MA 201: Partial Differential Equations
Lecture - 3

Recall:

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y). \tag{1}$$

The IVP for first-order linear PDE asks for a solution of (1) which has given values on a curve Γ in $\mathbb{R}^2$.
Let the initial curve Γ be described parametrically by the equations

$$x = x_0(s), \quad y = y_0(s); \quad s \in I, \tag{2}$$

where $x_0(s)$, $y_0(s)$ are in $C^1(I)$. Let $u_0(s) = u(x_0(s), y_0(s))$ be a given function in $C^1(I)$.
Consider the following three examples.

Example (Existence of a unique solution)

$$\text{PDE: } u_x = cu + d(x,y), \;\; c \in \mathbb{R}; \quad \text{IC}: \; u(0,y) = y. \tag{3}$$

The solution of the PDE is given by

$$u(x,y) = e^{cx}\left[\int_0^x e^{-c\xi} d(\xi, y)\,d\xi + u(0,y)\right]. \tag{4}$$

Note that the Cauchy data is prescribed on the y-axis. Thus, the unique solution is

$$u(x, y) = e^{cx} \left[\int_0^x e^{-c\xi} d(\xi, y) d\xi + y \right].$$

Example (Non-uniqueness of solutions)

$$\text{PDE:} \;\; u_x = cu, \;\; c \in \mathbb{R}; \quad \text{IC}: \;\; u(x, 0) = 2e^{cx}. \tag{5}$$

This Cauchy problem has infinitely many solutions:

$$u(x, y) = e^{cx} g(y).$$

Now $g(y)$ should satisfy $g(0) = 2$. Thus, every function $g(y)$ satisfying $g(0) = 2$ will be a solution to the IVP (5).

Example (Non-existence of solutions)

$$\text{PDE: } u_x = cu, \ \ c \in \mathbb{R}; \quad \text{IC : } u(x,0) = \sin x. \tag{6}$$

The solution must satisfy

$$\sin x = u(x,0) = e^{cx} g(0), \ \ \forall x \in \mathbb{R}.$$

The above cannot hold and hence this Cauchy problem has no solution.

Remark: These examples clearly tell us that we cannot prescribe Cauchy data on arbitrary curves in the xy-plane.

The method of characteristics for an IVP

Let the initial curve Γ be given parametrically as:

$$x = x(s), \quad y = y(s), \quad u = u(s) \quad \text{for } s \in I. \tag{7}$$

Every value of s fixes a point on Γ through which a unique characteristic curve passes. The family of characteristic curves determined by the points of Γ may be parametrized as

$$x = x(t, s), \quad y = y(t, s), \quad u = u(t, s)$$

with $t = 0$ corresponding to the initial curve Γ.
That is, we have

$$x(0, s) = x(s), \quad y(0, s) = y(s), \quad u(0, s) = u(s).$$

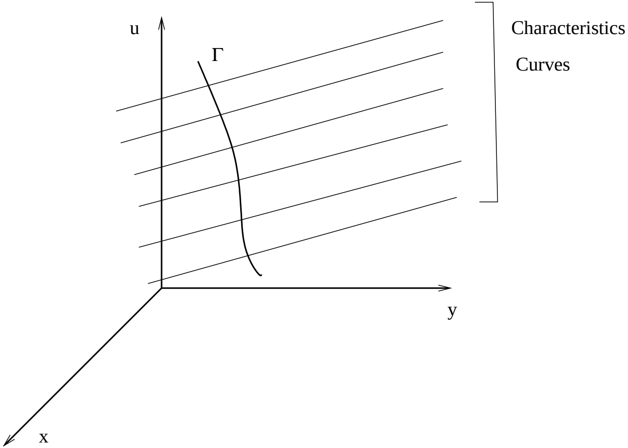

Figure : Characteristic curves and construction of the integral surface

In other words, we note that the functions $x(t, s)$ and $y(s, t)$ are the solutions of the characteristic system (for each fixed s)

$$\frac{d}{dt}x(t, s) = a(x(t, s), y(t, s)), \quad \frac{d}{dt}y(t, s) = b(x(t, s), y(t, s)) \qquad (8)$$

with given initial values $x(0, s)$ and $y(0, s)$.
Suppose that

$$u(x(0, s), y(0, s)) = g(s), \qquad (9)$$

where $g(s)$ is a given function. We obtain $u(x(t, s), y(t, s))$ as follows:
Let

$$\begin{aligned} &u(t, s) = u(x(t, s), y(t, s)), \ c(t, s) = c(x(t, s), y(t, s)), \\ &d(t, s) = d(x(t, s), y(t, s)). \end{aligned}$$

and

$$\mu(t, s) = \exp\left[-\int_0^t c(t, s)dt\right]. \qquad (10)$$

That is, for each fixed s, we obtain

$$\boxed{u(t,s) = \frac{1}{\mu(t,s)}\left[\int_0^t \mu(t,s)d(t,s)dt + g(s)\right].} \tag{11}$$

$u(t,s)$ is the value of u at the point $(x(t,s), y(t,s))$.

Note: As s and t vary, the point (x,y,u) in the xyu-space, given by

$$\boxed{x = x(t,s), \quad y = y(t,s), \quad u = u(t,s),} \tag{12}$$

traces out the surface of the graph of the solution u of PDE (1) which meets the initial curve (9).
The equations (12) constitute the parametric form of the solution of (1) satisfying the initial condition (9) (i.e., a surface in (x,y,u)-space that contains the initial curve Γ).

