2.4 Modeling of Free Oscillations of a Mass—Spring System

Linear ODEs with constant coefficients have important applications in mechanics, as we show in this section as well as in Sec. 2.8, and in electrical circuits as we show in Sec. 2.9.

In this section we model and solve a basic mechanical system consisting of a mass on an elastic spring (a so-called "mass–spring system," Fig. 33), which moves up and down.

Setting Up the Model

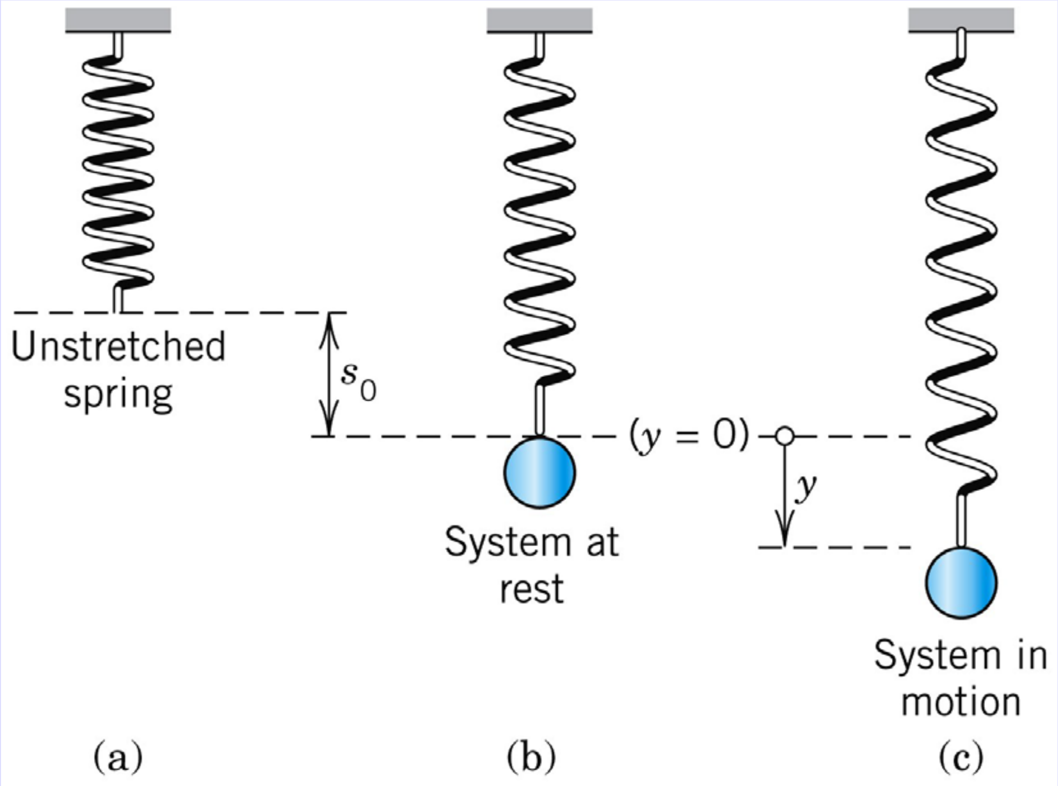
We take an ordinary coil spring that resists extension as well as compression.

We suspend it vertically from a fixed support and attach a body at its lower end, for instance, an iron ball, as shown in Fig. 33.

We let $y = 0$ denote the position of the ball when the system is at rest (Fig. 33b).

Furthermore, we choose ***the downward direction as positive,*** thus regarding downward forces as *positive* and upward forces as *negative*.

Setting Up the Model *(continued 1)*



Setting Up the Model *(continued 2)*

We now let the ball move, as follows. We pull it down by an amount $y > 0$ (Fig. 33c).

This causes a spring force

(1) $$F_1 = -ky$$ **(Hookes's law)**

proportional to the stretch y, with k (> 0) called the **spring constant**.

The minus sign indicates that F_1 points upward, against the displacement.

It is a *restoring force*: It wants to restore the system, that is, to pull it back to $y = 0$.

Stiff springs have large k.

