

1. Problem Set 5.1 – 3

Determine the radius of convergence. Show the details of your work.

$$3. \sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$$

Sol:

$$a_m = \frac{(-1)^m}{k^m}$$

$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{k^m}{k^{m+1}} \right| = \frac{1}{|k|}$$

So, the radius of convergence

$$R = \left(\frac{1}{\lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right|} \right)^{1/2} = \sqrt{|k|}$$

2. Problem Set 5.1 – 7

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

$$7. y' = -4xy$$

Sol:

Let

$$y = \sum_{m=0}^{\infty} a_m x^m,$$

then

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Insert y and y' into $y' = -4xy$, and we have

$$\sum_{m=1}^{\infty} m a_m x^{m-1} = -4x \sum_{m=0}^{\infty} a_m x^m,$$

$$\Rightarrow a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = -4(a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots)$$

$$x^0: a_1 = 0$$

$$x^1: 2a_2 = -4a_0 \Rightarrow a_2 = \frac{-4}{2} a_0 = -2a_0$$

$$x^2: 3a_3 = -4a_1 \Rightarrow a_3 = 0$$

$$x^3: 4a_4 = -4a_2 \Rightarrow a_4 = \frac{-4}{4} a_2 = \frac{-2}{2} \cdot -2a_0 = \frac{(-2)^2}{2!} a_0$$

$$x^4: 5a_5 = -4a_3 \Rightarrow a_5 = 0$$

$$x^5: 6a_6 = -4a_4 \Rightarrow a_6 = \frac{-4}{6}a_4 = \frac{-2}{3} \cdot \frac{-2}{2} \cdot -2a_0 = \frac{(-2)^3}{3!}a_0$$

$$\vdots$$

So, we have

$$\begin{aligned} y &= a_0 + -2a_0x^2 + \frac{(-2)^2}{2!}a_0x^4 + \frac{(-2)^3}{3!}a_0x^6 + \cdots \\ &= a_0 \left[1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \cdots \right] \\ &= a_0 e^{-2x^2} \end{aligned}$$

3. Problem Set 5.1 – 9

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

9. $y'' + y = 0$

Sol:

Let

$$y = \sum_{m=0}^{\infty} a_m x^m,$$

then

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1},$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2},$$

Insert y and y'' into $y'' + y = 0$, and we have

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0,$$

$$\begin{aligned} \Rightarrow (2 \cdot 1 a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + 5 \cdot 4 a_5 x^3 + 6 \cdot 5 a_6 x^4 + \cdots) + (a_0 + a_1 x + a_2 x^2 \\ + a_3 x^3 + a_4 x^4 + \cdots) = 0 \end{aligned}$$

$$x^0: 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-1}{2!} a_0$$

$$x^1: 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow a_3 = \frac{-1}{3!} a_1$$

$$x^2: 4 \cdot 3 a_4 + a_2 = 0 \Rightarrow a_4 = \frac{-1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0$$

$$x^3: 5 \cdot 4 a_5 + a_3 = 0 \Rightarrow a_5 = \frac{-1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1$$

$$\vdots$$

So, we have

$$\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\
&= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) \\
&= a_0 \cos x + a_1 \sin x
\end{aligned}$$

4. Problem Set 5.1 – 13

Find a power series solution in powers of x . Show the details.

$$13. y'' + (1 + x^2)y = 0$$

Sol:

Let

$$y = \sum_{m=0}^{\infty} a_m x^m,$$

then

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1},$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2},$$

Insert y and y'' into $y'' + (1 + x^2)y = 0$, and we have

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

$$x^0: 2 \cdot 1 a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{1}{2} a_0$$

$$x^1: 3 \cdot 2 a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{1}{6} a_1$$

$$x^2: 4 \cdot 3 a_4 + a_2 + a_0 = 0 \Rightarrow a_4 = -\frac{1}{12} (a_2 + a_0) = -\frac{1}{24} a_0$$

$$x^3: 5 \cdot 4 a_5 + a_3 + a_1 = 0 \Rightarrow a_5 = -\frac{1}{20} (a_3 + a_1) = -\frac{1}{24} a_1$$

$$x^4: 6 \cdot 5 a_6 + a_4 + a_2 = 0 \Rightarrow a_6 = -\frac{1}{30} (a_4 + a_2) = -\frac{13}{720} a_0$$

$$x^5: 7 \cdot 6 a_7 + a_5 + a_3 = 0 \Rightarrow a_7 = -\frac{1}{42} (a_5 + a_3) = -\frac{5}{1008} a_1$$

\vdots

So, we have

$$\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \\
&= a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{13}{720}x^6 + \cdots \right) + a_1 \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^5 + \frac{5}{1008}x^7 + \cdots \right)
\end{aligned}$$

For following problems, find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions. Show the details of your work.

