PART A

Ordinary Differential Equations (ODEs)

CHAPTER 1

First-Order ODEs

1.1 Basic Concepts. Modeling

The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling**.

Many physical concepts, such as velocity and acceleration, are derivatives.
A model is very often an equation containing derivatives of an unknown function.
Such a model is called a **differential equation.**

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t).
The equation may also contain y itself, known functions of x (or t), and constants.

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation.
The concept of order gives a useful classification into ODEs of first order, second order, and so on.
Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**.
Such equations contain only the first derivative y' and may contain y and any given functions of x.
Hence we can write them as

(4) $$F(x, y, y') = 0$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4).
For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

A function

$$y = h(x)$$

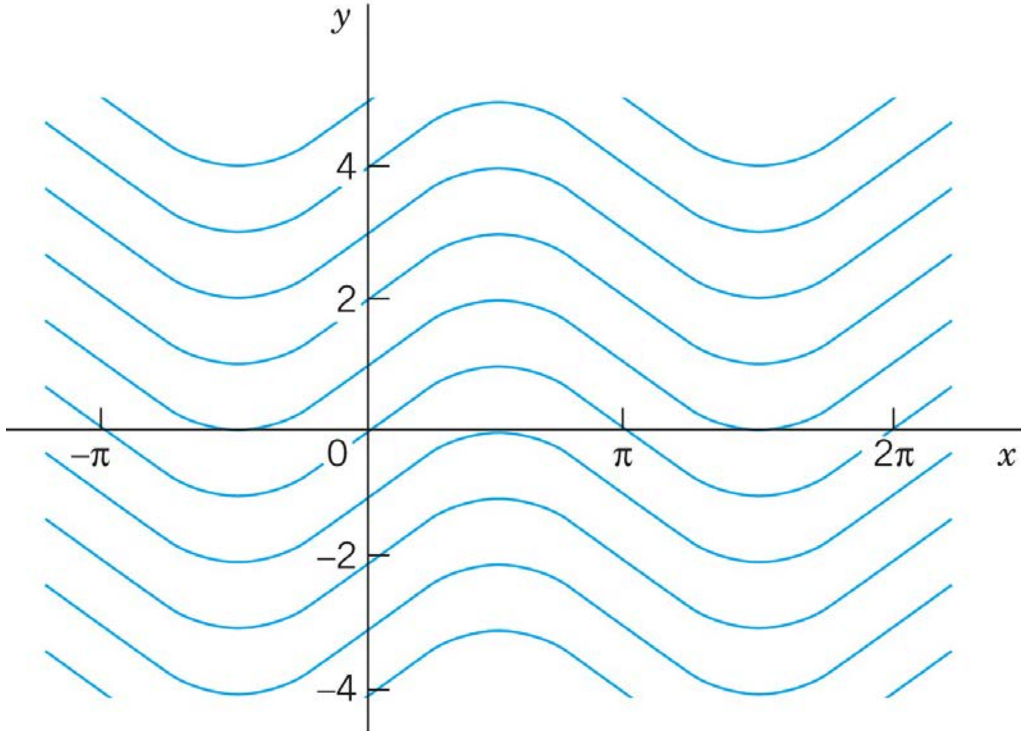
is called a **solution** of a given ODE (4) in some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively.
The curve (the graph) of h is called a **solution curve**.
Here, **open interval** means that the endpoints a and b are not regarded as points belonging to the interval.
Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

EXAMPLE 2

Solution by Calculus. Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x\, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions***. Each value of c, for instance, 2.75 or 0 or −8, gives one of these curves.

Figure 3 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.



EXAMPLE 3A

(A)Exponential Growth.

