1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

A first-order ODE is said to be **linear** if it can be brought into the form

(1) $$y' + p(x)y = r(x),$$

by algebra, and **nonlinear** if it cannot be brought into this form.

Example:

$$y' \cos x + y \sin x = x \text{ is a linear ODE,}$$

divide the equation by cos x to get the **standard form**

$$y' + y \tan x = x \sec x.$$

Homogeneous Linear ODE.

We want to solve (1) on some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

(2) $$y' + p(x)y = 0$$

and is called **homogeneous.**

By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

The general solution of the homogeneous ODE (2) is

(3) $$y(x) = ce^{-\int p(x)dx} \qquad (c = \pm e^{c^*} \text{ when } y >/< 0);$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Nonhomogeneous Linear ODE

We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero on the interval J considered. Then the ODE (1) is called **nonhomogeneous**.

(1) has a property: it has an integrating factor depending only on x

We multiply (1) by $F(x)$, obtaining

$$Fy' + pFy = rF. \tag{1*}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h'e^h y = e^h y' + (e^h)'y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r \, dx + c$$

Dividing by e^h, we obtain the desired solution formula

(4) $$y(x) = e^{-h}\left(\int e^h r \, dx + c\right), \qquad h = \int p(x)\, dx.$$

The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

(4*) $$y(x) = e^{-h}\int e^h r \, dx + c\, e^{-h},$$

we see the following:

(5) Total Output = Response to the Input r + Response to the Initial Data.

EXAMPLE 1

First-Order ODE, General Solution, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution.

Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$ and

$$h = \int p\, dx = \int \tan x\, dx = \ln|\sec x|.$$

From this we see that in (4),

$$e^{h} = \sec x,\ e^{-h} = \cos x,\ e^{h} r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\, dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$, thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$.

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**

(9) $$y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}).$$

If $a = 0$ or $a = 1$, Equation (9) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (9), obtaining

$$u' = (1 - a)y^{-a}y' = (1 - a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1 - a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

(10) $$u' + (1 - a)pu = (1 - a)g.$$

Example 4

Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**)

(11) $$y' = Ay - By^2$$

Solution. Write (11) in the form (9), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$ so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (11),

$$u' = -y^2 y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}.$$

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

$$u' + Au = B.$$

The general solution is [by (4)]

$$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx.$$

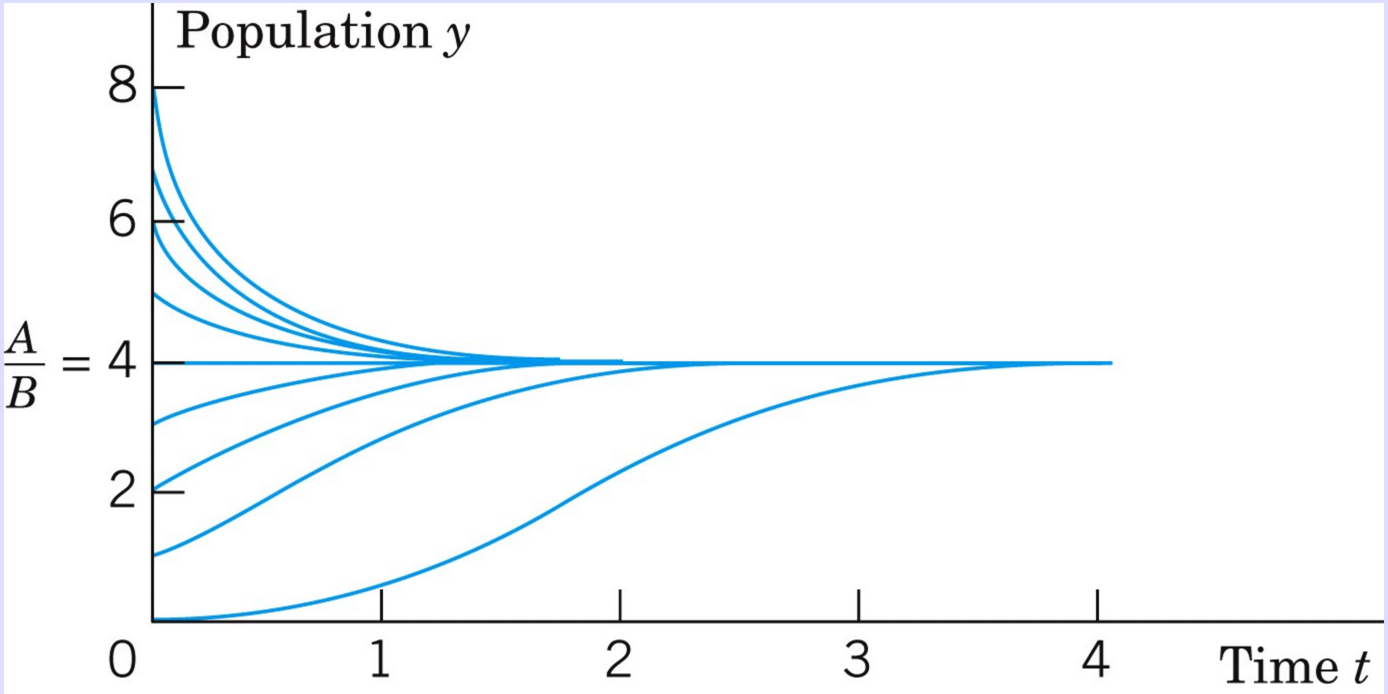
$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (11),

