2.5 Euler—Cauchy Equations

Euler–Cauchy equations are ODEs of the form

(1) $$x^2y'' + axy' + by = 0$$

with given constants a and b and unknown function $y(x)$.
We substitute

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

into (1). This gives

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

and we now see that $y = x^m$ was a rather natural choice because we have obtained a common factor x^m. Dropping it, we have the auxiliary equation $m(m-1) + am + b = 0$ or

(2) $$m^2 + (a-1)m + b = 0. \qquad (\textit{Note}\text{: } a-1, \text{ not } a.)$$

Euler–Cauchy equations *(continued)*

Hence $y = x^m$ is a solution of (1) if and only if m is a root of (2). The roots of (2) are

(3) $$m_1 = \tfrac{1}{2}(1-a) + \sqrt{\tfrac{1}{4}(1-a)^2 - b}, \qquad m_2 = \tfrac{1}{2}(1-a) - \sqrt{\tfrac{1}{4}(1-a)^2 - b}.$$

Case I. Real different roots m_1 and m_2 give two real solutions

$$y_1(x) = x^{m_1} \quad \text{and} \quad y_2(x) = x^{m_2}.$$

These are **linearly independent** since their quotient is not constant. Hence they constitute **a basis of solutions** of (1) for all x for which they are real. The corresponding general solution for all these x is

(4) $$y_1(x) = c_1 x^{m_1} + c_2 x^{m_2} \qquad (c_1, c_2 \text{ arbitrary}).$$

EXAMPLE 1

General Solution in the Case of Different Real Roots

The Euler–Cauchy equation $x^2y'' + 1.5xy' - 0.5y = 0$ has the auxiliary equation $m^2 + 0.5m - 0.5 = 0$. The roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0).$$

Case II. A real double root

$m_1 = \frac{1}{2}(1-a)$ occurs if and only if $b = \frac{1}{4}(a-1)^2$ because then (2) becomes $[m + \frac{1}{2}(a-1)]^2$, as can be readily verified. Then a solution is $\boldsymbol{y_1 = x^{(1-a)/2}}$, and (1) is of the form

$$\text{(5)} \quad x^2 y'' + axy' + \frac{1}{4}(1-a)^2 y = 0 \qquad \text{or} \qquad y'' + \frac{a}{x} y' + \frac{(1-a)^2}{4x^2} y = 0.$$

Case II. *(continued)*

A second linearly independent solution can be obtained by the method of reduction of order from Sec. 2.1, as follows. Starting from $y_2 = uy_1$, we obtain for u the expression (9) Sec. 2.1, namely,

$$u = \int U\,dx \qquad \text{where} \qquad U = \frac{1}{y_1^2}\exp\left(-\int p\,dx\right).$$

From (5) in standard form (second ODE) we see that $p = a/x$ (not ax; this is essential!). Hence $\exp\int(-p\,dx) = \exp(-a\ln x) = \exp(\ln x^{-a}) = 1/x^a$. Division by $y_1^2 = x^{1-a}$ gives $U = 1/x$, so that $u = \ln x$ by integration.

Thus, $y_2 = uy_1 = y_1 \ln x$, and y_1 and y_2 are linearly independent since their quotient is not constant. The general solution corresponding to this basis is

(6) $$y = (c_1 + c_2 \ln x)\,x^m, \qquad m = \frac{1}{2}(1-a).$$

EXAMPLE 2

General Solution in the Case of a Double Root

The Euler–Cauchy equation $x^2y'' - 5xy' + 9y = 0$ has the auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is

$$y = (c_1 + c_2 \ln x)\, x^3. \qquad \blacksquare$$

Case III. Complex conjugate roots are of minor practical importance, and we discuss the derivation of real solutions from complex ones just in terms of a typical example.