From calculus we know that $y = ce^{0.2t}$ has the derivative

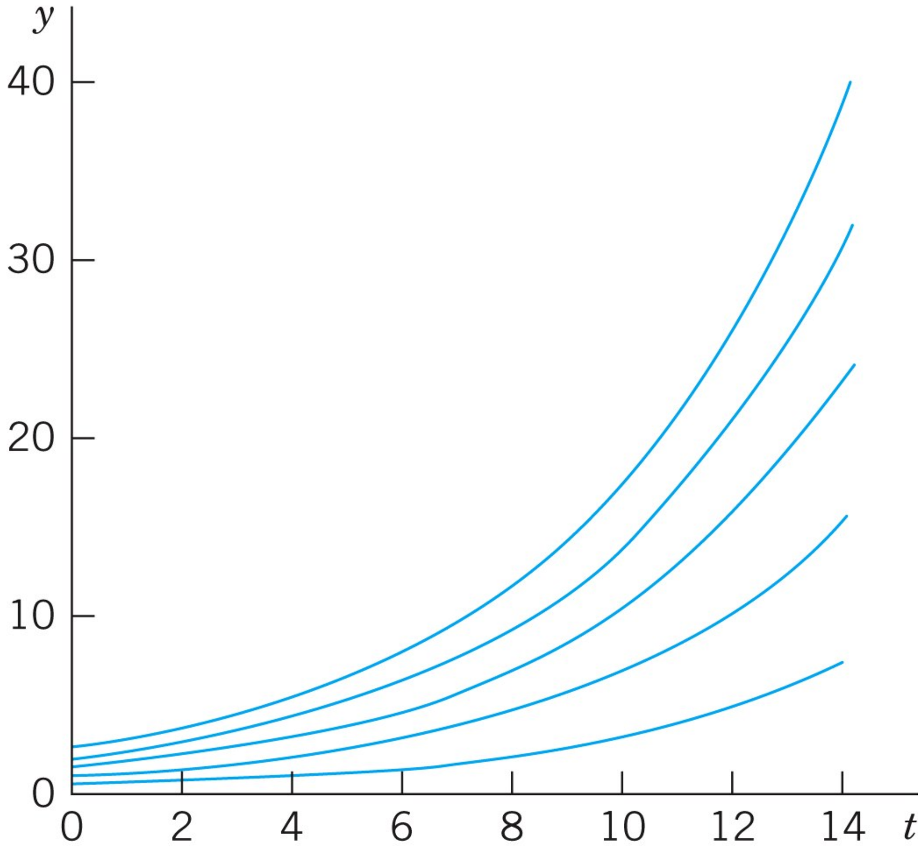
$$y' = \frac{dy}{dx} = 0.2e^{0.2t} = 0.2y$$

Hence y is a solution of $y' = 0.2y$ (Fig. 4A). This ODE is of the form $y' = ky$.

With positive-constant k, it can model exponential growth, for instance, of colonies of bacteria or populations of animals. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as **Malthus's law**.

EXAMPLE 4A (continued)

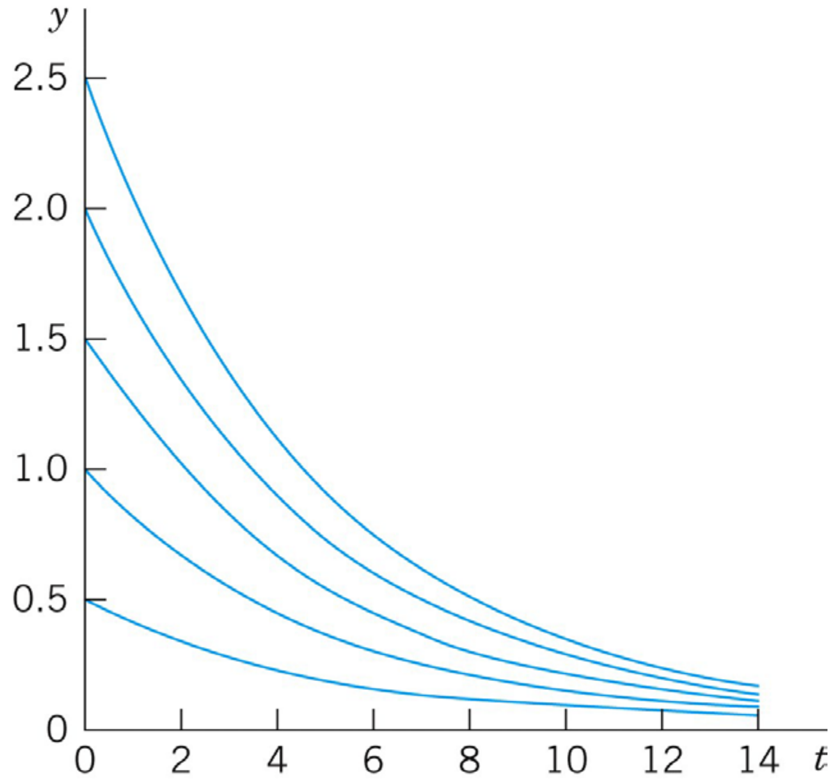
(A) Exponential Growth.



