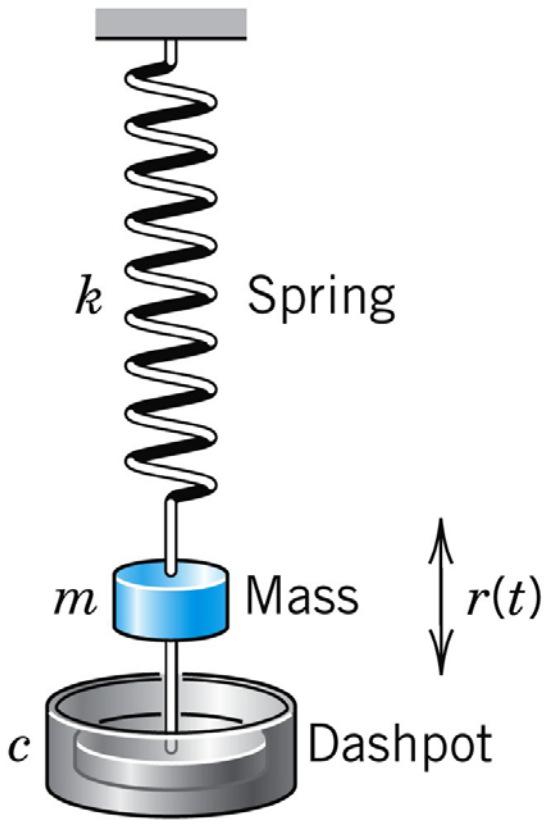
2.8 Modeling: Forced Oscillations. Resonance

In Sec. 2.4 we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring, as in Figs. 33 and 53) and modeled it by the ***homogeneous*** linear ODE

(1) $$my'' + cy' + ky = 0.$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest.



The mass–spring system of Sec. 2.4 exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion.
The internal forces are forces within the system. They are the force of inertia my'', the damping force cy' (if $c > 0$), and the spring force ky, a restoring force.

We now extend our model by including an additional force, that is, the external force $r(t)$, on the right. Then we have

$$my'' + cy' + ky = r(t). \tag{2*}$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$.
The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the nonhomogeneous ODE

(2)
$$my'' + cy' + ky = F_0 \cos \omega t.$$

Its solution will reveal facts that are fundamental in engineering mathematics and allow us to model resonance.

Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution y_h of the homogeneous ODE (1) plus any solution y_p of (2). To find y_p, we use the method of undetermined coefficients (Sec. 2.7), starting from

(3) $$y_p(t) = a \cos \omega t + b \sin \omega t.$$

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2) + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2) + \omega^2 c^2}.$$

Solving the Nonhomogeneous ODE (2) *(continued)*

If we set $\sqrt{k/m} = \omega_0 \ (>0)$ as in Sec. 2.4, then $k = m\omega_0^{\,2}$ and we obtain

$$\textbf{(5)} \quad a = F_0 \frac{m(\omega_0^{\,2} - \omega^2)}{m^2(\omega_0^{\,2} - \omega^2) + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^{\,2} - \omega^2) + \omega^2 c^2}.$$

We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

$$(6) \qquad y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE (1) and y_p is given by (3) with coefficients (5).

Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$ Hence (3) becomes (use $\omega_0^2 = k/m$)

(7) $$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t.$$

Here we must assume that $\omega^2 \neq \omega_0^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4].

Case 1. Undamped Forced Oscillations. Resonance *(continued 1)*

From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

(8) $$y(t) = C \cos\ (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0{}^2 - \omega^2)} \cos \omega t.$$

We see that this output is a ***superposition of two harmonic oscillations*** *of the frequencies just mentioned.*