(12) $$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A} \qquad \text{(Fig. 21)}$$

Directly from (11) we see that $y = 0$ ($y(t) = 0$ for all t) is also a solution.

Example 4 (continued)



We see that in the logistic equation (11) the independent variable t does not occur explicitly. An ODE $y' = f(t, y)$ in which t does not occur explicitly is of the form

(13) $$y' = f(y)$$

and is called an **autonomous ODE**. Thus the logistic equation (11) is autonomous.

Equation (13) has constant solutions, called **equilibrium solutions** or **equilibrium points**. These are determined by the zeros of $f(y)$, because $f(y) = 0$ gives $y' = 0$ by (13); hence y = const. These zeros are known as **critical points** of (13). An equilibrium solution is called **stable** if solutions close to it for some t remain close to it for all further t. It is called **unstable** if solutions initially close to it do not remain close to it as t increases.

1.6 Orthogonal Trajectories.

Optional

An important type of problem in physics or geometry is to find a family of curves that intersect a given family of curves at right angles. The new curves are called **orthogonal trajectories** of the given curves (and conversely). Examples are curves of equal temperature **(isotherms)** and curves of heat flow, curves of equal altitude **(contour lines)** on a map and curves of steepest descent on that map, curves of equal potential (**equipotential curves,** curves of equal voltage-the ellipses in Fig. 24, next slide) and curves of electric force (the parabolas in Fig. 24).

Here the **angle of intersection** between two curves is defined to be the angle between the tangents of the curves at the intersection point. *Orthogonal* is another word for *perpendicular.*

(continued)

In many cases orthogonal trajectories can be found using ODEs. In general, if we consider $G(x, y, c) = 0$ to be a given family of curves in the xy-plane, then each value of c gives a particular curve. Since c is one parameter, such a family is called a **one-parameter family of curves**.

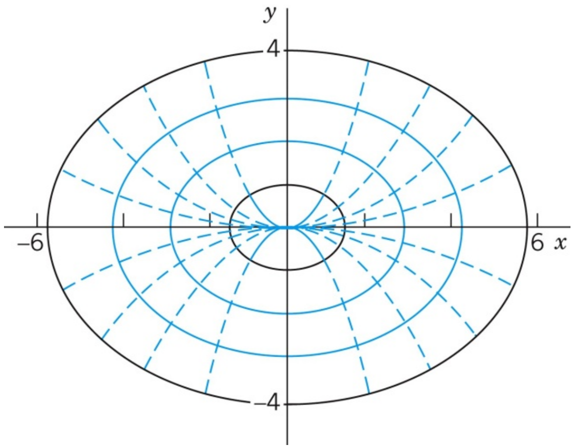


Fig.24 Electrostatic field between two ellipses (elliptic cylinders in space): Elliptic equipotential curves (equipotential surfaces) and orthogonal trajectories (parabolas)

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

The initial value problem

$$|y'| + |y| = 0, \qquad y(0) = 1$$

has no solution because $y = 0$ (that is, $y(x) = 0$ for all x) is the only solution of the ODE. The initial value problem

$$y' = 2x, \qquad y(0) = 1$$

has precisely one solution, namely, $y = x^2 + 1$. The initial value problem

$$xy' = y - 1, \qquad y(0) = 1$$

has infinitely many solutions, namely, $y = 1 + cx$, where c is an arbitrary constant because $y(0) = 1$ for all c.