EXAMPLE 4B

(B) Exponential Decay

Similarly, $y' = -0.2y$ (with a minus on the right) has the solution $y = ce^{-0.2t}$, (Fig. 4B) modeling **exponential decay**, as, for instance, of a radioactive substance (see Example 5).



We see that each ODE in these examples has a solution that contains an arbitrary constant c.

Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

(We shall see that c is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.)

We shall develop methods that will give general solutions *uniquely* (perhaps except for notation). Hence we shall say ***the*** general solution of a given ODE (instead of ***a*** general solution).

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c. If we choose a specific c (e.g., $c = 6.45$ or 0 or -2.01), we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

In most cases, general solutions exist, and every solution not containing an arbitrary constant is obtained as a particular solution by assigning a suitable value to c. Exceptions to these rules occur but are of minor interest in applications.

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane.

An ODE together with an initial condition is called an **initial value problem**.

Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

(5) $$y' = f(x, y), \qquad y(x_0) = y_0.$$

EXAMPLE 5

Radioactivity. Exponential Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Physical Information. Experiments show that at each instant a radioactive substance decomposes—and is thus decaying in time—proportional to the amount of substance present.

EXAMPLE 5 (continued)

Step 1. Setting up a mathematical model of the physical process.

Denote by $y(t)$ the amount of substance still present at any time t.
By the physical law, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives the **first-order ODE**

$$\frac{dy}{dx} = -ky \tag{6}$$

where the constant k is positive, so that, because of the minus, we do get decay (as in [B] of Example 3).
The value of k is known from experiments for various radioactive substances (e.g., $k = 1.4 \cdot 10^{-11}$ sec^{-1}, approximately, for radium ${}_{88}\text{Ra}^{226}$).

EXAMPLE 5 (continued)

Step 1. *(continued)* ***Setting up a mathematical model of the physical process.***

Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$.

Then we have the **initial condition** $y(0) = 0.5$.

This is the instant at which our observation of the process begins. It motivates the term *initial condition* (which, however, is also used when the independent variable is not time or when we choose a t other than $t = 0$.).

Hence the mathematical model of the physical process is the **initial value problem**

$$\frac{dy}{dx} = -ky, \qquad y(0) = 0.5. \tag{7}$$

EXAMPLE 5 (continued)

Step 2. Mathematical solution.