Case 1. Undamped Forced Oscillations. Resonance *(continued 2)*

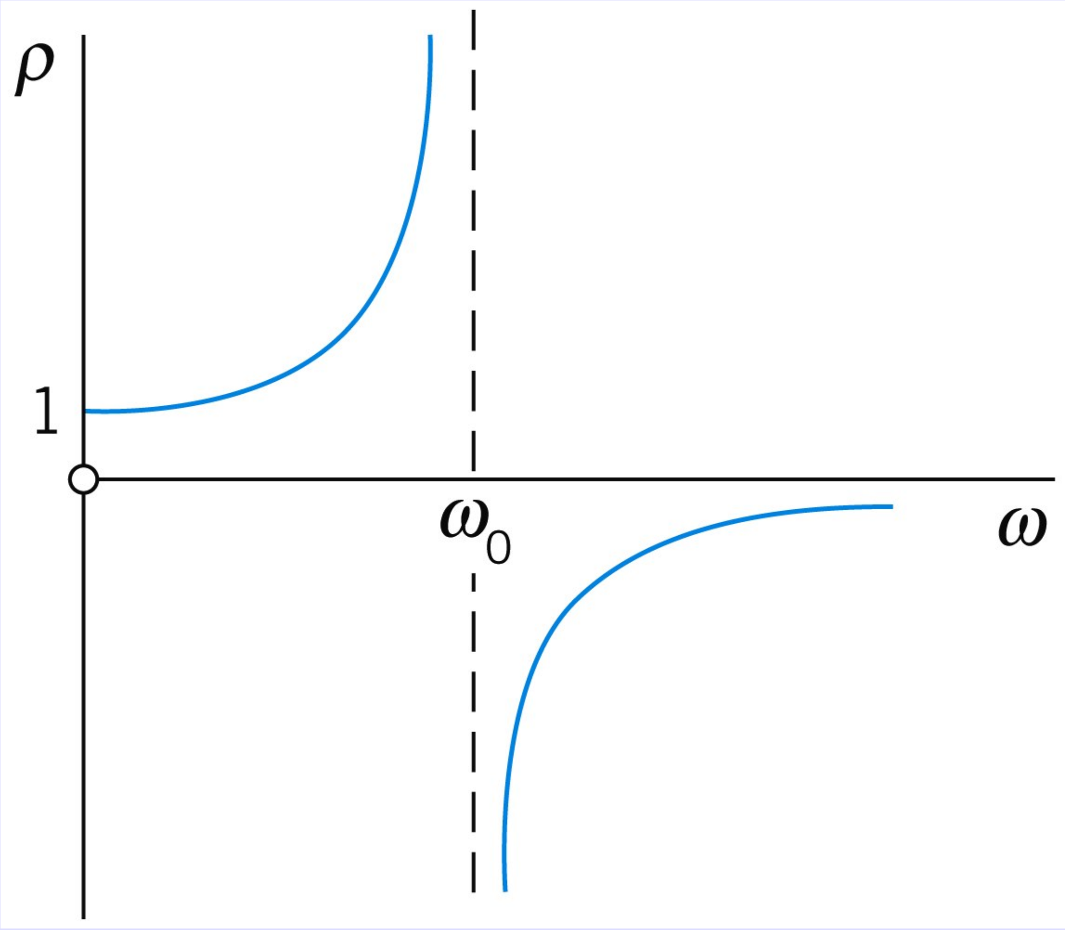
Resonance. We discuss (7). We see that the maximum amplitude of y_p is (put $\cos \omega t = 1$)

(9) $$a_0 = \frac{F_0}{k}\rho \quad \text{where} \quad \rho = \frac{1}{1-(\omega/\omega_0)^2}.$$

a_0 depends on ω and ω_0. If $\omega \to \omega_0$, then ρ and a_0 tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance**. ρ is called the **resonance factor** (Fig. 54), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution y_p and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

Case 1. Undamped Forced Oscillations. Resonance *(continued 3)*

Resonance. *(continued)*



Case 1. Undamped Forced Oscillations. Resonance *(continued 4)*

In the case of resonance the nonhomogeneous ODE (2) becomes

$$(10) \qquad y'' + \omega_0^{\,2} y = \frac{F_0}{m} \cos \omega_0 t.$$

Then (7) is no longer valid, and from the Modification Rule in Sec. 2.7, we conclude that a particular solution of (10) is of the form

