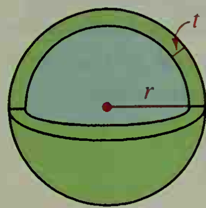
The relationship on the bottom of the previous page holds if there are just a few discs approximating each solid or very many discs. If there are very many discs, their total volume will be practically the same as the volume of the solid. Thus:

$$\begin{aligned}\text{Volume of sphere} &= \text{volume of cylinder} - \text{volume of double cone}\\ &= \pi r^2 \cdot 2r - 2(\tfrac{1}{3}\pi r^2 \cdot r)\\ &= 2\pi r^3 - \tfrac{2}{3}\pi r^3\\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

Justification of the Area Formula (Optional)

Imagine a rubber ball with inner radius r and rubber thickness t. To find the volume of the rubber, we can use the formula for the volume of a sphere. We just subtract the volume of the inner sphere from the volume of the outer sphere.



$$\begin{aligned}\text{Exact volume of rubber} &= \tfrac{4}{3}\pi(r + t)^3 - \tfrac{4}{3}\pi r^3\\ &= \tfrac{4}{3}\pi[(r + t)^3 - r^3]\\ &= \tfrac{4}{3}\pi[r^3 + 3r^2t + 3rt^2 + t^3 - r^3]\\ &= 4\pi r^2 t + 4\pi r t^2 + \tfrac{4}{3}\pi t^3\end{aligned}$$

The volume of the rubber can be found in another way as well. If we think of a small piece of the rubber ball, its approximate volume would be its outer area A times its thickness t. The same thing is true for the whole ball.

$$\begin{aligned}\text{Volume of rubber} &\approx \text{Surface area} \cdot \text{thickness}\\ V &\approx A \cdot t\end{aligned}$$

Now we can equate the two formulas for the volume of the rubber:

$$A \cdot t \approx 4\pi r^2 t + 4\pi r t^2 + \tfrac{4}{3}\pi t^3$$

If we divide both sides of the equation by t, we get the following result:

$$A \approx 4\pi r^2 + 4\pi r t + \tfrac{4}{3}\pi t^2$$

This approximation for A gets better and better as the layer of rubber gets thinner and thinner. As t approaches zero, the last two terms in the formula for A also approach zero. As a result, the surface area gets closer and closer to $4\pi r^2$. Thus:

$$A = 4\pi r^2$$

This is exactly what we would expect, since the surface area of a ball clearly depends on the size of the radius, not on the thickness of the rubber.