Setting Up the Model *(continued 3)*

Note that an additional force $-F_0$ is present in the spring, caused by stretching it in fastening the ball, but F_0 has no effect on the motion because it is in equilibrium with the weight W of the ball,

$$-F_0 = W = mg,$$

where $g = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 = 32.17 \text{ ft/sec}^2$ is the **constant of gravity at the Earth's surface** (not to be confused with the *universal gravitational constant* $G = gR^2/M = 6.67 \cdot 10^{-11} \text{ nt m}^2/\text{kg}^2$, which we shall not need; here $R = 6.37 \cdot 10^6$ m and $M = 5.98 \cdot 10^{24}$ kg are the Earth's radius and mass, respectively).

Setting Up the Model *(continued 4)*

The motion of our mass–spring system is determined by **Newton's second law**

(2) Mass × Acceleration = my'' = Force

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the ball.

ODE of the Undamped System

Every system has damping. Otherwise it would keep moving forever.
But if the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping.
Then Newton's law with $F = F_1$ gives the model
$my'' = F_1 = -ky$; thus

(3) $$my'' + ky = 0.$$

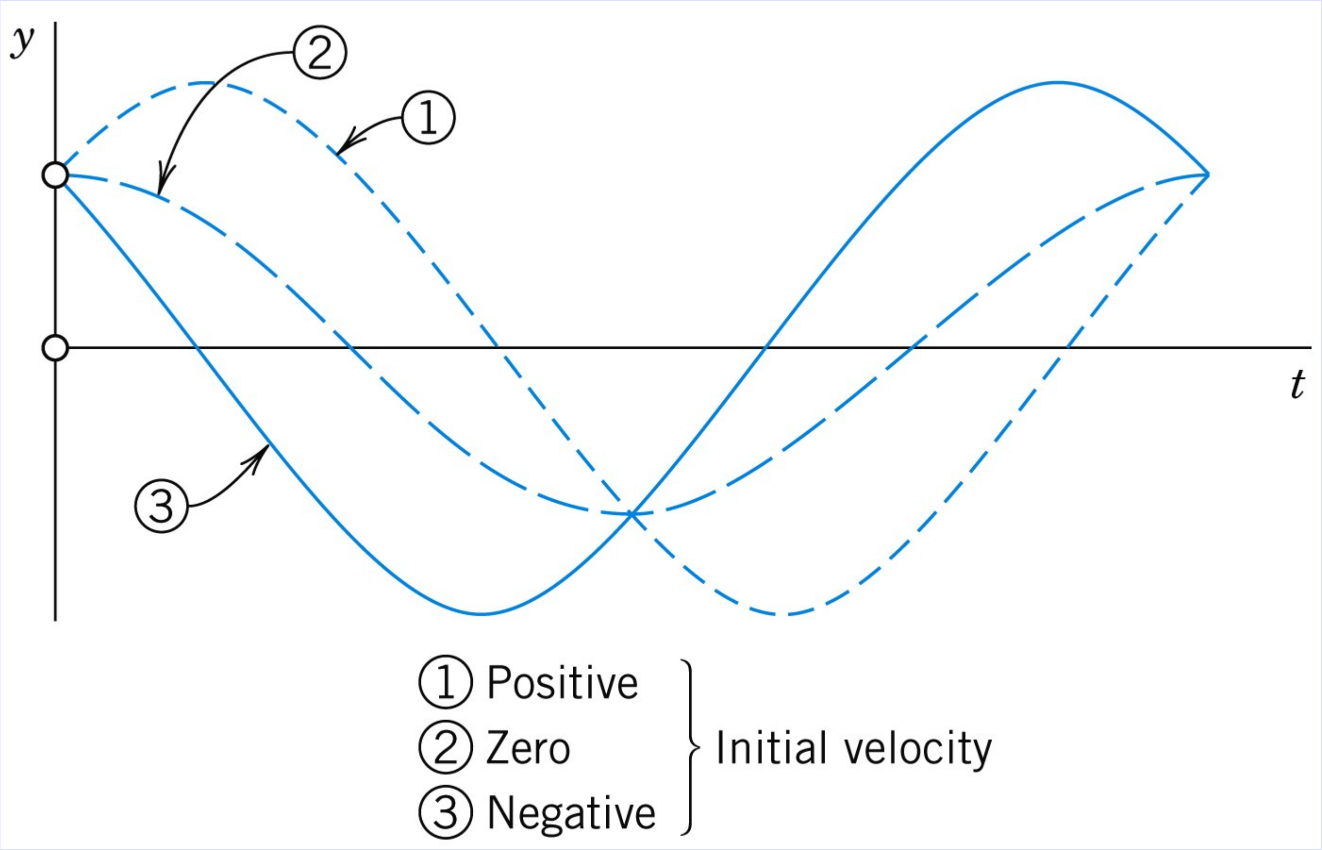
ODE of the Undamped System *(continued 1)*