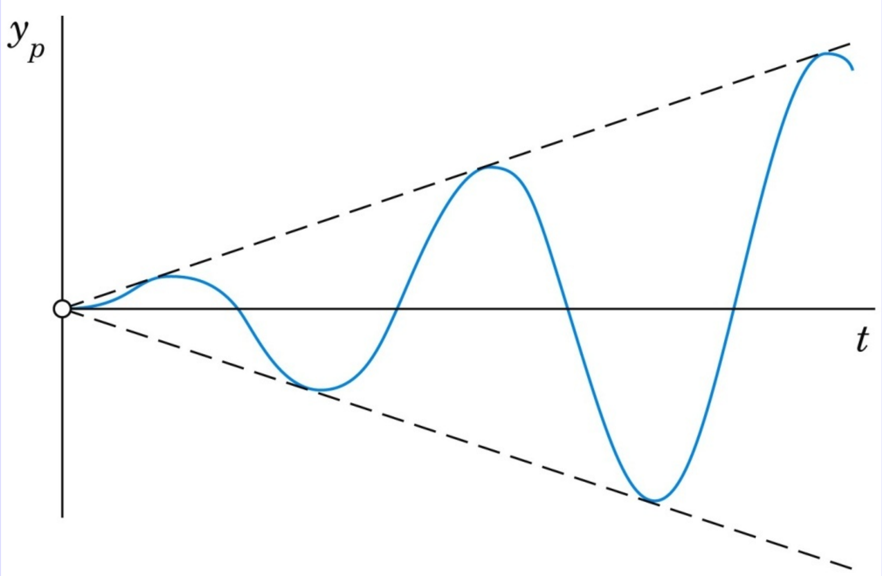
$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

Case 1. Undamped Forced Oscillations. Resonance *(continued 5)*

By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 55)

(11) $$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

Case 1. Undamped Forced Oscillations. Resonance *(continued 6)*



We see that, because of the factor t, the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system.

Case 1. Undamped Forced Oscillations. Resonance *(continued 7)*

Beats. Another interesting and highly important type of oscillation is obtained if ω is close to ω_0. Take, for example, the particular solution [see (8)]

$$(12)\qquad y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t) \qquad (\omega \neq \omega_0).$$

Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

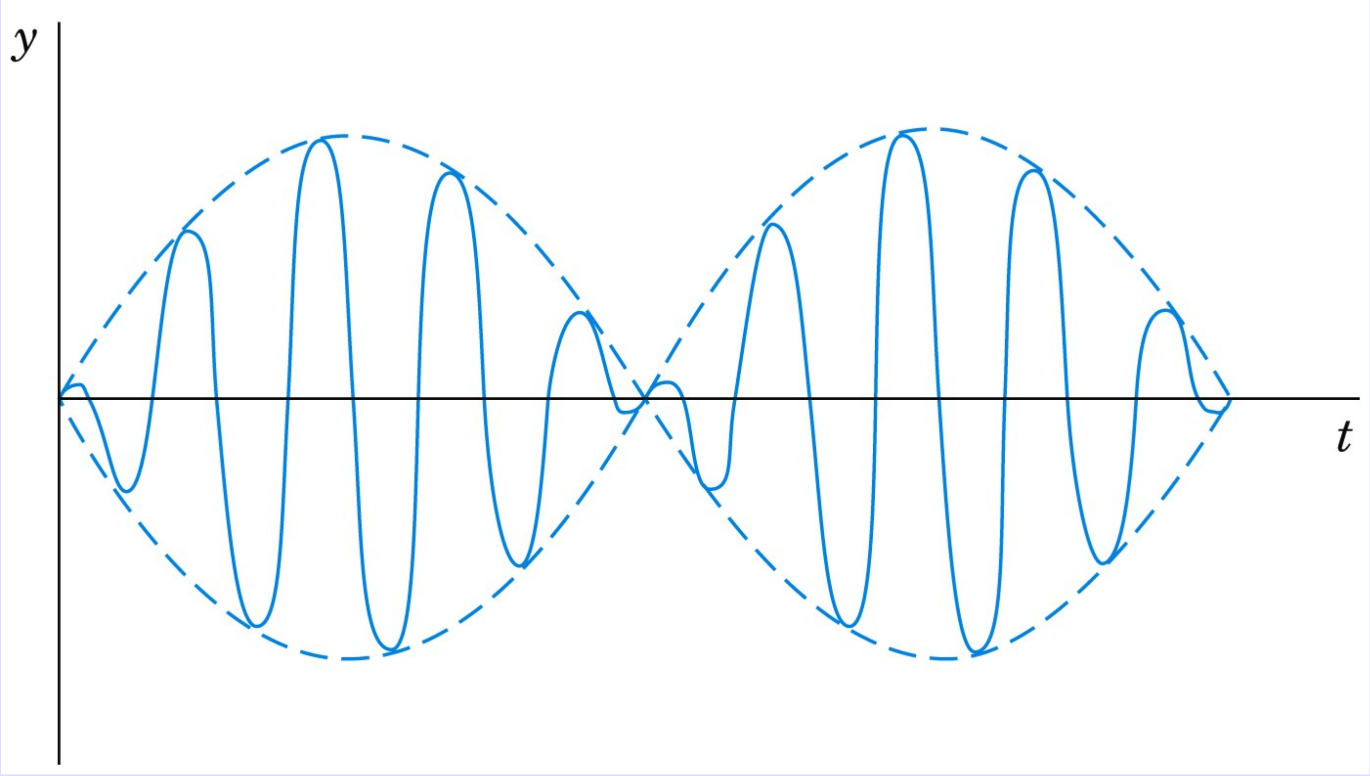
Case 1. Undamped Forced Oscillations. Resonance *(continued 8)*

