Introduction to Tropical Arithmetic

Ask any third grader what $3 \cdot 3$ is or $2+2$ is and they'll respond with 9 and 4, maybe almost immediately. But who says multiplication and addition have to work like this? Some of you may have seen those competition math questions like "Define $a \diamond b = 3a + 5b$, compute $5 \diamond 2$." Well in tropical arithmetic, we do just that! Let

$$A \oplus B = min(a,b) \text{ and } A \otimes B = a+b$$

Let's try some of these to get a feel for it:

$$1 \oplus 4$$

$$12 \oplus 2$$

$$10000 \oplus 2$$ [1]

$$1 \otimes 4$$

$$2 \otimes 2$$

$$9 \otimes 2$$ [2]

Just like in normal arithmetic, we have *identities*, which (in normal arithmetic) are numbers such that when added or multiplied to anything, just leave the value as is. Try to find out what the identities for tropical arithmetic are![3]

And by the way, choosing min instead of max for tropical addition seems weird, and totally arbitrary. So what if we chose max instead? Well, in fact, choosing max instead of min doesn't change much about tropical addition, and many mathematicians choose to use max instead of min. Do you see why this is the case? (Hint: what can you do to all the numbers to make it work?[4])

Of course, we can also do algebra and geometry with these definitions of plus and times, try converting some of your favorite graphs into tropical graphs and see how they might look! One place to start is trying

$$x^2 + 2x + 1$$

(If you want to get *even more* advanced, try the following, and graph only the points which reach the minimum twice)

$$p(x,y) = 1 \otimes x \otimes y \oplus x \oplus y \oplus 0$$

Is this even useful though? Yes! Tropical arithmetic and geometry have a lot of applications in optimization problems, and have been used in the past for scheduling, train networks, and other problems involving optimization.

If you're curious on why it's called tropical geometry, it used to be called max-plus (or min-plus) geometry until French mathematicians decided to call it tropical geometry to honor the work of Imre Simon, who worked on the subject in Brazil.

[1] 1, 2, and 2

[2] 5, 4, and 11

[3] Additive: infinity, Multiplicative: 0

[4] Just map every x to $-x$, and the properties all still hold!

If you want to find out more about tropical geometry, check out some of the following resources:

https://www-users.cse.umn.edu/ kell1642/documents/tropicalGeometry_FundamentalTheorem.pdf
https://www.ocf.berkeley.edu/ bmt/wp-content/uploads/2022/03/PowS2017S.pdf
https://en.wikipedia.org/wiki/Tropical_geometry