

# Lambda-conductors for group rings

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June 7, 2018

## 1 Introduction.

This paper is part of a project which aims to provide a method for computing the Nil groups of the group rings of finite abelian groups, by refining some of the techniques used in [1] and [2] in such a way that the allowed coefficient rings include polynomial rings. For the refinement of the  $p$ -adic logarithm discussed in [3] and [4] it is assumed that the rings involved have a structure of  $\lambda$ -ring; we refer to these papers for generalities about  $\lambda$ -rings. Thus it is useful to extend as much as possible of the other techniques to the context of  $\lambda$ -rings. In this paper we investigate how to describe a group ring of a finite abelian group as a pull back of a diagram of rings which are more accessible to calculations in algebraic K-theory.

Let be given a commutative ring  $S$  and subring  $R$ . For each ideal  $I$  of  $S$  which is contained in  $R$  one has a cartesian square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & S/I \end{array}$$

thus describing  $R$  as a pull back of rings for which the  $K$ -theory is hopefully better understood. By taking for  $I$  the sum of all such ideals one finds a diagram where the rings on the bottom row are as small as possible.

We modify this construction by assuming that  $R$  has a structure of  $\lambda$ -ring and considering only ideals  $I$  stable under the  $\lambda$ -operations. We call the resulting ideal the  $\lambda$ -conductor of  $S$  into  $R$ . In particular we are interested in the case that  $R$  is the group ring  $\mathbf{Z}[G]$  of a finite abelian group, and  $S$  is its normal closure in  $R \otimes \mathbf{Q}$ , which splits as a direct sum of rings  $S_i = \mathbf{Z}[\chi_i]$  associated to equivalence classes of characters  $\chi_i: G \rightarrow \mathbf{C}$ .

In this situation  $R$  is a  $\lambda$ -ring such that  $\psi^n(g) = g^n$  for every  $n \in \mathbf{N}$  and  $g \in G$ . In general  $S$  is not stable under the  $\lambda$ -operations on  $R \otimes \mathbf{Q}$ , but it is stable under the associated Adams operations  $\psi^n$  since they are ring homomorphisms.

We will prove that in case  $G$  is a primary group its  $\lambda$ -conductor is precisely the intersection of the classical conductor and the augmentation ideal. We do this by exhibiting generators of the classical conductor and examining their behavior under the fundamental  $\lambda$ -operations.

## 2 The primary case

Throughout this section  $G$  is a group of order  $n = p^e$ , where  $p$  is prime. We consider representations  $\rho: G \rightarrow \mathbf{C}^*$ . We say that  $\rho$  is of level  $k$  if the image of  $\rho$  has  $p^k$  elements.

Two representations  $\tau_1$  and  $\tau_2$  are called equivalent if they have the same kernel. That means that there must be some  $m \in \mathbf{Z}$  prime to  $p$  such that  $\tau_2(x) = \tau_1(x)^m$  for all  $x \in G$ . Obviously equivalent representations have the same level.

Given a representation  $\tau$  of level  $k > 0$  one gets a representation  $\psi\tau$  of level  $k - 1$  by the formula  $(\psi\tau)(x) = \tau(x^p)$  for  $x \in G$ . If  $\psi\tau_1$  and  $\psi\tau_2$  are equivalent then we may replace  $\tau_2$  by an equivalent representation  $\tau'_2$  so that  $\psi\tau'_2 = \psi\tau_1$ . So we may choose a representation in each class in such a way that  $\psi\tau$  and  $\rho$  coincide if they are equivalent.

Let  $\rho$  be a representation of level  $k > 0$  and write  $\omega = \exp(2\pi i/p)$ . We define an element  $b_\rho \in \mathbf{Z}[G]$  by the formula

$$b_\rho = \sum_{\rho x=1} x - \sum_{\rho \xi=\omega} \xi.$$

If we choose  $y_\rho \in G$  such that  $\rho(y) = \omega$  then we get

$$b_\rho = \left( \sum_{\rho x=1} x \right) (1 - y_\rho)$$

The only representation of level 0 is the trivial representation, which we denote by 1, and it gives rise to  $b_1 = \sum_{x \in G} x$ .