Beats. ***(continued)***

Since is ω close to ω_0, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 56, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

Case 1. Undamped Forced Oscillations. Resonance *(continued 9)*

Beats. *(continued)*



Case 2. Damped Forced Oscillations

If the damping of the mass–spring system is not negligibly small, we have $c > 0$ and a damping term cy' in (1) and (2). Then the general solution y_h of the homogeneous ODE (1) approaches zero as t goes to infinity, as we know from Sec. 2.4.
Practically, it is zero after a sufficiently long time.
Hence the "**transient solution**" (6) of (2), given by $y = y_h + y_p$, approaches the "**steady-state solution**" y_p.
This proves the following theorem.

THEOREM 1

Steady-State Solution

After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force [*see* (2)] *will practically be a harmonic oscillation whose frequency is that of the input.*

Amplitude of the Steady-State Solution. Practical Resonance

Whereas in the undamped case the amplitude of y_p approaches infinity as ω approaches ω_0, this will not happen in the damped case. In this case the amplitude will always be finite. But it may have a maximum for some ω depending on the damping constant c.

This may be called **practical resonance**.

It is of great importance because if c is not too large, then some input may excite oscillations large enough to damage or even destroy the system.

To study the amplitude of y_p as a function of ω, we write (3) in the form

(13) $$y_p(t) = C^* \cos(\omega t - \eta).$$

C^* is called the **amplitude** of y_p and η the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

(14)
$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^{\,2} - \omega^2)^2 + \omega^2 c^2}}$$

