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Integer Linear Programming: Branch and Bound Method

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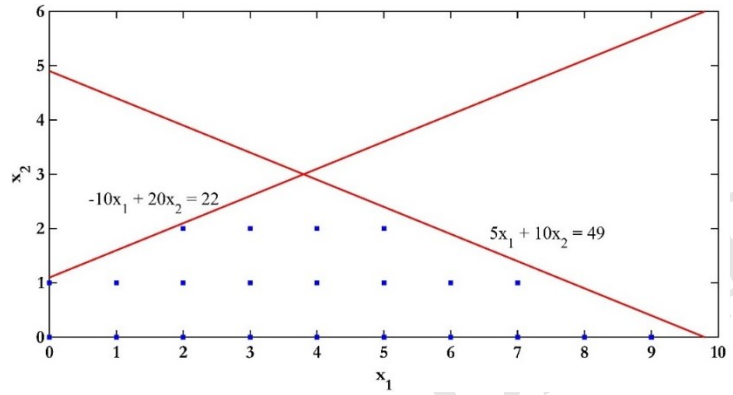
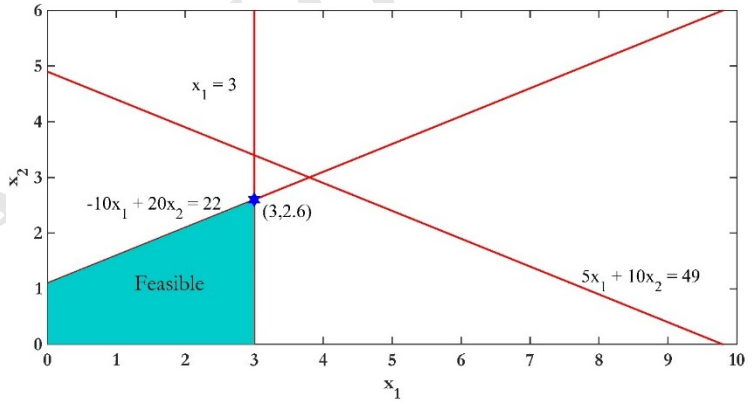
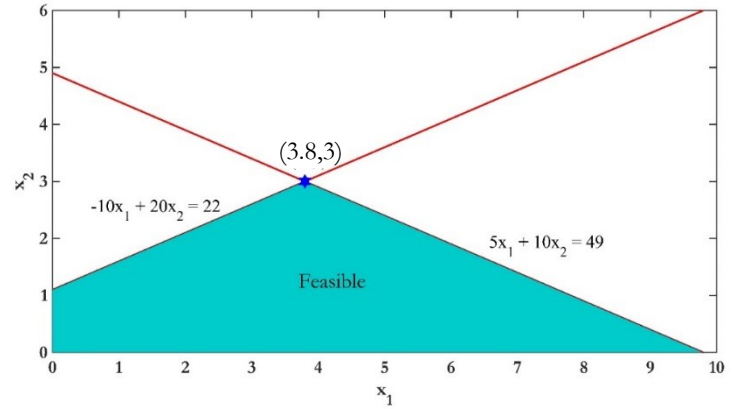
Simplex Method for LP: https://www.youtube.com/watch?v=VsyFFhzQVZM
Branch & Bound Method for MILP: https://www.youtube.com/watch?v=g1Xtmd94zns
Additional resources: tinyurl.com/sksopti, tinyurl.com/sksoptivid

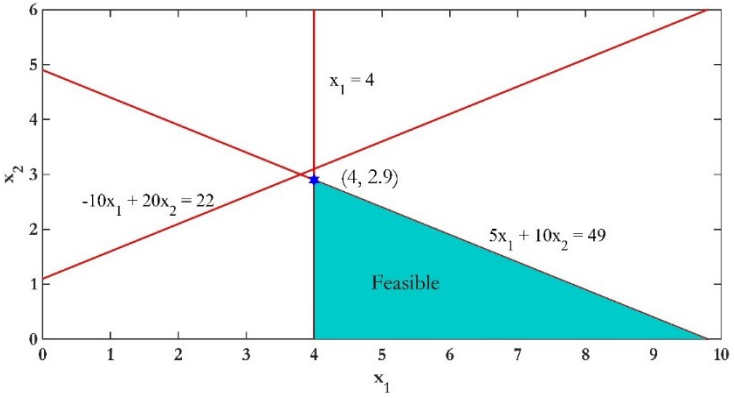
Integer Linear Programming (ILP)

- Divide and conquer method.
- Relaxed linear programming model ILP model without integer constraints.
- LP relaxation is used to estimate the optimal solution of an ILP problem.
- Two general methods: Branch-and-Bound method and Cutting-plane method



