Engineering Mathematics
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Homework #3 Solution Due: *December 4, 2020*

1. Problem Set 5.1 – 3

Determine the radius of convergence. Show the details of your work.

3. $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$

Sol:

$$a_m = \frac{(-1)^m}{k^m}$$

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{k^m}{k^{m+1}}\right| = \frac{1}{|k|}$$

So, the radius of convergence

$$\mathrm{R} = \left(\frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|}\right)^{1/2} = \sqrt{|k|}$$

2. Problem Set 5.1 – 7

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

7. $y' = -4xy$

Sol:

Let

$y = \sum_{m=0}^{\infty} a_m x^m$,

then

$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$.

Insert y and y' into $y' = -4xy$, and we have

$\sum_{m=1}^{\infty} m a_m x^{m-1} = -4x \sum_{m=0}^{\infty} a_m x^m$,

$$\Rightarrow \; a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = -4(a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots)$$

x^0: $a_1 = 0$

x^1: $2a_2 = -4a_0 \Rightarrow a_2 = \frac{-4}{2}a_0 = -2a_0$

x^2: $3a_3 = -4a_1 \Rightarrow a_3 = 0$

x^3: $4a_4 = -4a_2 \Rightarrow a_4 = \frac{-4}{4}a_2 = \frac{-2}{2} \cdot -2a_0 = \frac{(-2)^2}{2!}a_0$

x^4: $5a_5 = -4a_3 \Rightarrow a_5 = 0$

$$x^5:\ 6a_6 = -4a_4 \Rightarrow a_6 = \frac{-4}{6}a_4 = \frac{-2}{3}\cdot\frac{-2}{2}\cdot -2a_0 = \frac{(-2)^3}{3!}a_0$$

$$\vdots$$

So, we have

$$\begin{aligned} y &= a_0 + -2a_0x^2 + \frac{(-2)^2}{2!}a_0x^4 + \frac{(-2)^3}{3!}a_0x^6 + \cdots \\ &= a_0\left[1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \cdots\right] \\ &= a_0e^{-2x^2} \end{aligned}$$

3. Problem Set 5.1 – 9

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

9. $y'' + y = 0$

<u>Sol:</u>

Let

$y = \sum_{m=0}^{\infty} a_m x^m$,

then

$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$,

$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$,

Insert y and y'' into $y'' + y = 0$, and we have

$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$,

$$\Rightarrow (2\cdot 1a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + 6\cdot 5a_6x^4 + \cdots) + (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0$$

$$x^0:\ 2\cdot 1a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-1}{2!}a_0$$

$$x^1:\ 3\cdot 2a_3 + a_1 = 0 \Rightarrow a_3 = \frac{-1}{3!}a_1$$

$$x^2:\ 4\cdot 3a_4 + a_2 = 0 \Rightarrow a_4 = \frac{-1}{4\cdot 3}a_2 = \frac{1}{4!}a_0$$

$$x^3:\ 5\cdot 4a_5 + a_3 = 0 \Rightarrow a_5 = \frac{-1}{5\cdot 4}a_3 = \frac{1}{5!}a_1$$

$$\vdots$$

So, we have

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ &= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \ a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= a_0 \cos x + a_1 \sin x \end{aligned}$$

4. Problem Set 5.1 – 13

Find a power series solution in powers of x. Show the details.

13. $y'' + (1 + x^2)y = 0$

Sol:

Let

$y = \sum_{m=0}^{\infty} a_m x^m$,

then

$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$,

$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$,

Insert y and y'' into $y'' + (1 + x^2)y = 0$, and we have

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

x^0: $2 \cdot 1 a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-1}{2} a_0$

x^1: $3 \cdot 2 a_3 + a_1 = 0 \Rightarrow a_3 = \frac{-1}{6} a_1$

x^2: $4 \cdot 3 a_4 + a_2 + a_0 = 0 \Rightarrow a_4 = \frac{-1}{12}(a_2 + a_0) = \frac{-1}{24} a_0$

x^3: $5 \cdot 4 a_5 + a_3 + a_1 = 0 \Rightarrow a_5 = \frac{-1}{20}(a_3 + a_1) = \frac{-1}{24} a_1$

x^4: $6 \cdot 5 a_6 + a_4 + a_2 = 0 \Rightarrow a_6 = \frac{-1}{30}(a_4 + a_2) = \frac{13}{720} a_0$

x^5: $7 \cdot 6 a_7 + a_5 + a_3 = 0 \Rightarrow a_7 = \frac{-1}{42}(a_5 + a_3) = \frac{5}{1008} a_1$

$\vdots$

So, we have

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ &= a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{13}{720}x^6 + \cdots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5 + \frac{5}{1008}x^7 + \cdots\right) \end{aligned}$$

For following problems, find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions. Show the details of your work.