EXAMPLE 3

Real General Solution in the Case of Complex Roots

The Euler–Cauchy equation $x^2y'' + 0.6xy' + 16.04y = 0$ has the auxiliary equation $m^2 - 0.4m + 16.04 = 0.$ The roots are complex conjugate

$$m_1 = 0.2 + 4i \text{ and } m_2 = 0.2 - 4i, \text{ where } i = \sqrt{-1}.$$

writing $x = e^{\ln x}$ and obtain

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2}e^{(4 \ln x)i},$$
$$x^{m_2} = x^{0.2-4i} = x^{0.2}(e^{\ln x})^{-4i} = x^{0.2}e^{-(4 \ln x)i}.$$

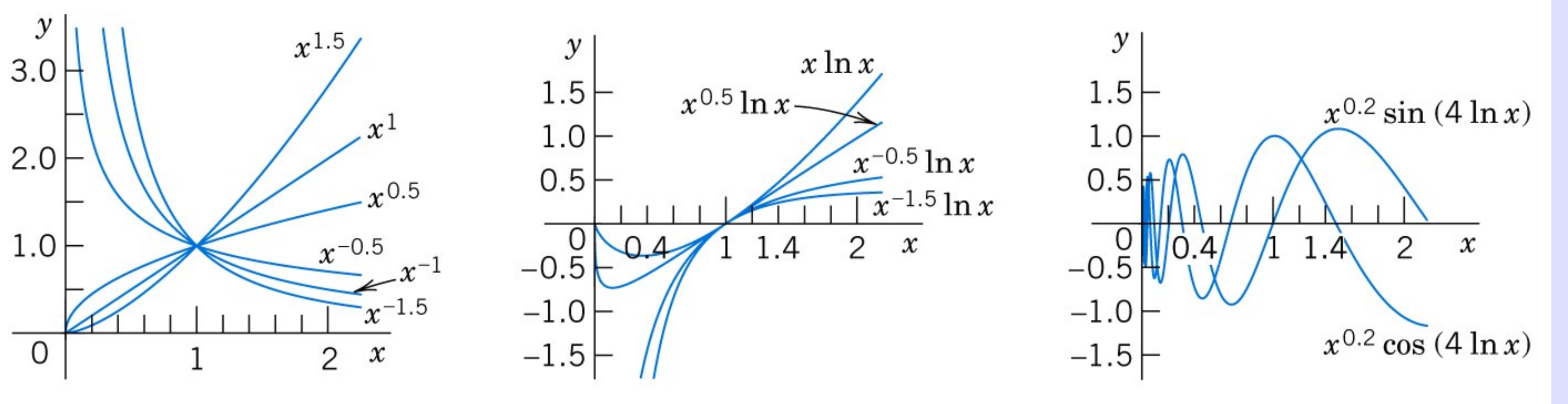
$$x^{m_1} = x^{0.2}[\cos(4 \ln x) + i \sin(4 \ln x)],$$
$$x^{m_2} = x^{0.2}[\cos(4 \ln x) - i \sin(4 \ln x)].$$

$$x^{0.2} \cos(4 \ln x) \quad \text{and} \quad x^{0.2} \sin(4 \ln x)$$

The corresponding real general solution for all positive x is

$$y = x^{0.2}[A \cos(4 \ln x) + B \sin(4 \ln x)]. \qquad (8)$$

Figure 48 shows typical solution curves in the three cases discussed, in particular the real basis functions in Examples 1 and 3.



Case I: Real roots

Case II: Double root

Case III: Complex roots

2.6 Existence and Uniqueness of Solutions. Wronskian

In this section we shall discuss the general theory of homogeneous linear ODEs

(1) $$y'' + p(x)y' + q(x)y = 0$$

with continuous, but otherwise arbitrary, ***variable coefficients*** *p* and *q*.
This will concern the existence and form of a general solution of (1) as well as the uniqueness of the solution of initial value problems consisting of such an ODE and two initial conditions

(2) $$y(x_0) = K_0, \qquad y'(x_0) = K_1$$

with given x_0, K_0, and K_1.

The two main results will be Theorem 1, stating that such an initial value problem always has a solution which is unique, and Theorem 4, stating that a general solution

$$y = c_1y_1 + c_2y_2 \qquad (c_1, c_2 \text{ arbitrary}) \tag{3}$$

includes all solutions. Hence *linear* ODEs with continuous coefficients have no *"singular solutions"* (solutions not obtainable from a general solution).