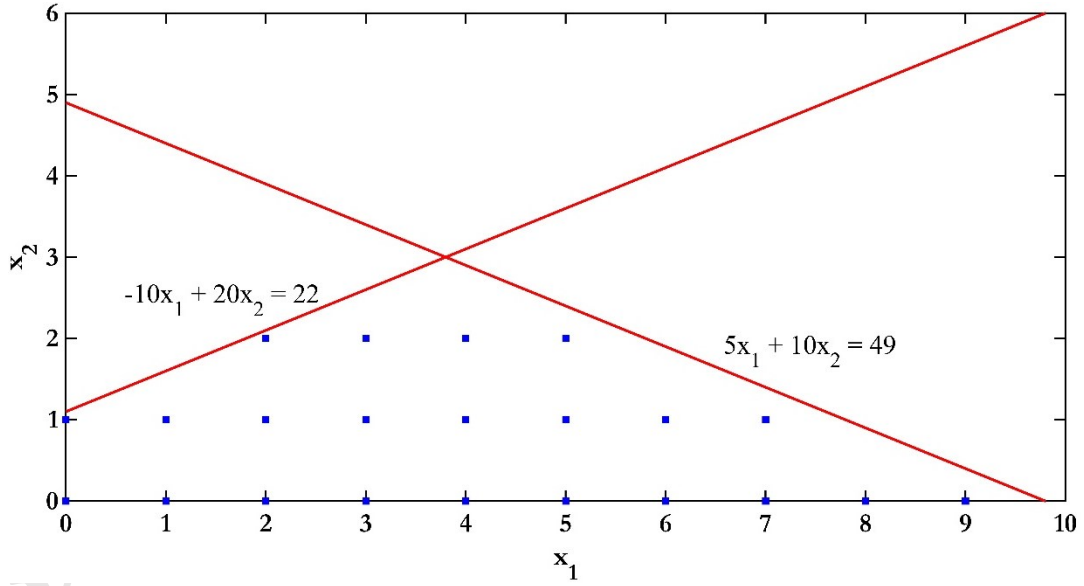




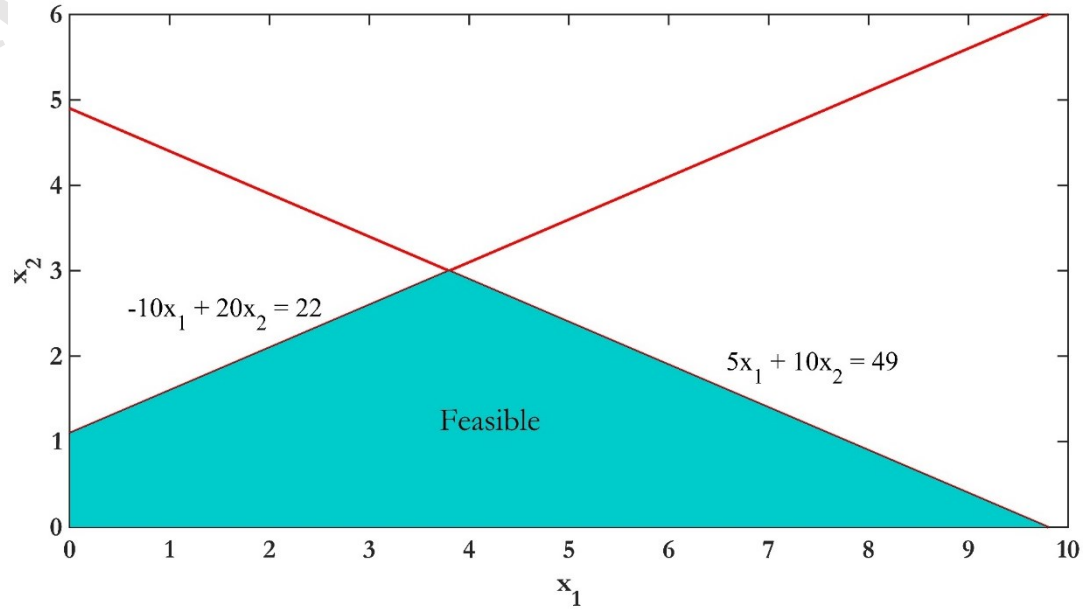


Branch and Bound: Relaxation

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$



$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



Rules in Branch and Bound (Maximization)

Condition	Operation
LP_j is infeasible	**Prune** the node j
$Z_{LP,j} \leq Z_I$	**Prune** the node j
$Z_{LP,j} > Z_I$, optimal LP_j has integer solutions	**Update** Z_I , **Prune** the node j
$Z_{LP,j} > Z_I$, optimal LP_j does not have integer solutions	**Branch** node j into two candidate problems