5. Problem Set 5.3 – 3

$$3. \quad xy'' + 2y' + xy = 0$$

Sol:

Let

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r},$$

then

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1},$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2},$$

Insert y , y' and y'' into $xy'' + 2y' + xy = 0$, and we obtain

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1} + 2 \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r+1} = 0$$

The smallest power is x^{r-1} ; by equating the sum of its coefficients to zero we have

$$x^{r-1}: \quad r(r-1)a_0 + 2ra_0 = 0 \Rightarrow r(r+1) = 0 \Rightarrow r = 0, -1$$

$$x^r: \quad (r+1)ra_1 + 2(r+1)a_1 = 0 \Rightarrow a_1 = 0$$

$$x^{s+r-1}: \quad (s+r)(s+r-1)a_s + 2(s+r)a_s + a_{s-2} = 0 \Rightarrow a_s = \frac{-a_{s-2}}{(s+r)(s+r+1)}$$

$$\text{Since } a_1 = 0, \Rightarrow a_3 = a_5 = a_7 = \dots = 0$$

$$1) \text{ For } r = r_1 = 0, \quad a_s = \frac{-a_{s-2}}{s(s+1)}$$

$$a_2 = \frac{-a_0}{2 \cdot 3} = \frac{-a_0}{3!}, a_4 = \frac{-a_2}{4 \cdot 5} = \frac{a_0}{5!}, a_6 = \frac{-a_4}{6 \cdot 7} = \frac{-a_0}{7!}, \dots, a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}$$

$$\Rightarrow y_1 = x^0 a_0 \left(1 - \frac{1}{3!} x^2 + \frac{1}{5!} x^4 - \frac{1}{7!} x^6 + \dots \right) = a_0 \frac{\sinh x}{x}$$

$$2) \text{ For } r = r_2 = -1, \quad a_s = \frac{-a_{s-2}}{(s-1)s}$$

$$a_2 = \frac{-a_0}{1 \cdot 2} = \frac{-a_0}{2!}, a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{4!}, a_6 = \frac{-a_4}{5 \cdot 6} = \frac{-a_0}{6!}, \dots, a_{2m} = \frac{(-1)^m a_0}{(2m)!}$$

$$\Rightarrow y_2 = x^{-1} a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right) = a_0 \frac{\cosh x}{x}$$

Taking $a_0 = 1$,

$$\Rightarrow y_1 = \frac{\sinh x}{x}, \quad y_2 = \frac{\cosh x}{x}$$

6. Problem Set 5.3 – 5

$$5. x^2 y'' + x(2x - 1)y' + (x + 1)y = 0$$

Sol:

Let

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r},$$

then

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1},$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2},$$

Insert y , y' and y'' into $x^2 y'' + x(2x - 1)y' + (x + 1)y = 0$, and we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + 2 \sum_{m=0}^{\infty} (m+r) a_m x^{m+r+1} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} \\ + \sum_{m=0}^{\infty} a_m x^{m+r+1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \end{aligned}$$

The smallest power is x^r ; by equating the sum of its coefficients to zero we have the indicial equation:

$$x^r: r(r-1)a_0 - ra_0 + a_0 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$$

For the coefficients of x^{s+r} , we find the below relation

$$\begin{aligned} (s+r)(s+r-1)a_s + 2(s+r-1)a_{s-1} - (s+r)a_s + a_{s-1} + a_s = 0 \\ \Rightarrow a_s = \frac{-2(s+r) + 1}{(s+r-1)^2} a_{s-1} \end{aligned}$$

$$\text{For } r = r_1 = 1, a_s = \frac{-2s-1}{s^2} a_{s-1}$$

$$a_1 = \frac{-2 \cdot 1 - 1}{1^2} a_0 = \frac{(-3)}{1^2} a_0,$$

$$a_2 = \frac{-2 \cdot 2 - 1}{2^2} a_1 = \frac{(-3)(-5)}{1^2 \cdot 2^2} a_0$$

$$a_3 = \frac{-2 \cdot 3 - 1}{3^2} a_2 = \frac{(-3)(-5)(-7)}{1^2 \cdot 2^2 \cdot 3^2} a_0$$

\vdots

$$a_m = \frac{(-1)^m 3 \cdot 5 \cdot 7 \cdots (2m+1)}{1^2 \cdot 2^2 \cdot 3^2 \cdots m^2} a_0$$

So, we have (taking $a_0 = 1$)

$$y_1 = x(1 - 3x + \frac{3 \cdot 5}{1^2 \cdot 2^2} x^2 - \frac{3 \cdot 5 \cdot 7}{1^2 \cdot 2^2 \cdot 3^2} x^3 + \cdots)$$

$$\text{Let } y_2 = y_1 \ln x + x(A_1x + A_2x^2 + A_3x^3 + \cdots) = y_1 \ln x + (A_1x^2 + A_2x^3 + A_3x^4 + \cdots)$$

$$\Rightarrow y_2' = y_1' \ln x + \frac{y_1}{x} + 2A_1x + 3A_2x^2 + 4A_3x^3 + \cdots$$

$$\Rightarrow y_2'' = y_1'' \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2} + 2A_1 + 3 \cdot 2A_2x + 4 \cdot 3A_3x^2 + \cdots$$

$$\text{Insert them into the ODE: } x^2y_2'' + x(2x - 1)y_2' + (x + 1)y_2 = 0$$

$$x^2: -12 + 3 + 2A_1 + 2 + 3 - 2A_1 + A_1 = 0 \Rightarrow A_1 = 4$$

$$x^3: \Rightarrow A_2 = \frac{-29}{4}$$

$$x^4: \Rightarrow A_3 = \frac{27}{4}$$

$$\vdots$$

$$y_2 = y_1 \ln x + x \left(4x - \frac{29}{4}x^2 + \frac{27}{4}x^3 + \cdots \right)$$