Theorem 1

Existence and Uniqueness Theorem for Initial Value Problems

If $\boldsymbol{p(x)}$ *and* $\boldsymbol{q(x)}$ *are* ***continuous*** *functions* ***on some open interval I*** (see Sec. 1.1) *and* $\boldsymbol{x_0}$ ***is on I****, then the initial value problem consisting of* (1) *and* (2) *has* ***a unique solution*** $y(x)$ *on the interval I.*

Linear Independence of Solutions

A general solution on an open interval I is made up from a **basis** y_1, y_2 on I, that is, from a pair of linearly independent solutions on I. Here we call y_1, y_2 **linearly independent** on I if the equation

(4) $\quad k_1y_1(x) + k_2y_2(x) = 0$ on I implies $\mathbf{k_1 = 0, k_2 = 0}$.

We call y_1, y_2 **linearly dependent** on I if this equation also holds for constants **k_1, k_2 not both 0**. In this case, and only in this case, y_1 and y_2 are proportional on I, that is (see Sec. 2.1),

(5) $\quad$ (a) $y_1 = ky_2$ $\quad$ or $\quad$ (b) $y_2 = ly_1$ $\quad$ for all on I.

Theorem 2

Linear Dependence and Independence of Solutions

Let the ODE (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval* I*. Then two solutions* y_1 and y_2 *of* (1) *on* I *are* ***linearly dependent*** *on* I *if and only if their "***Wronskian***"*

(6) $$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is 0 *at some* x_0 *in* I*. Furthermore, if* $W = 0$ *at an* $x = x_0$ *in* I*, then* $W = 0$ *on* I*; hence, if there is an* x_1 *in* I *at which* ***W is not 0****, then* y_1, y_2 *are* ***linearly independent*** *on* I*.*

Remark.

Determinants. Students familiar with second-order determinants may have noticed that

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

This determinant is called the *Wronski determinant* or, briefly, the **Wronskian**, of two solutions y_1 and y_2 of (1), as has already been mentioned in (6). Note that its four entries occupy the same positions as in the linear system (7).

EXAMPLE 1

Illustration of Theorem 2

The functions $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of $y'' + \omega^2 y = 0$. Their Wronskian is

$$W(\cos \omega x, \sin \omega x) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = y_1 y_2' - y_2 y_1' = \omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega.$$

Theorem 2 shows that these solutions are linearly independent if and only if $\omega \neq 0$. Of course, we can see this directly from the quotient $y_2/y_1 = \tan \omega x$. For $\omega = 0$ we have $y_2 = 0$, which implies linear dependence (why?). ■

A General Solution of (1) Includes All Solutions

Theorem 3

Existence of a General Solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then (1) *has a general solution on I.*

Theorem 4

A General Solution Includes All Solutions

If the ODE (1) *has continuous coefficients* $p(x)$ *and* $q(x)$ *on some open interval I, then every solution* $y = Y(x)$ *of* (1) *on I is of the form*

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{8}$$

where y_1, y_2 *is any basis of solutions of* (1) *on I and* C_1, C_2 *are suitable constants.*

Hence (1) *does not have* **singular solutions** *(that is, solutions not obtainable from a general solution).*

2.7 Nonhomogeneous ODEs

We now advance from homogeneous to nonhomogeneous linear ODEs.

Consider the second-order nonhomogeneous linear ODE

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

where $r(x) \neq 0$. We shall see that a "general solution" of (1) is the sum of a general solution of the corresponding homogeneous ODE

(2) $$y'' + p(x)y' + q(x)y = 0$$

and a "particular solution" of (1). These two new terms "general solution of (1)" and "particular solution of (1)" are defined as follows.

DEFINITION

General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

THEOREM 1

Relations of Solutions of (1) to Those of (2)

(a) *The sum of a solution* y *of* (1) *on some open interval* I *and a solution* $\tilde{y}$ *of* (2) *on* I *is a solution of* (1) *on* I. *In particular,* (3) *is a solution of* (1) *on* I.

