Overview of the QM-AM-GM-HM Inequality

In your math class, you might have learned about inequalities and how to solve them. They seem pretty straightforward, but there are a lot of interesting and famous inequalities. One example is called the mean inequality chain, or the QM-AM-GM-HM inequality

Starting from the middle, the AM-GM inequality itself is one of the most famous algebraic inequalities. It states that the arithmetic mean of a set of nonnegative numbers is at least as big as the geometric mean of the numbers. The 2-number version stems from the trivial inequality, which states that if x is a real number, then $x^2 \geq 0$. We start by the known statement $(x-y)^2 \geq 0$. Expanding, we get $(x^2 - 2xy + y^2) \geq 0$. After adding $4xy$ to both sides of the inequality, we get $(x^2 + 2xy + y^2) \geq 4xy$, or $(x+y)^2 \geq 4xy$ after factoring. Taking the square root of both sides and simplifying gives us what we want, or $\frac{(x+y)}{2} \geq \sqrt{xy}$. However, if expanding and rearranging perfect square trinomials was all there was to AM-GM, this inequality wouldn't be so famous! Using induction, we can prove that this works for all powers of 2, which can be seen by repeatedly applying the 2-number AM-GM inequality.

Expanding on this, we now go to the AM - GM - HM inequality. HM stands for harmonic mean, which is defined as the reciprocal of the sum of the reciprocals of n real numbers (that's a lot of reciprocals!). If we apply the AM - GM inequality to the harmonic mean, we get $\frac{(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n})}{n} \geq \sqrt[n]{(\frac{1}{x_1})(\frac{1}{x_2})\dots(\frac{1}{x_n})}$. Taking the reciprocal of both sides gives us that $\frac{n}{(\frac{1}{x_1})+(\frac{1}{x_2})+\cdots+(\frac{1}{x_n})} \leq \sqrt[n]{x_1 x_2 x_3 \cdots x_n}$, or the geometric mean.

The final part is to add in the QM, or quadratic mean, which is the square root of the average of the squares of n real numbers. To prove this, we use another famous inequality, the Cauchy-Schwartz inequality. While we cannot do a proof here, this inequality states that $(x_1^2 + x_2^2 \cdots x_n^2)(y_1^2 + y_2^2 \cdots y_n^2) \geq (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2$. If we let $x_1, x_2 \dots, x_n$ be the n numbers in our sequence, we can let $y_1 = y_2 = y_3 = \cdots = y_n = 1$. This gives us $(x_1^2 + x_2^2 \cdots x_n^2) \cdot n \geq (x_1 + x_2 + \cdots + x_n)^2$. Dividing both sides by n^2 and taking the square root of both sides gives us $\sqrt{\frac{(x_1^2+x_2^2 \cdots x_n^2)}{n}} \geq \frac{(x_1+x_2+\cdots+x_n)}{n}$, thus completing our mean inequality chain.

In all, the mean inequality chain states that $\sqrt{\frac{(x_1^2+x_2^2 \cdots x_n^2)}{n}} \geq \frac{(x_1+x_2+\cdots+x_n)}{n} \geq \sqrt[n]{x_1 x_2 x_3 \cdots x_n} \geq \frac{n}{(\frac{1}{x_1})+(\frac{1}{x_2})+\cdots+(\frac{1}{x_n})}$ where $x_1, x_2, \dots, x_n$ are positive real numbers.