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Article *in* Review of Quantitative Finance and Accounting · February 2003

DOI: 10.1023/A:1023699711805 · Source: RePEc

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Review of Quantitative Finance and Accounting, 20: 207–243, 2003

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Abstract. This paper analyzes the optimal dynamic asset allocation problem in economies with infrequent events and where the investment opportunities are stochastic and predictable. Analytical approximations are obtained, with which a thorough comparative study is performed on the impacts of jumps upon the dynamic decision. The model is then calibrated to the U.S. equity market. The comparative analysis and the calibration exercise both show that jump risk not only makes the investor's allocation more conservative overall but also makes her dynamic portfolio rebalancing less dramatic over time.

Key words: asset allocation, jumps, non-normality, time-varying investment opportunities, predictability

JEL Classification: G11

1. Introduction

Traditional asset allocation theory and practice are challenged by two distinct features of today's financial markets: (1) *jumps:* asset prices move discontinuously; (2) *predictability:* investment opportunities are time varying and, more importantly, predictable. Jumps generate more extreme realizations than implied by a normal distribution. Traditional mean-variance analysis is hence no longer enough: higher moments also play important roles. Predictability, on the other hand, implies the existence of an intertemporal hedging demand.

This paper investigates the dynamic asset allocation problem in economies with both features. Analytical approximations are obtained to solutions of the problem, with which a thorough comparative study is performed on the impacts of jumps upon both the myopic demand and the intertemporal hedging demand, as well as their interactions with each other. I then calibrate the model to the U.S. equity market and assess the quantitative impacts of the jumps under such a dynamic environment. Both the comparative analysis and the calibration exercises demonstrate that jump risk not only makes the investor's allocation more conservative overall but also makes her dynamic portfolio rebalancing less dramatic over time.

The *isolated* impacts of jumps or predictability on asset allocation have been studied in the literature a long while ago. Merton (1971), for example, considers the myopic asset allocation problem when the risky asset has a probability of default. The default event is captured by a Poisson jump equal to the negative of the current price. Das and Uppal (1998) consider a similar static problem with multiple risky assets with perfectly correlated jumps. On predictability, Kim and Omberg (1996) solve a dynamic problem analytically where the risky asset return follows a geometric Brownian motion and the risk premium follows an Ornstein-Uhlenbeck process with mean reversion. This paper combines these two strands of literature and investigates the impacts of jumps on the dynamic decision. I obtain several interesting results that are absent from the isolated analyses.

A key result of the paper is that the impact of jumps depends upon the investor's overall position in the risky asset. The net effect of jumps is to reduce the investor's overall position, long or short, in the asset. For example, when the investor has a big, long position in the asset, jump risk incurs a negative demand to reduce the long position. On the other hand, when the investor has a big, short position in the asset, jump risk incurs a positive demand to reduce the short position. The more involved the investor is in the asset, the bigger the impact of the jump becomes. Such a dependence structure implies that the intertemporal hedging demand and the jump effect are intertwined even if the jump and the state variable are independent from each other. Time varying investment opportunities call for active portfolio management. Depending upon the conditional information on the investment opportunities, one needs to update one's portfolio constantly, sometimes switching from a long position to a short one, or vice versa. Not only the presence of jump risk refrains the investor from becoming overly involved in the risky asset, but it also makes the portfolio updating or rebalancing less dramatic over time.

Another contribution of the paper is to link the dynamic asset allocation decision to moment analysis. In Section 3, the paper applies a Taylor expansion to the Euler equation and obtains the optimal decision rule as a function of mean, variance, skewness, and kurtosis of the excess return. The mean-variance result is well-known, but the effect of skewness and kurtosis is new to the literature. It also confirms intuition: fat tails (positive kurtosis) imply extra risk in addition to those captured by variance and hence reduce the investor's overall position in the risky asset. The effect of skewness, on the other hand, is direction sensitive and is analogous to that of the mean: positive skewness increases the demand while negative skewness reduces it.

The article further shows that the effects of jumps and predictability on the portfolio decision are inherently linked to their impacts on the conditional moments of the asset return. In particular, jumps reduce the investor's overall demand for the asset not only because they generate skewness (negative in case of a negative mean jump magnitude) and positive kurtosis, but also because they generate additional volatility to the asset return. Time-varying investment opportunities can increase or decrease the conditional variance of the asset return, depending on the direction of predictability. In particular, large negative correlation between the state variable and the asset return reduces the conditional variance (risk) and therefore incurs a positive intertemporal hedging demand.

I also calibrate the model to the U.S. stock market index (S&P 500) and assess the quantitative impact of jumps under a dynamic setting. Calibration to more than 36 years of daily S&P 500 index return data indicates that the index has an 18% probability of having one jump or more within each year and the jumps account for about 12% of the total variance of daily returns. Therefore, jumps are an inherent part of the asset price movement. To capture the time-varying and predictable nature of the index return, we use log dividend-price ratio as the stochastic variable to predict the expected excess return and investigate the impacts of jumps over different investment horizons and at different states. The calibration exercise illustrates that, on average, for an investor with a relative risk aversion of four, taking into account the jump risk reduces her position in the stock market by about 14%, a very significant reduction. The time-varying nature of the investment opportunities calls for the investor to rebalance her portfolio based on the updates on the expected excess return. Depending on the swing of the updates, the portfolio adjustment can be dramatic, sometimes goes from a long position to short one, or vice versa. Considering jump risk makes the investor's dynamic decision more conservative overall and the portfolio rebalancing less dramatic over time.

As mentioned earlier, the paper is essentially an integration of the static jump model of Merton (1971) and Das and Uppal (1998) and the dynamic diffusion model of Kim and Omberg (1996). Portfolio allocation problems under other static nonnormal specifications such as the general Lévy process (Kallsen, 2000; and Benth, Karlsen and Reikvam, 2001) have also been studied in the literature. Most recently, Liu, Longstaff and Pan (2002) study the implications of jumps in both price and volatility on investment strategies. Other recent works addressing the issue of dynamic portfolio choice facing predictability include, among others, Ang and Bekaert (1999), Balduzzi and Lynch (1999), Barberis (1999), Brandt (1999), Brennan, Schwartz and Lagnado (1997), Campbell and Viceira (1999, 2001), Liu (1998), Lynch (1999), and Rishel (1999). This paper integrate both features of the data and investigate their interactions.

The paper is structured as follows. The next section discusses the evidence and modeling of non-normality in financial asset returns. Section 3 develops the basic intuition on how non-normality as captured by higher moments varies the investor's investment decision. Section 4 formally sets up the model and solves the dynamic portfolio allocation problem in the presence of both jumps and predictability. Section 5 studies the comparative statics, illustrating the impacts of jumps on both the myopic demand and the intertemporal hedging demand. Section 6 calibrates the model to the United States stock market. Section 7 provides some final thoughts.

2. Evidence and modeling of nonnormality

It is widely documented and generally agreed upon that returns on financial assets are not normally distributed. We focus on returns on the U.S. stock market and use S&P 500 index as a proxy. The data are retrieved from the CRSP database (The Center for Research in Securities). The data are daily, from July 3rd, 1962 to December 31st, 1997 (8938 observations).

Table 1 summarizes the statistical properties of the index return with different time aggregations. Daily returns exhibit significant negative skewness (-1.31) and extreme kurtosis (34.70), both of which should be zero for a normal distribution. Time aggregation reduces the magnitude of non-normality, but in a speed slightly slower than implied by i.i.d. innovations, that is, $1/\sqrt{n}$ for skewness and $1/n$ for kurtosis, with n denoting number of days in aggregation.

In continuous time finance, a popular way to generate discontinuity is to apply the compound Poisson jump model of Press (1967). We follow suit in this paper and specify that the

Table 1. Properties of S&P 500 index daily returns

n	Mean	St. Dev.	Skewness	Kurtosis
1 day	12.64	13.66	−1.31	34.70
6 days	12.71	14.70	−0.49	7.16
11 days	12.69	14.52	−0.56	6.09
16 days	12.73	14.46	−0.53	4.37
21 days	12.76	14.54	−0.42	3.29
26 days	12.76	14.51	−0.40	2.91
31 days	12.76	14.45	−0.45	2.44
36 days	12.76	14.51	−0.46	2.43
41 days	12.77	14.56	−0.42	2.56
46 days	12.77	14.60	−0.38	2.49
51 days	12.79	14.63	−0.34	2.34
56 days	12.80	14.64	−0.28	2.14

Notes: Entries are sample moments computed for aggregated daily returns on S&P 500 index. The data are from CRSP, run from July 3rd, 1962 to December 31st, 1997 (8938 observations). The first column indicates number of days of aggregation. Mean is the sample mean (annualized percentage), St. Dev. the sample standard deviation (annualized percentage). The skewness and kurtosis are defined, specifically, in terms of central moments μ_j: $\gamma_1 = m_3/m_2^{3/2}$ and $\gamma_2 = m_4/m_2^2 - 3$, where m_3 and m_4 are the third and fourth central moments, respectively. Our estimates replace population moments with sample moments.

time-t price, P_t, of the risky asset follows a jump-diffusion process of the following form:

$$\frac{dP_t}{P_t} = (\mu_t - \lambda g)\,dt + \sigma\,dZ_t + (e^q - 1)\,dQ(\lambda), \tag{1}$$

where Z_t is a standard Brownian motion. We allow the drift μ_t to be stochastic to capture the time-varying investment opportunities. $dQ(\lambda)$ defines a Poisson jump process with intensity λ: $\Pr(dQ = 1) = \lambda\,dt$. The probability of having n jumps over investment horizon τ is characterized by the Poisson probability

$$\Pr\,(n \text{ jumps over } \tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}\,. \tag{2}$$

The term $g = E(e^q - 1)$ captures the mean percentage jump in the asset price conditional on one jump happening and q is assumed to be normally distributed with $N(\mu_q, \sigma_q^2)$. In the special case when jump represents complete default or bankruptcy, as in Merton (1971), $q = -\infty$, asset prices jump to zero.

Jumps in the asset return process generate nonnormality. When both μ_t and σ are constants (constant investment environment), the mean of return $\ln(P_{t+\tau}/P_t)$ over time horizon τ is $\mu\tau - \sigma^2\tau/2 - \lambda(g - \mu_q)\tau$. Its variance and higher moments are, respectively,

$$\begin{aligned}
\kappa_2 &= \left[\sigma^2 + \lambda\left(\mu_q^2 + \sigma_q^2\right)\right]\tau;\\
\kappa_3 &= \lambda\mu_q\left(\mu_q^2 + 3\sigma_q^2\right)\tau;\\
\kappa_4 &= \lambda\left(\mu_q^4 + 6\mu_q^2\sigma_q^2 + 3\sigma_q^4\right)\tau,
\end{aligned} \tag{3}$$

where κ_j is the jth cumulant of the return. See Appendix A for the derivation. Cumulants relate to the central moments, m_j, by,

$$m_2 = \kappa_2; \quad m_3 = \kappa_3; \quad m_4 = \kappa_4 + 3m_2^2.$$

Skewness and kurtosis are defined, respectively, as the normalized third and fourth cumulant:

$$\gamma_1 = \frac{\lambda\mu_q\left(\mu_q^2 + 3\sigma_q^2\right)}{\left[\sigma^2 + \lambda\left(\mu_q^2 + \sigma_q^2\right)\right]^{3/2}\sqrt{\tau}};$$
$$\gamma_2 = \frac{\lambda\left(\mu_q^4 + 6\mu_q^2\sigma_q^2 + 3\sigma_q^4\right)}{\left[\sigma^2 + \lambda\left(\mu_q^2 + \sigma_q^2\right)\right]^2\tau}.$$

The addition of jumps creates nonzero skewness and kurtosis, both of which are zero for a normal distribution. It also increases the variance of the asset return. The return variance increases with the jump frequency λ, the absolute mean jump size $|\mu_q|$, and the conditional variance of the jump size σ_q^2. The asymmetry, as captured by its skewness, depends on the mean direction of the jump μ_q. Another feature of the jump diffusion process is that skewness and kurtosis die away as the investment horizon increases. Skewness decreases with the square root of the horizon ($\sqrt{\tau}$) while kurtosis decreases with τ.