7. Problem Set 5.3 – 7

$$7. y'' + \left(x - \frac{1}{2}\right)y = 0$$

Sol:

$$y'' + (x - \frac{1}{2})y = 0$$

$$\text{Let } y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^{m+1} - \sum_{m=0}^{\infty} \frac{1}{2} a_m x^m = 0$$

$$\sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s + \sum_{s=1}^{\infty} a_{s-1} x^s - \sum_{s=0}^{\infty} \frac{1}{2} a_s x^s = 0$$

$$x^s (s=0): \quad 2 \cdot 1 a_2 - \frac{1}{2} a_0 = 0 \Rightarrow a_2 = \frac{1}{4} a_0$$

$$x^s (s=1, 2, \dots): \quad (s+2)(s+1) a_{s+2} + a_{s-1} - \frac{1}{2} a_s = 0$$

$$a_{s+2} = \frac{\frac{1}{2} a_s - a_{s-1}}{(s+2)(s+1)} \quad (s=1, 2, \dots)$$

$$a_2 = \frac{\frac{1}{2} a_0 - a_{-1}}{2 \cdot 1} = \frac{1}{4} a_0 \quad \text{Let } a_0 = a_1 = 1, \quad a_2 = \frac{1}{4}$$

$$a_3 = \frac{\frac{1}{2} a_1 - a_0}{3 \cdot 2} = -\frac{1}{12}, \quad a_4 = \frac{\frac{1}{2} a_2 - a_1}{4 \cdot 3} = -\frac{1}{48}$$

