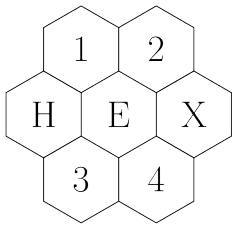
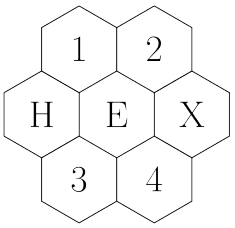


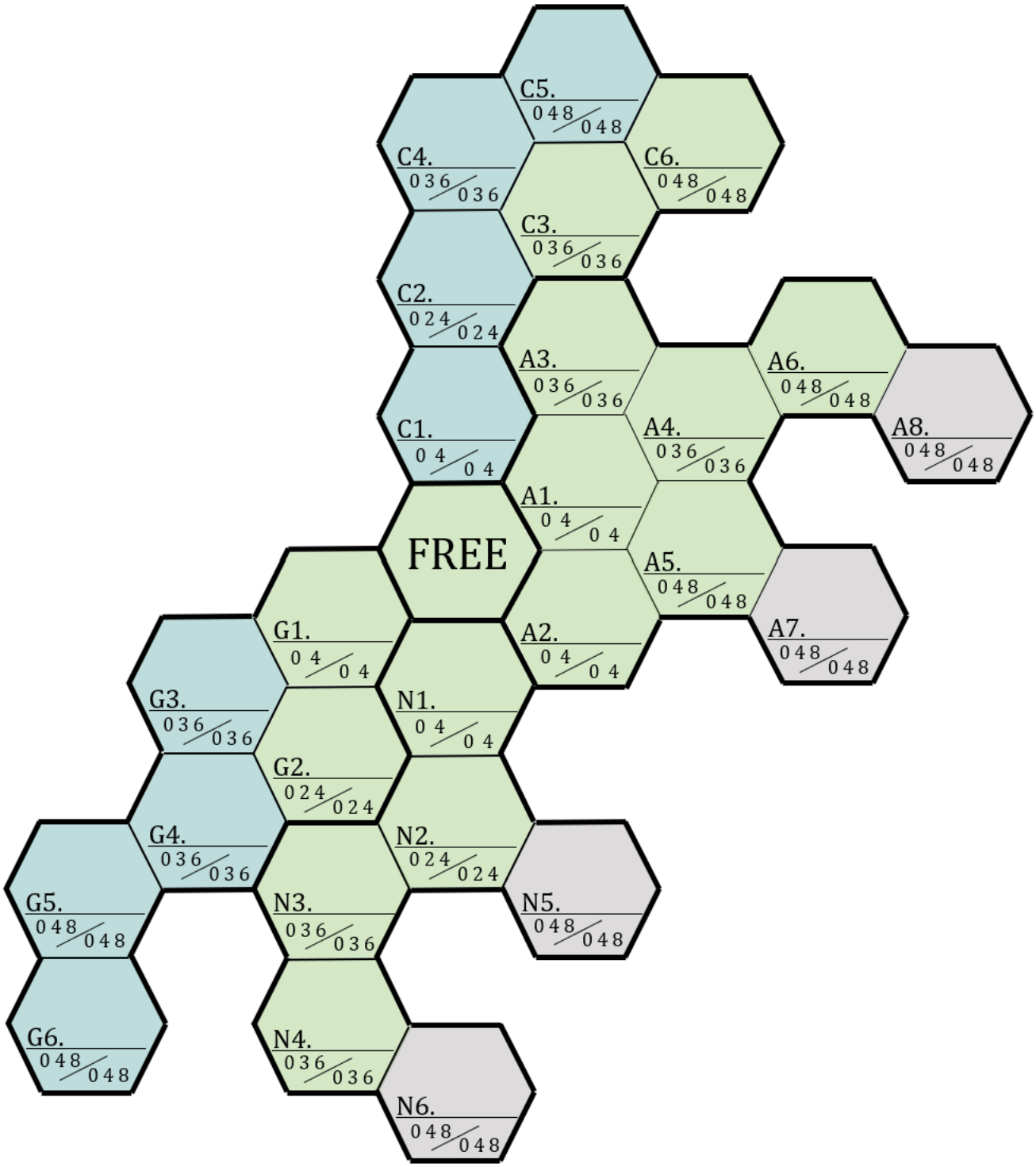
Mustang Math Tournament 2023

Herding Hexes Stallion Round



Basic Format

- This round contains 26 problems to be solved in 45 minutes.
- Each problem corresponds to a hexagon on the answer grid (backside). Write the answer to each question inside its corresponding hexagon.
- The grid is separated by subject into 8 Algebra, 6 Combinatorics, 6 Geometry, 6 Number Theory, and 1 Free tile.
- Problems are identified as [Subject][Problem #] (e.g. Algebra Problem 5 is A5)
- A correct answer will grant 2 points for Problems 1 to 2, 3 points for Problems 3 to 4, and 4 points for all other problems in any given set.
- The score of a hexagonal tile is doubled if it can be connected back to the Free tile through other tiles (not necessarily of the same subject) that only contain correct answers.
- **Do not** write below the provided answer blank inside each hexagon (the space is for grading purposes).
- *Although this is a hex round, please write your answers in base-10 unless otherwise specified.*



Algebra

A1. [2] Let $a \diamond b$ represent the value $ab + 100a$. What is the value of $18 \diamond 17 - 17 \diamond 18$?

Connects to FREE, A2, A3, A4, A5, C1

A2. [2] The average of the six numbers $1, 3, 5, 7, 9, N$ is equal to the average of the five numbers $2, 4, 6, 8, 10$. What is N?

Connects to FREE, A1, A5, N1

A3. [3] Let $n!$ denote the product of the positive integers from 1 to n. Given that $8! = 40320$, what is the value of $8! + 9! + 10!$?

Connects to A1, A4, C1, C2, C3

A4. [3] There are 2023 numbers written on a blackboard. The first number is $\frac{1}{2023}$. The second number is $\frac{1}{2023} + \frac{1}{2022}$. The third number is $\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021}$, and so on until the last number, $\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021} + \cdots + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}$. What is the average of the 2023 numbers?

Connects to A1, A3, A5, A6

A5. [4] Let a and b be real numbers, and define the function $f(x) = ax + b$. Given that $f(f(f(0))) = 2023$ and $f(f(f(1))) = 2031$, what is $f(0)$?

Connects to A1, A2, A4, A7

