp \int R(x)dx.$$

**Note:**

**(12)** $$P(x, y)\, dx + Q(x, y)\, dy = 0,$$

**(16)** $$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

Similarly, if $F^* = F^*(y)$, then instead of (16) we get

$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{18}$$

# Theorem 2

## Integrating Factor $F^*(x)$

*If* (12) *is such that the right side $R^*$ of* (18) *depends only on y, then* (12) *has an integrating factor $F^* = F^*(y)$, which is obtained from* (18) *and taking exponents on both sides.*

**(19)**
$$F^*(y) = \exp \int R^*(y)\,dy$$

**Note:**

**(12)**
$$P(x, y)\,dx + Q(x, y)\,dy = 0,$$

**(18)**
$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

# EXAMPLE 5

**Application of Theorems 1 and 2. Initial Value Problem**

Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

$$(e^{x+y} + ye^y)\,dx + (xe^y - 1)\,dy = 0, \quad y(0) = -1 \tag{20}$$

**Solution.** *Step 1. Nonexactness.* The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^y) = e^{x+y} + e^y + ye^y \qquad \text{but} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y - 1) = e^y.$$

*Step 2. Integrating factor. General solution.* Theorem 1 fails because $R$ [the right side of (16)] depends on both $x$ and $y$.

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (18) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1.$$

Hence (19) gives the integrating factor $F^*(y) = e^{-y}$. From this result and (20) you get the exact equation

$$(e^x + y)\,dx + (x - e^{-y})\,dy = 0.$$

# EXAMPLE 5 (continued)

Test for exactness; you will get 1 on both sides of the exactness condition. By integration, using (4a),

$$u = \int (e^x + y)\,dx = e^x + xy + k(y).$$

Differentiate this with respect to $y$ and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x - e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

***Setp 3. Particular solution.*** The initial condition $y(0) = -1$ gives $u(0, -1) = 1 + 0 + e = 3.72$. Hence the answer is $e^x + xy + e^{-y} = 1 + e = 3.72$. Figure 18 shows several particular solutions obtained as level curves of $u(x, y) = c$, obtained by a CAS, a convenient way in cases in which it is impossible or difficult to cast a solution into explicit form. Note the curve that (nearly) satisfies the initial condition.

***Step 4. Checking.*** Check by substitution that the answer satisfies the given equation as well as the initial condition.

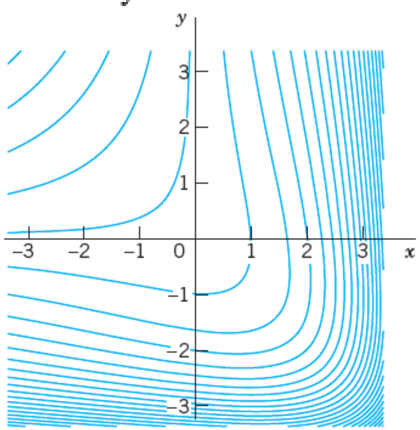



**Fig. 18.** Particular solutions in Example 5