$$\tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^{\,2} - \omega^2)}.$$

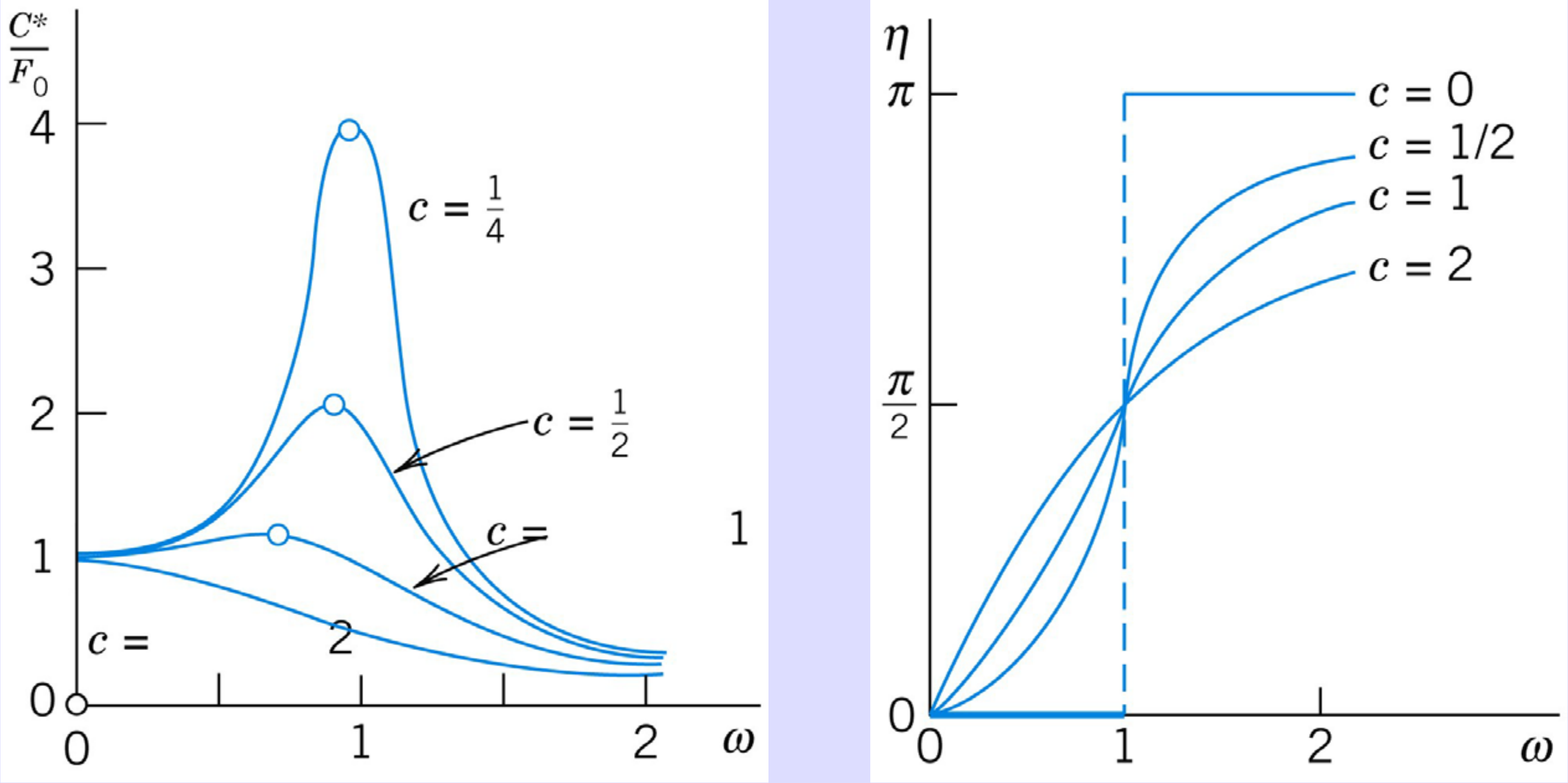
$$C^*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^{\,2} - c^2}} \qquad (16)$$

We see that $C^*(\omega_{max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2\omega_0^{\,2} - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as c^2 ($<2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1. Figure 57 shows the **amplification** C^*/F_0 (ratio of the amplitudes of output and input) as a function of ω for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant c.

Figure 58 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.



2.9 Modeling: Electric Circuits

Figure 61 shows an ***RLC*-circuit**, as it occurs as a basic building block of large electric networks in computers and elsewhere.

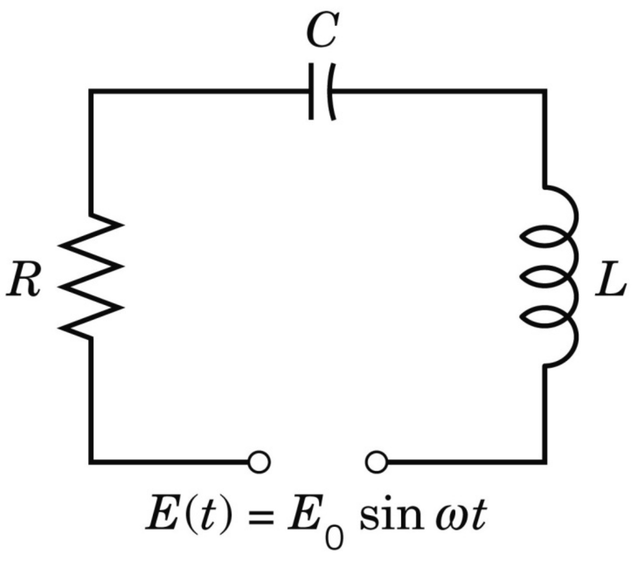
An *RLC*-circuit is obtained from an *RL*-circuit by adding a capacitor. Recall Example 2 on the *RL*-circuit in Sec. 1.5: The model of the *RL*-circuit is $LI' + RI = E(t)$. It was obtained by **KVL** (Kirchhoff's Voltage Law)* by equating the voltage drops across the resistor and the inductor to the EMF (electromotive force).