This is a homogeneous linear ODE with constant coefficients. A general solution is obtained, namely

(4) $$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

This motion is called a **harmonic oscillation** (Fig. 34, see next slide).

ODE of the Undamped System *(continued 2)*



ODE of the Undamped System *(continued 3)*

Its *frequency* is $f = \omega_0/2\pi$ Hertz (= cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$.
The frequency f is called the **natural frequency** of the system.

ODE of the Undamped System *(continued 4)*

An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

(4*) $$y(t) = C \cos(\omega_0 t - \delta)$$

with $C = \sqrt{A^2 + B^2}$

and phase angle δ, where $\tan \delta = B/A$.

ODE of the Damped System

To our model $my'' = -ky$ we now add a damping force

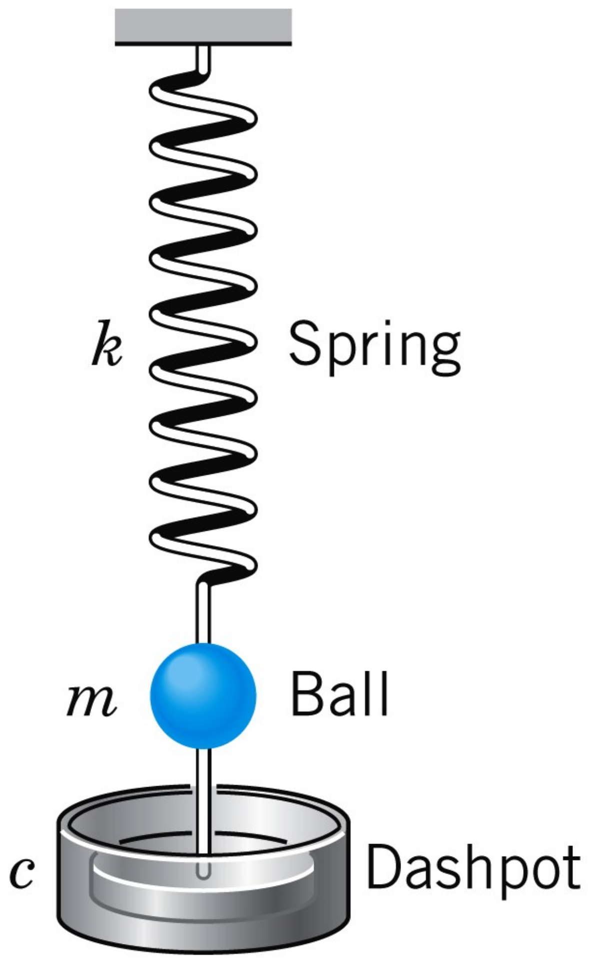
$$F_2 = -cy',$$

obtaining $my'' = -ky - cy'$;
thus the ODE of the damped mass–spring system is

(5) $$my'' + cy' + ky = 0. \qquad \text{(Fig. 36)}$$

Physically this can be done by connecting the ball to a dashpot; see Fig. 36 *(next slide)*.
We assume this damping force to be proportional to the velocity $y' = dy/dt$. This is generally a good approximation for small velocities.

ODE of the Damped System *(continued 1)*



ODE of the Damped System *(continued 2)*

The constant c is called the *damping constant*. Let us show that c is positive.

Indeed, the damping force $F_2 = -cy'$ acts *against* the motion; hence for a downward motion we have $y' > 0$, which for positive c makes F negative (an upward force), as it should be.

Similarly, for an upward motion we have $y' < 0$, which for $c > 0$ makes F_2 positive (a downward force).