Maximum likelihood estimation of jump-diffusion models may break down in practice when one approximates the density with a truncated number of normals. See Kiefer (1978) and Hamilton (1994) for standard arguments on the unboundedness of the likelihood function for mixtures of normals. Honore (1998) proposes a two-step maximum likelihood estimation method to get around the problem. We implement a just-identified generalized methods of moments (GMM) estimation to the jump diffusion model: we choose the five parameters (μ, σ^2, λ, μ_q, σ_q^2) to match the first five moments of the index returns. The first four cumulants are given in (3), the fifth cumulant of the jump-diffusion process is

$$\kappa_5 = \lambda\left(15\sigma_q^4\mu_q + 10\sigma_q^2\mu_q^3 + \mu_q^5\right)\tau.$$

It relates to the fifth central moment (m_5) by $\kappa_5 = (m_5 - 10\kappa_3\kappa_2)$.

Since the five moments are highly non-linear functions of the five parameters, to increase the speed of convergence, we obtain the first stage estimates by the following sequence: (i) given λ, we estimate μ_q and σ_q^2 from the third and fourth cumulants of the asset returns; (ii) given μ_q and σ_q^2, we choose λ to match the fifth cumulant; and finally, (iii) given λ, μ_q, and σ_q, we choose μ and σ to match the mean and variance of the return. The breaking down of the moment conditions is based on the key observation that higher cumulants (κ_i, $i \geq 3$) come solely from the Poisson jump process and are independent of the diffusion parameters (μ, σ).

For the second stage estimation, we construct the weighting matrix following Newey and West (1987) with 2 lags and de-meaned moment conditions. Bekaert and Urias (1996) find that weighting matrices with de-meaned moments tends to perform better than

Table 2. Calibrating S&P 500 index to a jump-diffusion process

	(A. Daily Data)		(B. Monthly Data)	
Parameter	Estimates	Std. Errors	Estimates	Std. Errors
μ	0.1264	0.0248	0.1285	0.0241
σ	0.1292	0.0024	0.1289	0.0090
λ	0.2029	0.1444	0.4478	0.3984
μ_q	−0.0560	0.0123	−0.0432	0.0174
σ_q	0.0886	0.0055	0.0879	0.0205

Notes: Entries are the just-identified GMM estimates (and standard errors) for the following jump-diffusion process on S&P 500 Index:

$$dP/P = (\mu - \lambda g)\,dt + \sigma\,dZ(t) + (e^q - 1)\,dQ(\lambda).$$

We choose the parameters to match the first five moments of the log daily (Panel A) and monthly (Panel B) return on S&P 500 Index. Parameters are annualized. Standard errors are computed following Newey and West (1987) with 2 lags for daily return data and 0 lags for monthly data. The number of lags are optimally chosen following Andrews (1991). The data used for the calibration are retrieved from the CRSP database, daily from July 3rd, 1962 to December 31st, 1997 (8938 observations).

un-demeaned ones. The number of lags is chosen optimally following Andrews (1991) assuming a VAR(1) specification for the moment conditions.

Panel A of Table 2 calibrates the jump-diffusion process in (1) to the S&P 500 daily return data while assuming constant mean drift μ and diffusion volatility σ. Parameters are annualized. With daily returns, we set $\tau = 1/252$, assuming approximately 252 business days a year. The estimated jump intensity is $\lambda = 0.203$, which implies a 18.4% probability of having one or more jumps within one year. Under the estimated parameters, the addition of jumps account for 11.8% of the total variance of the daily returns. The overall drift of the asset price is driven down by 8.1% due to the jumps, which are on average negative. The negative mean jumps also generate the negative skewness ($\gamma_1 = -1.85$) observed in S&P 500 index returns. Both the mean and the variance of the jumps contribute to the large excess kurtosis ($\gamma_2 = 49$) observed in the daily log returns of S&P 500 Index.

Panel B of Table 2 calibrates the model to monthly S&P 500 index returns for the same period. The optimal lag in the Newey-West weighting matrix is zero in this case. The estimates are very close to those obtained from the daily data, indicating that the methods of moments procedure yields robust estimates.

The calibration exercise illustrates that discontinuity in asset price movements directly generates asset return nonormality. In the next section, we use a simple one-period model to demonstrate intuitively how asset return nonnormality, in particular skewness and kurtosis, affects the investor's demand for the asset.

3. Intuitions from a one-period model

In this section, we apply a Taylor expansion to the Euler equation and approximate the portfolio decision rule as a function of the mean, variance, skewness, and kurtosis of the excess return. The solution, simple as it is, is new to the literature. It provides us with the simple intuition and insight on how non-normality varies investors' demand for the financial asset. The solution says that fat tails (kurtosis) in the return distribution reduce investors' position in the asset. When the distribution is asymmetric, positive skewness increases the investors' demand while negative skewness reduces it.

Specifically, we assume that an investor maximizes her expected utility of next period wealth,

$$\max_{\theta_t} \quad E_t u(W_{t+1}),$$

by investing between a riskfree asset and a risky asset,

$$W_{t+1} = W_t[\theta_t(R_{1,t+1} - R_f) + R_f],$$

where $R_{1,t}$ is the gross return to the risky asset, R_f is the gross return to the riskfree asset, and θ_t is the allocation weight to the risky asset.

Assuming that the utility function $u(\cdot)$ is concave, the following first order condition is a sufficient condition for the optimization problem:

$$E_t[u'(W_{t+1})(R_{1,t+1} - R_f)] = 0. \tag{4}$$

The solution to this static model is the investor's myopic demand for the risky asset. We can drop the subscript t and use unconditional expectation if the investment opportunities are constant over time. Otherwise, at each period the investor can actively rebalance her portfolio based on her conditional information. In this section, we do not assume any particular distribution for the asset return but only assume that it has finite moments. We expand the investor's marginal utility function $u'(W_{t+1})$ as a Taylor series around the expected next period wealth $E_t[W_{t+1}]$. Substituting the expansion into the first order condition (4) results

$$\begin{aligned}0 &= u^{(1)}(E_t[W_{t+1}])x_t + u^{(2)}(E_t[W_{t+1}])W_t\theta_t m_{2t} + \frac{1}{2}u^{(3)}(E_t[W_{t+1}])W_t^2\theta_t^2(m_{3t} + m_{2t}x_t)\\ &\quad + \frac{1}{6}u^{(4)}(E_t[W_{t+1}])W_t^3\theta_t^3(m_{4t} + m_{3t}x_t) + \mathbf{O}(m_{5t}),\end{aligned} \tag{5}$$

where $x_t = E_t R_{1,t+1} - R_f$ is the expected excess return, m_{nt} is the nth central moment of $R_{1,t+1}$, and $u^{(n)}(\cdot)$ is the nth derivative of the utility function. The residual term, $\mathbf{O}(m_{5t})$, is a sum of even-higher moments.

The traditional mean-variance result can be obtained as a first degree approximation from equation (5):

$$\theta_t \cong -\frac{u^{(1)}(E_t[W_{t+1}])x_t}{u^{(2)}(E_t[W_{t+1}])W_t m_{2t}} \cong \frac{x_t}{\gamma m_{2t}}, \tag{6}$$

where $\gamma = -u^{(2)}W/u^{(1)}$ is the Arrow-Pratt measure of relative risk aversion. Higher moments, i.e., skewness and kurtosis, have higher-order effects. If we further assume that the utility on wealth takes the form of constant-relative-risk-aversion (CRRA):

$$u(W) = \frac{W^{1-\alpha}}{1-\alpha}, \quad \alpha > 0,$$

the optimal allocation weight θ becomes the solution to the following cubic function:

$$0 \cong a\theta^3 + b\theta^2 + c\theta + d, \tag{7}$$

with

$$\begin{aligned}
a &= -\frac{1}{6}\alpha(1+\alpha)(2+\alpha)(\gamma_2+3)\sigma^4 + \frac{1}{6}\alpha(1-\alpha^2)\gamma_1\sigma^3 x - \frac{1}{2}\alpha(1-\alpha)\sigma^2 x^2 + x^4;\\
b &= R_f\left[\frac{1}{2}\alpha(1+\alpha)\gamma_1\sigma^3 - \frac{1}{2}\alpha(3-\alpha)\sigma^2 x + 3x^3\right];\\
c &= R_f^2[-\alpha\sigma^2 + 3x^2];\\
d &= R_f^3 x,
\end{aligned}$$

and where σ^2, γ_1, γ_2 denote, respectively, the variance, skewness, and kurtosis of the excess return.

The allocation weight θ can then be readily solved from the cubic equation:

$$\theta \cong \frac{-b + K + (b^2 - 3ac)/K}{3a}, \tag{8}$$

where

$$\begin{aligned}
K &= \left(\frac{A + \sqrt{4(-b^2+3ac)^3 + A^2}}{2}\right)^{1/3};\\
A &= -2b^3 + 9abc - 27a^2 d.
\end{aligned}$$

The result in (8) is new to the literature. The closest is by Chunha, chinda, Dandapani, Hamid and Prakash (1997), who investigate the impact of skewness on portfolio selection using a similar approach.

To see how nonnormality impacts the investor's portfolio decision, we apply the Implicit Function Theorem to get the partial derivatives of the allocation weight with respect to skewness (γ_1) and kurtosis (γ_2). Let $f(\gamma_1, \gamma_2)$ denote the right hand side of equation (7). Assume θ and x are small, we have

$$\begin{aligned}
\frac{\partial\theta}{\partial\gamma_1} &\equiv -\frac{\partial f/\partial\gamma_1}{\partial f/\partial\theta} \cong \frac{1}{2}(1+\alpha)\sigma\theta^2 > 0;\\
\frac{\partial\theta}{\partial\gamma_2} &\equiv -\frac{\partial f/\partial\gamma_2}{\partial f/\partial\theta} \cong -\frac{1}{6}(1+\alpha)(2+\alpha)\sigma^2\theta^3.
\end{aligned} \tag{9}$$

The partial derivatives illustrate that (1) the optimal allocation to the risky asset increases with positive skewness and decreases with excess kurtosis, (2) the impacts of nonnormality increases with the investor's relative risk aversion.

As shown in Table 1, non-normality decreases rapidly with time aggregation. As such, some argue that when the investment horizon is long, say, portfolio managers only rebalance their portfolio quarterly due to transaction cost considerations, skewness and kurtosis may not be a big issue anymore because the magnitudes of skewness and kurtosis are much smaller. This argument is actually not true because, as can be seen from (9), the impact of skewness increases with the standard deviation, and the impact of kurtosis increases with the variance. Therefore, for an independently and identically distributed return series, while skewness and kurtosis decrease with the investment horizon $\sqrt{n}$ and n, respectively, their impacts increase, too, with $\sqrt{n}$ and n, respectively. As a net result, the impact of non-normality on the portfolio decision does not change with the investment horizon n.

A simple calibration exercise summarized in Table 3 illustrates the relative impacts of nonnormality. In the exercise, we assume that an investor has a relative risk aversion of four and is making her portfolio decision between a 5% riskfree bond and a mutual fund mimicking the return of S&P 500 index. The exercise indicates that both the negative skewness and the positive kurtosis observed in the U.S. stock market reduce the investor's demand for the stock portfolio. Taking both into account, the investor reduces her investment

Table 3. Investment decision on S&P 500

n	θ_1 Normal	θ_2 Skewness	θ_3 Kurtosis	θ_4 Non-Normality	$\Delta\theta/\theta$ Change
1 day	102.32	99.55	101.01	98.40	−3.84
6 days	89.23	87.21	88.06	86.10	−3.51
11 days	91.30	88.21	89.48	86.55	−5.20
16 days	92.41	88.92	90.54	87.21	−5.63
21 days	91.83	88.76	90.07	87.06	−5.19
26 days	92.20	89.02	90.34	87.18	−5.44
31 days	92.96	89.17	91.16	87.37	−6.01
36 days	92.12	88.10	90.13	86.13	−6.51
41 days	91.54	87.73	89.23	85.42	−6.69
46 days	91.09	87.65	88.66	85.15	−6.52
51 days	90.96	87.80	88.52	85.19	−6.35
56 days	91.07	88.50	88.70	85.82	−5.76

Notes: Entries are optimal static allocation weight (in percentage) to S&P 500 index with different investment horizons n (in business days). The investor has CRRA utility with relative risk aversion $\alpha = 4$ and maximizes her next period wealth by making her portfolio decisions between S&P 500 and a 5% riskfree bond, based on sample moments provided in Table 1. The optimal allocation weight is computed based on the static decision formula in (8). θ_1 is the allocation weight assuming normality ($\gamma_1 = \gamma_2 = 0$), θ_2 is the allocation weight assuming zero kurtosis, θ_3 is the allocation weight assuming zero skewness, and θ_4 is the allocation weight taking into account all the first four moments. The last column presents the percentage change in allocation weight with and without considering non-normality:

$$\Delta\theta/\theta = 100 \times \frac{\theta_4 - \theta_1}{\theta_1}.$$

in the stock portfolio by about 6%. Further, as analyzed before, the impacts of non-normality do not decrease with the investment horizon.