Branch and Bound: Solution of Relaxed Problem

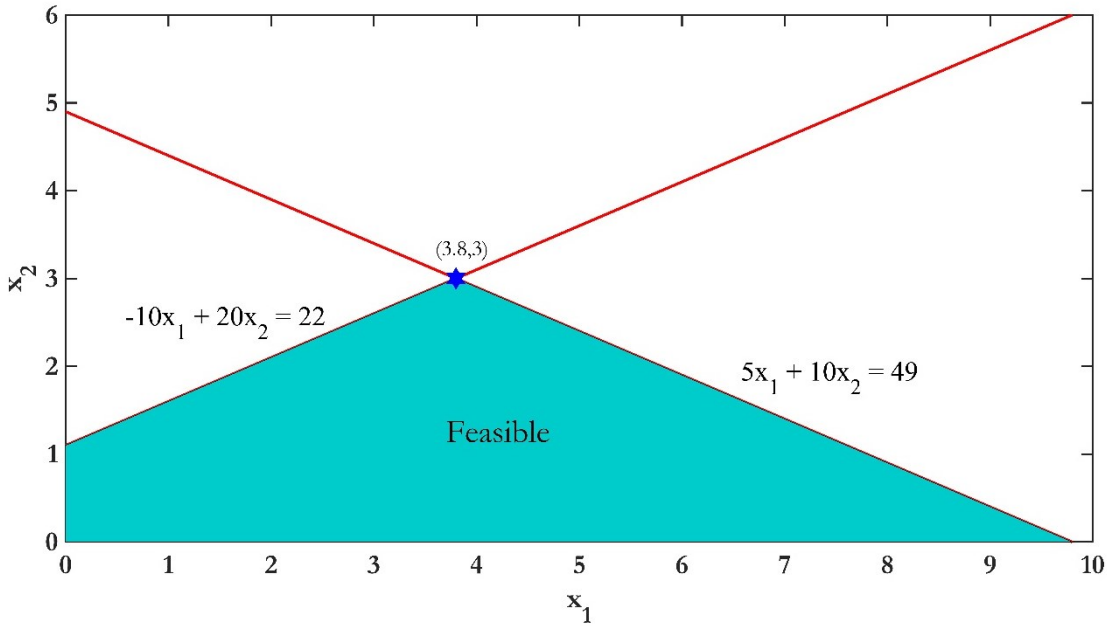
LP 1

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

LP1

$Z = 8.2,\ x = (3.8, 3)$

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$



Optimal solution of LP1 is not integer, branch node into two candidate problems

Branch and Bound: Branching

LP 2

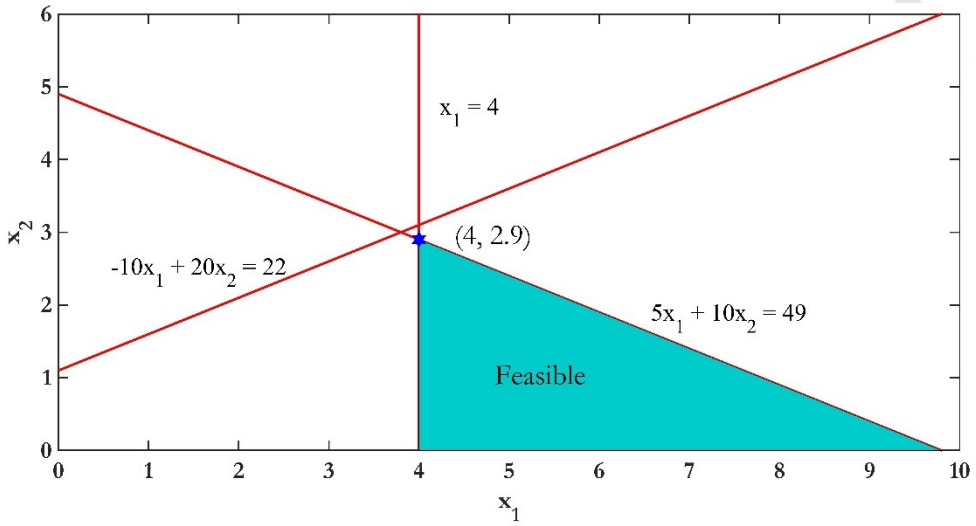
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \ge 4 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

LP1
$Z = 8.2,\ x = (3.8,\ 3)$

$x_1 \ge 4$

LP2
$Z = 7.6,\ x = (4,\ 2.9)$

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$



Optimal solution of LP2 is not integer, branch node into two candidate problems

Branch and Bound: Infeasible Node

LP 3

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \ge 4 \\
& x_2 \ge 3 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

LP1
Z = 8.2, x = (3.8, 3)

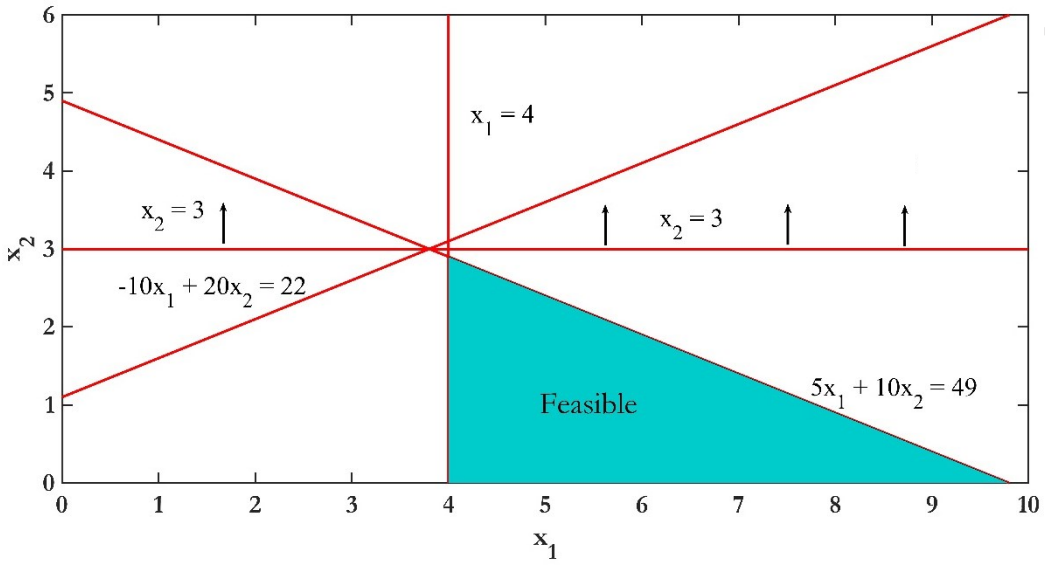
$x_1 \ge 4$

LP2
Z = 7.6, x = (4, 2.9)