**Proposition 1.** *If  $\rho$  and  $\tau$  are not equivalent then  $b_\rho b_\tau = 0$ . Furthermore  $b_\rho^2 = p^{e-k} (1 - y_\rho) b_\rho$  for  $\rho$  of level  $k > 0$ .*

*Proof.* If  $\ker(\rho) \neq \ker(\tau)$  we may assume that there is  $g \in G$  with  $\rho(g) = 1$  but  $\tau(g) = \omega$ . If  $g$  has order  $m$  then  $\sum_{\rho(x)=1} x$  and thus  $b_\rho$  is a multiple of  $\sum_{j=0}^{m-1} g^j$ , whereas  $b_\tau$  is a multiple of  $1 - g$ . The product of these two factors is 0.

The second part follows from  $(\sum_{\rho(\xi)=1} \xi)(\sum_{\rho(x)=1} x) = p^{n-k} \sum_{\rho(x)=1} x$  which is true because each  $\xi$  gives the same contribution and there are  $p^{e-k}$  of them.  $\square$

Every representation  $\rho$  of level  $k$  gives rise to a homomorphism  $j_\rho$  from  $\mathbf{Z}[G]$  to  $S_\rho = \mathbf{Z}[\omega_k]$ , where  $\omega_k = \exp(2\pi i/p^k)$ .

**Proposition 2.** *If  $\rho$  and  $\tau$  are not equivalent then  $j_\rho(b_\tau) = 0$ . Furthermore  $j_1(b_1) = p^e$ , and  $j_\rho(b_\rho) = p^{e-k} (1 - \omega)$  if  $\rho$  is of level  $k > 0$ .*

*Proof.* The second part is obvious since every  $x$  in the definition of  $b_\rho$  maps to 1, and  $y_\rho$  maps to  $\omega$ . The first part follows since  $j_\rho(b_\rho)j_\rho(b_\tau) = 0$  by Proposition 1 and  $S_\rho$  is a domain.  $\square$

It is well known that the maps  $j_\rho$  (one from each equivalence class) combine to an embedding from  $R$  into its integral closure  $S = \oplus_\rho S_\rho$ .

**Proposition 3.** *The  $b_\rho$  generate the conductor ideal  $I$  of  $S$  into  $R$ .*

*Proof.* By the theorem of Jacobinski (Theorem 27.8 in [5]) the conductor is  $\oplus_\rho n\mathcal{D}_\rho^{-1} \subset \oplus_\rho S_\rho = S$ , where  $\mathcal{D}_\rho^{-1}$  is the lattice in  $\mathbf{Q}[\omega_k]$  dual to  $\mathbf{Z}[\omega_k]$  under the trace form. It is a simple exercise that this fractional ideal is in fact generated by  $p^{-k}(1 - \omega)$ , which means that  $n\mathcal{D}_\rho^{-1}$  is just  $j(b_\rho)S = j(B_\rho)S_\rho = j(b_\rho R)$ .  $\square$

We remind the reader that in particular  $nS$  is contained in the conductor.

The  $\lambda$ -conductor  $I_\lambda$  from  $S$  into  $R$  is defined as the largest ideal of  $S$  contained in  $R$  which is stable under the fundamental  $\lambda$ -operations  $\theta^\ell$ . It is of course a subset of the largest ideal of  $S$  contained in  $R$ , which is the ordinary conductor  $I$  described above. Thus we have to investigate the behaviour of the operations  $\theta^\ell$  on the generators  $b_\rho$ .