Non-normality creates risk and/or benefit that are not captured by traditional mean-variance analysis. Both fat tails and negative skewness, as observed in the U.S. stock market, imply additional risk to the investor and thus reduce the investor's demand. Increasing investment horizon, while reduces the magnitude of skewness and kurtosis, does not reduce its general impacts on the portfolio decision. Such an illustration, while intuitive and straightforward, is merely an approximation of the investor's myopic decision. In what follows, we formally build up a dynamic model that captures both the discontinuity of the asset price movements and the time-varying (and more importantly, predictable) nature of the investment opportunities. We are then equipped to analyze how nonnormality affects the investor's myopic decision as well as her dynamic hedging behavior.

4. Dynamic portfolio decision

This section sets up the model and solves the dynamic portfolio decision problem in the presence of both jumps and predictability. Formally, we assume that an investor makes her portfolio choice between two assets: one riskfree asset and one risky asset. The investor maximizes her terminal wealth over investment horizon $\tau = T - t$. The riskfree asset is assumed to have a constant continuously compounded rate of return, r_f. The price, P_t, of the risky asset follows a jump-diffusion process, as specified in (1),

$$\frac{dP_t}{P_t} = (\mu_t - \lambda g)\, dt + \sigma\, dZ_t + (e^q - 1)\, dQ(\lambda).$$

Recall that $g = E(e^q - 1)$ is the mean percentage jump in the asset price conditional on one jump happening and λ denotes the jump intensity.

Following Kim and Omberg (1996), we allow the investment environment to be stochastic and characterize its variation by modeling the fluctuation of x_t, which is defined as,

$$x_t = \frac{\mu_t - \lambda g - r_f}{\sigma}. \tag{10}$$

Since $\mu_t - \lambda g - r_f$ captures the excess return to the diffusion part of the process while σ is the diffusion volatility, x_t captures the risk premium pertaining to the diffusion part of the asset return process. We therefore label x_t as the *diffusion risk premium*. As in Kim and Omberg (1996), we assume that this risk premium follows a simple Ornstein-Uhlenbeck process with mean reversion,

$$dx_t = -\kappa_x (x - \mu_x)\, dt + \sigma_x\, dZ_{xt}, \tag{11}$$

where μ_x is the long run mean of x, κ_x controls the speed of mean reversion, the volatility parameter σ_x is assumed be a constant, and Z_{xt} is a second standard Brownian motion. The correlation of the two Brownian motion processes is given by

$$E[dZ\,dZ_x] = \rho\,dt. \tag{12}$$

The jump process is assumed to be uncorrelated with either of the Brownian process. The predictability of the investment opportunities is captured by the correlation between the two diffusion processes.

Implicit in the set-up is the assumption that we cannot predict the occurrence of jumps. The movement of the state variable gives no more information on either the probability or the magnitudes of the jumps than those we know unconditionally. We can regard these jumps as big, unpredictable events or catastrophes. The state variable only provides information on the diffusion risk. A set-up of this kind focuses our attention on the effects of "normal-time" predictability and the effects of unpredictable jumps on the portfolio allocation.

Such a set-up, on paper, would essentially separate the effects of jumps from that of predictability. This, however, is not true. The reason lies in the key observation of the paper: the effect of jumps depends upon the overall position the investor takes in the risky asset. Therefore, even with the independence assumption between jumps and the state variable, the predictability effect is still intertwined with the jump effect due to its direct impact on the investor's overall position.

Let W_t denote the investor's current wealth and θ_t denote her fraction of wealth allocated to the risky asset. Assume that there is no intermediate consumption or labor income, the investor's wealth dynamics can be written as,

$$dW_t = r_f W_t\,dt + \theta_t W_t[\sigma x_t\,dt + \sigma\,dZ_t + (e^q - 1)\,dQ(\lambda)]. \tag{13}$$

The investor, at time t, maximizes her terminal wealth over finite time horizon $\tau = T - t$ subject to the wealth process (13) and the risk premium process (11):

$$J(W, x, \tau) = \max_{\theta_t} E_t[e^{-r_f\tau} U(W_T)].$$

Following standard procedures, e.g., Merton (1969, 1971), we obtain the Hamilton-Bellman-Jacobi equation:

$$0 = \max_{\theta_t}\Big\{ - J_\tau + \lambda E_t[J(W', x, \tau) - J(W, x, \tau)] + J_W r_f W_t + J_W\theta_t\sigma x_t W_t \\ + \frac{1}{2}J_{WW}\theta_t^2\sigma^2 W_t^2 - J_x\kappa_x(x - \mu_x) + \frac{1}{2}J_{xx}\sigma_x^2 + \theta_t W_t J_{Wx}\sigma\sigma_x\rho\Big\},$$

where $W' = W_t[1 + \theta_t(e^q - 1)]$ is the wealth level conditional on one jump occurring. In the special case when the jump represents a complete default or bankruptcy, $q = -\infty$, the investor's wealth reduces to $W' = W(t)[1 - \theta(t)]$. Assuming the investor's utility function satisfies the Inada conditions: $U'(0) = \infty$, and $U'(\infty) = 0$, then the investor would never invest fully in the risky asset, $\theta_t < 100\%$ for all t, to guarantee the positivity of her wealth.

The optimal portfolio decision is obtained from the first order condition:

$$\theta^*(W, x, t) = \left(\frac{-J_W}{J_{WW}W(t)}\right)\frac{x_t}{\sigma} + \left(\frac{-J_{Wx}}{J_{WW}W_t}\right)\frac{\sigma_x\rho}{\sigma} + \frac{\lambda E_t[J_{W'}(W', x, \tau)(e^q - 1)]}{-J_{WW}W_t\sigma^2}. \quad (14)$$

It is comprised of three parts. The first part is the myopic demand due to the risk premium x_t. The second part is the intertemporal hedging demand due to the predictability of the investment opportunities, as captured by the correlation ρ. The last part is induced by the Poisson jump process in the asset price movement. We label it as the *jump demand*. Note that equation (14) is actually an implicit function of the portfolio decision θ^* since W' in the jump demand contains θ^*. Indeed, it illustrates the key point that the magnitude of the jump demand depends upon the overall position of the investor in the risky asset. It is this interdependence that essentially prevents us from obtaining an analytical solution to the problem. Nevertheless, assuming CRRA utility, we are able to come to an approximate semi-analytical form that facilitates both the comparative analysis and the calibration. In particular, the optimal expected utility has the following approximate functional form

$$J(W, x, \tau) \doteq \Phi(x, \tau)U(e^{r\tau}W), \quad (15)$$

where

$$\Phi(x, \tau) = \exp(A(\tau) + B(\tau)x + C(\tau)x^2/2)$$

with the boundary condition: $A(0) = B(0) = C(0) = 0$. $A(\tau)$, $B(\tau)$, and $C(\tau)$ can be solved from the following three first-order nonlinear ordinary differential equations (ODE):

$$\begin{aligned}
\frac{dC}{d\tau} &= aC^2 + bC + c;\\
\frac{dB}{d\tau} &= aBC + \frac{b}{2}B + \kappa_x\mu_x C;\\
\frac{dA}{d\tau} &= \frac{a}{2}B^2 + \frac{1}{2}\sigma_x^2 C + \kappa_x\mu_x B + D_t,
\end{aligned} \quad (16)$$

with $a = \sigma_x^2(1 - c\,\rho^2)$, $b = 2(-\kappa_x + c\,\rho\sigma_x)$, and $c = (1-\alpha)/\alpha$. The approximation is due to the term D_t, which is given in Appendix B. It is related to the effect of jumps on the (marginal) indirect utility and hence depends upon the optimal allocation decision. Since neither $B(\tau)$ nor $C(\tau)$ depends on D_t, its effect on the optimal decision rule is minimal. Given (15), the optimal portfolio decision becomes

$$\theta^*(x, t, \tau) \doteq \frac{x_t\sigma + \lambda\hat{g}_t + \rho\sigma\sigma_x[C(\tau)x_t + B(\tau)]}{\alpha\sigma^2}, \quad (17)$$

where $\hat{g}_t = E_t[1 + \theta^*(x, t)(e^q - 1)]^{-\alpha}(e^q - 1)$, capturing the marginal utility of wealth change conditional on one jump occurring. Note that $\hat{g}_t$ is also a function of the optimal allocation $\theta^*(x,t)$. So equation (17) is actually an implicit function of $\theta^*(x,t)$. Nevertheless, it greatly facilitates our comparative analysis. Calibration to real time data is also straightforward: Starting with $\theta = 0$, the optimal allocation can be obtained rapidly with a few iteration on (17).

$B(\tau)$ and $C(\tau)$ in the optimal portfolio allocation can be solved from the above ODEs. When the investor has log utility: $\alpha = 1$, both $B(\tau)$ and $C(\tau)$ are equal to zero, the intertemporal hedging demand thus becomes zero. When $\alpha > 1$, $a > 0$ and $c < 0$, the discriminant $b^2 - 4ac > b^2 > 0$. Denote $\eta = \sqrt{b^2 - 4ac}$, we have

$$C(\tau) = \frac{2c(1-e^{-\eta\tau})}{2\eta - (\eta + b)(1-e^{-\eta\tau})};$$
$$B(\tau) = \frac{4c\kappa_x\mu_x(1-e^{-\eta\tau/2})^2}{\eta[2\eta - (\eta + b)(1-e^{-\eta\tau})]}.$$

See Appendix B for the derivation. Here we focus on one single risky asset and one state variable. That enables us to solve for the allocation decision parameters $B(\tau)$ and $C(\tau)$ analytically and hence do the following comparative analysis. The framework, however, can be readily extended to a setting with multiple assets and multiple state variables. Similar ODEs in matrix notation will be obtained. In general, we cannot solve for the parameters in analytical form anymore; nevertheless, ODEs can be solved easily numerically.

PROPOSITION 1. *Under the economy specified in this section, the optimal fraction of wealth invested in the risky asset is approximately given by the implicit equation:*

$$\theta(x,t,\tau) \doteq \frac{x_t}{\alpha\sigma} + \frac{\lambda\hat{g}_t}{\alpha\sigma^2} + \frac{\rho\sigma_x[C(\tau)x_t + B(\tau)]}{\alpha\sigma},$$

where

$$\hat{g}_t = E_t[1+\theta(x,t)(e^q - 1)]^{-\alpha}(e^q - 1);$$
$$C(\tau) = \frac{2c(1-e^{-\eta\tau})}{2\eta - (\eta - b)(1-e^{-\eta\tau})};$$
$$B(\tau) = \frac{4c\kappa_x\mu_x(1-e^{-\eta\tau/2})^2}{\eta[2\eta - (\eta + b)(1-e^{-\eta\tau})]}.$$

A key property for the coefficients $B(\tau)$ and $C(\tau)$ is summarized in the following Lemma:

LEMMA 1. *When $\alpha \geq 1$, both $B(\tau)$ and $C(\tau)$ are nonpositive, so are their partial derivatives to τ:*

$$B(\tau), B'(\tau), C(\tau), C'(\tau) \leq 0 \quad \textit{if } \alpha \geq 1.$$

The equality holds when the investor has log utility: $\alpha = 1$.

See Appendix C for the proof. From the lemma, we obtain an important result regarding the intertemporal hedging demand,

PROPOSITION 2. *For moderately risk averse investors ($\alpha > 1$), the intertemporal hedging demand is positive and increases with the investment horizon τ if and only if the risk premium is negatively correlated with the return process.*

The proof follows directly from Lemma 1. Intuitively, a negative correlation implies that when the asset return, and therefore the investor's wealth, experiences a negative shock, the diffusion risk premium, x_t, is more likely to have a positive shock. The negative shock to the asset price is therefore partially compensated by the positive shock to the risk premium. In essence, the negative correlation provides an insurance mechanism for the investor, who thus regards the risky asset as "less" risky and decides to invest more in it. The opposite is the case when the correlation is positive.