$$a_5 = \frac{\frac{1}{2} a_2 - a_1}{5 \cdot 4} = -\frac{1}{40}, \quad a_6 = \frac{\frac{1}{2} a_3 - a_2}{6 \cdot 5} = \frac{1}{240}, \quad a_7 = \frac{\frac{1}{2} a_4 - a_3}{7 \cdot 6} = \frac{1}{504}, \quad a_8 = \frac{\frac{1}{2} a_5 - a_4}{8 \cdot 7} = \frac{1}{4480}, \quad a_9 = \frac{\frac{1}{2} a_6 - a_5}{9 \cdot 8} = \frac{1}{40320}, \quad a_{10} = \frac{\frac{1}{2} a_7 - a_6}{10 \cdot 9} = \frac{1}{362880}, \quad a_{11} = \frac{\frac{1}{2} a_8 - a_7}{11 \cdot 10} = \frac{1}{3326400}, \quad a_{12} = \frac{\frac{1}{2} a_9 - a_8}{12 \cdot 11} = \frac{1}{30240000}, \quad a_{13} = \frac{\frac{1}{2} a_{10} - a_9}{13 \cdot 12} = \frac{1}{272160000}, \quad a_{14} = \frac{\frac{1}{2} a_{11} - a_{10}}{14 \cdot 13} = \frac{1}{2425440000}, \quad a_{15} = \frac{\frac{1}{2} a_{12} - a_{11}}{15 \cdot 14} = \frac{1}{21240000000}, \quad a_{16} = \frac{\frac{1}{2} a_{13} - a_{12}}{16 \cdot 15} = \frac{1}{182160000000}, \quad a_{17} = \frac{\frac{1}{2} a_{14} - a_{13}}{17 \cdot 16} = \frac{1}{1519200000000}, \quad a_{18} = \frac{\frac{1}{2} a_{15} - a_{14}}{18 \cdot 17} = \frac{1}{12168000000000}, \quad a_{19} = \frac{\frac{1}{2} a_{16} - a_{15}}{19 \cdot 18} = \frac{1}{91440000000000}, \quad a_{20} = \frac{\frac{1}{2} a_{17} - a_{16}}{20 \cdot 19} = \frac{1}{612000000000000}, \quad a_{21} = \frac{\frac{1}{2} a_{18} - a_{17}}{21 \cdot 20} = \frac{1}{3060000000000000}, \quad a_{22} = \frac{\frac{1}{2} a_{19} - a_{18}}{22 \cdot 21} = \frac{1}{15300000000000000}, \quad a_{23} = \frac{\frac{1}{2} a_{20} - a_{19}}{23 \cdot 22} = \frac{1}{76500000000000000}, \quad a_{24} = \frac{\frac{1}{2} a_{21} - a_{20}}{24 \cdot 23} = \frac{1}{382500000000000000}, \quad a_{25} = \frac{\frac{1}{2} a_{22} - a_{21}}{25 \cdot 24} = \frac{1}{1912500000000000000}, \quad a_{26} = \frac{\frac{1}{2} a_{23} - a_{22}}{26 \cdot 25} = \frac{1}{9562500000000000000}, \quad a_{27} = \frac{\frac{1}{2} a_{24} - a_{23}}{27 \cdot 26} = \frac{1}{47812500000000000000}, \quad a_{28} = \frac{\frac{1}{2} a_{25} - a_{24}}{28 \cdot 27} = \frac{1}{239062500000000000000}, \quad a_{29} = \frac{\frac{1}{2} a_{26} - a_{25}}{29 \cdot 28} = \frac{1}{1195312500000000000000}, \quad a_{30} = \frac{\frac{1}{2} a_{27} - a_{26}}{30 \cdot 29} = \frac{1}{5976562500000000000000}, \quad a_{31} = \frac{\frac{1}{2} a_{28} - a_{27}}{31 \cdot 30} = \frac{1}{29882812500000000000000}, \quad a_{32} = \frac{\frac{1}{2} a_{29} - a_{28}}{32 \cdot 31} = \frac{1}{149414062500000000000000}, \quad a_{33} = \frac{\frac{1}{2} a_{30} - a_{29}}{33 \cdot 32} = \frac{1}{747070312500000000000000}, \quad a_{34} = \frac{\frac{1}{2} a_{31} - a_{30}}{34 \cdot 33} = \frac{1}{3735351562500000000000000}, \quad a_{35} = \frac{\frac{1}{2} a_{32} - a_{31}}{35 \cdot 34} = \frac{1}{18676757812500000000000000}, \quad a_{36} = \frac{\frac{1}{2} a_{33} - a_{32}}{36 \cdot 35} = \frac{1}{93383789062500000000000000}, \quad a_{37} = \frac{\frac{1}{2} a_{34} - a_{33}}{37 \cdot 36} = \frac{1}{466918945312500000000000000}, \quad a_{38} = \frac{\frac{1}{2} a_{35} - a_{34}}{38 \cdot 37} = \frac{1}{2334594726562500000000000000}, \quad a_{39} = \frac{\frac{1}{2} a_{36} - a_{35}}{39 \cdot 38} = \frac{1}{11672973632812500000000000000}, \quad a_{40} = \frac{\frac{1}{2} a_{37} - a_{36}}{40 \cdot 39} = \frac{1}{58364868164062500000000000000}, \quad a_{41} = \frac{\frac{1}{2} a_{38} - a_{37}}{41 \cdot 40} = \frac{1}{291824340820312500000000000000}, \quad a_{42} = \frac{\frac{1}{2} a_{39} - a_{38}}{42 \cdot 41} = \frac{1}{1459121704101562500000000000000}, \quad a_{43} = \frac{\frac{1}{2} a_{40} - a_{39}}{43 \cdot 42} = \frac{1}{7295608520507812500000000000000}, \quad a_{44} = \frac{\frac{1}{2} a_{41} - a_{40}}{44 \cdot 43} = \frac{1}{36478042602539062500000000000000}, \quad a_{45} = \frac{\frac{1}{2} a_{42} - a_{41}}{45 \cdot 44} = \frac{1}{182390213012695312500000000000000}, \quad a_{46} = \frac{\frac{1}{2} a_{43} - a_{42}}{46 \cdot 45} = \frac{1}{911951065063476562500000000000000}, \quad a_{47} = \frac{\frac{1}{2} a_{44} - a_{43}}{47 \cdot 46} = \frac{1}{4559755325317382812500000000000000}, \quad a_{48} = \frac{\frac{1}{2} a_{45} - a_{44}}{48 \cdot 47} = \frac{1}{22798776626586914062500000000000000}, \quad a_{49} = \frac{\frac{1}{2} a_{46} - a_{45}}{49 \cdot 48} = \frac{1}{113993883132934570312500000000000000}, \quad a_{50} = \frac{\frac{1}{2} a_{47} - a_{46}}{50 \cdot 49} = \frac{1}{569969415664672851562500000000000000}, \quad a_{51} = \frac{\frac{1}{2} a_{48} - a_{47}}{51 \cdot 50} = \frac{1}{2849847078323364257812500000000000000}, \quad a_{52} = \frac{\frac{1}{2} a_{49} - a_{48}}{52 \cdot 51} = \frac{1}{14249235391616821289062500000000000000}, \quad a_{53} = \frac{\frac{1}{2} a_{50} - a_{49}}{53 \cdot 52} = \frac{1}{71246176958084106445312500000000000000}, \quad a_{54} = \frac{\frac{1}{2} a_{51} - a_{50}}{54 \cdot 53} = \frac{1}{356230884790420532226562500000000000000}, \quad a_{55} = \frac{\frac{1}{2} a_{52} - a_{51}}{55 \cdot 54} = \frac{1}{1781154423952102661132812500000000000000}, \quad a_{56} = \frac{\frac{1}{2} a_{53} - a_{52}}{56 \cdot 55} = \frac{1}{8905772119760513305664062500000000000000}, \quad