$x_2 \ge 3$

LP3
Infeasible

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

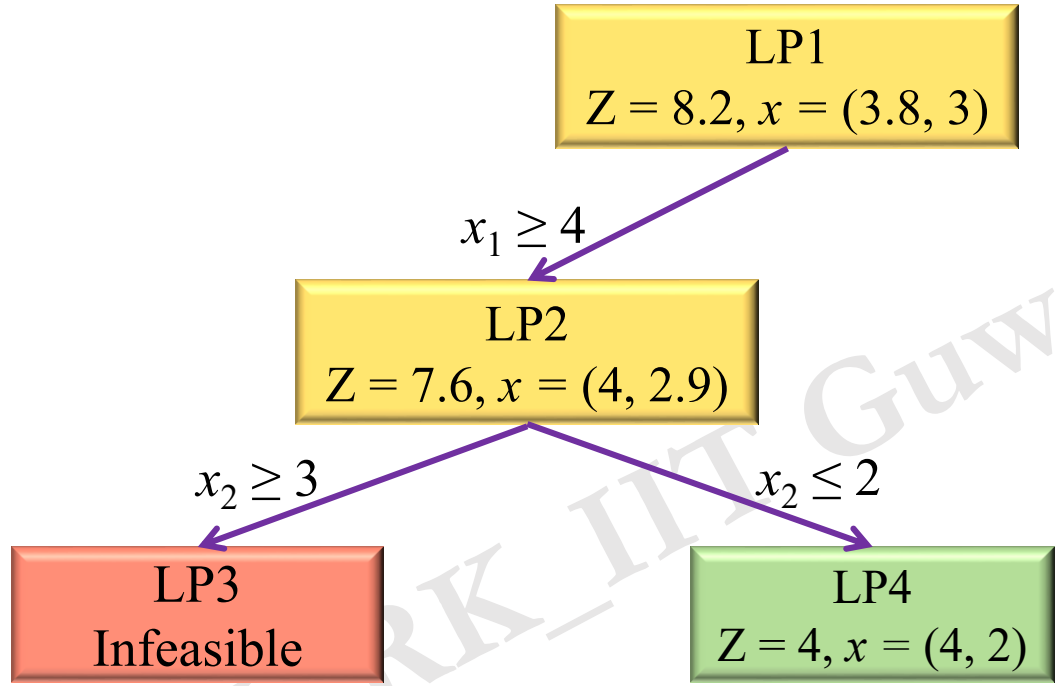


LP3 is infeasible, prune the node

Branch and Bound: Integer Feasible Solution

LP 4

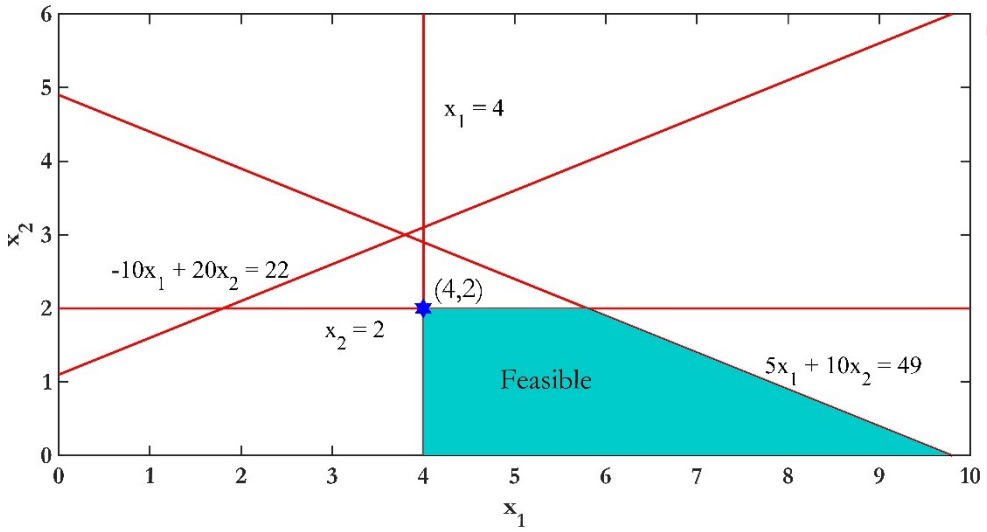
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \ge 4 \\
& x_2 \le 2 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 4$

Integer feasible solution

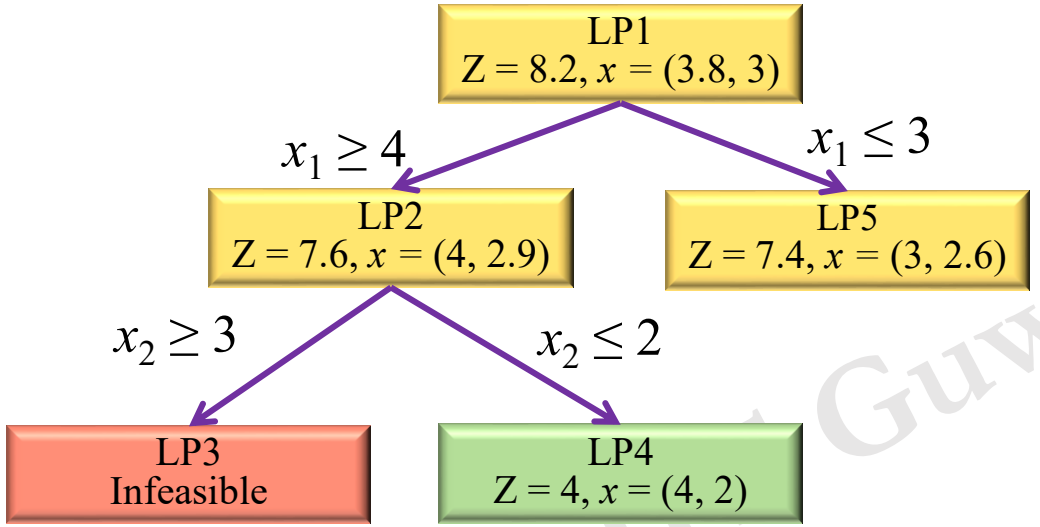


Optimal solution of LP4 is integer, Accept solutions and prune the node.

Branch and Bound: Branching

LP 5

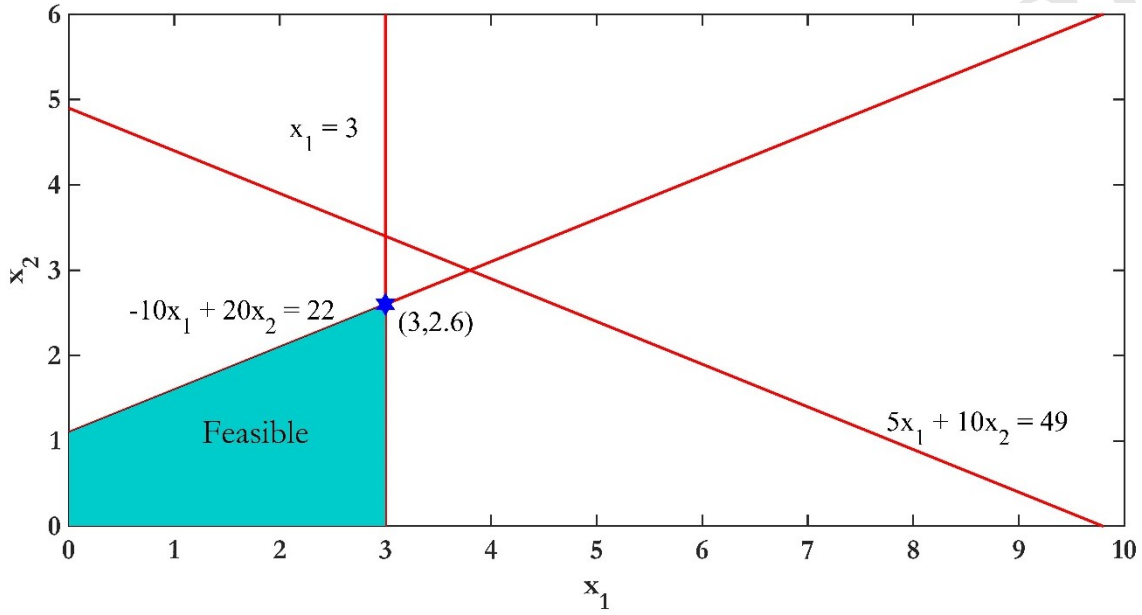
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \le 3 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