The effect of the correlation ρ on the "riskiness" of the asset can be seen even more clearly through the following moment analysis. With constant volatility σ and stochastic risk premium x_t in (11), the conditional cumulants for the return $r_{t+\tau} = \ln(P_{t+\tau}/P_t)$ over investment horizon τ are

$$\begin{aligned}
\kappa_1 &= \left(\mu_r - \frac{1}{2}\sigma^2 - \lambda(g - \mu_q)\right)\tau + (\mu(t) - \mu_r)\frac{1 - e^{-\kappa_x \tau}}{\kappa_x};\\
\kappa_2 &= N\sigma^2 + \lambda\tau\left(\mu_q^2 + \sigma_q^2\right);\\
\kappa_3 &= \lambda\mu_q\left(\mu_q^2 + 3\sigma_q^2\right)\tau;\\
\kappa_4 &= \lambda\left(\mu_q^4 + 6\mu_q^2\sigma_q^2 + 3\sigma_q^4\right)\tau,
\end{aligned}$$

where $\mu_r = r_f + \lambda g + \mu_x\sigma$ is the long-run mean of μ_t, and

$$N = \tau\left(1 + \frac{2\rho\sigma_x}{\kappa_x} + \frac{\sigma_x^2}{\kappa_x^2}\right) - (1 - e^{-\kappa_x\tau})\left(\frac{2\rho\sigma_x}{\kappa_x^2} + \frac{\sigma_x^2}{\kappa_x^3}\right) - (1 - e^{-\kappa_x\tau})^2\frac{\sigma_x^2}{2\kappa_x^3}. \tag{18}$$

Refer to Appendix A for the derivation. We see that the third and fourth cumulants here are the same as in the case when the investment environment is constant. They are solely generated by the jump process. However, the stochastic drift does have an impact on the conditional variance of the asset return. The impact is summarized in the following remark,

Remark 1. The incorporation of stochastic drift in the asset return process in general increases the conditional variance of the log return $r_{t+\tau} = \ln P_{t+\tau}/P_t$. This effect is enhanced when the asset return process and the stochastic drift are positively correlated, but is mitigated in the presence of a negative correlation. When the correlation is so negative such that

$$\rho < -\sigma_x/\kappa_x,$$

the variance of the asset return will actually be smaller than that in the constant drift case and will decrease with the horizon τ.

The proof is given in Appendix C. The remark says that the stochastic drift in general adds additional uncertainty and thus increases the conditional variance. A positive correlation enhances such an effect while a negative correlation mitigates it. It is exactly due to this property of the conditional variance that the intertemporal hedging demand depends crucially on the correlation. A negative correlation provides an insurance mechanism for the investor. It reduces uncertainty by mitigating the variance increase. As a result, a positive hedging demand is induced by such a negative correlation. The opposite is true for $\rho > 0$.

The effect of the investment horizon also comes through the intertemporal hedging demand. According to Lemma 1, for $\alpha > 1$, $B(\tau)$ and $C(\tau)$ start from zero and decrease monotonically to their steady state value, as τ approaches infinity,

$$C(\infty) = \frac{2c}{\eta + b}; \quad B(\infty) = \frac{4c\kappa_x \mu_x}{\eta(\eta - b)}.$$

The magnitude of the intertemporal hedging demand therefore increases monotonically with the investor's investment horizon and eventually levels off to the steady state value.

The above analysis is based on the assumption that the investor is moderately risk averse: $\alpha > 1$. When the investor is less risk averse: $\alpha < 1$, the sign of the discriminant is not clear. When the discriminant is negative, redefine $\eta = \sqrt{4ac - b^2}$, we obtain a tangent solution:

$$C(\tau) = \frac{\eta}{2a} \tan\left(\frac{\eta\tau}{2} + \arctan\left(\frac{b}{\eta}\right)\right).$$

However, the tangent solution for $C(\tau)$ explodes as the investment horizon τ goes to infinity. We focus our analysis on the stationary case when the investor is moderately risk averse, i.e., $\alpha > 1$.

Also note from (17) that, although the skewness and kurtosis both decrease with investment horizon τ, the jump demand, as captured by $\hat{g}_t$, does not decrease with the investment horizon. This confirms with our analysis in Section 3. Nevertheless, under a stochastic investment environment, the impact of jumps will be affected by investment horizon through its dependence on the optimal overall allocation θ^*. A horizon-dependent hedging demand makes overall allocation θ^*, and therefore the *jump demand,* horizon-dependent. In what follows, we investigate how jump alters the portfolio decision and how the intertemporal hedging demand interacts with it.

5. Impact of jumps and predictability

We analyze the impacts of jumps on both the myopic demand and the intertemporal hedging demand, as well as their interaction with one another. Note that in the set-up of the model, the mean return to the risky asset μ_t is independent of the jump presence. Through the analysis, when we vary the jump intensity and jump magnitude distribution we adjust the diffusion drift simultaneously by λg such that the mean return remains μ_t. Later we also fix the total variance of the asset return so that we can focus on the pure effects of higher moments generated by jumps.

To begin with, we define

$$V_t(\theta^*) \equiv \theta^*(x,t) - \frac{x_t\sigma + \lambda\hat{g}_t}{\alpha\sigma^2} - \frac{\rho\sigma_x[C(\tau)x_t + B(\tau)]}{\alpha\sigma}$$
$$= \theta^*(x,t) - \frac{\mu_t - r_f + \lambda(\hat{g}_t - g)}{\alpha\sigma^2} - \frac{\rho\sigma_x[C(\tau)(\mu_t - r_f - \lambda g) + B(\tau)\sigma]}{\alpha\sigma^2}.$$

We can solve $V(\theta^*) = 0$ to get the optimal allocation. Applying the Implicit Function Theorem, we have the partial derivative of the optimal allocation with respect to the jump intensity λ:

$$\frac{\partial\theta^*}{\partial\lambda} \equiv \frac{\partial V_t/\partial\lambda}{\partial V_t/\partial\theta^*} = \frac{(\hat{g}_t - g)}{\alpha\sigma^2 - \lambda\partial\hat{g}_t/\partial\theta} - \frac{\rho\sigma_x C(\tau)g}{\alpha\sigma^2 - \lambda\partial\hat{g}_t/\partial\theta}, \tag{19}$$

The first term in equation (19) defines the effect of the probability of a jump on the myopic decision while the second term defines its effect on the intertemporal hedging demand. The following Lemma summarizes the property of $\hat{g}_t$:

LEMMA 2. *$\hat{g}_t$ equals $g = E(e^q - 1)$ when the investor is risk-neutral. For risk averse investors ($\alpha > 0$), (i) $\partial\hat{g}_t/\partial\theta < 0$; (ii) both $\hat{g}_t - g$ and $\partial(\hat{g}_t - g)/\partial\alpha$ have opposite signs to that of the allocation weight θ.*

Refer to Appendix C for the proof. Since $\partial\hat{g}_t/\partial\theta < 0$, the denominator in (19) is positive. The effects of jumps depend only on the numerator.

A. Effect on the myopic demand

As seen from (19), the effect of a jump on the myopic demand is determined by the sign of $(\hat{g}_t - g)$, whose properties are summarized in Lemma 2.

PROPOSITION 3. *For risk averse investors ($\alpha > 0$), the risk of a jump occurring, regardless of the direction of the jump, reduces the investor's myopic position (long or short) in the risky asset. This effect becomes more prominent when the investor is more risk averse.*

The proof is self-evident from Lemma 2.

Figure 1 illustrates the effects of jumps on the investor's myopic demand by setting the correlation ρ equal to zero. The top graph in Figure 1 shows that the presence of jumps, regardless of its direction (upward or downward), reduces the absolute position of the investor's myopic demand (long or short). When the jump magnitude becomes extremely

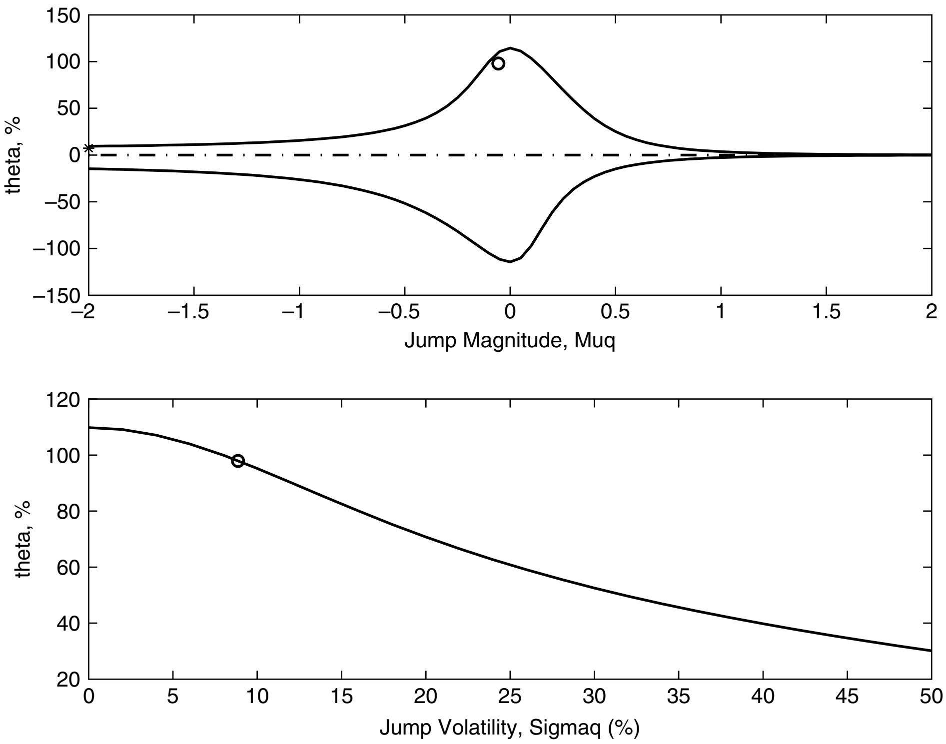


Figure 1. Effect of jump risk on the investor's myopic demand. (Lines depict the impacts of mean jump magnitudes (μ_q, top) and jump volatility (σ_q, bottom) on the allocation weight (θ, %) to the risky asset assuming constant investment opportunities. Risk aversion is set to $\alpha = 4$. The riskfree asset return $r_f = 5\%$. All other relevant parameters are chosen from panel A of Table 2: $\mu = 0.1264$, $\sigma = 0.1296$, $\lambda = 0.2029$. For the top graph, we set $\sigma_q = 0$ while varying μ_q. To generate a symmetric short position in the risky asset, we flip the sign of the two returns μ and r_f. For the bottom graph, we set $\mu_q = -0.0560$ while varying σ_q. The two circles "o" on the lines represent the optimal myopic allocation to S&P 500 based on parameters from panel A of Table 2. The star "$*$" denotes the allocation when jump represents complete default: $q = -\infty$.)

large, the investor will simply avoid getting involved in this risky asset: $\theta \to 0$. In the special case when the jump represents a complete default (asset price falls to zero, $\mu_q = -\infty$), holding everything else the same, the investor reduces her position in the risky asset from 114.4% in the absence of jumps ($\mu_q = 0$) to 7.27%. Therefore, the potential of a default on an asset, even with modest probability, will greatly reduce its attraction to the investor.

The bottom graph in Figure 1 shows that, holding everything else equal, increasing jump volatility also reduces the investor's demand for the risky asset. Moment analysis demonstrates that increasing jump volatility (σ_q) not only increases kurtosis and negative skewness (when $\mu_q < 0$) but also increases the variance of the asset return. It therefore represents another source of risk to the investor. As a result, when the volatility of the jump (σ_q) increases from 0 to 50%, the investor reduces her position in the asset from 109.8% to 30.1%.

It is understandable that investors avoid taking large positions on risky assets when the asset price has a probability of jumping down to zero in case of default, but it is somewhat counter-intuitive that the direction of the jump has no effect on the impact. The top graph in Figure 1 says that even when the magnitude of the jump is certain and it is an upward jump, the investor still tries to avoid it in their myopic demand. If the upward jump is infinitely large, $q = \infty$, the asset price goes infinitely high in case of the jump, yet the investor refuse to allocate anything to this asset ($\theta^* = 0$).