***Kirchhoff's Current Law (KCL)**:
At any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.

Hence we obtain the model of the *RLC*-circuit simply by adding the voltage drop Q/C across the capacitor.
Here, C F(farads) is the capacitance of the capacitor.
Q coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt}, \qquad \text{equivalently} \qquad Q(t) = \int I(t)\,dt.$$

See also Fig. 62. Assuming a sinusoidal EMF as in Fig. 61, we thus have the model of the *RLC*-circuit



Name	Symbol	Notation		Unit	Voltage Drop
Ohm's Resistor		R	Ohm's Resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L\dfrac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

(1′)
$$LI' + RI + \frac{1}{C}\int I\,dt = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate (1′) with respect to t, obtaining

(1)
$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega \cos \omega t.$$

This shows that the current in an RLC-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients. In connection with initial value problems, we shall occasionally use

(1″)
$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

obtained from (1′) and $I = Q'$.

Solving the ODE (1) for the Current in an *RLC*-Circuit

A general solution of (1) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to (1) and is a particular solution of (1).

We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$
\begin{aligned}
I_p &= a \cos \omega t + b \sin \omega t \\
I_p' &= \omega(-a \sin \omega t + b \cos \omega t) \\
I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t)
\end{aligned}
\tag{2}
$$

into (1).

Solving the ODE (1) for the Current in an *RLC*-Circuit *(continued 1)*

Then we collect the cosine terms and equate them to $E_0\omega \cos \omega t$ on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$L\omega^2(-a) + R\omega b + a/C = E_0\omega \qquad \text{(Cosine terms)}$$

$$L\omega^2(-b) + R\omega(-a) + b/C = 0 \qquad \text{(Sine terms).}$$

Before solving this system for a and b, we first introduce a combination of L and C, called the **reactance**

(3)
$$S = \omega L - \frac{1}{\omega C}.$$

Solving the ODE (1) for the Current in an *RLC*-Circuit *(continued 2)*

We can solve for a and b,

$$(4) \qquad a = \frac{-E_0 S}{R^2 + S^2}, \quad b = \frac{E_0 S}{R^2 + S^2}.$$

Equation (2) with coefficients a and b given by (4) is the desired particular solution I_p of the nonhomogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal electromotive force.

Solving the ODE (1) for the Current in an *RLC*-Circuit *(continued 3)*

Using (4), we can write I_p in terms of "physically visible" quantities, namely, amplitude I_0 and phase lag θ of the current behind the EMF, that is,

(5) $$I_p(t) = I_0 \sin(\omega t - \theta)$$

where [see (14) in App. A3.1]

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $\sqrt{R^2 + S^2}$ is called the **impedance**. Our formula shows that the impedance equals the ratio E_0/I_0. This is somewhat analogous to $E/I = R$ (Ohm's law), and because of this analogy, the impedance is also known as the **apparent resistance.**

Solving the ODE (1) for the Current in an *RLC*-Circuit *(continued 4)*

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

Solving the ODE (1) for the Current in an *RLC*-Circuit *(continued 5)*

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, where

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this it follows that I_h approaches zero, theoretically $t \to \infty$, as but practically after a relatively short time. Hence the transient current $I = I_h + I_p$ tends to the steady-state current I_p, and after some time the output will practically be a harmonic oscillation, which is given by (5) and whose frequency is that of the input (of the electromotive force).