a_{57} = \frac{\frac{1}{2} a_{54} - a_{53}}{57 \cdot 56} = \frac{1}{44528860598802566528320312500000000000000}, \quad a_{58} = \frac{\frac{1}{2} a_{55} - a_{54}}{58 \cdot 57} = \frac{1}{222644302994012832641601562500000000000000}, \quad a_{59} = \frac{\frac{1}{2} a_{56} - a_{55}}{59 \cdot 58} = \frac{1}{1113221514970064163208007812500000000000000}, \quad a_{60} = \frac{\frac{1}{2} a_{57} - a_{56}}{60 \cdot 59} = \frac{1}{5566107574850320816040039062500000000000000}, \quad a_{61} = \frac{\frac{1}{2} a_{58} - a_{57}}{61 \cdot 60} = \frac{1}{27830537874251604080200195312500000000000000}, \quad a_{62} = \frac{\frac{1}{2} a_{59} - a_{58}}{62 \cdot 61} = \frac{1}{139152689371258020401000976562500000000000000}, \quad a_{63} = \frac{\frac{1}{2} a_{60} - a_{59}}{63 \cdot 62} = \frac{1}{695763446856290102005004882812500000000000000}, \quad a_{64} = \frac{\frac{1}{2} a_{61} - a_{60}}{64 \cdot 63} = \frac{1}{3478817234281450510025024414062500000000000000}, \quad a_{65} = \frac{\frac{1}{2} a_{62} - a_{61}}{65 \cdot 64} = \frac{1}{17394086171407252550125122070312500000000000000}, \quad a_{66} = \frac{\frac{1}{2} a_{63} - a_{62}}{66 \cdot 65} = \frac{1}{86970430857036262750625610351562500000000000000}, \quad a_{67} = \frac{\frac{1}{2} a_{64} - a_{63}}{67 \cdot 66} = \frac{1}{434852154285181313753128051757812500000000000000}, \quad a_{68} = \frac{\frac{1}{2} a_{65} - a_{64}}{68 \cdot 67} = \frac{1}{2174260771425906568765640258789062500000000000000}, \quad a_{69} = \frac{\frac{1}{2} a_{66} - a_{65}}{69 \cdot 68} = \frac{1}{10871303857129532843828201293945312500000000000000}, \quad a_{70} = \frac{\frac{1}{2} a_{67} - a_{66}}{70 \cdot 69} = \frac{1}{54356519285647664219141006469726562500000000000000}, \quad a_{71} = \frac{\frac{1}{2} a_{68} - a_{67}}{71 \cdot 70} = \frac{1}{271782596428238321095705032348632812500000000000000}, \quad a_{72} = \frac{\frac{1}{2} a_{69} - a_{68}}{72 \cdot 71} = \frac{1}{1358912982141191605478525161743164062500000000000000}, \quad a_{73} = \frac{\frac{1}{2} a_{70} - a_{69}}{73 \cdot 72} = \frac{1}{6794564910705958027392625808715820312500000000000000}, \quad a_{74} = \frac{\frac{1}{2} a_{71} - a_{70}}{74 \cdot 73} = \frac{1}{33972824553529790136963129043579101562500000000000000}, \quad a_{75} = \frac{\frac{1}{2} a_{72} - a_{71}}{75 \cdot 74} = \frac{1}{169864122767648950684815645217895507812500000000000000}, \quad a_{76} = \frac{\frac{1}{2} a_{73} - a_{72}}{76 \cdot 75} = \frac{1}{849320613838244753424078226089477539062500000000000000}, \quad a_{77} = \frac{\frac{1}{2} a_{74} - a_{73}}{77 \cdot 76} = \frac{1}{4246603069191223767120391130447387695312500000000000000}, \quad a_{78} = \frac{\frac{1}{2} a_{75} - a_{74}}{78 \cdot 77} = \frac{1}{21233015345956118835601955652236938476562500000000000000}, \quad a_{79} = \frac{\frac{1}{2} a_{76} - a_{75}}{79 \cdot 78} = \frac{1}{106165076729780594178009778261184692382812500000000000000}, \quad a_{80} = \frac{\frac{1}{2} a_{77} - a_{76}}{80 \cdot 79} = \frac{1}{530825383648902970890048891305923461914062500000000000000}, \quad a_{81} = \frac{\frac{1}{2} a_{78} - a_{77}}{81 \cdot 80} = \frac{1}{2654126918244514854450244456529617309570312500000000000000}, \quad a_{82} = \frac{\frac{1}{2} a_{79} - a_{78}}{82 \cdot 81} = \frac{1}{13270634591222574272251222282648086547851562500000000000000}, \quad a_{83} = \frac{\frac{1}{2} a_{80} - a_{79}}{83 \cdot 82} = \frac{1}{66353172956112871361256111413240432739257812500000000000000}, \quad a_{84} = \frac{\frac{1}{2} a_{81} - a_{80}}{84 \cdot 83} = \frac{1}{331765864780564356806280557066202163696289062500000000000000}, \quad a_{85} = \frac{\frac{1}{2} a_{82} - a_{81}}{85 \cdot 84} = \frac{1}{1658829323902821784031402785331010818481445312500000000000000}, \quad a_{86} = \frac{\frac{1}{2} a_{83} - a_{82}}{86 \cdot 85} = \frac{1}{8294146619514108920157013926655054092407226562500000000000000}, \quad a_{87} = \frac{\frac{1}{2} a_{84} - a_{83}}{87 \cdot 86} = \frac{1}{41470733097570544600785069633275270462036132812500000000000000}, \quad a_{88} = \frac{\frac{1}{2} a_{85} - a_{84}}{88 \cdot 87} = \frac{1}{207353665487852723003925348166376352310180664062500000000000000}, \quad a_{89} = \frac{\frac{1}{2} a_{86} - a_{85}}{89 \cdot 88} = \frac{1}{1036768327439263615019626740831881761550903320312500000000000000}, \quad a_{90} = \frac{\frac{1}{2} a_{87} - a_{86}}{90 \cdot 89} = \frac{1}{5183841637196318075098133704159408807754516601562500000000000000}, \quad a_{91} = \frac{\frac{1}{2} a_{88} - a_{87}}{91 \cdot 90} = \frac{1}{25919208185981590375490668520797044038772583007812500000000000000}, \quad a_{92} = \frac{\frac{1}{2} a_{89} - a_{88}}{92 \cdot 91} = \frac{1}{129596040929907951877453342603985220193862915039062500000000000000}, \quad a_{93} = \frac{\frac{1}{2} a_{90} - a_{89}}{93 \cdot 92} = \frac{1}{647980204649539759387266713019926100969314575195312500000000000000}, \quad a_{94} = \frac{\frac{1}{2} a_{91} - a_{90}}{94 \cdot 93} = \frac{1}{3239901023247698796936333565099630504846572875976562500000000000000}, \quad a_{95} = \frac{\frac{1}{2} a_{92} - a_{91}}{95 \cdot 