The intuition behind this "counter-intuitive" result is that a positive jump increases the mean return to the asset by $\lambda g \tau$. Since we are keeping the total mean drift of the asset return $\mu(t)$ independent of the jump by subtracting the drift term of the diffusion process by λg, the reduction in demand due to this mean adjustment more than offsets the demand increase due to a positive jump, i.e., $|g| > |\hat{g}_t|$ when $g = E(e^q - 1)$ is positive. In the case of a downward jump ($g < 0$), the drift term is adjusted upward; however, the upward adjustment is not enough to compensate the downward effect of the jump on the myopic demand, i.e., $|g| < |\hat{g}_t|$ when g is negative. Therefore, *the net effect of a jump, regardless of its direction, always reduces the myopic demand*. This asymmetry of the jump effect is introduced by the investor's moderate risk aversion α. Only when the investor is risk neutral: $\alpha = 0$, do the two effects cancel out, $\hat{g}_t = g$.

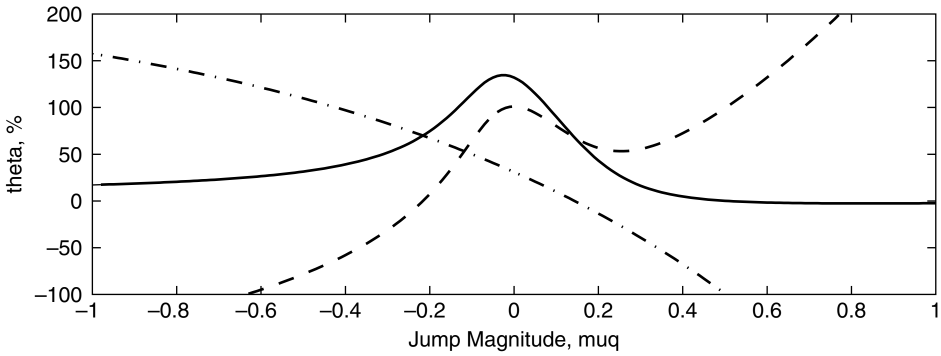
B. Effect on the intertemporal hedging demand

As has been well-known and re-confirmed by our analytical solution to the dynamic problem, the intertemporal hedging demand is generated by predictability. Since we assume that only diffusion risk premium is predictable but jump risk is not, jump shall have no direct impact on the hedging demand. Of course, correlation can be built between jumps (either its magnitude q or its intensity λ) in the asset return process and the state variable. An intertemporal hedging demand will then be incurred due to the "predictability" of the jump risk. The direction of the demand will be analogous to the hedging demand due to the diffusion risk. That is, under the mechanism of hedging, positive correlation incurs a negative demand and vice versa.

But even more interesting is the result that, even in the absence of predictability for jumps, the intertemporal hedging demand and the impacts of jump risk are intrinsically linked to each other. A minor link is that jumps affect the hedging demand through the diffusion drift adjustment term λg. As illustrated in equation (19), a positive mean jump (g) decreases the absolute magnitude of the intertemporal hedging demand while a negative mean jump (g) increases it. A more important link between the two, however, is that the impact of the jump risk depends upon the investor's overall position in the asset. The intertemporal hedging demand varies the impact of jump risk through its contribution to the overall position.

C. The overall impact

Since the magnitude and direction of the jump impact depend directly upon the investor's overall position in the risky asset, the myopic demand, the intertemporal hedging demand,



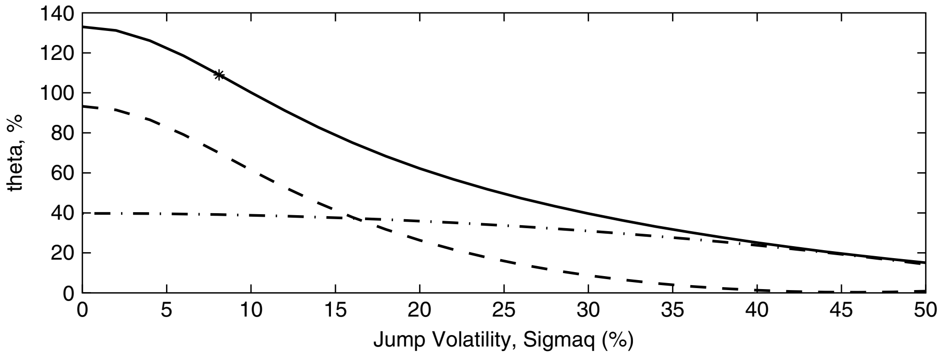


Figure 2. Effect of jump magnitude on portfolio allocation. (Solid lines are the optimal allocation (θ, in percentage) to the risky asset with an investment horizon of 10 year ($\tau = 10$), dashed lines are its myopic component, while dash-dotted lines are its intertemporal hedging demand component. The top panel depicts its dependence on the mean jump magnitude μ_q by setting the jump volatility σ_q to zero. The bottom panel depicts its dependence on jump volatility σ_q while keeping $\mu_q = -0.0450$. Investor's relative risk aversion is $\alpha = 4$. Riskfree rate is $r_f = 5\%$. All other parameters are from Table 4: $\rho = -0.9424$, $\kappa_x = 0.0462$, $\sigma_x = 0.0309$, $\mu_x = 0.7201$, $\sigma = 0.1409$, and $\lambda = 0.5201$. The current state is set to the mean: $x_t = \mu_x$. The star "$*$" in the bottom panel represents the case for S&P 500 Index (Table 4) with $\sigma_q = 0.0831$.)

and the effect of the jump risk are all intertwined. It is therefore necessary to analyze the effect of jump risk on the investor's overall position in the risky asset under different dynamic scenarios. In Figure 2, we use parameters estimated from S&P 500 index returns (Table 4) and demonstrate how both the myopic demand and the intertemporal hedging demand are affected by the jump risk. The solid lines depict the overall optimal allocation θ to the risky asset. The dashed lines are the myopic component while the dash-dotted lines are the intertemporal hedging demand component.

On the top panel of Figure 2, we set the variance of the jump σ_q to zero and study the effect of the mean jump magnitude μ_q. Given a negative correlation between the asset return and the risk premium process ($\rho = -0.9242$), the intertemporal hedging demand is positive in absence of jumps ($\mu_q = 0$). This hedging demand decreases with the mean jump

Table 4. Calibrating S&P 500 index to the dynamic jump-diffusion model

Parameter	Estimates	Std. Errors
A. VAR(1) Estimates:		
$B(1, 1)$	0.0368	0.0272
$B(1, 2)$	0.0082	0.0080
$B(2, 1)$	−0.0145	0.0287
$B(2, 1)$	0.9962	0.0085
σ_1	0.0417	0.0026
σ_2	0.0444	0.0044
ρ_{12}	−0.9424	0.0822
λ	0.5201	0.3466
μ_q	−0.0450	0.0359
σ_q	0.0831	0.0247
B. Implied Parameters:		
σ	0.1409	0.0088
μ_x	0.7201	0.0725
κ_x	0.0462	0.1018
σ_x	0.0309	0.0295
ρ	−0.9424	0.0822

Notes: Entries in Panel A are just-identified generalized methods of moments estimates and standard errors of the following VAR(1) system:

$$\begin{bmatrix} r_{t+1} \\ d_{t+1} \end{bmatrix} = B \begin{bmatrix} 1 \\ d_t \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon_{1,t+1} \\ \sigma_2 \varepsilon_{2,t+1} \end{bmatrix},$$

where r_{t+1} is the log return on S&P 500 Index (including all distributions), d_t is the log dividend-price ratio defined as the log of the ratio of the cumulative dividend for the last 12 months (from $t-11$ to t) to the current price, B is a 2×2 matrix. The correlation between the residual is $\rho_{12} = corr(\varepsilon_1, \varepsilon_2)$. The non-normality of the residual ε_{t+1} is modeled by a Poisson jump $q\,dQ(\lambda)$, with normally distributed jump magnitude $q \sim N(\mu_q, \sigma_q)$. Moment conditions for the VAR(1) system is self-obvious and we choose the jump parameters to match the 3rd, 4th, and 5th cumulants of the residual $e_{t+1} = r_{t+1} - B_{11} - B_{12} d_t$. Standard errors are computed following Newey and West (1987) with an optimal lag of 2. Panel B are implied (from Panel A) parameters for the dynamic jump-diffusion model specified in (1), (11), and (12). Standard errors for Panel B are computed using the delta method. The data are monthly, from July 1962 to December, 1997 (426 observations).

magnitude due to the diffusion drift adjustment. The change in the hedging demand also has an indirect impact on the myopic demand. As stated in Lemma 2, $\hat{g}_t(\theta)$ is a decreasing function of θ. As μ_q decreases, θ increases by way of the hedging demand, $\hat{g}_t(\theta)$ therefore decreases and $(\hat{g}_t - g)$ becomes more negative: The negative impact of the jumps on the myopic demand becomes more pronounced as μ_q becomes more negative. The opposite is true when μ_q becomes more positive. Therefore, *as the jump magnitude μ_q becomes larger in either direction, both the myopic demand and the intertemporal hedging demand become very large, yet with opposite signs.* The net result is that the investor reduces her position in the risky asset as the jump magnitude becomes large in either direction. Under the parameters of the graph, in absence of jumps ($\mu_q = 0$), the optimal allocation to the risky asset is $\theta = 132.06\%$: the investor borrow money to invest in the risky asset. However, as the mean jump magnitude increases to $\mu_q = 1$, the investor only has a small short position in the risky asset: $\theta = -2.44\%$. When the mean jump magnitude is big and negative: $\mu_q = -1$, the investor has a long position, but also very small: $\theta = 17.17\%$. The bottom line is, *whenever the investor expects large jumps of either direction, she reduces her position in the risky asset drastically*.

The bottom panel of Figure 2 demonstrates the effect of jump volatility (σ_q) on the optimal allocation. Similar to Figure 1, the investor's myopic demand decreases with increasing jump volatility. In addition, with negative correlation, increasing jump volatility also reduces the hedging demand. Since the impacts of jump volatility on both demands are negative, the overall demand decreases with increasing jump volatility: Under the parameter values used in this graph, the overall demand for the risky asset decreases from 133% to 15% as jump volatility increases from 0 to 50%.

In summary, jump risk affects not only the myopic demand but also the intertemporal hedging demand even if the jump risk is not predictable. Overall, an investor reduces her position significantly in the risky asset whenever she expects (i) a high probability of having jumps, (ii) large jump magnitudes of either direction, and/or (iii) great uncertainty regarding the jump magnitudes.

D. *Control for the volatility change*

Preliminary moment analysis illustrates that the presence of jump risk reduces investors' overall position in the risky asset not only because of the addition of negative skewness and kurtosis, but also because of the increase in variance for the asset return. In order to see the net effects of higher moments, this section repeats the comparative analysis with fixed overall volatility.

Let us define the asset return process without jump as

$$\frac{dP_t}{P_t} = \mu_t\, dt + \hat{\sigma}(\tau)\, dZ_t,$$

where the volatility parameter is adjusted to match that of the jump-diffusion process:

$$\hat{\sigma}(\tau) = \sqrt{\sigma^2 + \lambda\tau\left(\mu_q^2 + \sigma_q^2\right)/N}.$$

Recall N, as defined in (18), is given by

$$N = \tau\left(1 + \frac{2\rho\sigma_x}{\kappa_x} + \frac{\sigma_x^2}{\kappa_x^2}\right) - (1 - e^{-\kappa_x\tau})\left(\frac{2\rho\sigma_x}{\kappa_x^2} + \frac{\sigma_x^2}{\kappa_x^3}\right) - (1 - e^{-\kappa_x\tau})^2\frac{\sigma_x^2}{2\kappa_x^3}.$$

The conditional variance for return $\ln P_{t+\tau}/P_t$ over time interval τ is therefore

$$\hat{\kappa}_2(\tau) = \hat{\sigma}(\tau)^2 N = N\sigma^2 + \lambda\tau\left(\mu_q^2 + \sigma_q^2\right),$$

which matches exactly the conditional variance of the return with jumps and with diffusion volatility parameter equal to σ. Note that the adjusted volatility $\hat{\sigma}(\tau)$, though not stochastic, is time-varying. It depends on the investment horizon. This effect comes from the time-nonlinearity of the conditional volatility in asset return processes with stochastic drift.