5. Problem Set 5.3 – 3

3. $xy'' + 2y' + xy = 0$

<u>Sol:</u>

Let

$y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$,

then

$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$,

$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$,

Insert y, y' and y'' into $xy'' + 2y' + xy = 0$, and we obtain

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1} + 2\sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r+1} = 0$$

The smallest power is x^{r-1}; by equating the sum of its coefficients to zero we have

$$x^{r-1}: \quad r(r-1)a_0 + 2ra_0 = 0 \ \Rightarrow \ r(r+1) = 0 \ \Rightarrow \ r = 0, -1$$

$$x^r: \quad (r+1)ra_1 + 2(r+1)a_1 = 0 \ \Rightarrow \ a_1 = 0$$

$$x^{s+r-1}: \quad (s+r)(s+r-1)a_s + 2(s+r)a_s + a_{s-2} = 0 \ \Rightarrow \ a_s = \frac{-a_{s-2}}{(s+r)(s+r+1)}$$

Since $a_1 = 0$, $\Rightarrow$ $a_3 = a_5 = a_7 = \cdots = 0$

1) For $r = r_1 = 0$, $a_s = \frac{-a_{s-2}}{s(s+1)}$

$$a_2 = \frac{-a_0}{2 \cdot 3} = \frac{-a_0}{3!}, a_4 = \frac{-a_2}{4 \cdot 5} = \frac{a_0}{5!}, a_6 = \frac{-a_4}{6 \cdot 7} = \frac{-a_0}{7!}, \cdots, a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}$$

$$\Rightarrow \quad y_1 = x^0 a_0 \left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots\right) = a_0 \frac{\sinh x}{x}$$

2) For $r = r_2 = -1$, $a_s = \frac{-a_{s-2}}{(s-1)s}$

$$a_2 = \frac{-a_0}{1 \cdot 2} = \frac{-a_0}{2!}, a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{4!}, a_6 = \frac{-a_4}{5 \cdot 6} = \frac{-a_0}{6!}, \cdots, a_{2m} = \frac{(-1)^m a_0}{(2m)!}$$

$$\Rightarrow \quad y_2 = x^{-1} a_0 \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) = a_0 \frac{\cosh x}{x}$$

Taking $a_0 = 1$,

$$\Rightarrow \quad y_1 = \frac{\sinh x}{x}, \qquad y_2 = \frac{\cosh x}{x}$$

6. Problem Set 5.3 – 5

5. $x^2y'' + x(2x-1)y' + (x+1)y = 0$

Sol:

Let

$y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$,

then

$y' = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}$,

$y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$,

Insert y, y' and y'' into $x^2y'' + x(2x-1)y' + (x+1)y = 0$, and we obtain

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + 2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r+1} - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$

$$+\sum_{m=0}^{\infty} a_m x^{m+r+1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

The smallest power is x^r; by equating the sum of its coefficients to zero we have the indicial equation:

$$x^r: \quad r(r-1)a_0 - ra_0 + a_0 = 0 \;\Rightarrow\; r^2 - 2r + 1 = 0 \;\Rightarrow\; r = 1,1$$

For the coefficients of x^{s+r}, we find the below relation

$$(s+r)(s+r-1)a_s + 2(s+r-1)a_{s-1} - (s+r)a_s + a_{s-1} + a_s = 0$$

$$\Rightarrow \quad a_s = \frac{-2(s+r)+1}{(s+r-1)^2}a_{s-1}$$

For $r = r_1 = 1$, $a_s = \frac{-2s-1}{s^2}a_{s-1}$

$$a_1 = \frac{-2\cdot 1-1}{1^2}a_0 = \frac{(-3)}{1^2}a_0,$$

$$a_2 = \frac{-2\cdot 2-1}{2^2}a_1 = \frac{(-3)(-5)}{1^2\cdot 2^2}a_0$$

$$a_3 = \frac{-2\cdot 3-1}{3^2}a_2 = \frac{(-3)(-5)(-7)}{1^2\cdot 2^2\cdot 3^2}a_0$$

$$\vdots$$

$$a_m = \frac{(-1)^m 3\cdot 5\cdot 7\cdot \cdots \cdot (2m+1)}{1^2\cdot 2^2\cdot 3^2\cdot \cdots \cdot m^2}a_0$$