(b) *The difference of two solutions of* (1) *on* I *is a solution of* (2) *on* I.

THEOREM 2

A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, *and the function* $r(x)$ *in* (1) *are continuous on some open interval* I, *then every solution of* (1) *on* I *is obtained by assigning suitable values to the arbitrary constants* c_1 *and* c_2 *in a general solution* (3) *of* (1) *on* I.

Method of Undetermined Coefficients

To solve the nonhomogeneous ODE (1) *or an initial value problem for* (1), *we have to solve the homogeneous ODE* (2) *and find any solution* y_p *of* (1), so that we obtain a general solution (3) of (1).

How can we find a solution y_p of (1)?

One method is the so-called **method of undetermined coefficients**. It is much simpler than another, more general, method (given in Sec. 2.10). Since it applies to models of vibrational systems and electric circuits to be shown in the next two sections, it is frequently used in engineering.

Method of Undetermined Coefficients (continued)

More precisely, the method of undetermined coefficients is suitable for linear ODEs with ***constant coefficients a and b***

(4) $$y'' + ay' + by = r(x)$$

when $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea.

We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. Table 2.1 on p. 82 shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Choice Rules for the Method of Undetermined Coefficients

(a) Basic Rule. *If* $r(x)$ *in* (4) *is one of the functions in the first column in Table* 2.1, *choose* y_p *in the same line and determine its undetermined coefficients by substituting* y_p *and its derivatives into* (4).

(b) Modification Rule. *If a term in your choice for* y_p *happens to be a solution of the homogeneous ODE corresponding to* (4), ***multiply this term by*** $\boldsymbol{x}$ (*or by* $\boldsymbol{x^2}$ *if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE*).

(c) Sum Rule. *If* $r(x)$ *is a sum of functions in the first column of Table* 2.1, *choose for* y_p *the sum of the functions in the corresponding lines of the second column.*

The Basic Rule applies when $r(x)$ is a single term.
The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first.
The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$. (Verify!)

The method is self-correcting. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \; (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$\left.\right\} K\cos \omega x + M\sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left.\right\} e^{\alpha x}(K\cos \omega x + M\sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

EXAMPLE 2 Application of the Modification Rule (b)

Solve the initial value problem

(6) $$y'' + 3y' + 2.25y = -10e^{-1.5x}, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. Step 1. General solution of the homogeneous ODE.

The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}.$$

EXAMPLE 2 (continued) Application of the Modification Rule (b)

Solution. *(continued)*

Step 2. Solution y_p of the nonhomogeneous ODE.

The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a *double root* of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$yp = Cx^2 e^{-1.5x}.$$

Then $y_p' = C(2x - 1.5x^2) e^{-1.5x}$, $y_p'' = C(2 - 3x - 3x + 2.25x^2) e^{-1.5x}$.

EXAMPLE 2 (continued) Application of the Modification Rule (b)

Solution. *(continued)*

Step 2. *(continued)*

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = 10.$$

Comparing the coefficients of x^2, x, x^0 gives

$$0 = 0,\ 0 = 0,\ 2C = -10, \text{ hence } C = -5.$$

This gives the solution $y_p = -5x^2e^{-1.5x}$.

Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x}.$$

EXAMPLE 2 (continued) Application of the Modification Rule (b)

Solution. *(continued)*

Step 3. Solution of the initial value problem.

Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives

$$y' = (c_2 - 1.5c_1 - 1.5c_2 x)e^{-1.5x} - 10x\, e^{-1.5x} + 7.5x^2 e^{-1.5x}.$$

From this and the second initial condition we have

$$y'(0) = c_2 - 1.5c_1 = 0. \text{ Hence } c_2 = 1.5c_1 = 1.5.$$

This gives the answer (Fig. 51)

$$y = (1 + 1.5x)\, e^{-1.5x} - 5x^2\, e^{-1.5x} = (1 + 1.5x - 5x^2)\, e^{-1.5x}.$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

EXAMPLE 2 (continued) Application of the Modification Rule (b)

Solution. *(continued)*

Step 3. *(continued)*