With this set-up, we can work through the portfolio decision problem analogously and obtain the optimal portfolio decision for the jump-free return process,

$$\theta_s^*(x, t, \tau) = \frac{x_s(t)}{\alpha\hat{\sigma}(\tau)} + \frac{\rho\sigma_x[C(\tau)x_s(t) + B(\tau)]}{\alpha\hat{\sigma}(\tau)}, \tag{20}$$

with the risk premium redefined as

$$x_s(t) = \frac{\mu_t - r_f}{\hat{\sigma}(\tau)}.$$

$C(\tau)$ and $B(\tau)$ are the same as before. The subscript s denotes a "smooth" jump-free diffusion. In the following calibration exercises, for each case, the model is calibrated to the data twice, with and without considering jumps. As a result, both the mean and the variance of the asset returns are the same. The allocation difference captures the net impact of higher moments introduced by jumps.

6. Calibration exercises

We calibrate the model to the U.S. equity market, proxyed by S&P 500 Index. The index data are retrieved from the CRSP database (The Center for Research in Security Prices). The data period covers from July 3rd, 1962 to December 31st, 1997. Table 1 summarizes the statistical properties of the return data with different time aggregations. While the U.S. stock market is the most sophisticated and most liquid stock market around the world, we do observe, every now and then, very large, abrupt price movements in the market, most notably the October 1987 stock market crash and a similar event 10 years later. As a result, the returns on the index are highly non-normal. As shown in Table 1, daily return exhibit a excess kurtosis of 34.7 and a negative skewness of –1.31. Table 2 presents the parameter estimates for a jump-diffusion model, as specified in (1), on daily (Panel A) and monthly (Panel B) log returns of S&P 500 Index. The non-normalities in the return data are captured by the Poisson jump parameters. In this section, through the calibration exercise, we intend to obtain a quantitative idea of how big an impact such jumps have on the investor's portfolio decision.

For the exercise, we assume that an investor makes her portfolio choice between an index fund, which mimics the S&P 500 Index, and a riskfree asset, which has a constant continuously compounded return of $r_f = 5\%$. Throughout the exercise, we assume a relative risk aversion of $\alpha = 4$ for the investor. First, we consider the case when the investment opportunities are assumed constant, from which we investigate the impact of jumps on the investor's myopic demand for the index fund. Secondly, we use log dividend-price ratio to predict the fund's return and investigate, on average, how jump risk interacts with predictability in their impacts on the investor's dynamic portfolio decision. Lastly, we investigate the comparative dynamics: the interaction of the time-varying investment opportunities with the impacts of jumps over time.

A. *Constant investment opportunities*

Under constant investment opportunities, $\mu_t = \mu$, the optimal allocation to S&P 500 Index reduces to the myopic decision,

$$\theta_m = \frac{\mu - r_f + \lambda(\hat{g}_t - g)}{\alpha\sigma^2}.$$

Based on parameters from Panel A of Table 2, the optimal allocation weight to the index fund is $\theta_m = 97.95\%$. If we ignore the presence of jump, the mean drift remains at $\mu = 12.64\%$, but the volatility is adjusted to $\hat{\sigma} = \sqrt{\sigma^2 + \lambda(\mu_q^2 + \sigma_q^2)} = 13.75\%$. The allocation weight is

$$\theta_{ms} = \frac{\mu - r_f}{\alpha\hat{\sigma}^2} = 101\%\,.$$

Compare θ_m and θ_{ms}, we see a reduction in allocation to the index fund by a little more than 3%, comparable to the results in Table 3, which are obtained from a one-period approximation model in Section 3.

Applying estimates from monthly data (Panel B of Table 2) yields similar results. The optimal allocation considering jump risk is $\theta_m = 90.48\%$, that ignoring jump risk is $\theta_{ms} = 93.75\%$. Again, considering jump risk reduces the demand for the index fund by a little more than 3%.

The calibration exercise in the static setting tells us that investors regard jumps observed in the stock market as extra risk, in addition to those captured by the stock volatility, and thus reduce their allocation to the stock market accordingly. This result confirms to what we have obtained from the one-period approximation model in Section 3: both negative skewness and positive kurtosis imply additional sources of risk. Since the investment opportunities are assumed to be constant, the investment decision does not depend on the investment horizon; nor does the impact of jumps.

B. *Predictability of dividend yields*

A large stream of literature has documented the predictability of stock returns by variables such as lagged returns, dividend yields, term structure of interest rates, etc. Our dynamic portfolio decision model can accommodate predictability very nicely. In the model, we allow the expected risk premium for the asset return to be stochastic and correlated with the return innovation. We can therefore use the above variables documented in the literature to forecast the expected risk premium and then fit the risk premium x_t to an AR(1) process, a discrete-time counterpart of the Ornstein-Uhlenbeck process specified in (11).

For the exercise, we choose the log dividend-price ratio, d_t, as the forecasting variable. d_t is defined as the log of the ratio of the cumulative dividend over the past 12 months to the current stock price. Campbell and Shiller (1988), Fama and French (1988), and Hodrick (1992), among others, have found that the log dividend-price ratio is a good predictor of the stock return. Similar to Campbell and Viceira (1999), we estimate the following restricted VAR(1) model on monthly data:

$$\begin{bmatrix} r_{t+1} \\ d_{t+1} \end{bmatrix} = B \begin{bmatrix} 1 \\ d_t \end{bmatrix} + \begin{bmatrix} \sigma_1 \varepsilon_{1,t+1} \\ \sigma_2 \varepsilon_{2,t+1} \end{bmatrix}, \tag{21}$$

where r_{t+1} is the log return on S&P 500 Index (including all distributions), B is a 2×2 matrix, and the variance-covariance matrix of the residuals is denoted as

$$\Omega = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Jumps in the asset return process $q\,dQ(\lambda)$ is captured by higher moments (nonnormality) in the residual $\varepsilon_{1,t+1}$.

In such a set-up, the expected return is forecasted by the log dividend-price ratio: $\mu_t = B_{11} + B_{12} d_t$, where B_{ij} denotes the ijth element of the B matrix. The expected return is then fitted to an AR(1) process through the AR(1) specification of the log dividend-price ratio. Table 4 reports the just-identified GMM estimation results of the system. Moment conditions for the VAR(1) system is self-obvious. It essentially amounts to an equation-by-equation ordinary least square regression. The jump parameters $(\lambda, \mu_q, \sigma_q)$ are chosen to match the non-normality (the 3rd, 4th, and 5th cumulants) of the regression residual: $e_{1,t+1} = r_{t+1} - B_{11} - B_{12} d_t$. We obtain the first stage estimates for the VAR(1) system via linear regression and estimates for the jump parameters via the fast-converging sequence described in Section 2. The weighting matrix for the second stage GMM estimation is constructed by the Newey-West method with 2 lags and de-meaned moment conditions. The number of lags is chosen optimally following Andrews (1991) assuming a VAR(1) specification for the moment conditions. The estimates, together with the standard errors, indicate two potential problems of the restricted VAR(1) system in (21). Firstly, the predicting power of the log dividend-price ratio is rather weak. The estimate on $B(1, 2)$ is not significantly different from zero and the R^2 for the ordinary least square regression is merely 0.0014. Secondly, the log dividend-price ratio is close to a unit root process. The focus of the paper, however, is neither on whether stock returns are predictable nor on how to predict stock returns, but instead on the impact of jump risk on the dynamic portfolio decision. We therefore do not pursue these issues further, but rather take the VAR(1) system in (21) as given, regard the estimates in Table 4 as true values for the parameters, and proceed to investigate the impact of jump risk under such a dynamic system.

The regression residual $\varepsilon_{1,t+1}$ has slightly higher skewness (in absolute value) and kurtosis. As a result, the estimated jump frequency ($\lambda = 0.5201$) implies a 40.55% chance of having one or more jumps each year, higher than those directly estimated from the return data (Table 2). As will be shown later, in presence of predictability, jumps also have a much bigger impact on investors' portfolio decision.

Parameters in (1), (11), and (12) that define our dynamic jump-diffusion model can be recovered from the VAR(1) system in (21) in the following sequence:

$$
\begin{aligned}
\rho &= \rho_{12};\\
\kappa_x &= -\frac{1}{\tau}\ln(B_{22});\\
\sigma_\mu &= \sigma\sigma_x = |B_{12}|\sigma_2/\tau;\\
\sigma &= \sqrt{\sigma_1^2/\tau - \lambda\left(\mu_q^2+\sigma_q^2\right)+\Theta_1} - \Theta_2;\\
\sigma_x &= \sigma_\mu/\sigma;\\
\mu_x &= \frac{\left[B_{11}+B_{12}B_{21}/(1-B_{22}) - r_f\tau + \frac{1}{2}\sigma^2\tau - \lambda\mu_q\tau\right]}{\sigma\tau}.
\end{aligned}
$$

with

$$
\begin{aligned}
\Theta_1 &= \frac{\rho^2\sigma_\mu^2}{\kappa_x^2}\frac{(\tau-z)^2}{\tau^2} - \frac{\sigma_\mu^2}{\tau\kappa_x^2}\left(\tau - z + \frac{1}{2}\kappa_x z^2\right);\\
\Theta_2 &= -\frac{\rho\sigma_\mu}{\kappa_x}\frac{(\tau-z)}{\tau};\\
z &= \frac{(1-e^{-\kappa_x\tau})}{\kappa_x}.
\end{aligned}
$$

Panel B of Table 4 reports these implied parameters. Standard errors are computed using the delta method. The instantaneous correlation between the log dividend-price ratio and the asset return is negative: $\rho = -0.9424$, which implies a positive intertemporal hedging demand.

B.1. Dynamic decisions at different horizons. With the estimates in Table 4, we compute the optimal allocation weight to the index fund at different investment horizons. With time-varying investment opportunities, the allocation weight depends upon the current state x_t. We first investigate the average case by setting the current state at its mean: $x_t = \mu_x$. For comparison, we also compute the allocation weight ignoring the jump risk. For that purpose, we assume that the residuals of the VAR(1) system in (21) are normally distributed. The implied parameters σ, σ_x, and μ_x are then adjusted as follows:

$$\sigma = \sqrt{\sigma_1^2/\tau + \Theta_2} - \Theta_2;$$
$$\sigma_x = \sigma_\mu/\sigma;$$
$$\mu_x = \frac{\left[B_{11} + B_{12}B_{21}/(1 - B_{22}) - r_f\tau + \frac{1}{2}\sigma^2\tau\right]}{\sigma\tau}.$$

Table 5 reports the allocation results at different investment horizons. For each case (with and without jumps), we present the myopic demand (θ_m) and the intertemporal hedging demand (θ_h) along with the optimal overall allocation (θ). The subscript s denotes the case when jump risk is ignored. Within each category, we compute the percentage impact of the jump risk, defined by

$$\Delta\theta/\theta, \% = 100 \times \frac{\theta - \theta_s}{\theta_s}.$$

Table 5 illustrates a well-known result on predictability. With a negative instantaneous correlation between the index return and the state variable, the investor has a positive

Table 5. Optimal portfolio decisions with different investment horizons

Horizon	Myopic			Hedging			Overall		
Yrs	θ_{ms}	θ_m	$\Delta\theta_m/\theta_m$	θ_{hs}	θ_h	$\Delta\theta_h/\theta_h$	θ_s	θ	$\Delta\theta/\theta$
1	93.78	78.82	−15.96	2.22	3.09	39.67	96.00	81.91	−14.68
2	93.78	77.86	−16.98	4.83	6.74	39.60	98.61	84.60	−14.21
3	93.78	76.76	−18.15	7.78	10.86	39.53	101.57	87.63	−13.73
4	93.78	75.54	−19.45	11.02	15.37	39.46	104.81	90.92	−13.26
5	93.78	74.20	−20.88	14.50	20.22	39.39	108.29	94.42	−12.81
6	93.78	72.76	−22.42	18.18	25.33	39.31	111.96	98.08	−12.40
7	93.78	71.22	−24.06	22.02	30.66	39.22	115.80	101.87	−12.03
8	93.78	69.59	−25.80	25.99	36.16	39.13	119.77	105.75	−11.71
9	93.78	67.88	−27.62	30.06	41.80	39.03	123.84	109.67	−11.44
10	93.78	66.10	−29.52	34.21	47.53	38.93	127.99	113.62	−11.23

Notes: Entries are proportions (in percentage) of the investor's wealth invested in the index fund when the investor is facing different investment horizons. The investor's demand for the index fund is dissected into two parts: the myopic demand (θ_m) and the intertemporal hedging demand (θ_h). We compare two cases (1) the jump risk is taken into account and (2) the jump risk is ignored, denoted by a subscript s. $\Delta\theta/\theta$ captures the percentage difference between the two:

$$\Delta\theta/\theta = 100 \times \frac{\theta - \theta_s}{\theta_s}.$$

The allocations are computed based on estimates in Table 4. The state variable x_t is assumed to be at its mean value: $x_t = \mu_x$. Investor is assumed to have a relative risk aversion of $\alpha = 4$. The riskfree asset is assumed to have a return of $r_f = 5\%$.

intertemporal hedging demand that increases with the investment horizon. For an investor with a one-year horizon, the hedging demand is small: 3.09% with jump and 2.22% without jump. However, when the investment horizon increases to 10 years, the hedging demand increases to 47.53% with jump and 34.21% without jump. Similar findings are also documented by, among others, Barberis (1999) and Campbell and Viceira (1999), and are related to the argument of Siegel (1994) that long run investors should not try to time the stock market, but should buy and hold large equity positions because these positions involve little risk at long horizons.