Integer feasible solution

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 4$

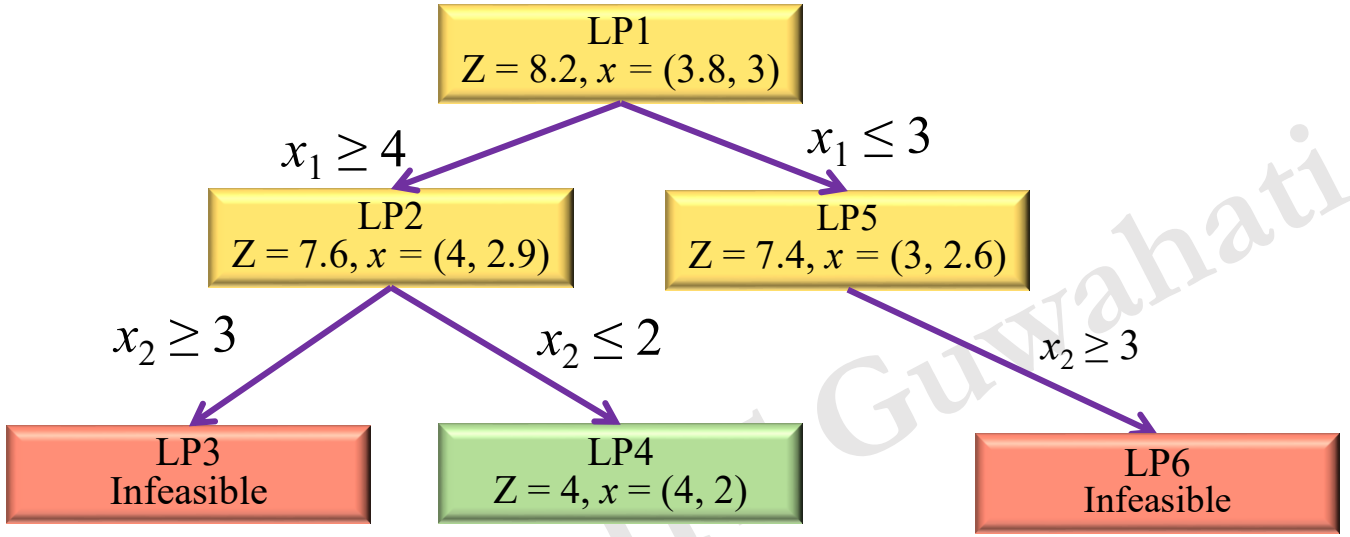


Optimal solution of LP5 is not integer and $Z_{LP5} > Z_I$, branch node into two candidate problems

Branch and Bound: Infeasible Node

LP 6

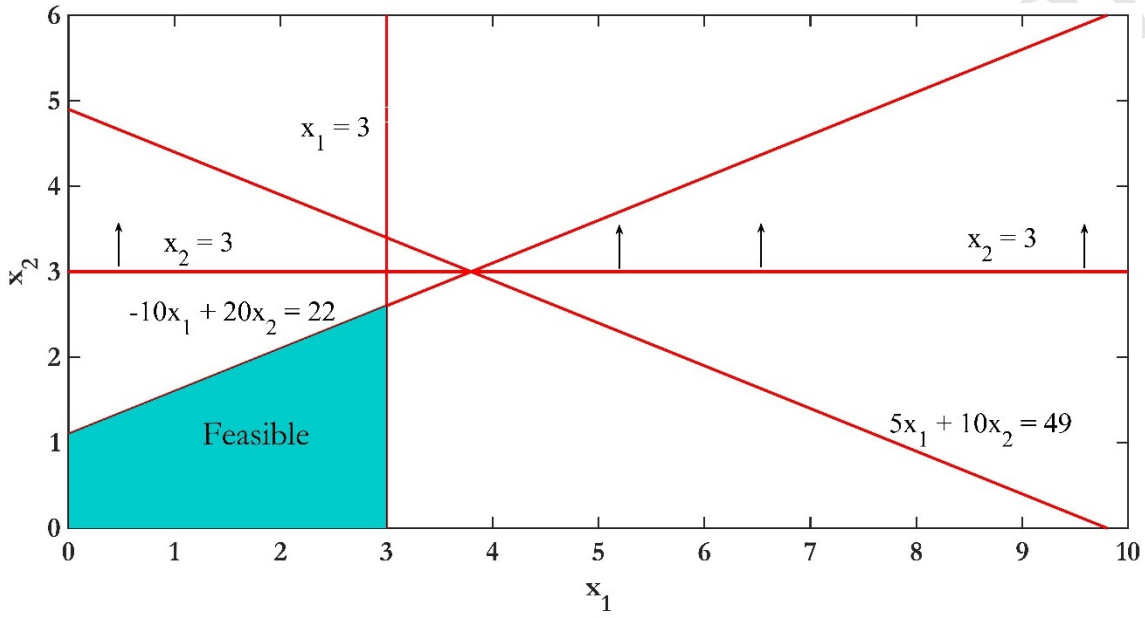
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \le 3 \\
& x_2 \ge 3 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 4$