So, we have (taking $a_0 = 1$)

$$y_1 = x\left(1 - 3x + \frac{3\cdot 5}{1^2\cdot 2^2}x^2 - \frac{3\cdot 5\cdot 7}{1^2\cdot 2^2\cdot 3^2}x^3 + \cdots\right)$$

Let $y_2 = y_1 \ln x + x(A_1x + A_2x^2 + A_3x^3 + \cdots) = y_1 \ln x + (A_1x^2 + A_2x^3 + A_3x^4 + \cdots)$

$$\Rightarrow \; y_2' = y_1' \ln x + \frac{y_1}{x} + 2A_1x + 3A_2x^2 + 4A_3x^3 + \cdots$$

$$\Rightarrow \; y_2'' = y_1'' \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2} + 2A_1 + 3\cdot 2A_2x + 4\cdot 3A_3x^2 + \cdots$$

Insert them into the ODE: $x^2y_2'' + x(2x-1)y_2' + (x+1)y_2 = 0$

$x^2:\; -12 + 3 + 2A_1 + 2 + 3 - 2A_1 + A_1 = 0 \Rightarrow \quad A_1 = 4$

$x^3: \Rightarrow \quad A_2 = \dfrac{-29}{4}$

$x^4: \Rightarrow \quad A_3 = \dfrac{27}{4}$

$\vdots$

$$y_2 = y_1 \ln x + x\left(4x - \frac{29}{4}x^2 + \frac{27}{4}x^3 + \cdots\right)$$

7. Problem Set 5.3 – 7

7. $y'' + \left(x - \frac{1}{2}\right)y = 0$

<u>Sol:</u>

$$y'' + (x - \tfrac{1}{2})y = 0$$

Let $y = \sum_{m=0}^{\infty} a_m x^m$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^{m+1} - \sum_{m=0}^{\infty} \tfrac{1}{2} a_m x^m = 0$$

$$\sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s + \sum_{s=1}^{\infty} a_{s-1} x^s - \sum_{s=0}^{\infty} \tfrac{1}{2} a_s x^s = 0$$

$x^0 (s=0)$: $\quad 2 \cdot 1\, a_2 - \frac{1}{2} a_0 = 0 \Rightarrow a_2 = \frac{1}{4} a_0$

$x^s (s=1,2,\cdots)$: $\quad (s+2)(s+1)a_{s+2} + a_{s-1} - \frac{1}{2} a_s = 0$

$$a_{s+2} = \frac{\frac{1}{2}a_s - a_{s-1}}{(s+2)(s+1)} \qquad (s=1,2,\cdots)$$

$\cancel{a_3 = \frac{\frac{1}{2}a_1 - a_0}{3 \cdot 2}}$ $\quad \cancel{\text{let } a_0 = a_1 = 1, \ a_2 = \frac{1}{4}}$

$\cancel{a_3 = \frac{\frac{1}{2}a_1 - a_0}{3 \cdot 2} = \frac{1}{12}, \ a_4 = \frac{\frac{1}{2}a_2 - a_1}{4 \cdot 3}}$

$a_3 = \frac{\frac{1}{2}a_1 - a_0}{3 \cdot 2} = \frac{1}{12}a_1 - \frac{1}{6}a_0$. $\quad a_4 = \frac{\frac{1}{2}a_2 - a_1}{4 \cdot 3} = \frac{1}{24}a_2 - \frac{a_1}{12} = \frac{1}{96}a_0 - \frac{1}{12}a_1$