What is new is the interaction between predictability and the impact of jump risk. As expected, jump risk reduces the myopic demand. But a more interesting result is that *the presence of a time-varying and predictable investment environment increases the jump impact on the myopic demand and, furthermore, makes it horizon-dependent*. While the percentage jump impact on the myopic demand is merely about 3% when the investment opportunities are constant (previous subsection), it is now 15.96% for a one-year horizon and increases to 29.52% for a ten year horizon. As a result, the myopic demand with jumps (θ_m) decreases with the investment horizon. With a one-year horizon, the myopic demand for the index fund is 78.82%. As the investment horizon increases to 10 years, the myopic demand is decreased to 66.10% due to the increasing impact of jump risk.

Jump risk also has a direct impact on the intertemporal hedging demand. Jump risk increases the hedging demand. While the percentage impact is about constant at less than 40% across different investment horizons, the absolute impact increases with the increase of the hedging demand and hence with investment horizon.

Overall, *considering jump risk reduces the investor's optimal allocation to the index fund.* Across all investment horizons, jump risk reduces the investor's overall demand by about 11–15%. Furthermore, *this impact becomes larger in presence of time-varying and predictable investment opportunities.* In the previous subsection, by focusing on the unconditional distribution under the assumption of constant investment environment, the jump reduces the demand for the index fund by only 3%. Similar results are obtained through the one-period approximation model in Section 3. However, when we explicitly take into account the time-varying investment environment, the impact of jump increases to 11–15%. Note that such interaction between predictability and jump impact is obtained under the independence assumption between the jump risk and the state variable. This interactive result is new to the literature.

B.2. Comparative dynamics. By setting the state variable to its mean, the above analysis can be regarded as a comparative static analysis on the investor's portfolio decision at an average time. This section focuses on the comparative dynamics, that is, how the impact of jump risk changes with the time-varying investment opportunities, as captured by the state variable x_t.

For the comparative dynamics, we assume that an investor comes to the stock market at the beginning of 1993 and expects to actively manage her portfolio thereafter in the sense that her portfolio will be rebalanced every month. The investor expects to consume her cumulative wealth in 5 years, i.e., at the end of 1997. Therefore, her objective will be to maximize her utility of her terminal wealth at the end of 1997. Each month, the investor rebalances her portfolio based on (i) her updated conditional information about the financial market and (ii) her decreasing investment horizon.

The investment decision is still between an index fund mimicking S&P 500 Index and a riskfree asset with a return $r_f = 5\%$. The investment environment is also allowed to be stochastic and predictable by the log dividend-price ratio, as specified in (21). At the beginning of 1993, the VAR(1) system in (21) will be estimated using historical data from July 1962 to December 1992 and the expected return on the index fund will be forecasted by $\mu_t = B_{11} + B_{12}d_t$. From then on, each month the VAR(1) system will be re-estimated using data from July 1962 to the most recent date and the forecast will be made correspondingly. We do not, however, update the jump parameters $(\lambda, \mu_q, \sigma_q)$ but instead use the full-sample estimates in Table 2 for two reasons: (1) Since we are assuming that the jump frequency λ and the distribution for jump magnitude q are constant over time, we want to use the same jump parameters all through the dynamic analysis. (2) Since jump parameters capture extreme events, in general a long sample is needed to obtain robust estimates.

Based on the forecast, the optimal portfolio decision (θ_s) are computed from (17). Along the way, we also compute the corresponding portfolio decision disregarding jump risk (θ_s) from (20) and compare the difference between the two decisions over the dynamic process of 5 years. With CRRA utility, the level of initial wealth or intermediate wealth do not affect the portfolio decision.

The results are summarized in Figure 3. The bottom graph depicts the movement of the forecasting variable, the log dividend-price ratio d_t. It reached a peak during 93–94 but has been decreasing ever since. Since the expected return increases with log dividend-price ratio $(B_{12} > 0)$, the optimal allocation to the index fund follows closely to the fluctuation of the log dividend-price ratio, as illustrated by the top graph in Figure 3. The impacts of the jumps are depicted in the middle graph, as captured by the difference between θ and θ_s, allocations with and without considering jump risk. As expected, the overall investment $\theta(t)$ in the index fund and the impacts of jump, $\Delta\theta(t)$ always move in opposite directions: Jump reduces the holding of the index fund whenever the position is long and increases the holding whenever the position is short. The impact is biggest when the investor is most deeply involved (long or short) in the stock market. For example, when the allocation to the index fund is about 30% between 93 and 94, the jump risk reduces the allocation by more than 6%. When the exposure to the stock market is minimal, so is the impact of jump risk, as is the case around 1997 when the log dividend-price ratio is small and when the investor's investment horizon is shrinking. Depending on the fluctuation of log dividend-price ratio, there are times when the investor actually takes a short position in the stock market. Then the impact of jump risk becomes positive, that is, to reduce this short position. Overall, *in the presence of jumps, the investor becomes more conservative with her position in the risky asset and her dynamic rebalancing of the portfolio becomes less dramatic.*

We use a very simple VAR(1) system to forecast the expected return (drift). It illustrate how our framework can readily incorporate stochastic investment environments. In addition to the log dividend-price ratio example, we can easily incorporate other well-documented forecasting variables to the system and investigate their dynamic interaction with the jump risk.

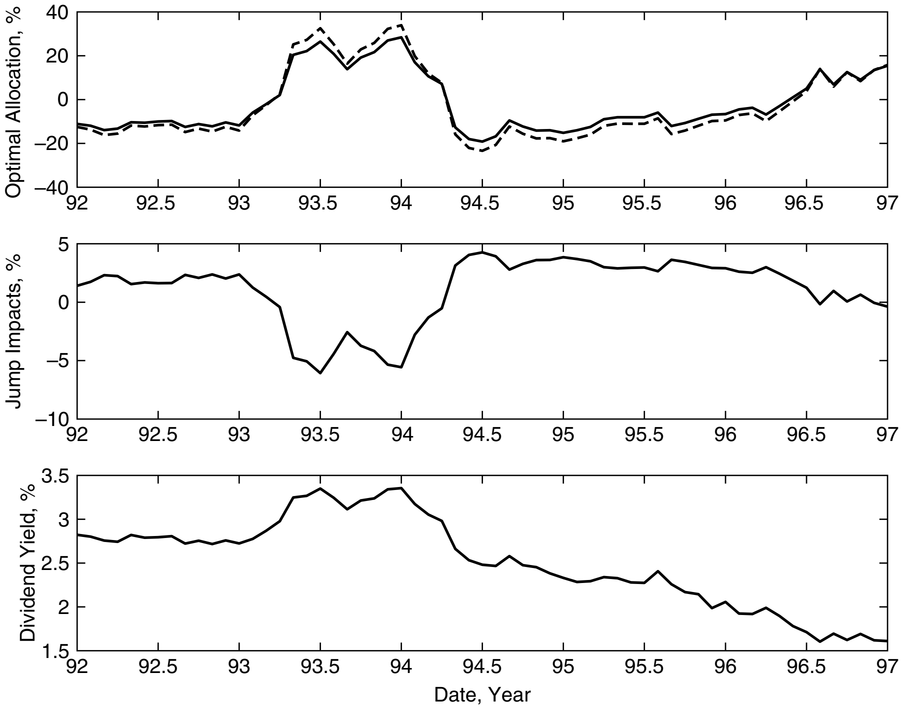


Figure 3. Dynamic asset allocation decisions: 1993–1997. (The top graph depicts the optimal allocation to the S&P 500 index fund over time for an investor who, with a relative risk aversion of $\alpha = 4$, a riskfree rate of $r_f = 5\%$, maximizes her expected utility of terminal wealth at the end of 1997. The solid line takes jump risk into consideration (θ) while the dashed line ignores it (θ_s). The middle graph depicts the impacts of jumps over time, as captured by $\Delta\theta = \theta - \theta_s$. The bottom graph illustrates the movement of the forecasting variable, the log dividend-price ratio d_t.)

As mentioned before, the U.S. stock market is probably the most liquid and sophisticated stock market in the world; yet we still observe large discontinuous movements in the stock index every now and then. As illustrated by the above calibration exercise, these discontinuous movements can have significant impacts on portfolio managers' dynamic decision. When we turn our eyes to individual stocks or to the international stock market, particularly the emerging markets, where even larger discontinuities happen at a much more frequent interval, the jump risk would presumably have a much bigger impact on the investment decision.

7. Final thoughts

Discontinuous movements, or jumps, in asset prices have important implications for the investors' dynamic portfolio decisions, particularly when the investment opportunities are time-varying. This paper provides a fairly general yet rather simple framework to analyze the impacts of jumps on the dynamic investment decision. The analyses shall bear even more significance when investing in the emerging markets where the discontinuity of asset price movements is all the more obvious and the predictability of asset returns all the more pronounced. See, for example, Bekaert, Erb, Harvey and Viskanta (1998) and Harvey(1995) for empirical characterizations of the emerging stock markets.

Incorporating jumps in the stock price movements not only alters the dynamic portfolio decision, but also has great implications on asset pricing. For example, Nietert (1997) argues that both firm-specific jumps and market jumps enter the fundamental pricing relation and hence contain market risk, which in turn implies that the market price of risk for even firm-specific jumps is not zero, as often assumed in option pricing literature (e.g., Merton 1976). The key reason is that, as we argued in the paper, the impact of jump risk depends on the overall position of the investor and hence is decisively intertwined with the hedging demand.

This paper focuses on the effect of jumps in a single risky asset, or a single portfolio. An important question for future research is how jumps in different risky assets correlate with each other and how higher moments diversify by forming portfolios? Contagion observed in financial crises would imply that jumps in asset prices, particularly negative jumps, are more correlated than diffusions. If this is true, the relative impact of jumps will not be able to be reduced, if not increased, through diversification.

Appendix A. Non-normality of asset returns in a jump-diffusion process

Jump-diffusion processes generate non-normality, as captured by non-zero skewness and kurtosis. The appendix derives the conditional characteristic function of the log price $p_{t+\tau} = \ln P_{t+\tau}$: $\psi_t(p_t, \tau; s) = E_t[\exp(isp_{t+\tau}]$, from which we can obtain its conditional cumulants by the following differentiation:

$$\kappa_j = \left. \frac{\partial^j \psi_t(p_t, \tau; s)}{\partial (is)^j} \right|_{s=0}.$$

In what follows, we derive the characteristic functions of asset returns using the Kolmogorov backward equation (KBE). The methodology used here has become a standard practice.

