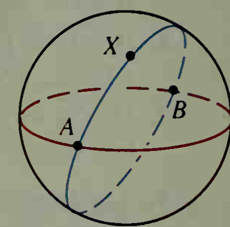
Euclidean geometry — Through a point outside a line, there is *exactly one* line parallel to the given line.

Hyperbolic geometry — Through a point outside a line, there is *more than one* line parallel to the given line. (This geometry was discovered by Bolyai, Lobachevsky, and Gauss.)

Elliptic geometry — Through a point outside a line, there is *no* line parallel to the given line. (This geometry was discovered by Riemann and is used by ship and airplane navigators.)

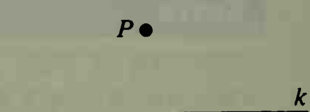
To see a model of a no-parallel geometry, visualize the surface of a sphere. Think of a line as being a great circle of the sphere, that is, the intersection of the sphere and a plane that passes through the center of the sphere. On the sphere, through a point outside a line, there is no line parallel to the given line. All lines, as defined, intersect. In the figure, for example, X is a point not on the red great circle. A line has been drawn through X, namely the great circle shown in blue. You can see that the two lines intersect in *two* points, A and B.



To see how statement (B) follows from our postulates, notice that Postulates 10 and 11 play a crucial role in the following proof. In fact, without such assumptions about parallels there couldn't be a proof. Before the discovery of non-Euclidean geometries people didn't know that this was the case and tried, without success, to find a proof that was independent of any assumption about parallels.

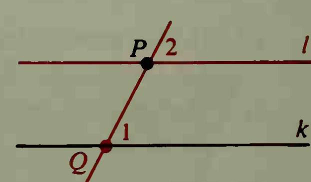
Given: Point P outside line k.

Prove: (1) There is a line through P parallel to k.
(2) There is only one line through P parallel to k.



Key steps of proof of (1):

1. Draw a line through P and some point Q on k. (Postulates 5 and 6)
2. Draw line l so that $\angle 2$ and $\angle 1$ are corresponding angles and $m\angle 2 = m\angle 1$. (Protractor Postulate)
3. $l \parallel k$, so there is a line through P parallel to k. (Postulate 11)



Indirect proof of (2):

Assume temporarily that there are at least two lines, x and y, through P parallel to k. Draw a line through P and some point R on k. $\angle 4 \cong \angle 3$ and $\angle 5 \cong \angle 3$ by Postulate 10, so $\angle 5 \cong \angle 4$. But since x and y are different lines we also have $m\angle 5 > m\angle 4$. This is impossible, so our assumption must be false, and it follows that there is only one line through P parallel to k.