$a_5 = \frac{\frac{1}{2}a_3 - a_2}{5 \cdot 4} = \frac{a_3}{40} - \frac{a_2}{20} = \frac{a_1}{480} - \frac{a_0}{60}$. $\quad a_6 = \cdots$

$$\therefore y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + \tfrac{1}{4}a_0 x^2 + (\tfrac{1}{12}a_1 - \tfrac{1}{6}a_0)x^3 + (\tfrac{1}{96}a_0 - \tfrac{1}{12}a_1)x^4 + (\tfrac{1}{480}a_1 - \tfrac{1}{60}a_0)x^5 + \cdots$$

$$= a_0(1 + \tfrac{1}{4}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{96}x^4 - \tfrac{1}{60}x^5 + \cdots) + a_1(x + \tfrac{1}{12}x^3 - \tfrac{1}{12}x^4 + \tfrac{1}{480}x^5 + \cdots)$$

8. Problem Set 5.3 – 9

   9. $2x(x-1)y'' - (x+1)y' + y = 0$

Sol:

$$2x(x-1)y'' - (x+1)y' + y = 0$$

Let $y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r} - \sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r-1} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$

$$- \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{s=1}^{\infty} 2(s+r-1)(s+r-2) a_{s-1} x^{s+r-1} - \sum_{m=0}^{\infty} 2(s+r)(s+r-1) a_s x^{s+r-1} - \sum_{s=1}^{\infty} (s+r-1) a_{s-1} x^{s+r-1}$$

$$- \sum_{s=0}^{\infty} (s+r) a_s x^{s+r-1} + \sum_{s=1}^{\infty} a_{s-1} x^{s+r-1} = 0$$

$x^{r-1}\ (s=0)$: $\quad [-2r(r-1) - r]a_0 = 0 \Rightarrow a_0 \neq 0$

$$(-2r^2 + r) = 0 \Rightarrow r = \frac{1}{2},\ 0$$

$x^{s+r-1}\ (s=1, 2, \cdots)$: $\quad 2(s+r-1)(s+r-2)a_{s-1} - 2(s+r)(s+r-1)a_s - (s+r-1)a_{s-1}$

$$- (s+r)a_s + a_{s-1} = 0$$

$$a_s = \left[\frac{(s+r-1)(2s+2r-5)+1}{(s+r)(2s+2r-1)}\right] a_{s-1}$$

① $r = \frac{1}{2}$: $\quad a_s = \left[\frac{(s-\frac{1}{2})(2s-4)+1}{(s+\frac{1}{2})\cdot 2s}\right] a_{s-1}, \ (s = 1, 2, \cdots)$

$\because a_1 = 0, \quad \therefore a_2 = a_3 = a_4 = \cdots = 0$

Let $a_0 = 1, \quad \therefore y_1 = x^{\frac{1}{2}}$

② $r=0$; $a_s = \left[\frac{(s-1)(2s-3)+1}{s(2s-1)}\right] a_{s-1}$ $(s=1, 2, 3, \cdots)$

$a_1 = a_0$, $a_2 = 0$ $\therefore a_3 = a_4 = \cdots = 0$

Let $a_0 = 1$, $a_1 = 1$

$\therefore y_2 = x^0 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$= 1 + x$

$\therefore y = c_1 y_1 + c_2 y_2$

9. Problem Set 5.3 – 11

 11. $xy'' + (2 - 2x)y' + (x - 2)y = 0$

 <u>Sol:</u>

$xy'' + (2-2x)y' + (x-2)y = 0.$

Let $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$

$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + \sum_{m=0}^{\infty} 2(m+r)a_m x^{m+r-1} - \sum_{m=0}^{\infty} 2(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1} - \sum_{m=0}^{\infty} 2a_m x^{m+r} = 0$$

$$\sum_{s=0}^{\infty}(s+r)(s+r-1)a_s x^{s+r-1} + \sum_{s=0}^{\infty} 2(s+r)a_s x^{s+r-1} - \sum_{s=1}^{\infty} 2(s+r-1)a_{s-1} x^{s+r-1} + \sum_{s=2}^{\infty} a_{s-2} x^{s+r-1} - \sum_{s=1}^{\infty} 2a_{s-1} x^{s+r-1} = 0$$