Remarks.

- By implicit function theorem, if the Jacobian

$$\begin{aligned} J &= \frac{\partial(x,y)}{\partial(t,s)} \\ &= \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial s} \\ &= \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} \neq 0 \qquad (13) \end{aligned}$$

  on Γ, where $(x_0)_s = \frac{dx_0}{ds}$, $(y_0)_s = \frac{dy_0}{ds}$, then $x = x(t,s)$ and $y = y(t,s)$ can be inverted to give s and t as (smooth) functions of x and y, i.e., $s = s(x,y)$ and $t = t(x,y)$.
  The resulting function $U(x,y) = u(t(x,y), s(x,y))$ satisfies PDE (1) in a neighbourhood of the curve Γ (in view of $u'(t) - c(t)u(t) = d(t)$ and the initial condition (7)) and is the unique solution of the IVP.
- The condition (13) is called transversality condition.

Example

Determine the solution the following IVP:

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x),$$

where $f(x)$ is a given function and c is a constant.

Solution.

- **Step 1.**(Finding characteristic curves)
  To apply the method of characteristics, parametrize the initial curve C as follows: as follows:

  $$x = s, \quad y = 0, \quad u = f(s). \tag{14}$$

  The family of characteristic curves $x((t,s), y(t,s))$ are determined by solving the ODEs

  $$\frac{d}{dt}x(t,s) = c, \quad \frac{d}{dt}y(t,s) = 1.$$

  The solution of the system is

  $$x(t,s) = ct + c_1(s) \ \text{ and } \ y(t,s) = t + c_2(s).$$

- **Step 2.** (Applying IC)
  Using the initial conditions

$$x(0, s) = s, \quad y(0, s) = 0,$$

  we find that

$$c_1(s) = s, \quad c_2(s) = 0,$$

  and hence

$$x(t, s) = ct + s \;\text{ and }\; y(t, s) = t.$$

- **Step 3.** (Writing the parametric form of the solution)
  Comparing with (1), we have $c(x, y) = 0$ and $d(x, y) = 0$.
  Therefore, using (10) and (11), we find that

$$d(t, s) = 0, \quad \mu(t, s) = 1.$$

  Since $u(x(0, s), y(0, s)) = u(s, 0) = g(s) = f(s)$, we obtain $u(t, s) = f(s)$.
  Thus, the parametric form of the solution of the problem is given by

$$x(t, s) = ct + s, \quad y(t, s) = t, \quad u(t, s) = f(s).$$

- **Step 4.** (Expressing $u(s,t)$ in terms of $U(x,y)$)
  Expressing s and t as $s = s(x,y)$ and $t = t(x,y)$, we have

$$s = x - cy, \quad t = y.$$

  We now write the solution in the explicit form as

$$U(x,y) = u(t(x,y), s(x,y)) = f(x - cy).$$

  Clearly, if $f(x)$ is differentiable, the solution $U(x,y) = f(x - cy)$ satisfies the given PDE as well as the initial condition.

- **Remarks.**
  The above example characterizes unidirectional wave motion with velocity c.
  If $c > 0$, the entire initial wave form $f(x)$ moves to the right without changing its shape with speed c (if $c < 0$, the direction of motion is reversed).

Example

$$\begin{aligned} \text{PDE:} \qquad & -yu_x + xu_y = 0 \\ \text{Side Condition:} \qquad & u(s, s^2) = s^3, \quad (s > 0). \end{aligned}$$

Solution.

- **Step 1.** (Finding characteristic curves $(x(t,s), y(t,s))$)
  Solve
  $$\frac{d}{dt}x(t,s) = -y(t,s), \quad \frac{d}{dt}y(t,s) = x(t,s)$$
  with initial conditions $x(0,s) = s, \quad y(0,s) = s^2$.
  The general solution is
  $$x(t,s) = c_1(s)\cos(t) + c_2(s)\sin(t), \quad y(t,s) = c_1(s)\sin(t) - c_2(s)\cos(t).$$

- **Step 2.** (Applying IC)
  Using ICs, we find that

$$c_1(s) = s, \quad c_2(s) = -s^2,$$

and hence

$$x(t,s) = s\cos(t) - s^2\sin(t) \quad \text{and} \quad y(t,s) = s\sin(t) + s^2\cos(t).$$

- **Step 3.** (Writing the parametric form of the solution)
  Comparing with (1), we note that $c(x,y) = 0$ and $d(x,y) = 0$. Therefore, using (10) and (11), it follows that

$$d(t,s) = 0, \quad \mu(t,s) = 1.$$

In view of the given condition curve and $u = u(t,s)$, we obtain

$$u(x(s,0), y(s,0)) = u(s, s^2) = g(s) = s^3, \quad u(t,s) = s^3.$$

Thus, the parametric form of the solution of the problem is given by

$$x(t,s) = s\cos(t) - s^2\sin(t), \quad y(t,s) = s\sin(t) + s^2\cos(t), \quad u(t,s) = s^3.$$

- **Step 4.** (Expressing $u(s,t)$ in terms of $U(x,y)$)
  It is left as an exercise to show that
  $$U(x,y) = \frac{1}{\sqrt{8}}\left[-1+\sqrt{1+4(x^2+y^2)}\right]^{3/2}.$$