As in (B) of Example 3 we conclude that the ODE (6) models exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(t) = ce^{-kt}. \tag{8}$$

We now determine c by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is (cf. Fig. 5)

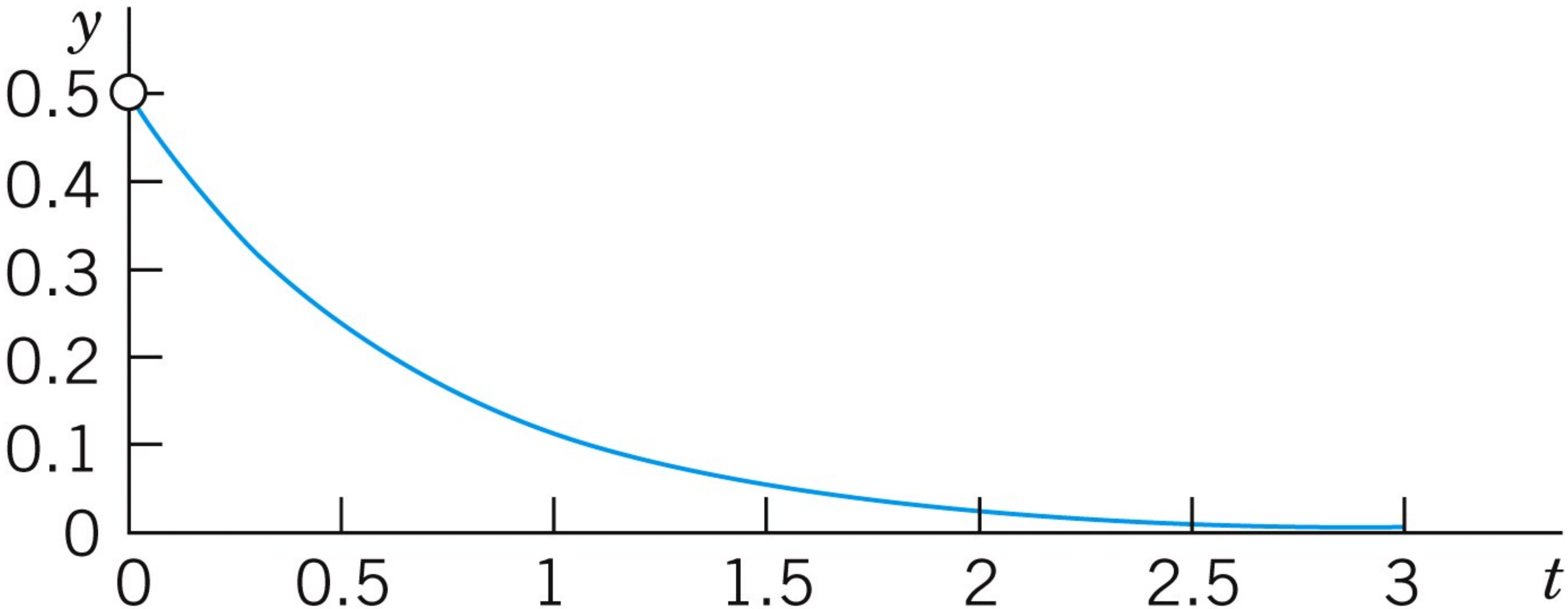
$$y(t) = 0.5e^{-kt} \qquad (k > 0). \tag{9}$$

Always check your result—it may involve human or computer errors! Verify by differentiation (chain rule!) that your solution (9) satisfies (7) as well as $y(0) = 0.5$:

$$dy/dt = -0.5ke^{-kt} = -k \cdot 0.5e^{-kt} = -ky, \qquad y(0) = 0.5e^{0} = 0.5.$$

EXAMPLE 5 (continued)

Step 3. Interpretation of result. Formula (9) gives the amount of radioactive substance at time t. It starts from the correct initial amount and decreases with time because k is positive. The limit of y as $t \to \infty$ is zero.