ODE of the Damped System *(continued 3)*

The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by m)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

(6) $\lambda_1 = -\alpha + \beta, \lambda_1 = -\alpha - \beta$

where $\alpha = \dfrac{c}{2m}$ and $\beta = \dfrac{1}{2m}\sqrt{c^2 - 4mk}.$

ODE of the Damped System *(continued 4)*

It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping, or little damping—three types of motions occur, respectively:

Case I. $c^2 > 4mk$. *Distinct real roots* λ_1, λ_2. **(Overdamping)**

Case II. $c^2 = 4mk$. *A real double root.* **(Critical damping)**

Case III. $c^2 < 4mk$. *Complex conjugate roots.* **(Underdamping)**

Discussion of the Three Cases

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots.

In this case the corresponding general solution of (5) is

(7) $$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate.

For $t > 0$ both exponents in (7) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$.

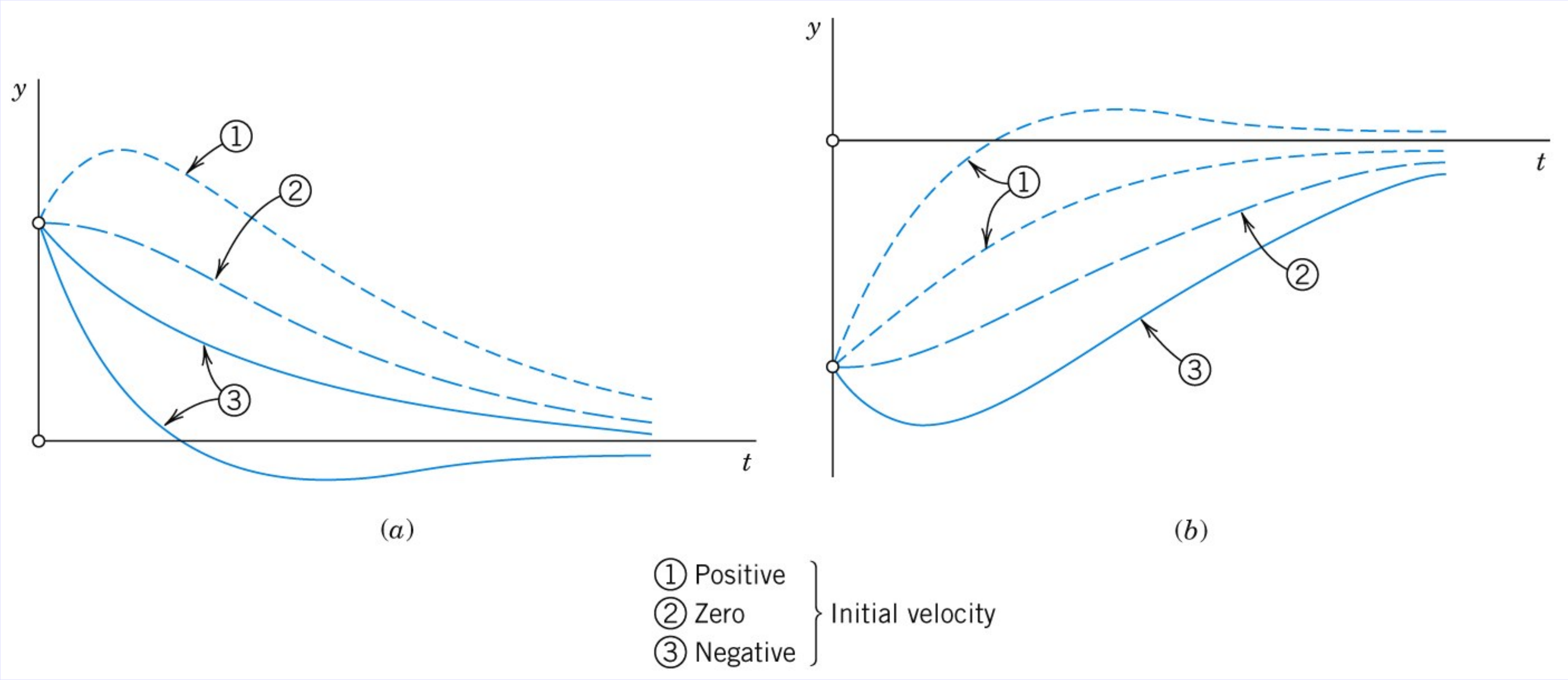
Hence both terms in (7) approach zero as $t \to \infty$.