**Lemma 1.** *If  $\rho$  is of level  $k > 0$  and there is no  $\tau$  with  $\psi\tau = \rho$  then  $\psi^p b_\rho = 0$ .*

*Proof.* Write  $G$  as a direct product of cyclic groups, with generators  $g_i$ . If the order of  $\rho(g_i)$  is strictly smaller than the order of  $g_i$  for all  $i$  then one can find a suitable  $\tau$  by taking for each  $\tau(g_i)$  a  $p$ -th root of  $\rho(g_i)$ . If however the orders are the same for some  $i$  then there is certainly some  $h \in G$  such that  $h^p = 1$  and  $\rho(h) = \omega$ . By definition of  $b_\rho$  we have

$$\psi^p b_\rho = \sum_{\rho\xi=1} \xi^p - \sum_{\rho\eta=\omega} \eta^p$$

Here the term in the second sum associated to  $\eta = h\xi$  cancels the term in the first sum associated to  $\xi$ .  $\square$

**Proposition 4.** *If  $\rho$  is of level  $k > 0$  then*

$$\psi^p(b_\rho) = \sum_{\psi\tau=\rho} pb_\tau$$

*Proof.* By definition we have

$$\sum_{\psi\tau=\rho} b_\tau = \sum_{\psi\tau=\rho} \sum_{\tau x=1} x - \sum_{\psi\tau'=\rho} \sum_{\tau'x=\omega} x$$

We claim that all terms with  $x \notin G^p$  cancel. To prove this assume that the class of  $x$  in  $G/G^p$  is nontrivial. Then there exists a homomorphism  $\sigma: G/G^p \rightarrow \mathbf{C}^*$  such that  $\sigma(x) = \omega$ . Now the term associated to  $\tau$  in the first sum equals the term associated to  $\tau' = \tau \cdot \sigma$  in the second sum.

So we only have to consider terms of the form  $x = \xi^p$  with  $\xi \in G$ . The condition  $\tau(x) = 1$  is then independent of  $\tau$  (since it is equivalent to  $\rho(\xi) = 1$ ) and the sum over all  $\tau$  with  $\psi\tau = \rho$  reduces to a multiplication with the number

of equivalence classes of such  $\tau$ . By the Lemma we may assume that this number is nonzero. Now  $\tau_1$  and  $\tau_2$  with  $\psi\tau_1 = \rho = \psi\tau_2$  are equivalent iff  $\tau_2 = \tau_1^{1+mp^k}$  for some  $m$  with  $0 \leq m < p$ . So this number is  $1/p$  times the number of homomorphisms  $\sigma: G/G^p \rightarrow \mathbf{C}^*$ , hence equals  $p^{r-1}$ , where  $r$  denotes the rank of  $G$ .

On the other hand we have

$$\psi^p b_\rho = \sum_{\rho\xi=1} \xi^p - \sum_{\rho\eta=\omega} \eta^p$$

Here the first sum is a certain factor times the sum over all  $x \in G$  for which there exists  $\xi \in G$  with  $x = \xi^p$  and which satisfy  $\tau(x) = 1$  for any (and thus all)  $\tau$  with  $\psi\tau = \rho$ . The factor is the number of  $\xi$  which satisfy these conditions, which equals  $p^r$ .  $\square$

We write  $h$  for the polynomial of degree  $p-2$  given by

$$h(t) = \frac{1}{1-t} \left( p - \frac{1-t^p}{1-t} \right)$$

**Proposition 5.**

$$\psi^p(b_1) = b_1 + \sum_{\tau \neq 1, \psi\tau=1} h(y_\tau) b_\tau$$

*Proof.* We have

$$b_\tau = \left( \sum_{\tau x=1} x \right) (1 - y_\tau)$$

and thus

$$h(y_\tau) b_\tau = \left( \sum_{\tau x=1} x \right) \left( p - \sum_{j=0}^{p-1} y_\tau^j \right) = p \left( \sum_{\tau x=1} x \right) - \left( \sum_{x \in G} x \right)$$

We must take the sum of  $\sum_{\tau x=1} x$  over all equivalence classes of  $\tau \neq 1$  with  $\psi\tau = 1$ . Interchange the sum over  $x$  and the sum over  $\tau$ . There are two cases:

- If  $x \in G^p$  then  $\tau x = 1$  for all  $\tau$ , and we must simply count the number of equivalence class of  $\tau$ . There are  $p^r - 1$  of them, with  $p-1$  in each class.
- If  $x \notin G^p$  then the number of  $\tau$  such that  $\tau x = 1$  is  $p^{r-1} - 1$ , with again  $p-1$  in each class.