Analogy of Electrical and Mechanical Quantities

Entirely different physical or other systems may have the same mathematical model. For instance, we have seen this from the various applications of the ODE $y' = ky$ in Chap. 1. Another impressive demonstration of this ***unifying power of mathematics*** is given by the ODE (1) for an electric *RLC*-circuit and the ODE (2) in the last section for a mass–spring system. Both equations

$$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega\cos\omega t$$

and

$$my'' + cy' + ky = F_0 \cos \omega t$$

are of the same form.

Analogy of Electrical and Mechanical Quantities *(continued 1)*

Table 2.2 shows the analogy between the various quantities involved. The inductance L corresponds to the mass m and, indeed, an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance R corresponds to the damping constant c, and a resistor causes loss of energy, just as a damping dashpot does. And so on.

This analogy is ***strictly quantitative*** in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

Analogy of Electrical and Mechanical Quantities *(continued 2)*

The ***practical importance*** of this analogy is almost obvious. The analogy may be used for constructing an "electrical model" of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

Analogy of Electrical and Mechanical Quantities *(continued 3)*

Table 2.2 Analogy of Electrical and Mechanical Quantities

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$

2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs, that is

(1) $$y'' + p(x)y' + q(x)y = r(x).$$

To obtain y_p when $r(x)$ is not too complicated, we can often use the *method of undetermined coefficients*.

However, since this method is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself (powers, exponential functions, etc.), it is desirable to have a method valid for more general ODEs (1), which we shall now develop. It is called the **method of variation of parameters** and is credited to Lagrange (Sec. 2.1). Here p, q, r in (1) may be variable (given functions of x), but we assume that they are continuous on some open interval I.

Lagrange's method gives a particular solution y_p of (1) on I in the form

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

(3) $$y'' + p(x)y' + q(x)y = 0$$

on I, and W is the Wronskian of y_1, y_2.

(4) $$W = y_1y_2' - y_2y_1' \qquad \text{(see Sec. 2.6).}$$

EXAMPLE 1

Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution. A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$y_p = -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx$$

$$= \cos x \ln |\cos x| + x \sin x$$

(Fig. 70)

EXAMPLE 1 *(continued)*

Solution. ***(continued 1)***

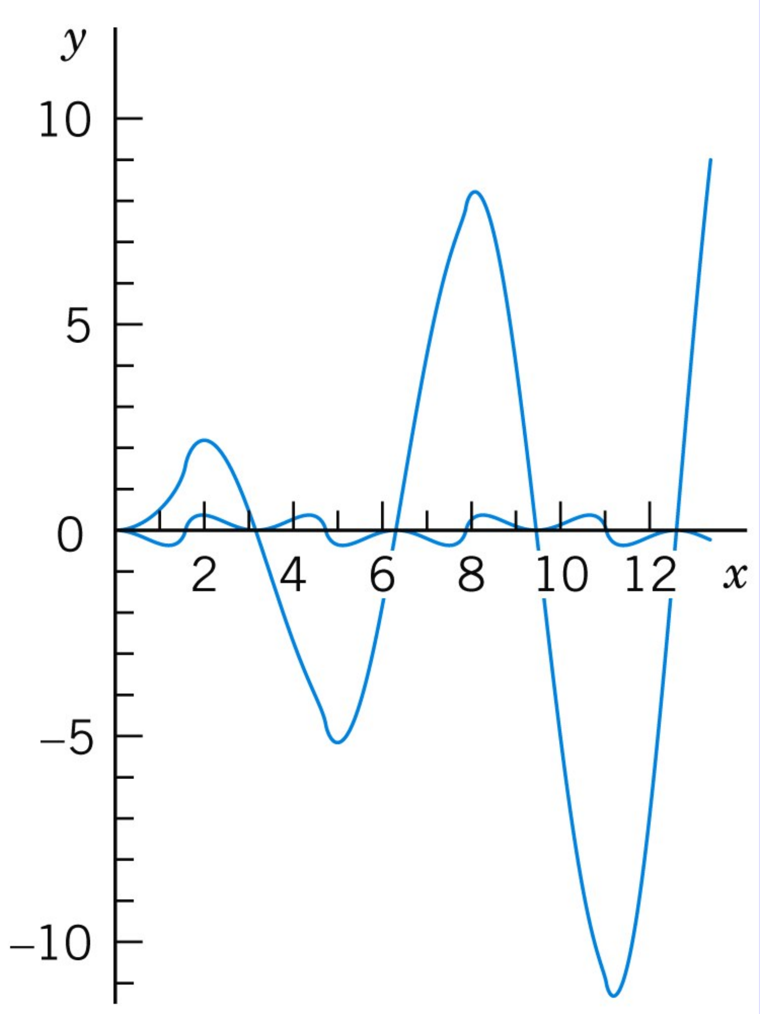
Figure 70 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. From y_p and the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous ODE we obtain the a*nswer*