Integer feasible solution

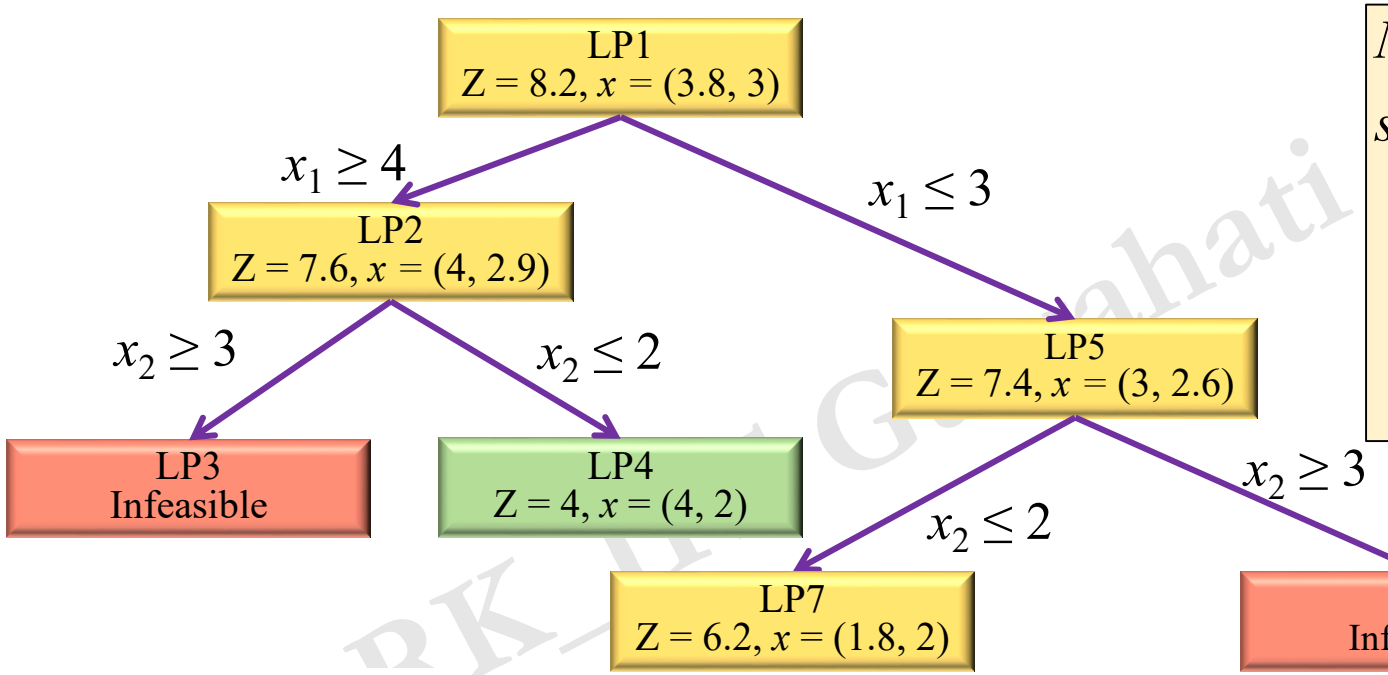


LP6 is infeasible, prune the node

Branch and Bound: Branching

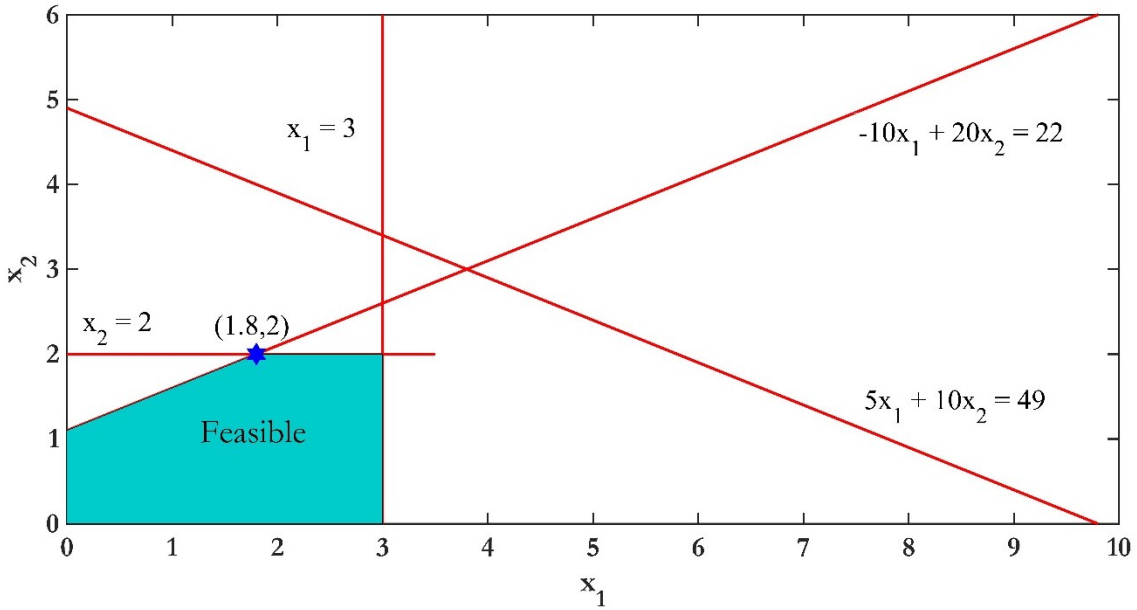
LP 7

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \le 3 \\
& x_2 \le 2 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 4$