A6. [4] A photograph is 5 inches wide and 8 inches tall. It is mounted in a frame with a non-zero border x inches wide on all sides. If the border's width were doubled, the area of the frame would increase by 150%. What is the original width x? Express your answer as a common fraction.

Connects to A4, A8

A7. [4] What is the sum of all possible unique 4-digit integers that can be formed using the digits $\{1, 3, 3, 7\}$?

Connects to A5

A8. [4] Let r_1, r_2, and r_3 be the roots of $x^3 - x + 1$. What is the value of the expression below? Express your answer as a common fraction.

$$\frac{1}{1 - r_1^3} + \frac{1}{1 - r_2^3} + \frac{1}{1 - r_3^3}$$

Connects to A6

Combinatorics

C1. [2] Sami swaps 2 (not necessarily adjacent) letters in the word $HOOFS$ to form a different string of letters. How many different strings of letters can result?

Connects to FREE, A1, A3, C2

C2. [2] Bob uses a random number generator to pick 3 (not necessarily distinct) digits from 0 to 9. What is the probability that the sum of the digits is 27? Express your answer as a common fraction.

Connects to A3, C1, C3, C4

C3. [3] The outside of a $10 \times 10 \times 10$ cube is painted blue. It is then chopped into 1000 unit cubes. One of these unit cubes is chosen at random, and it is rolled like a die so that each face is equally likely to come up. What is the probability that the face that comes up is blue? Express your answer as a common fraction.

Connects to A3, C2, C4, C5, C6

C4. [3] Angela is in charge of scheduling 5 meetings for her math club. The first meeting will be on Tuesday, September 3. The gap between adjacent meetings must be between 4 and 10 days, inclusive. Furthermore, meetings cannot be scheduled for Saturday nor Sunday. How many ways can Angela schedule the remaining 4 meetings? For example, this means that the second meeting must be on a weekday between September 7 and September 13.

Connects to C2, C3, C5

C5. [4] Emil adds 2 joker cards to a standard deck of 52 playing cards and shuffles the 54 total cards thoroughly. He then discards every card that lies between the two jokers. Given that the Ace of Spades, Ace of Clubs, and Ace of Diamonds were discarded, what is the probability that the Ace of Hearts was not discarded? Express your answer as a common fraction.

