2.2 Homogeneous Linear ODEs with Constant Coefficients

We shall now consider second-order homogeneous linear ODEs whose coefficients *a* and *b* are constant,

(1) $$y'' + ay' + by = 0.$$

These equations have important applications in mechanical and electrical vibrations.

To solve (1), we recall from Sec. 1.5 that the solution of the first-order linear ODE with a constant coefficient k

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$.
This gives us the idea to try as a solution of (1) the function

$$y = e^{\lambda x}. \tag{2}$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \text{ and } y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

(3) $$\lambda^2 + a\lambda + b = 0$$

then the exponential function (2) is a solution of the ODE (1).

Now from algebra we recall that the roots of this quadratic equation (3) are

(4) $$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \quad \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

(3) and (4) will be basic because our derivation shows that the functions

(5) $$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

are solutions of (1).

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I) *Two real roots if* $a^2 - 4b > 0$,

(Case II) *A real double root if* $a^2 - 4b = 0$,

(Case III) *Complex conjugate roots if* $a^2 - 4b < 0$.

Case I. Two Distinct Real-Roots λ_1 and λ_2

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant.

The corresponding general solution is

(6) $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

EXAMPLE 2

Initial Value Problem in the Case of Distinct Real Roots

Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution. Step 1. General solution.

The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

Its roots are

$$\lambda_1 = \frac{1}{2}(-1+\sqrt{9}) = 1 \quad \text{and} \quad \lambda_2 = \frac{1}{2}(-1-\sqrt{9}) = -2$$

so that we obtain the general solution

$$y = c_1 e^{x} + c_2 e^{-2x}.$$

EXAMPLE 2 (continued)

Initial Value Problem in the Case of Distinct Real Roots

Solution. ***(continued)***

Step 2. Particular solution.

Since $y'(x) = c_1e^x - 2c_2e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$

$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$and $c_2 = 3$. This gives the *answer* $y = e^x + 3e^{-2x}$.

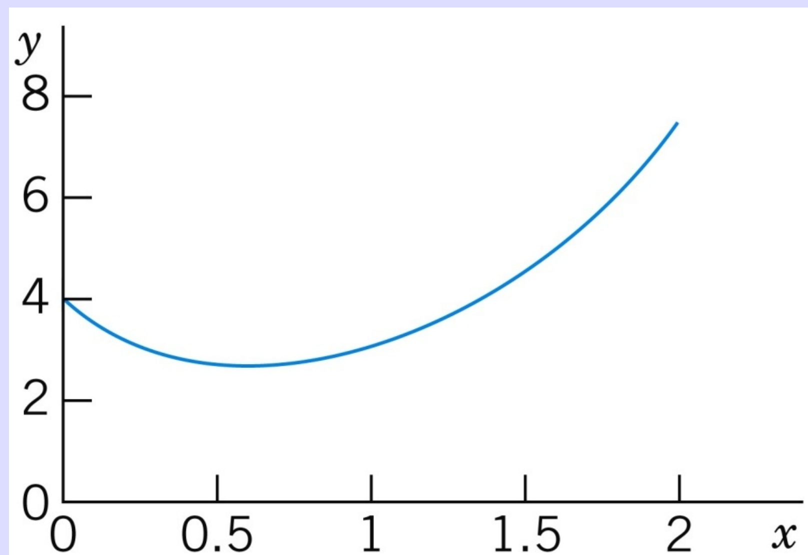
EXAMPLE 2 (continued)

Initial Value Problem in the Case of Distinct Real Roots

Solution. *(continued)*

Step 2. Particular solution. *(continued)*

Figure 30 shows that the curve begins at $y = 4$ with a negative slope (-5, but note that the axes have different scales!), in agreement with the initial conditions.



Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}$$

To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$.
Substituting this and its derivatives $y'_2 = u'y_1 + uy'_1$ and y''_2 into (1), we first have

$$(u''y_1 + 2u'y'_1 + uy''_1) + a(u'y_1 + uy'_1) + buy_1 = 0.$$

Case II. Real Double Root $\lambda = -a/2$ *(continued 1)*

Collecting terms in u'', u', and u, as in the last section, we obtain

$$u''y_1 + u'(2y'_1 + ay_1) + u(y''_1 + ay'_1 + by_1) = 0.$$

The expression in the last parentheses is zero, since y_1 is a solution of (1). The expression in the first parentheses is zero, too, since

$$2y'_1 = -ae^{-ax/2} = -ay_1.$$

We are thus left with $u''y_1 = 0$. Hence $u'' = 0$. By two integrations, $u = c_1x + c_2$.

To get a second independent solution $y_2 = uy_1$, we can simply choose $c_1 = 1$, $c_2 = 0$ and take $u = x$. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis.

Case II. Real Double Root $\lambda = -a/2$ *(continued 2)*

Hence in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2},\ xe^{-ax/2}.$$

The corresponding general solution is

(7) $$y = (c_1 + c_2 x)e^{-ax/2}.$$

WARNING! If λ is a *simple* root of (4), then $(c_1 + c_2 x)e^{\lambda x}$ with $c_2 \neq 0$ is ***not*** a solution of (1)

Example 4

Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Solution. The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2 x)e^{-0.5x}.$$

We need its derivative

$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0, \qquad y'(0) = c_2 - 0.5c_1 = 3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$.

Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative.
In this case, the roots of (3) are the complex $\lambda = (-\frac{1}{2})a \pm i\omega$ that give the complex solutions of the ODE (1).
However, we will show that we can obtain a basis of *real* solutions

$$(8) \qquad y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x \qquad (\omega > 0)$$

where $\omega^2 = b - (\frac{1}{4})a^2$.

Case III. Complex Root $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

(continued)

It can be verified by substitution that these are solutions in the present case.
They form a basis on any interval since their quotient cot ωx is not constant.
Hence a real general solution in Case III is

(9) $y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$ $(A, B \text{ arbitrary})$

Example 5

Complex Roots. Initial Value Problem

Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution. ***Step 1. General solution.*** The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x} \sin 3x$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x).$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

$$y = e^{-0.2x} \sin 3x.$$

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, \ e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
II	Real double root $\lambda = (-½)a$	$e^{-ax/2}, \ xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$.
III	Complex conjugate $\lambda_1 = (-½)a + i\omega$ $\lambda_2 = (-½)a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2} (A \cos \omega x + B \sin \omega x)$

Derivation in Case III.
Complex Exponential Function

(11) $e^{it} = \cos t + i \sin t,$

called the **Euler formula**.

$e^{-it} = \cos(-t) + i \sin(-t) = \cos t - i \sin t$

$\lambda_1 = \frac{1}{2}a + i\omega \qquad \lambda_2 = \frac{1}{2}a - i\omega.$

$$e^{\lambda_1 x} = e^{-(a/2)x + i\omega x} = e^{-(a/2)x}(\cos \omega x + i \sin \omega x)$$

$$e^{\lambda_2 x} = e^{-(a/2)x - i\omega x} = e^{-(a/2)x}(\cos \omega x - i \sin \omega x)$$

2.3 Differential Operators.

Optional

2.3 Differential Operators. *Optional*

Operational calculus means the technique and application of operators.

Here, an **operator** is a transformation that transforms a function into another function.

Hence differential calculus involves an operator, the **differential operator** D, which transforms a (differentiable) function into its derivative.

In operator notation we write $D = d/dx$

and

$$Dy = y' = dy/dx. \tag{1}$$

For a homogeneous linear ODE $y'' + ay' + by = 0$ with constant coefficients we can now introduce the **second-order differential operator**

$$L = P(D) = D^2 + aD + bI,$$

where I is the **identity operator** defined by $Iy = y$.
Then we can write that ODE as

$$(2) \qquad Ly = P(D)y = (D^2 + aD + bI)y = 0.$$

P suggests "polynomial." *L* is a **linear operator**.
By definition this means that if Ly and Lw exist (this is the case if y and w are twice differentiable), then $L(cy + kw)$ exists for any constants c and k, and

$$L(cy + kw) = cLy + kLw.$$

The point of this operational calculus is that $P(D)$ *can be treated just like an algebraic quantity.*