From these examples we see that an **initial value problem**

(1) $$y' = f(x, y), \qquad y(x_0) = y_0$$

may have no solution, precisely one solution, or more than one solution. This fact leads to the following two fundamental questions.

Problem of Existence

Under what conditions does an initial value problem of the form (1) *have at least one solution (hence one or several solutions)?*

Problem of Uniqueness

Under what conditions does that problem have at most one solution (hence excluding the case that it has more than one solution)?

Theorem 1

Existence Theorem

Let the right side $f(x, y)$ of the ODE in the initial value problem

(1) $\quad y' = f(x, y), \qquad y(x_0) = y_0$

be continuous at all points (x, y) in some rectangle

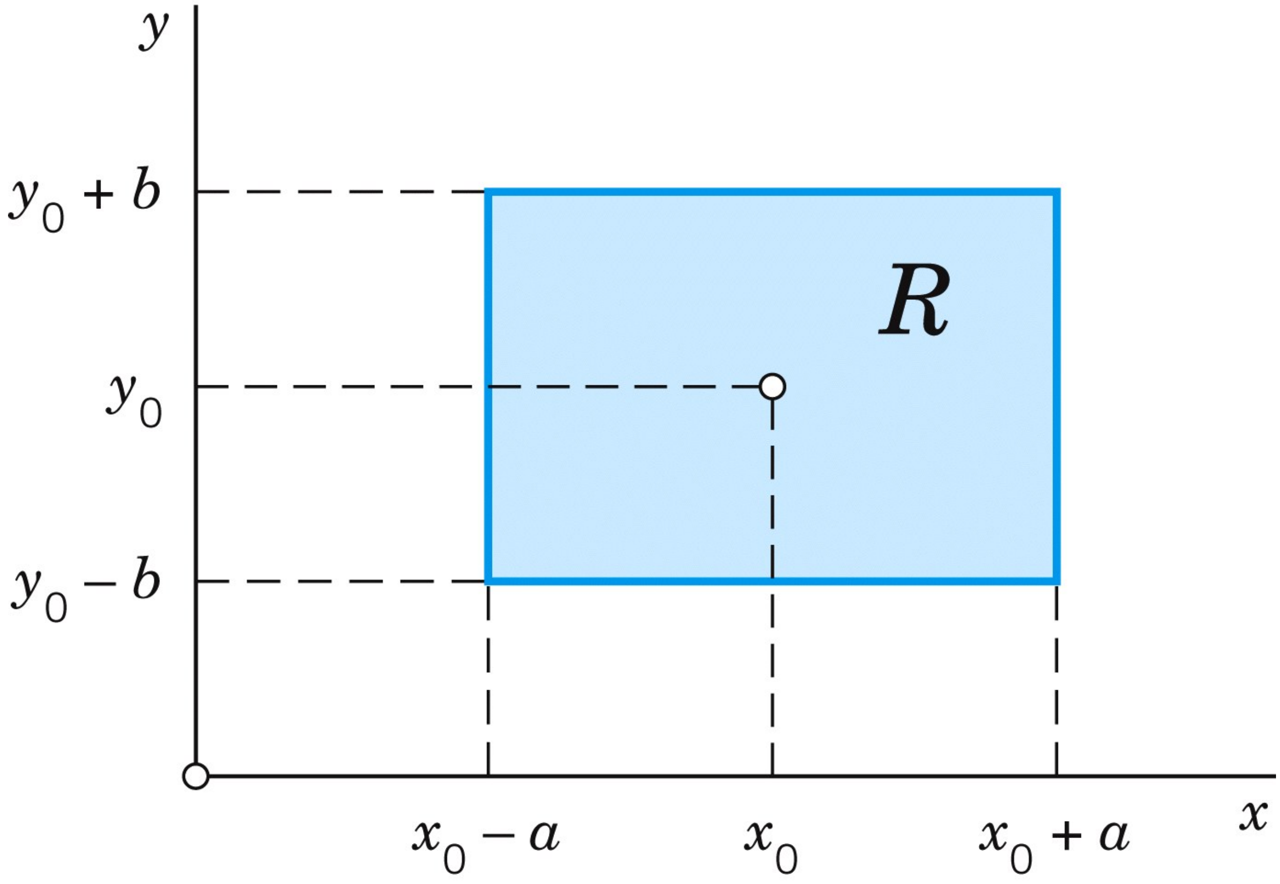
$$R: |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in R; that is, there is a number K such that*