So we get

$$\begin{aligned} \sum_{\tau \neq 1, \psi\tau=1} h(y_\tau) b_\tau &= p \left( \frac{p^r - 1}{p - 1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p - 1} \sum_{x \notin G^p} x \right) \\ &\quad - \left( \frac{p^r - 1}{p - 1} \sum_{x \in G^p} x + \frac{p^{r-1} - 1}{p - 1} \sum_{x \notin G^p} x \right) \\ &= p^r \sum_{x \in G^p} x - \sum_{x \in G} x = \sum_{\xi \in G} \xi^p - \sum_{x \in G} x = \psi^p b_1 - b_1 \end{aligned}$$

□

Now we consider the effect of Adams operations  $\psi^q$  for primes  $q \neq p$ . For any prime  $q$  we write  $f_q$  and  $g_q$  for the polynomials given by

$$f_q(t) = \frac{1-t^q}{1-t}, \quad g_q(t) = \frac{(1-t)^{q-1} - f_q(t)}{q}$$

**Proposition 6.** *If  $\rho$  is of level  $k > 0$  then*

$$\psi^q(b_\rho) = f_q(y_\rho)b_\rho$$

and  $\psi^q(b_1) = b_1$ .

*Proof.* We have  $\psi^q(b_1) = \psi^q(\sum_{x \in G} x) = \sum_{x \in G} x^q = \sum_{\xi \in G} \xi = b_1$  and

$$\begin{aligned} \psi^q(b_\rho) &= \psi^q\left(\left(\sum_{\rho x=1} x\right)(1-y_\rho)\right) = \left(\sum_{\rho x=1} x^q\right)(1-y_\rho^q) \\ &= \left(\sum_{\rho \xi=1} \xi\right)(1-y_\rho)f_q(y_\rho) = b_\rho f_q(y_\rho) \end{aligned}$$

□

**Corollary 1.** *For the idempotents  $e_\rho \in S_\rho$  one has*

$$\begin{aligned} \psi^p(e_\rho) &= \sum_{\psi\tau=\rho} e_\tau \quad \text{if } \rho \neq 1, \\ \psi^p(e_1) &= e_1 + \sum_{\tau \neq 1, \psi\tau=1} e_\tau \\ \psi^q(e_\rho) &= e_\rho \quad \text{if } q \neq p \end{aligned}$$

Since  $R$  has no  $\mathbf{Z}$ -torsion the Adams operations  $\psi^\ell$  determine the operations  $\theta^\ell$  and we find

**Proposition 7.** *If  $\rho$  has level  $k > 0$  then*

$$\begin{aligned} \theta^p(b_\rho) &= p^{(e-k)(p-1)-1}(1-y_\rho)^{p-1}b_\rho - \sum_{\psi\tau=\rho} b_\tau \quad \text{if } k < e \\ \theta^p(b_\rho) &= g_p(y_\rho)b_\rho \quad \text{if } k = e \\ \theta^q(b_\rho) &= \left(\frac{p^{(e-k)(q-1)} - 1}{q}(1-y_\rho)^{q-1} + g_q(y_\rho)\right)b_\rho \quad \text{for } q \neq p \end{aligned}$$

Moreover

$$\begin{aligned} \theta^p(b_1) &= p^{e(p-1)-1}b_1 - p^{-1}b_1 - p^{-1} \sum_{\tau \neq 1, \psi\tau=1} h(y_\tau)b_\tau \\ \theta^q(b_1) &= \frac{p^{e(q-1)} - 1}{q}b_1 \quad \text{for } q \neq p \end{aligned}$$

*Proof.* This is just a matter of combining the last three Propositions with the formula  $\ell\theta^\ell(a) = a^\ell - \psi^\ell a$ . Note that  $k = e$  can only happen if  $G$  is cyclic, in which case  $y_\rho^p = 1$ , which implies that  $b_\rho^p = (1 - y_\rho)^p = p(1 - y_\rho)g_p(y_\rho)$ .  $\square$