A jump-diffusion process with constant drift

Using the extended form (incorporating Poisson jumps) of Ito's lemma, from (1), we have the stochastic process for $\ln P_t$:

$$d \ln P_t = \left(\mu - \lambda g - \frac{1}{2}\sigma^2 \right) dt + \sigma \, dZ_t + q \, dQ(\lambda).$$

The characteristic function of the log return, $\psi(p_t, \tau; s) = E_t[\exp(isp_{t+\tau})]$, obeys the Kolmogorov backward equation:

$$0 = \psi_p\left(\mu - \lambda g - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\psi_{pp}\sigma^2 - \psi_\tau + \lambda E[\psi(p_t + q) - \psi(p_t)].$$

The solution has the following form,

$$\psi(p_t, \tau; s) = \exp[A(\tau; s) + p_t B(\tau; s)],$$

with the coefficients given by,

$$\begin{aligned} B(\tau; s) &= is; \\ A(\tau; s) &= is\left(\mu - \lambda g - \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}s^2\sigma^2\tau + \lambda E[\exp[is\,q] - 1]\tau. \end{aligned}$$

A jump-diffusion with stochastic drift

Given constant volatility σ, we can derive the stochastic process for the drift $\mu(t)$ from the diffusion risk premium process (11):

$$d\mu_t = -\kappa_x(\mu_t - r_f - \lambda g - \sigma\mu_x)\,dt + \sigma\,\sigma_x\,dZ_{xt}.$$

Similarly, denote the characteristic function for the asset return as $\psi(p_t, \mu_t, \tau; s)$ with p_t and μ_t as the initial value of the logarithm of the asset price and the drift term at time t. The Kolmogorov backward equation for the characteristic function becomes,

$$\begin{aligned} 0 = {} & \psi_p\left(\mu_t - \lambda g - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\psi_{pp}\sigma^2 - \psi_\tau + \lambda E[\psi(p_t + q) - \psi(p_t)] \\ & - \psi_\mu\,\kappa_x(\mu_t - r_f - \lambda g - \sigma\mu_x) + \frac{1}{2}\psi_{\mu\mu}\sigma^2\sigma_x^2 + \psi_{\mu p}\rho\sigma^2\sigma_x. \end{aligned}$$

The solution has the following form,

$$\psi(p_t, \mu, \tau; s) = \exp[A(\tau; s) + p_t B(\tau; s) + \mu_t C(\tau; s)],$$

where the coefficients are given by,

$$\begin{aligned} A(\tau; s) &= -\frac{1}{\kappa_x}\left[(a+b)y + \frac{1}{2}ay^2 - \kappa_x\tau(a+b+c)\right]; \\ B(\tau; s) &= is; \\ C(\tau; s) &= \frac{is}{\kappa_x}y, \end{aligned}$$

with

$$a = -\frac{s^2\sigma^2\sigma_x^2}{2\kappa_x^2};$$

$$b = is\,(r + \lambda g + \sigma\mu_x) - \frac{s^2\rho\sigma^{\,2}\sigma_x}{\kappa_x};$$

$$c = -is\,\lambda g - \frac{1}{2}is\sigma^2 - \frac{1}{2}s^2\sigma^2 + \lambda E[\exp[is\,q] - 1];$$

$$y = (1 - e^{-\kappa_x\tau}).$$

Appendix B. Solutions for the ordinary differential equations

The three ordinary differential equations (ODEs) in (16) for $A(\tau)$, $B(\tau)$, and $C(\tau)$ are, respectively,

$$\frac{dC}{d\tau} = aC^2 + bC + c;$$

$$\frac{dB}{d\tau} = aBC + \frac{b}{2}B + \kappa_x\mu_x C;$$

$$\frac{dA}{d\tau} = \frac{a}{2}B^2 + \frac{1}{2}\sigma_x^2 C + \kappa_x\mu_x B + D_t;$$

where a, b, and c are given in the text and D_t is a complicated function related to the jumps in the asset return process,

$$D_t = \lambda\hat{G}_t - \frac{1}{2}c\,\lambda^2\hat{g}_t^2;$$

$$\hat{G}_t = E_t[(1 + \theta(t)(e^q - 1))^{1-\alpha} - 1];$$

$$\hat{g}_t = E_t[(1 + \theta(t)(e^q - 1))^{-\alpha}(e^q - 1)].$$

The boundary conditions are: $A(0) = B(0) = C(0) = 0$.

The solution for $\mathbf{C(\tau)}$.

The first ODE is a Riccati equation with constant coefficients. Recasting it as the integral equation, we have

$$\int_0^\tau \frac{dC}{aC^2 + bC + c} = \tau.$$

The form of the solution depends on the sign of the discriminant $b^2 - 4ac$. When $\alpha > 1$, we have $c < 0$, $a > 0$, thus $b^2 - 4ac > b^2 > 0$. Let $\eta = \sqrt{b^2 - 4ac}$, together with the boundary condition $C(0) = 0$, we have,

$$\int_0^\tau \frac{dC}{aC^2 + bC + c} = -\frac{1}{\eta} \ln \left| \left(\frac{2aC(\tau) + b + \eta}{2aC(\tau) + b - \eta} \right) \left(\frac{b - \eta}{b + \eta} \right) \right| = \tau,$$

from which we have the solution for $C(\tau)$

$$C(\tau) = \frac{2c(1 - e^{-\eta\tau})}{2\eta - (\eta + b)(1 - e^{-\eta\tau})}.$$

Also, we see that for log utility ($\alpha = 1$), $c = (1 - \alpha)/\alpha = 0$, $\eta = \sqrt{b^2 - 4ac} = b$, $C(\tau) = 0$. Then, from the second ordinary differential equation, we see that $dB/d\tau = 0$ and $B(\tau) = 0$ for all τ. Thus, the intertemporal hedging demand is zero. On the other hand, when $\tau \to \infty$, we obtain the steady state value for $C(\infty)$:

$$C(\infty) = -\frac{\eta + b}{2a} < 0\,.$$

The solution for $\boldsymbol{B(\tau)}$.

From the second ODE, the solution for $C(\tau)$, and the boundary condition: $B(0) = 0$, we have the solution for $B(\tau)$:

$$B(\tau) = \frac{4c\kappa_x \mu_x (1 - e^{-\eta\tau/2})^2}{\eta[2\eta - (\eta + b)(1 - e^{-\eta\tau})]}.$$

The solution for $\mathbf{A(\tau)}$.

Substitute the solutions for $B(\tau)$ and $C(\tau)$ into the third ODE, together with the boundary condition: $A(0) = 0$, we have the solution for $A(\tau)$:

$$\begin{aligned} A(\tau) &= \int_0^\tau \left(\frac{a}{2} B^2 + \frac{1}{2} \sigma_x^2 C + \kappa_x \mu_x B + D_t \right) dt \\ &= \frac{4c\kappa_x^2 \mu_x^2 [(2b + \eta)e^{-\eta\tau} - 4be^{-\eta\tau/2} + 2b - \eta]}{\eta^3[(\eta - b) + (\eta + b)(1 - e^{-\eta\tau})]} \\ &\quad - \frac{\sigma_x^2}{2a} \ln \left| \frac{(\eta - b) + (\eta + b)(1 - e^{-\eta\tau})}{2\eta} \right| + c \left(\frac{2\kappa_x^2 \mu_x^2}{\eta^2} + \frac{\sigma_x^2}{\eta - b} \right) \tau + D_t \tau. \end{aligned}$$

Note that since D_t is a function of the optimal allocation decision $\theta(x, t)$. The ODE for $A(\tau)$ and hence the exponential-quadratic form for the indirect utility function are only approximations.

Appendix C. Proofs

Proof of Lemma 1. When $\alpha = 1$, $c = 0$, so both $B(\tau)$ and $C(\tau)$ are zero.
When $\alpha > 1$, $c < 0$, $a > 0$, $\eta = \sqrt{b^2 - 4ac} > |b|$,

$$2\eta - (\eta \pm b)(1 - e^{-\eta\tau}) > 2\eta - (\eta \pm b) > \eta \mp b > 0.$$

The denominators for both $B(\tau)$ and $C(\tau)$ are positive. The numerators are both negative since $c < 0$. Therefore, both $B(\tau)$ and $C(\tau)$ are negative.
Their derivatives with respect to τ are:

$$C'(\tau) = \frac{2c\eta e^{-\eta\tau}}{2\eta - (\eta - b)(1 - e^{-\eta\tau})} + \frac{2c\eta(1 - e^{-\eta\tau})(\eta + b)e^{-\eta\tau}}{[2\eta - (\eta + b)(1 - e^{-\eta\tau})]^2};$$
$$B'(\tau) = \frac{4c\kappa_x\mu_x(1 - e^{-\eta\tau/2})e^{-\eta\tau/2}}{2\eta - (\eta + b)(1 - e^{-\eta\tau})} + \frac{4c\kappa_x\mu_x(1 - e^{-\eta\tau/2})^2(\eta + b)e^{-\eta\tau}}{[2\eta - (\eta + b)(1 - e^{-\eta\tau})]^2}.$$

They are both negative since $c < 0$. ■

Proof of Remark 1. When the investment environment is constant, $N = \tau$. Otherwise, the difference between N and τ captures the impact of the stochastic drift on the conditional variance of the asset return. Equation (18) tells us that $N = 0$ when $\tau = 0$. For $\tau > 0$, the derivatives of N with respect to τ are:

$$\frac{\partial N}{\partial \tau} = 1 + \sigma(1 - e^{-\kappa_x\tau})\frac{2\rho\kappa_x + \sigma_x(1 - e^{-\kappa_x\tau})}{\kappa_x^2} = 1 + \sigma\frac{2\rho\kappa_x + \sigma_x}{\kappa_x^2}, \quad \text{when } \tau \to \infty;$$
$$\frac{\partial^2 N}{\partial \tau^2} = 2\sigma_x e^{-k_x\tau}\frac{\rho\kappa_x + \sigma_x(1 - e^{-\kappa_x\tau})}{\kappa_x} = 0 \quad \text{when } \tau \to \infty.$$

When $\rho = 0$, the second derivative is positive while the first derivative is greater than 1. It implies that for $\tau > 0$, $N > \tau$, and the difference is increasing at an increasing spread. A positive correlation both increases the slope and the curvature of $N(\tau)$ and thus increases the variance in a more pronounced manner. A negative correlation mitigates these effects. When the correlation is very negative such that

$$|\rho| > \sigma_x/\kappa_x,$$

the first derivative will be less than 1 and the second derivative will be negative. The variance of the asset return will actually be reduced. ■

Proof of Lemma 2. When $\alpha = 0$, $\hat{g}_t = E_t(e^q - 1) = g$. When $\alpha > 0$, (i) the partial derivative

$$\frac{\partial \hat{g}_t(\theta)}{\partial \theta} = -\alpha E_t[(1+\theta(t)(e^q - 1))^{-\alpha-1}(e^q - 1)^2] < 0.$$

Note that to guarantee positive wealth at all times, we require

$$(1+\theta_t(e^q - 1))^{-\alpha-1} > 0$$

for all q.

(ii) First we consider the case when the investor's position on the risky asset is long, i.e., $0 < \theta_t < 1$, we have

$$\begin{aligned}
\hat{g}_t - g &= E_t[(1+\theta_t(e^q - 1))^{-\alpha}(e^q - 1)] - E_t[(e^q - 1)] \\
&= E_t\{[(1+\theta_t(e^q - 1))^{-\alpha} - 1](e^q - 1)\} \\
&= E_t[Q] \leq 0,
\end{aligned}$$

and

$$\frac{\partial(\hat{g}_t - g)}{\partial \alpha} = -E_t \frac{(e^q - 1)\ln(1+\theta(t)(e^q - 1))}{(1+\theta(t)(e^q - 1))^{\alpha}} = E_t[P] \leq 0.$$

The equalities hold when $q = 0$, that is, when there is no jump. To see that the inequalities hold, we separate the cases when $q < 0$ and $q > 0$:

1. When $q > 0$,

$$\begin{aligned}
(1+\theta(t)(e^q - 1))^{-\alpha} &< 1, \\
\ln(1+\theta_t(e^q - 1)) &> 0.
\end{aligned}$$

   Therefore, $Q < 0$, $P < 0$.

2. When $q < 0$,

$$\begin{aligned}
(1+\theta_t(e^q - 1))^{-\alpha} &> 1, \\
\ln(1+\theta_t(e^q - 1)) &> 0.
\end{aligned}$$

   Therefore, $Q < 0$, $P < 0$.

Therefore, the expectations of Q and P are always nonpositive.

Similarly, when the investor's position on the risky asset is short ($\theta < 0$), we have

$$\hat{g}_t - g \geq 0, \quad \text{and} \quad \frac{\partial(\hat{g}_t - g)}{\partial \alpha} \geq 0.$$

Therefore, both $\hat{g}_t - g$ and $\partial(\hat{g}_t - g)/\partial\alpha$ have opposite signs to that of θ. ∎

Acknowledgments

The author thanks Cheng-few Lee (the editor), an anonymous referee, David Backus, Silverio Foresi, Anthony Lynch, Mathew Richardson, and seminar participants at New York University and Fordham University for helpful comments.

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