$x^{r-1}(s=0)$: $[r(r-1)+2r]a_0 = 0 \Rightarrow a_0 \neq 0$

$r^2 + r = 0 \Rightarrow r = 0, -1.$

$x^{r}(s=1)$: $[(r+1)r + 2(r+1)]a_1 - 2ra_0 - 2a_0 = 0$

$$a_1 = \frac{2}{(r+2)} a_0$$

$x^{s+r-1}(s=2,3\cdots)$: $(s+r)(s+r-1)a_s + 2(s+r)a_s - 2(s+r-1)a_{s-1} - a_{s-2} - 2a_{s-1} = 0$

$$a_s = \frac{[2(s+r)]a_{s-1} + a_{s-2}}{(s+r)(s+r+1)} \quad (s=2,3,\cdots)$$

① $r=0$; Let $a_0 = 1$, $a_1 = a_0 = 1$, $a_s = \frac{2s a_{s-1} + a_{s-2}}{s(s+1)}$ $(s=2,3\cdots)$

$a_2 = \frac{3}{2\cdot 3} = \frac{1}{2!}$, $a_3 = \frac{1}{3!}$, $a_4 = \frac{1}{4!}$.

$y = x^0 \sum_{m=0}^{\infty} a_m x^m = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x$

② ~~Let~~ $r=-1$, Let $a_0 = 1$. $a_1 = 1$. $a_s = \frac{[2(s-1)]a_{s-1} + a_{s-2}}{(s-1)s}$ $(s=2,3\cdots)$

$a_2 = \frac{1}{2!}$, $a_3 = \frac{1}{3!}$, $a_4 = \frac{1}{4!}$.

$\therefore y_2 = x^{-1}\sum_{m=0}^{\infty} a_m x^m = x^{-1}(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)$

$= x^{-1}e^x$

10. Problem Set 5.3 – 13

13. $xy'' + (2x+1)y' + (x+1)y = 0$

<u>Sol:</u>

$$xy'' + (2x+1)y' + (x+1)y = 0$$

Let $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1} + \sum_{m=0}^{\infty} 2(m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r+1} + \sum_{m=0}^{\infty} a_m x^{m+r} =$$

$$\sum_{s=0}^{\infty} (s+r)(s+r-1) a_s x^{s+r-1} + \sum_{s=1}^{\infty} 2(s+r-1) a_{s-1} x^{s+r-1} + \sum_{s=0}^{\infty} (s+r) a_s x^{s+r-1} + \sum_{s=2}^{\infty} a_{s-2} x^{s+r-1} + \sum_{s=1}^{\infty} a_{s-1} x^{s+r-1} = 0$$

x^{r-1} $(s=0)$: $\quad [r(r-1) + r] a_0 = 0 \quad \Rightarrow a_0 \neq 0$

$r^2 = 0 \quad \Rightarrow \quad r = 0$ (Double root)

x $(s=1)$: $\quad [r(r+1) + (r+1)] a_1 + [2r+1] a_0 = 0 \qquad (r=0)$

$a_1 = -a_0$

x^{s+r-1} $(s=2, 3 \ldots)$: $\quad [(s+r)(s+r-1) + (s+r)] a_s + [2(s+r-1)+1] a_{s-1} + a_{s-2} = 0$

$$a_s = \frac{-[2(s+r-1)+1] a_{s-1} - a_{s-2}}{(s+r)^2}$$

① $r = 0$: $\quad a_s = \dfrac{(-2s+1) a_{s-1} - a_{s-2}}{s^2} \qquad (s = 2, 3 \ldots)$

Let $a_0 = 1$, $a_1 = -1$; $\quad a_2 = \dfrac{+2}{2^2} = \dfrac{1}{2!} \qquad a_3 = \dfrac{-5a_2 - a_1}{3^2} = -\dfrac{1}{3!}$

$$a_m = \frac{(-1)^m}{m!}$$

$$y_1 = x^0 \sum \frac{(-1)^m}{m!} x^m = e^{-x}$$

Q. Let $y_2 = uy_1$

$y_2' = uy_1' + u'y_1$

$y_2'' = uy_1'' + 2u'y_1' + u''y_1$

$x(uy_1'' + 2u'y_1' + u''y_1) + (2x+1)(uy_1' + u'y_1) + (x+1)uy_1 = 0$

得 $u = \ln x$

$y_2 = uy_1 = e^{-x}\ln x$

$\therefore y = c_1 y_1 + c_2 y_2$