**Theorem 1.** *The  $b_\rho$  with  $\rho \neq 1$  generate the  $\lambda$ -conductor ideal  $I_\lambda$ . In other words  $I_\lambda$  is the intersection of the augmentation ideal and the ordinary conductor ideal  $I$ .*

*Proof.* Write  $J$  for the  $R$ -ideal generated by the  $b_\rho$  with  $\rho \neq 1$ . From Proposition 7 one reads off that  $\theta^\ell(b_\rho) \in J$  for  $\rho \neq 1$  and for every prim  $\ell$ . From the identity

$$\theta^\ell(ab) = \theta^\ell(a)b^\ell + \psi^\ell(a)\theta^\ell(b)$$

it then follows that  $\theta^\ell(Rb_\rho) \subset J$  for  $\rho \neq 1$  and all  $\ell$ . Finally from

$$\theta^\ell(u + v) = \theta^\ell(u) + \theta^\ell(v) + \sum_{i=1}^{\ell-1} \frac{1}{\ell} \binom{\ell}{i} u^i v^{\ell-i}$$

it follows that  $\theta^\ell(J) \subset J$  for every  $\ell$ . Since  $J \subset I$  by Proposition 3 this shows that  $J \subset I_\lambda$ .

Suppose that  $x \in I_\lambda$  and  $x \notin J$ . Then  $x \in I$ , so by Proposition 3 there are  $x_\rho \in R$  such that  $x = \sum_\rho x_\rho b_\rho$ . Since  $\sum_{\rho \neq 1} x_\rho b_\rho \in J \subset I_\lambda$  by the first half of the proof, it follows that  $x_1 b_1 \in I_\lambda$ . Since  $gb_1 = b_1$  for every  $g \in G$  we may assume that  $x_1 \in \mathbf{Z}$ . Moreover  $x_1 \neq 0$  which means that its  $p$ -valuation  $v_p(x_1)$  is a natural number. We may assume that  $x$  is chosen in such a way that  $v_p(x_1)$  is minimal. Now  $I_\lambda$  must also contain

$$\theta^p(x_1 b_1) = p^{-1} \left( x_1^p p^{e(p-1)} b_1 - x_1 (b_1 + \sum_{\tau \neq 1, \psi\tau=1} b_\tau) \right)$$

However the valuation of the coefficient of  $b_1$  is  $v_p(x_1^p p^{e(p-1)} - x_1) - 1 = v_p(x_1) - 1$ , in contradiction with the way  $x$  was chosen. Thus  $I_\lambda \subset J$ .  $\square$

### 3 Direct products of relatively prime order

Let  $G_1$  be a group of order  $n_1 = p^e$ , and let  $G_2$  a group of order  $n_2 = q^f$ , where  $p$  and  $q$  are different primes. We write  $R_1 = \mathbf{Z}[G_1]$  and  $R_2 = \mathbf{Z}[G_2]$ , and denote their normal closures by  $S_1$  and  $S_2$  respectively. Finally we write  $I_1$  for the conductor of  $S_1$  into  $R_1$  and  $I_2$  for the conductor of  $S_2$  into  $R_2$ . Since the  $S_i$  are free abelian groups, the same is true for the other additive groups involved, and we can view  $I_1 \otimes I_2$  as a subgroup of  $R_1 \otimes I_2$  and of  $R_1 \otimes R_2$ .