Practically speaking, after a sufficiently long time, the mass will be at rest at the *static equilibrium position* ($y = 0$).

Figure 37 shows (7) for some typical initial conditions.

Discussion of the Three Cases *(continued 1)*

Case I. Overdamping *(continued)*



(*a*) Positive initial displacement
(*b*) Negative initial displacement

Discussion of the Three Cases *(continued 2)*

Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$.

Then the corresponding general solution of (5) is

(8) $$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$

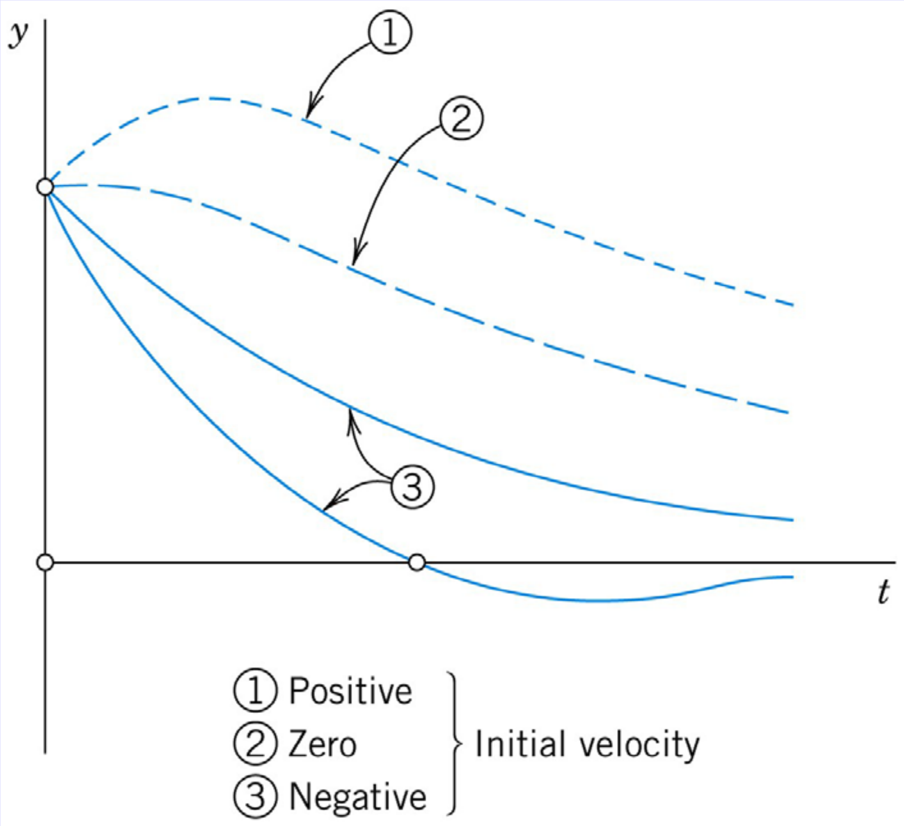
This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.

If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.

Discussion of the Three Cases *(continued 3)*

Case II. Critical damping *(continued)*

Figure 38 shows typical forms of (8). Note that they look almost like those in the previous figure (Figure 37).



Discussion of the Three Cases *(continued 4)*

Case III. Underdamping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$(9)\quad \beta = i\omega^* \quad \text{where } \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0).$$

(We now write ω^* to reserve ω for driving and electromotive forces in Secs. 2.8 and 2.9.)

Discussion of the Three Cases *(continued 5)*

Case III. Underdamping ***(continued 1)***

The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + i\omega^*, \ \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

(10) $$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

Discussion of the Three Cases *(continued 6)*

Case III. Underdamping *(continued 2)*

This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^*t - \delta$ is an integer multiple of π because these are the points at which $\cos(\omega^*t - \delta)$ equals 1 or −1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (>0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches

$$\omega_0 = \sqrt{k/m},$$

giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

Discussion of the Three Cases *(continued 7)*

Case III. Underdamping *(continued 3)*