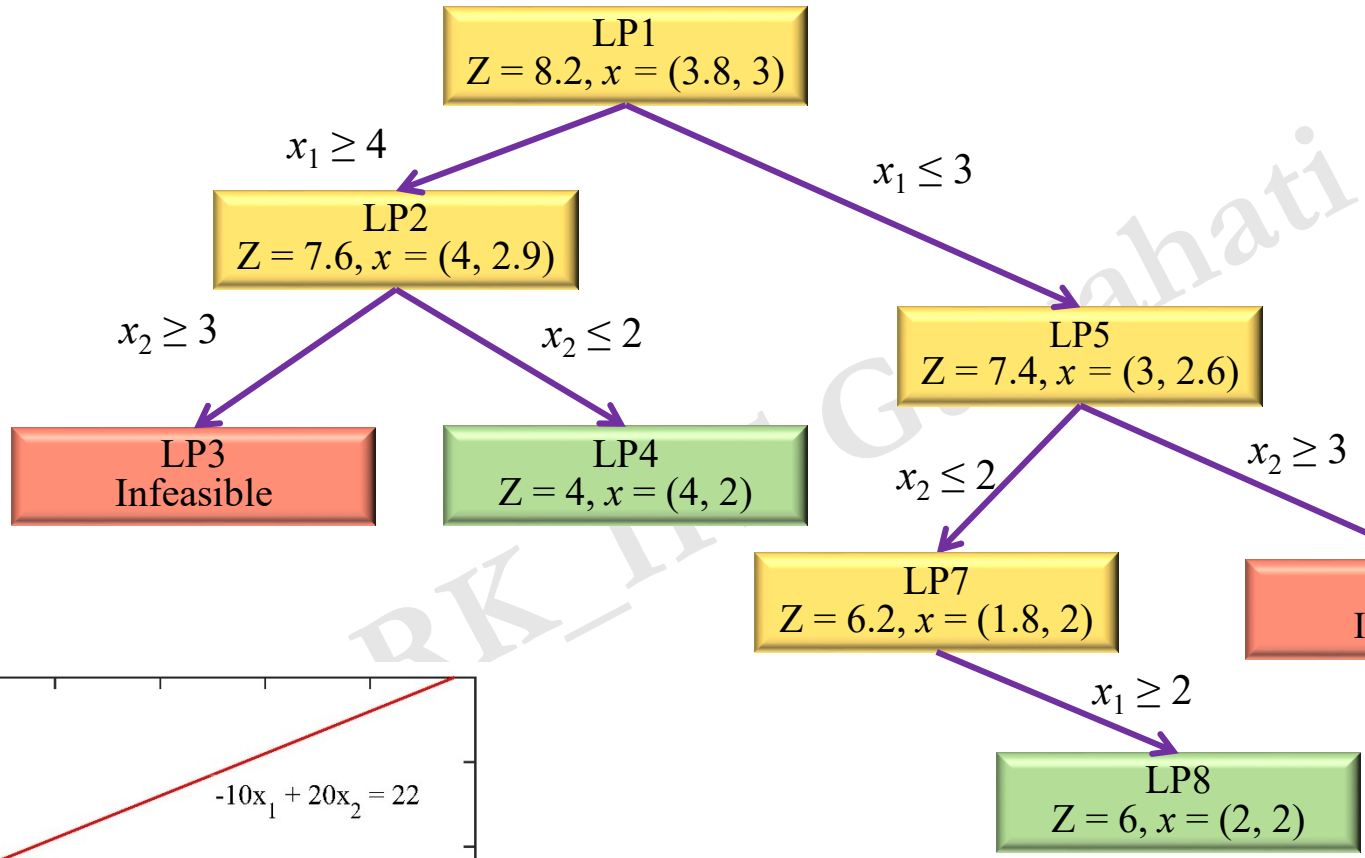


Optimal solution of LP7 is not integer and $Z_{LP7} > Z_I$, branch node into two candidate problems

Branch and Bound: Integer Feasible Solution

LP 8

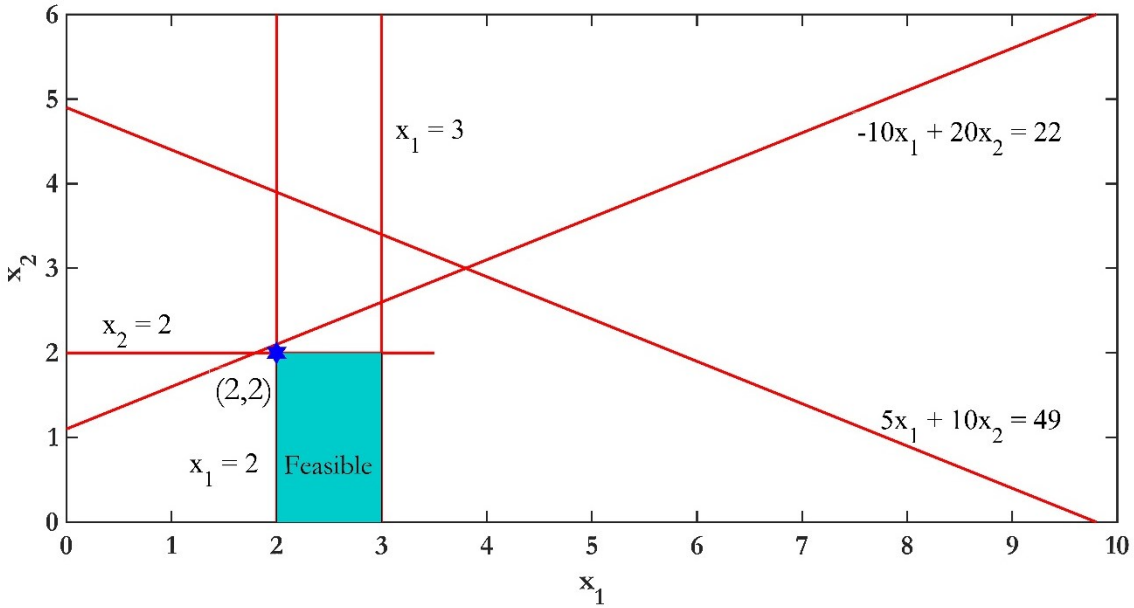
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \le 3 \\
& x_2 \le 2 \\
& x_1 \ge 2 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 6$