94} = \frac{1}{16199505116238493984681667825498152524232864379882812500000000000000}, \quad a_{96} = \frac{\frac{1}{2} a_{93} - a_{92}}{96 \cdot 95} = \frac{1}{80997525581192469923408339127490762621164321899414062500000000000000}, \quad a_{97} = \frac{\frac{1}{2} a_{94} - a_{93}}{97 \cdot 96} = \frac{1}{404987627905962349617041695637453813105821609497070312500000000000000}, \quad a_{98} = \frac{\frac{1}{2} a_{95} - a_{94}}{98 \cdot 97} = \frac{1}{2024938139529811748085208478187269065529108047485351562500000000000000}, \quad a_{99} = \frac{\frac{1}{2} a_{96} - a_{95}}{99 \cdot 98} = \frac{1}{10124690697649058740426042390936345327645540237426757812500000000000000}, \quad a_{100} = \frac{\frac{1}{2} a_{97} - a_{96}}{100 \cdot 99} = \frac{1}{50623453488245293702130211954681726638227701187133789062500000000000000}, \quad a_{101} = \frac{\frac{1}{2} a_{98} - a_{97}}{101 \cdot 100} = \frac{1}{253117267441226468510651059773408633191138505935668945312500000000000000}, \quad a_{102} = \frac{\frac{1}{2} a_{99} - a_{98}}{102 \cdot 101} = \frac{1}{1265586337206132342553255298867043165955692529678344726562500000000000000}, \quad a_{103} = \frac{\frac{1}{2} a_{100} - a_{99}}{103 \cdot 102} = \frac{1}{6327931686030661712766276494335215829778462648391723632812500000000000000}, \quad a_{104} = \frac{\frac{1}{2} a_{101} - a_{100}}{104 \cdot 103} = \frac{1}{31639658430153308563831382471676079148892313241958618164062500000000000000}, \quad a_{105} = \frac{\frac{1}{2} a_{102} - a_{101}}{105 \cdot 104} = \frac{1}{158198292150766542819156912358380395744461566209793090820312500000000000000}, \quad a_{106} = \frac{\frac{1}{2} a_{103} - a_{102}}{106 \cdot 105} = \frac{1}{790991460753832714095784561791901978722307831048965454101562500000000000000}, \quad a_{107} = \frac{\frac{1}{2} a_{104} - a_{103}}{107 \cdot 106} = \frac{1}{3954957303769163570478922808959509893611539155244827270507812500000000000000}, \quad a_{108} = \frac{\frac{1}{2} a_{105} - a_{104}}{108 \cdot 107} = \frac{1}{19774786518845817852394614044797549468057695776224136352539062500000000000000}, \quad a_{109} = \frac{\frac{1}{2} a_{106} - a_{105}}{109 \cdot 108} = \frac{1}{98873932594229089261973070223987747340288478881120681762695312500000000000000}, \quad a_{110} = \frac{\frac{1}{2} a_{107} - a_{106}}{110 \cdot 109} = \frac{1}{494369662971145446309865351119938736701442394405603408813476562500000000000000}, \quad a_{111} = \frac{\frac{1}{2} a_{108} - a_{107}}{111 \cdot 110} = \frac{1}{2471848314855727231549326755599693683507211972028017044067382812500000000000000}, \quad a_{112} = \frac{\frac{1}{2} a_{109} - a_{108}}{112 \cdot 111} = \frac{1}{12359241574278636157746633777998468417536059860140085220336914062500000000000000}, \quad a_{113} = \frac{\frac{1}{2} a_{110} - a_{109}}{113 \cdot 112} = \frac{1}{61796207871393180788733168889992342087680299300700426101684570312500000000000000}, \quad a_{114} = \frac{\frac{1}{2} a_{111} - a_{110}}{114 \cdot 113} = \frac{1}{308981039356965903943665844449961710438401496503502130508422851562500000000000000}, \quad a_{115} = \frac{\frac{1}{2} a_{112} - a_{111}}{115 \cdot 114} = \frac{1}{1544905196784829519718329222249808552192007482517510652542114257812500000000000000}, \quad a_{116} = \frac{\frac{1}{2} a_{113} - a_{112}}{116 \cdot 115} = \frac{1}{7724525983924147598591646111249042760960037412587553262710571289062500000000000000}, \quad a_{117} = \frac{\frac{1}{2} a_{114} - a_{113}}{117 \cdot 116} = \frac{1}{38622629919620737992958230556245213804800187062937766313552856445312500000000000000}, \quad a_{118} = \frac{\frac{1}{2} a_{115} - a_{114}}{118 \cdot 117} = \frac{1}{193113149598103689964791152781226069024000935314688831567764282226562500000000000000}, \quad a_{119} = \frac{\frac{1}{2} a_{116} - a_{115}}{119 \cdot 118} = \frac{1}{965565747990518449823955763906130345120004676573444157838821411132812500000000000000}, \quad a_{120} = \frac{\frac{1}{2} a_{117} - a_{116}}{120 \cdot 119} = \frac{1}{4827828739952592249119778819530651725600023382867220789194107055664062500000000000000}, \quad a_{121} = \frac{\frac{1}{2} a_{118} - a_{117}}{121 \cdot 120} = \frac{1}{24139143699762961245598894097653258628000116914336103945970535278320312500000000000000}, \quad a_{122} = \frac{\frac{1}{2} a_{119} - a_{118}}{122 \cdot 121} = \frac{1}{120695718498814806227994470488266293140000584571680519729852676391601562500000000000000}, \quad a_{123} = \frac{\frac{1}{2} a_{120} - a_{119}}{123 \cdot 122} = \frac{1}{603478592494074031139972352441331465700002922858402598649263381958007812500000000000000}, \quad a_{124} = \frac{\frac{1}{2} a_{121} - a_{120}}{124 \cdot 123} = \frac{1}{3017392962470370155699861762206657328500014614292012993246316909790039062500000000000000}, \quad a_{125} = \frac{\frac{1}{2} a_{122} - a_{121}}{125 \cdot 124} = \frac{1}{15086964812351850778499308811033286642500073071460064966231584548950195312500000000000000}, \quad a_{126} = \frac{\frac{1}{2} a_{123} - a_{122}}{126 \cdot 125} = \frac{1}{75434824061759253892496544055166433212500365357300324831157922744750976562500000000000000}, \quad a_{127} = \frac{\frac{1}{2} a_{124} - a_{1$$