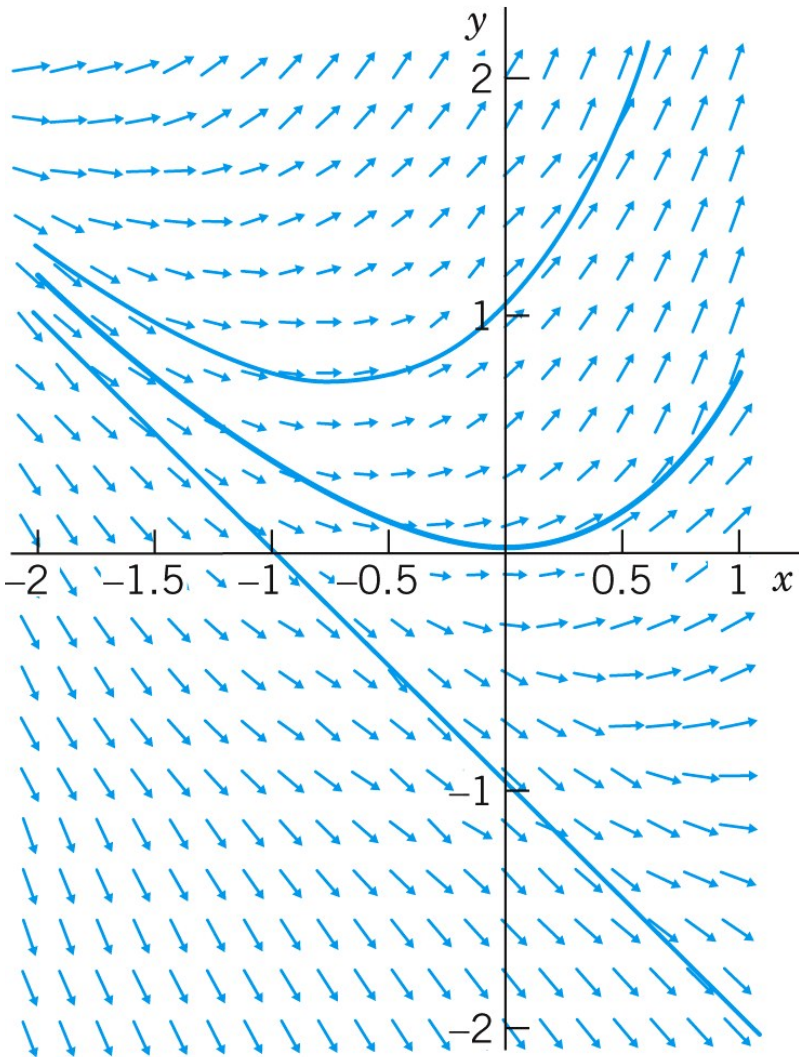
1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

Graphic Method of Direction Fields.
Practical Example Illustrated in Fig. 7.

We can show directions of solution curves of a given ODE (1) by drawing short straight-line segments (lineal elements) in the *xy*-plane.
This gives a **direction field** (or *slope field*) into which you can then fit (approximate) solution curves. This may reveal typical properties of the whole family of solutions.
Figure 7 shows a direction field for the ODE

$$y' = y + x \tag{2}$$

obtained by a **CAS (computer algebra system)** and some approximate solution curves fitted in.


y
2
1
–1
–2
–2
–1.5
–1
–0.5
0.5
1
x

If you have no CAS, first draw a few *level curves* $f(x, y) = const$ of $f(x, y)$, then parallel lineal elements along each such curve (which is also called an **isocline**, meaning ***a curve of equal inclination***), and finally draw approximation curves fit to the lineal elements.

Numeric Method by Euler

Given an ODE **(1)** $y' = f(x, y)$ and an initial value $y(x_0) = y_0$ **Euler's method** yields approximate solution values at equidistant x-values $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots$, namely,