Integer feasible solution

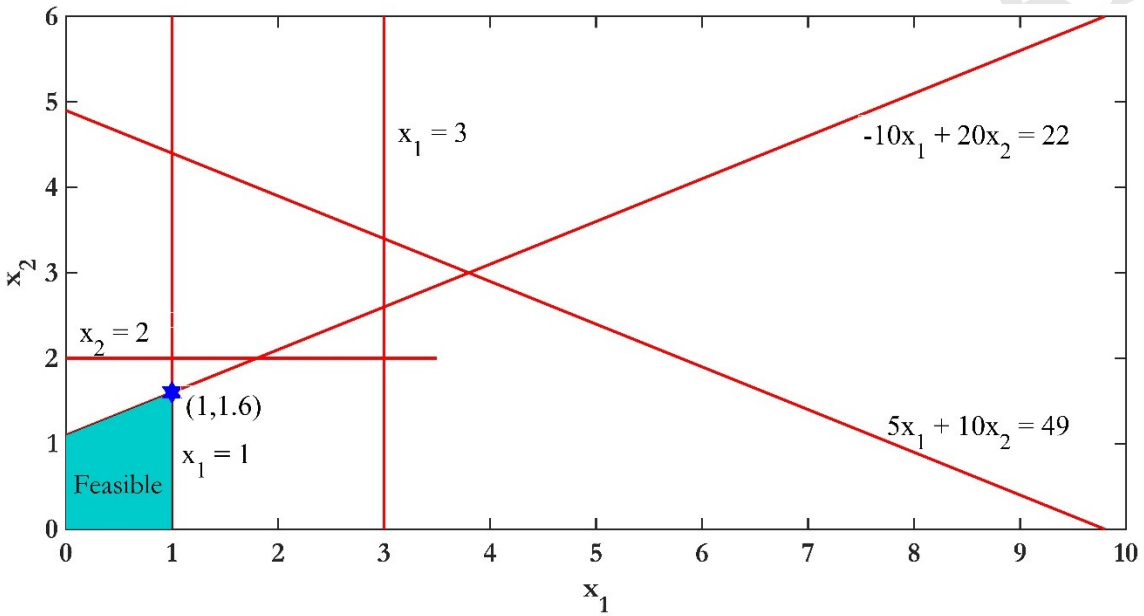


Optimal solution of LP8 is integer, Accept solutions and prune the node.

Branch and Bound: Optimal Solution

LP 9

$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1 \le 3 \\
& x_2 \le 2 \\
& x_1 \le 1 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



LP1
Z = 8.2, x = (3.8, 3)

$x_1 \ge 4$ — LP2
Z = 7.6, x = (4, 2.9)

$x_1 \le 3$ — LP5
Z = 7.4, x = (3, 2.6)

LP2: $x_2 \ge 3$ — LP3 Infeasible

LP2: $x_2 \le 2$ — LP4
Z = 4, x = (4, 2)

LP5: $x_2 \le 2$ — LP7
Z = 6.2, x = (1.8, 2)

LP5: $x_2 \ge 3$ — LP6 Infeasible

LP7: $x_1 \le 1$ — LP9
Z = 5.4, x = (1, 1.6)

LP7: $x_1 \ge 2$ — LP8
Z = 6, x = (2, 2)

Prune

Integer feasible solution

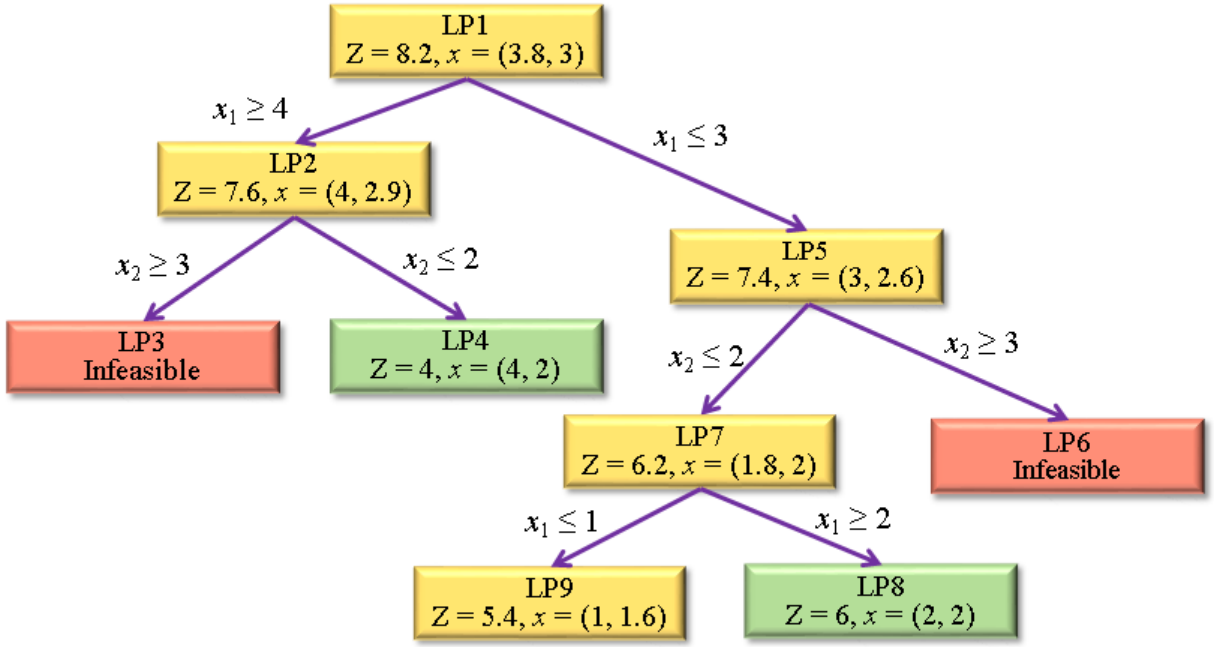
$$
\begin{aligned}
Max \quad & Z = -x_1 + 4x_2 \\
st \quad & -10x_1 + 20x_2 \le 22 \\
& 5x_1 + 10x_2 \le 49 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ are integers}
\end{aligned}
$$

$Z_I = 6$

Optimal solution of LP9 is not integer and $Z_{LP9} < Z_I$, prune the node

Rules in Branch and Bound

Condition	Operation
LP_j is infeasible	**Prune** the node j
$Z_{LP,j} \leq Z_I$	**Prune** the node j
$Z_{LP,j} > Z_I$, optimal LP_j has integer solutions	**Update** Z_I , **Prune** the node j
$Z_{LP,j} > Z_I$, optimal LP_j does not have integer solutions	**Branch** node j into two candidate problems



Thank You !!!