Sol:

$$2x(x-1)y'' - (x+1)y' + y = 0$$

$$\text{Let } y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r} - \sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r-1} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} \\ - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{s=1}^{\infty} 2(s+r-1)(s+r-2) a_{s-1} x^{s+r-1} - \sum_{m=0}^{\infty} 2(s+r)(s+r-1) a_s x^{s+r} - \sum_{s=1}^{\infty} (s+r-1) a_{s-1} x^{s+r-1} \\ - \sum_{s=0}^{\infty} (s+r) a_s x^{s+r} + \sum_{s=1}^{\infty} a_{s-1} x^{s+r-1} = 0$$

$$x^{r-1} (s=0): \quad [-2r(r-1) - r] a_0 = 0 \Rightarrow a_0 \neq 0 \\ (-2r^2 + r) = 0 \Rightarrow r = \frac{1}{2}, 0$$

$$x^{s+r-1} (s=1, 2, \dots): \quad 2(s+r-1)(s+r-2) a_{s-1} - 2(s+r)(s+r-1) a_s - (s+r-1) a_{s-1} \\ - (s+r) a_s + a_{s-1} = 0 \\ a_s = \left[\frac{(s+r-1)(2s+2r-5) + 1}{(s+r)(2s+2r-1)} \right] a_{s-1}$$

$$\textcircled{D} \quad r = \frac{1}{2}: \quad a_s = \left[\frac{(s-\frac{1}{2})(2s-4) + 1}{(s+\frac{1}{2}) \cdot 2s} \right] a_{s-1}, \quad (s=1, 2, \dots)$$

$$\therefore a_1 = 0, \quad \therefore a_2 = a_3 = a_4 = \dots = 0$$

$$\text{Let } a_0 = 1, \quad \therefore y_1 = x^{\frac{1}{2}}$$

$$\textcircled{2} \gamma=0; \quad a_s = \left[\frac{(s-1)(2s-3)+1}{s(2s-1)} \right] a_{s-1} \quad (s=1, 2, 3, \dots)$$

$$a_1 = a_0, \quad a_2 = 0, \quad \therefore a_3 = a_4 = \dots = 0$$

$$\text{Let } a_0 = 1, \quad a_1 = 1$$

$$\therefore y_2 = x^0 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$= 1 + x$$

$$\therefore y = C_1 y_1 + C_2 y_2$$

9. Problem Set 5.3 – 11

$$11. \quad xy'' + (2 - 2x)y' + (x - 2)y = 0$$

Sol:

$$xy'' + (2-x)y' + (x-2)y = 0$$

$$\text{Let } y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} + \sum_{m=0}^{\infty} 2(m+r) a_m x^{m+r-1} - \sum_{m=0}^{\infty} 2(m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1} - \sum_{m=0}^{\infty} 2a_m x^{m+r} = 0$$

$$\sum_{s=0}^{\infty} (s+r)(s+r-1) a_s x^{s+r-1} + \sum_{s=0}^{\infty} 2(s+r) a_s x^{s+r} - \sum_{s=1}^{\infty} 2(s+r-1) a_{s-1} x^{s+r-1} + \sum_{s=2}^{\infty} a_{s-2} x^{s+r-1} - \sum_{s=1}^{\infty} 2a_{s-1} x^{s+r-1} = 0$$

$$x^{r-1} (s=0): [r(r-1) + 2r] a_0 = 0 \Rightarrow a_0 \neq 0$$

$$r^2 + r = 0 \Rightarrow r = 0, -1$$

$$x^r (s=1): [(r+1)r + 2(r+1)] a_1 - 2ra_0 - 2a_0 = 0$$

$$a_1 = \frac{2}{(r+2)} a_0$$

$$x^{s+r-1} (s=2, 3, \dots): (s+r)(s+r-1) a_s + 2(s+r) a_s - 2(s+r-1) a_{s-1} - a_{s-2} - 2a_{s-1} = 0$$

$$a_s = \frac{[2(s+r)] a_{s-1} + a_{s-2}}{(s+r)(s+r+1)} \quad (s=2, 3, \dots)$$

$$\textcircled{1} r=0; \text{ Let } a_0=1, a_1=a_0=1, a_s = \frac{2s a_{s-1} + a_{s-2}}{s(s+1)} \quad (s=2, 3, \dots)$$

$$a_2 = \frac{3}{2 \cdot 3} = \frac{1}{2!}, a_3 = \frac{4}{3 \cdot 4} = \frac{1}{3!}, a_4 = \frac{1}{4!} \dots$$

$$y = x^0 \sum_{m=0}^{\infty} a_m x^m = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots = e^x$$

$$\textcircled{2} r=-1; \text{ Let } a_0=1, a_1=1, a_s = \frac{[2(s-1)] a_{s-1} + a_{s-2}}{(s-1)s} \quad (s=2, 3, \dots)$$

$$a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!} \dots$$

$$\therefore y_2 = x^{-1} \sum_{m=0}^{\infty} a_m x^m = x^{-1} (1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots)$$

$$= x^{-1} e^x$$

10. Problem Set 5.3 – 13

13. $xy'' + (2x+1)y' + (x+1)y = 0$

Sol:

$$xy'' + (2x+1)y' + (x+1)y = 0$$

$$\text{Let } y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} + \sum_{m=0}^{\infty} 2(m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1} = 0$$

$$\sum_{s=0}^{\infty} (s+r)(s+r-1) a_s x^{s+r-2} + \sum_{s=1}^{\infty} 2(s+r-1) a_{s-1} x^{s+r-1} + \sum_{s=0}^{\infty} (s+r) a_s x^{s+r} + \sum_{s=2}^{\infty} a_{s-2} x^{s+r-1} + \sum_{s=1}^{\infty} a_{s-1} x^{s+r} = 0$$

$$x^{r-2}(s=0): \quad [r(r-1) + r] a_0 = 0 \Rightarrow a_0 \neq 0$$

$$r^2 = 0 \Rightarrow r = 0 \quad (\text{Double root})$$

$$x^{s-1}(s=1): \quad [r(r+1) + (r+1)] a_1 + [2r+1] a_0 = 0 \quad (r=0)$$

$$a_1 = -a_0$$

$$x^{s+r-1}(s=2, 3, \dots): \quad [(s+1)(s+r-1) + (s+r)] a_s + [2(s+r-1) + 1] a_{s-1} + a_{s-2} = 0$$

$$a_s = \frac{-[2(s+r-1) + 1] a_{s-1} - a_{s-2}}{(s+r)^2}$$

$$\text{① } r=0: \quad a_s = \frac{(-2s+1)a_{s-1} - a_{s-2}}{s^2} \quad (s=2, 3, \dots)$$

$$\text{Let } a_0 = 1, a_1 = -1; \quad a_2 = \frac{1-2}{2^2} = -\frac{1}{2!}, \quad a_3 = \frac{-5a_0 - a_1}{3^2} = -\frac{1}{3!}$$

$$a_m = \frac{(-1)^m}{m!}$$

$$y_1 = x^0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = e^{-x}$$

$$\begin{aligned} \textcircled{2} \quad & \text{Let } y_2 = u y_1, \\ & y_2' = u y_1' + u' y_1, \\ & y_2'' = u y_1'' + 2u' y_1' + u'' y_1 \\ & x(u y_1'' + 2u' y_1' + u'' y_1) + (2x+1)(u y_1' + u' y_1) + (x+1)u y_1 = 0 \end{aligned}$$

$$\text{得 } u = \ln x$$

$$y_2 = u y_1 = e^{-x} \ln x$$

$$\therefore y = C_1 y_1 + C_2 y_2$$