(2) $\quad |f(x, y)| \le K \qquad$ *for all (x, y) in R.*

Then the initial value problem (1) has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$; here, α is the smaller of the two numbers a and b/K.

Theorem 1 (continued)



Theorem 2

Uniqueness Theorem

Let f *and its partial derivative* $f_y = \partial f/\partial y$ *be continuous for all* (x, y) *in the rectangle* R (Fig. 26) *and bounded, say,*

$$(3)\quad \text{(a)}\ |f(x, y)| \le K, \qquad \text{(b)}\ |f_y(x, y)| \le M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at most one solution* $y(x)$. *Thus, by* Theorem 1, *the problem has precisely one solution. This solution exists at least for all* x *in that subinterval* $|x - x_0| < \alpha$.

SUMMARY OF CHAPTER 1
First-Order ODEs

First-Order ODEs

This chapter concerns **ordinary differential equations (ODEs) of first order** and their applications. These are equations of the form

(1) $F(x, y, y') = 0$ or in explicit form $y' = f(x, y)$

involving the derivative $y' = dy/dx$ of an unknown function y, given functions of x, and, perhaps, y itself. If the independent variable x is time, we denote it by t.

In Sec. 1.1 we explained the basic concepts and the process of **modeling**, that is, of expressing a physical or other problem in some mathematical form and solving it. Then we discussed the method of direction fields (See. 1.2), solution methods and models (Sees. 1.3–1.6), and, finally, ideas on existence and uniqueness of solutions (Sec. 1.7).

(continued 1)

A first-order ODE usually has a **general solution**, that is, a solution involving an arbitrary constant, which we denote by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem**

(2) $\quad y' = f(x, y) \qquad y(x_0) = y_0 \qquad (x_0, y_0 \text{ given numbers})$

and its solution is a **particular solution** of the ODE. Geometrically, a general solution represents a family of curves, which can be graphed by using **direction fields** (Sec. 1.2). And each particular solution corresponds to one of these curves.

(continued 2)

A **separable ODE** is one that we can put into the form

(3) $$g(y)\, dy = f(x)\, dx \qquad \text{(Sec. 1.3)}$$

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides.

An **exact ODE** is of the form

(4) $$M(x, y)\, dx + N(x, y)\, dy = 0 \qquad \text{(Sec. 1.4)}$$

where $M\, dx + N\, dy$ is the **differential**

$$du = u_x\, dx + u_y\, dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. This method extends to nonexact ODEs that can be made exact by multiplying them by some function $F(x, y)$, called an **integrating factor** (Sec. 1.4).

(continued 3) **Linear ODEs**

(5) $$y' + p(x)y = r(x)$$

are very important. Their solutions are given by the integral formula (4). Sec. 1.5. Certain nonlinear ODEs can be transformed to linear form in terms of new variables.

This holds for the **Bernoulli equation**

$$y' + p(x)y = g(x)y^a \qquad \text{(Sec. 1.5).}$$

Applications and ***modeling*** are discussed throughout the chapter, in particular in Secs. 1.1, 1.3, 1.5 (***population dynamics***, etc.), and 1.6 (***trajectories***).

Picard's ***existence*** and ***uniqueness theorems*** are explained in Sec. 1.7 (and *Picard's iteration* in Problem Set 1.7).

Numeric methods for first-order ODEs can be studied in Secs. 21.1 and 21.2 immediately after this chapter, as indicated in the chapter opening.