$$y = y_h + y_p = (c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x.$$

Had we included integration constants $-c_1$, c_2 in (2), then (2) would have given the additional $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case.

EXAMPLE 1 *(continued)*

Solution. *(continued 2)*



SUMMARY OF CHAPTER 2
Second-Order ODEs

Second-Order ODEs

Second-order linear ODEs are particularly important in applications, for instance, in mechanics (Secs. 2.4, 2.8) and electrical engineering (Sec. 2.9). A second-order ODE is called **linear** if it can be written

(1) $$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(Sec. 2.1).}$$

(If the first term is, say, $f(x)y''$, divide by $f(x)$ to get the "**standard form**" (1) with y'' as the first term.) Equation (1) is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval; this is written $r \equiv 0$. Then

(2) $$y'' + p(x)y' + q(x)y = 0$$

Equation (1) is called **nonhomogeneous** if $r(x) \not\equiv 0$ (meaning $r(x)$ is not zero for some x considered).

For the homogeneous ODE (2) we have the important **superposition principle** (Sec. 2.1) that a linear combination $y = ky_1 + ly_2$ of two solutions y_1, y_2 is again a solution.

Two *linearly independent* solutions y_1, y_2 of (2) on an open interval I form a **basis** (or **fundamental system**) of solutions on I, and $y = c_1y_1 + c_2y_2$ with arbitrary constants c_1, c_2 a **general solution** of (2) on I. From it we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two **initial conditions**

(3) $\quad y(x_0) = K_0, \qquad y'(x_0) = K_1$

(x_0, K_0, K_1 given numbers; Sec. 2.1).

(2) and (3) together form an **initial value problem**. Similarly for (1) and (3).

(continued 2)

For a nonhomogeneous ODE (1) a **general solution** is of the form

(4) $$y = y_h + y_p$$ (Sec. 2.7).

Here y_h is a general solution of (2) and y_p is a particular solution of (1). Such a y_p can be determined by a general method (***variation of parameters***, Sec. 2.10) or in many practical cases by the ***method of undetermined coefficients***. The latter applies when (1) has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Sec. 2.7). Then we write (1) as

(5) $$y'' + ay' + by = r(x)$$ (Sec. 2.7).

The corresponding homogeneous ODE $y' + ay' + by = 0$ has solutions $y = e^{\lambda x}$ where λ is a root of

(6) $$\lambda^2 + a\lambda + b = 0.$$

(continued 3)

Hence there are three cases (Sec. 2.2):

Case	Type of Roots	General Solution
I	Distinct real λ_1, λ_2	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Double $-\frac{1}{2}a$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex $-\frac{1}{2}a \pm i\omega^*$	$y = e^{-ax/2}(A \cos \omega^* x + B \sin \omega^* x)$

Here ω^* is used since ω is needed in driving forces.
Important applications of (5) in mechanical and electrical engineering in connection with ***vibrations*** and ***resonance*** are discussed in Secs. 2.4, 2.7, and 2.8.

(continued 4)

Another large class of ODEs solvable "algebraically" consists of the **Euler–Cauchy equations**

(7) $$x^2y'' + axy' + by = 0$$ (Sec. 2.5).

These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

(8) $$m^2 + (a - 1)m + b = 0.$$

Existence and uniqueness of solutions of (1) and (2) is discussed in Secs. 2.6 and 2.7, and ***reduction of order*** in Sec. 2.1.