**Lemma 2.**

$$I_1 \otimes I_2 = (R_1 \otimes I_2) \cap (I_1 \otimes R_2)$$

*Proof.* There are  $m_1, m_2 \in \mathbf{Z}$  such that  $m_1 n_1 + m_2 n_2 = 1$ . If  $x$  is an element of the left hand side then  $x \in R_1 \otimes I_2$ , so  $n_1 x \in n_1 R_1 \otimes I_2 \subset n_1 S_1 \otimes I_2 \subset I_1 \otimes I_2$  and therefore  $m_1 n_1 x \in I_1 \otimes I_2$ . Similarly  $m_2 n_2 x \in I_1 \otimes I_2$  and thus  $x = m_1 n_1 x + m_2 n_2 x \in I_1 \otimes I_2$ . The other implication is obvious  $\square$

**Proposition 8.** *The conductor  $I$  of  $S_1 \otimes S_2$  into  $R_1 \otimes R_2$  is  $I_1 \otimes I_2$ .*

*Proof.* Suppose that  $x \in I$ , so that  $x(S_1 \otimes S_2) \subset R_1 \otimes R_2$ . We write  $x \in R_1 \otimes R_2$  as  $\sum x_g \otimes g$ , where  $g$  runs through  $G_2$ . For any  $a \in S_1$  we have  $\sum (x_g a) \otimes g = (\sum x_g \otimes g)(a \otimes 1) = x(a \otimes 1) \in R_1 \otimes R_2$ . Therefore  $x_g a \in I_1$  for any  $a \in S_1$ , which means that  $a_g \in I_1$  for all  $g \in G_1$ . Thus  $x \in I_1 \otimes R_2$ . Similarly  $x \in R_1 \otimes I_2$ . Thus  $x \in I_1 \otimes I_2$  by the Lemma. The other inclusion is obvious.  $\square$

We show now that for the  $\lambda$ -conductor a similar theorem holds:

**Theorem 2.** *The  $\lambda$ -conductor  $I_\lambda$  of  $S_1 \otimes S_2$  into  $R_1 \otimes R_2$  is the tensor product of the  $\lambda$ -conductors  $I_{1\lambda}$  of  $S_1$  into  $R_1$  and  $I_{2\lambda}$  of  $S_2$  into  $R_2$ .*

*Proof.* The  $\lambda$ -conductor  $I_\lambda$  is a subset of the classical conductor  $I$ , which is  $I_1 \otimes I_2$ . However  $I_1$  is the direct sum  $\mathbf{Z}b_1 \oplus I_{1\lambda}$  by theorem 1. and similarly for  $I_2$ . Thus any  $x \in I_\lambda$  can uniquely be written as

$$x = x_0(b_1 \otimes b_1) \oplus (x_1 \otimes b_1) \oplus (b_1 \otimes x_2) \oplus y$$

with  $x_0 \in \mathbf{Z}$ ,  $x_1 \in I_{1\lambda}$ ,  $x_2 \in I_{2\lambda}$ ,  $y \in I_{1\lambda} \otimes I_{2\lambda}$ . Since  $I_\lambda$  is an ideal of  $S_1 \otimes S_2$ , each of these four summands must be in  $I_\lambda$ .

Therefore we consider the intersection of  $I_\lambda$  with  $b_1 \otimes I_{2\lambda}$ . Suppose that  $a$  is an element of this intersection, say  $a = b_1 \otimes x$  with  $x \in I_{2\lambda}$ . Then  $\theta^p(a) \in I_\lambda$  too. We have

$$\theta^p(a) = p^{-1}(a^p - \psi^p a) = p^{-1}(p^{e(p-1)}b_1 \otimes x^p - (b_1 + \sum_{\tau \neq 1, \psi\tau=1} b_\tau) \otimes \psi^p x)$$

and thus  $p^{e(p-1)-1}b_1 \otimes x^p - p^{-1}b_1 \otimes \psi^p x$  should be in  $I_\lambda$ . Now the first term is a multiple of  $a$  and thus in  $I_\lambda$ . So the other term  $p^{-1}b_1 \otimes \psi^p x$  is in the aforementioned intersection. Since  $\psi^p$  is an automorphism (of finite order) of  $R_2$  this shows that the intersection is  $p$ -divisible. Since the intersection is a finitely generated abelian group this can only happen if it vanishes.

The same argument applies to the first and second summand of  $x$ . Thus  $x = y \in I_{1\lambda} \otimes I_{2\lambda}$  and we have shown that  $I_\lambda \subset I_{1\lambda} \otimes I_{2\lambda}$ . The other inclusion is obvious.  $\square$

## References

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