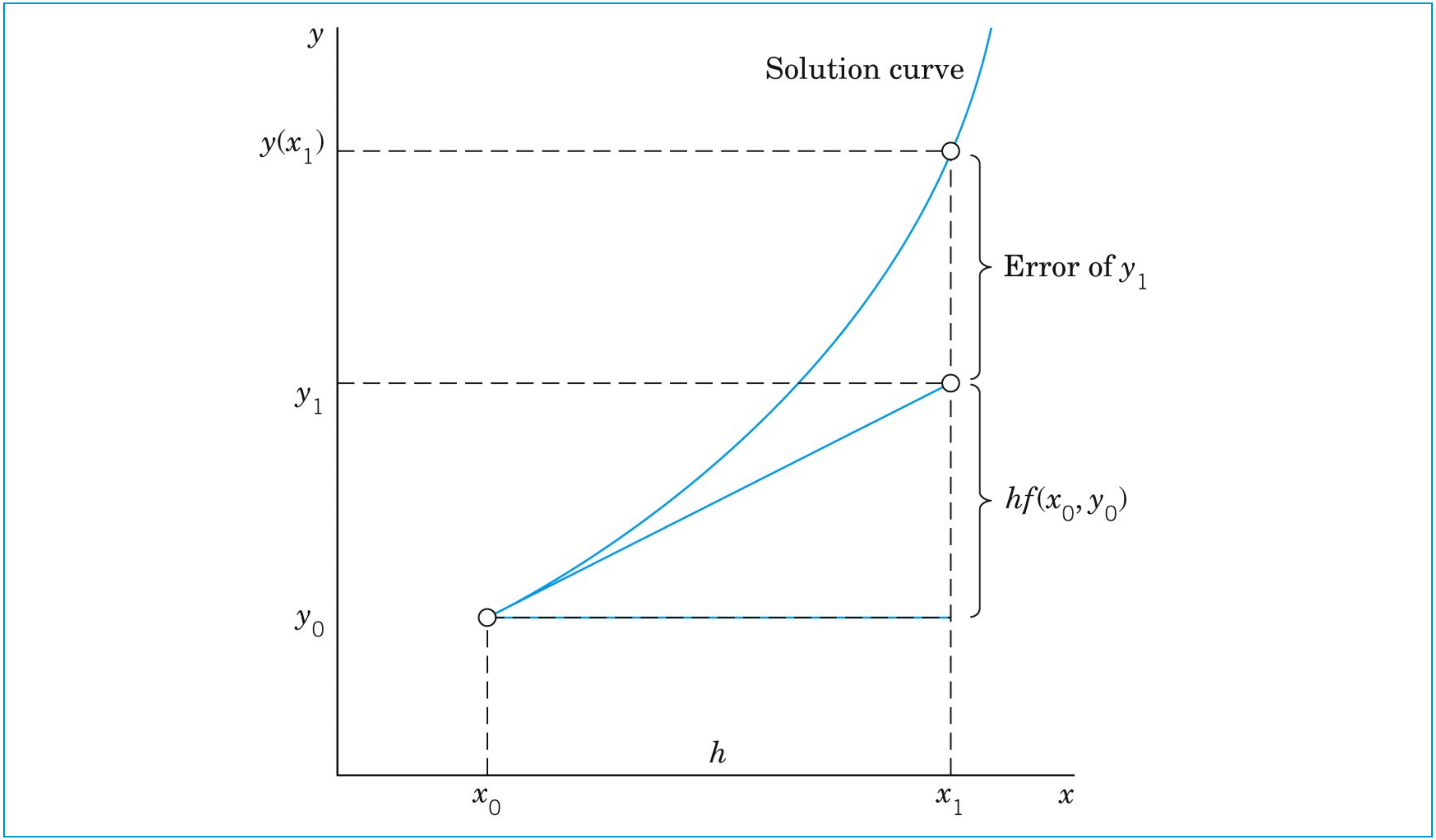
$$y_1 = y_0 + hf(x_0, y_0) \qquad \text{(Fig. 8)}$$

$$y_2 = y_1 + hf(x_1, y_1), \qquad \text{etc.}$$

In general,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

where the step h equals, e.g., 0.1 or 0.2 (see text pg. 11 for Table 1.1) or a smaller value for greater accuracy.


y
Solution curve
$y(x_1)$
Error of y_1
y_1
$hf(x_0, y_0)$
y_0
h
x_0
x_1
x