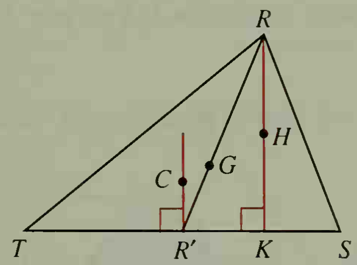
10–18. Work Exercises 19, 20 on page 538; Exercises 36–39 on page 552; Classroom Exercise 14 on page 554 and Exercise 35 on page 556; and Classroom Exercise 13 on page 595.

Constructions (Chapter 10)

Objective: Use transformations with concurrency and construction problems. (Requires understanding of Lessons 14-1 through 14-5.)

Concurrency Theorems 10-1 and 10-2 are proved by synthetic methods, while the proofs of concurrency Theorems 10-3 and 10-4 are delayed until coordinate methods can be used. However, all four theorems can be proved with coordinates or without. The following theorem states that three of the four concurrency points are collinear. A proof of this theorem with coordinates is quite involved. (See Exercises 11–14 of Lesson 13-9.) However, the transformational proof below is much shorter and more elegant.

Theorem: The orthocenter, centroid, and circumcenter of a triangle are collinear. (The line on which they lie is called *Euler's line*.)



Given: $\triangle RST$ with orthocenter H, centroid G, and circumcenter C.

Prove: H, G, and C are collinear.

Proof: Consider the dilation $D_{G,\,-\frac{1}{2}}$.

Because G divides each median in a 2:1 ratio, this dilation maps R to R', the midpoint of $\overline{TS}$. Also this dilation maps altitude $\overline{RK}$ to a parallel line through R'. But since $\overline{RK}$ is perpendicular to $\overline{TS}$, the image of $\overline{RK}$ must be perpendicular to $\overline{TS}$. Thus the image of altitude $\overline{RK}$ is the perpendicular bisector of $\overline{TS}$.

We have just proved that $D_{G,\,-\frac{1}{2}}$ maps an altitude to a perpendicular bisector. Similar reasoning shows that the dilation also maps the other two altitudes to perpendicular bisectors. Therefore the dilation maps the orthocenter H, which is on all three altitudes, to the circumcenter C, which is on all three perpendicular bisectors. And since the dilation $D_{G,\,-\frac{1}{2}}$ also maps G to itself, then H, G, and C must be collinear by the definition of a dilation.