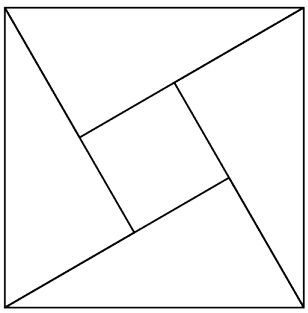
Connects to C3, C4, C6

C6. [4] How many ways are there to color each vertex of a 10-sided polygon red, green, or blue such that among any three consecutive vertices, exactly two distinct colors appear? Note that reflections and rotations are considered distinct.

Connects to C3, C5

Geometry

G1. [2] In the diagram below, 4 congruent right-angled triangles and a small square are arranged to form a large square. If the area of the small square is 5 and the area of each triangle is 19, what is the side length of the large square?

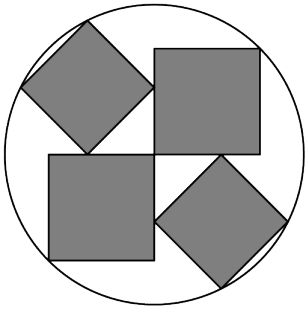


Connects to FREE, G2, G3, N1

G2. [2] Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let ω be a circle whose center lies on $\overline{CA}$. If ω passes through A and is tangent to $\overline{BC}$, what is the radius of ω? Express your answer as a common fraction.

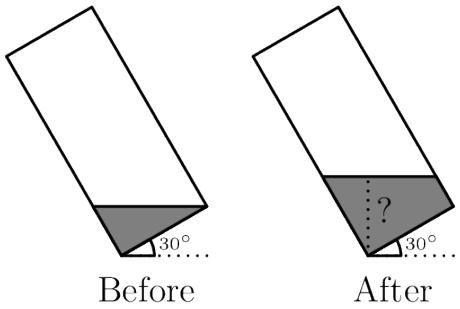
Connects to G1, G3, G4, N1, N2, N3

G3. [3] Four squares are inscribed in a circle as shown in the diagram. If the circle's radius is 1, what is the sum of the areas of all 4 squares? Express your answer as a common fraction.



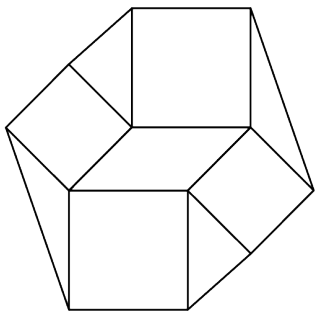
Connects to G1, G2, G4

G4. [3] A hollow rectangular prism has a square base that is 6 inches by 6 inches and has some water in it. When the prism is tilted on its edge by 30 degrees, the water is exactly deep enough to entirely cover the base of the prism, as shown in the diagram. If the amount of water in the prism is then doubled, what would the new depth of the water be? Express your answer as a common fraction.



Connects to G2, G3, G5, N3

G5. [4] In the diagram below, squares of length 6 and 8 have been constructed outside a parallelogram. The eight outer vertices are then connected to form an octagon with area of 302. What is the area of parallelogram?



Connects to G4, G6

G6. [4] Let $\triangle ABC$ be an equilateral triangle and P be a point on the incircle (circle inscribed in) $\triangle ABC$. If the distances from P to $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are $2, 5$, and a, what is the product of the possible values of a?

Connects to G5

Number Theory

N1. [2] Albert wrote down a number N. Betty wrote down the number that equals the sum of the digits of N. Carol wrote down the sum of the digits of Betty's number. If Carol wrote the number 11, what is the smallest possible value N could have been?

Connects to FREE, A2, G1, G2, N2

N2. [2] Let x be a 2022 digit number where every digit is 5 ($5555\cdots5$). Compute the greatest common divisor of x and 45.

Connects to G2, N1, N3, N5

N3. [3] There exists a four-digit positive integer $\overline{abcd}$ (a, b, c, d are not necessarily distinct) such that $\overline{abcd} + 2 \cdot \overline{dcba} = 22221$. Find $\overline{abcd}$.

Connects to G2, G4, N2, N4

N4. [3] What is the smallest positive integer n such that n^n is divisible by 2023^{2023} but n is not divisible by 2023?

Connects to N3, N6

N5. [4] Compute the sum of all positive integers $n \leq 100$ such that $\frac{30^n}{n!}$ is also an integer.

Connects to N2

N6. [4] Find the sum of all integers n between 1 and 105, inclusive, such that 105 evenly divides $n^2 - 4n + 3$.

Connects to N4