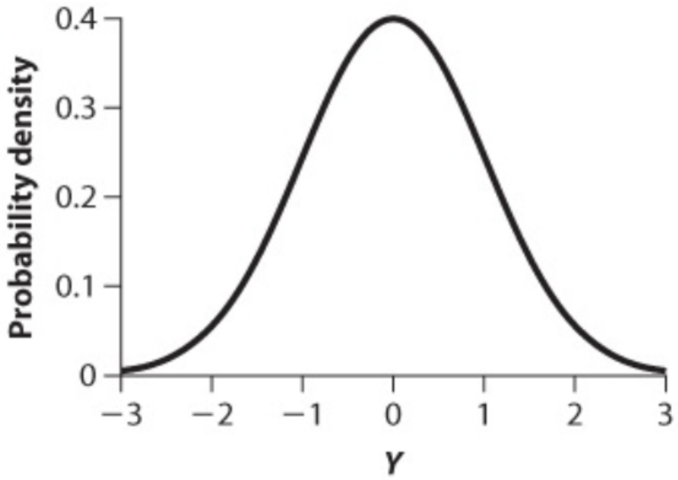
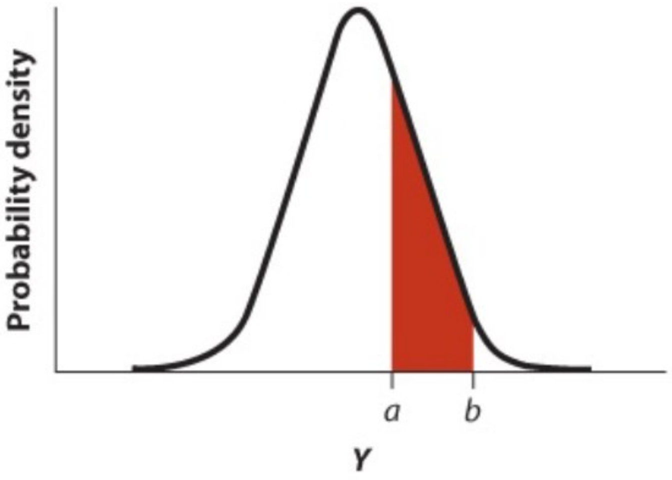
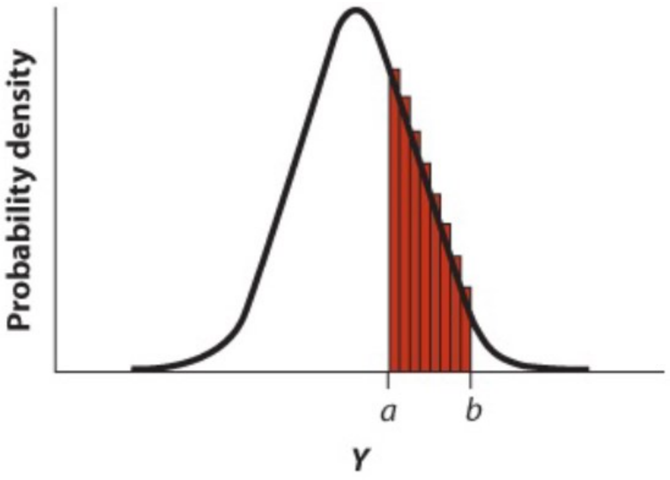
Continuous Probability Distributions

Continuous Probability Distribution

Unlike discrete variables, continuous numerical variables can take on any real number value within some range. Between any two values of a continuous variable, an infinite number of other values are possible. We describe a continuous probability distribution with a curve whose height is **probability density**. A probability density allows us to describe the probability of any range of values for a random variable X.







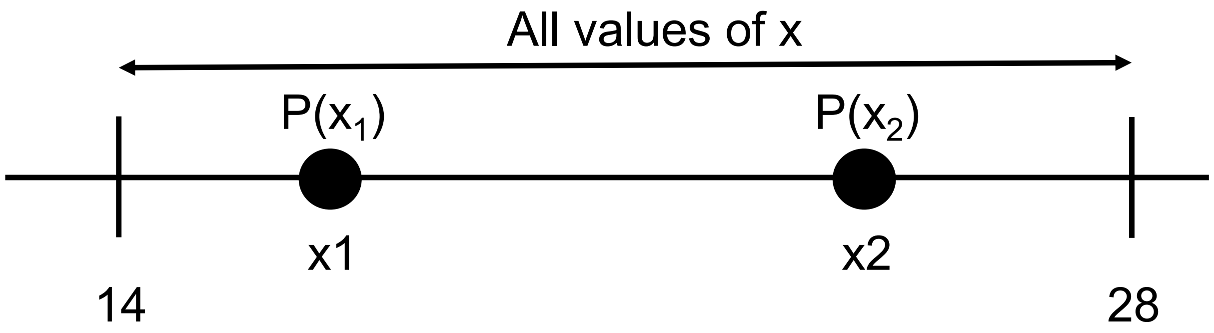
Because a continuous probability distribution covers an infinite number of possible outcomes, the probability of obtaining any specific outcome is infinitesimally small and therefore zero.

The probability of obtaining a value of *Y* within some range is indicated by the area under the curve.

Continuous Probability Distribution

Let X: weight of school-going child, that is X is a continuous random variable. For example, 14 kg ≤ X ≤ 28 kg.

How to calculate the probability of P(X=16 kg) ?



Then

$$\sum_{\forall x_i} p(X = x_i) = 1$$

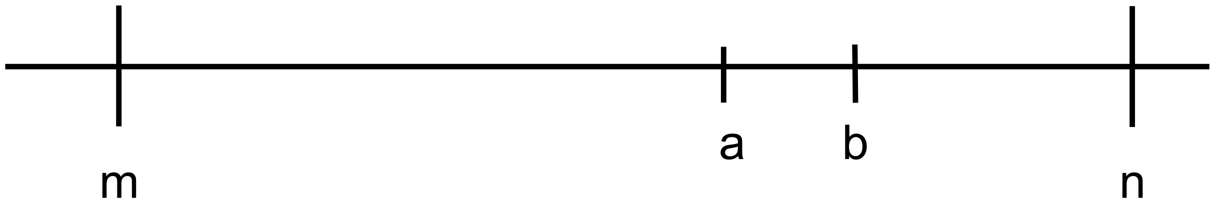
It implies that

$$p(X = x_i) = 0 \qquad p(X = 16kg) = 0$$

Continuous Probability Distribution

A continuous random distribution does not have a PMF, rather it has Probability Density Function (PDF).

$$m \leq X \leq n$$



$$p(a \leq \mathrm{X} \leq b) = ?$$

$$e.g.\, p(17 \leq \mathrm{X} \leq 19) = ?$$

Let $f_X(x)$: is a Probability Density Function, then

$$p(a \leq X \leq b) = \int_a^b f_X(x)\, \mathrm{d}x$$

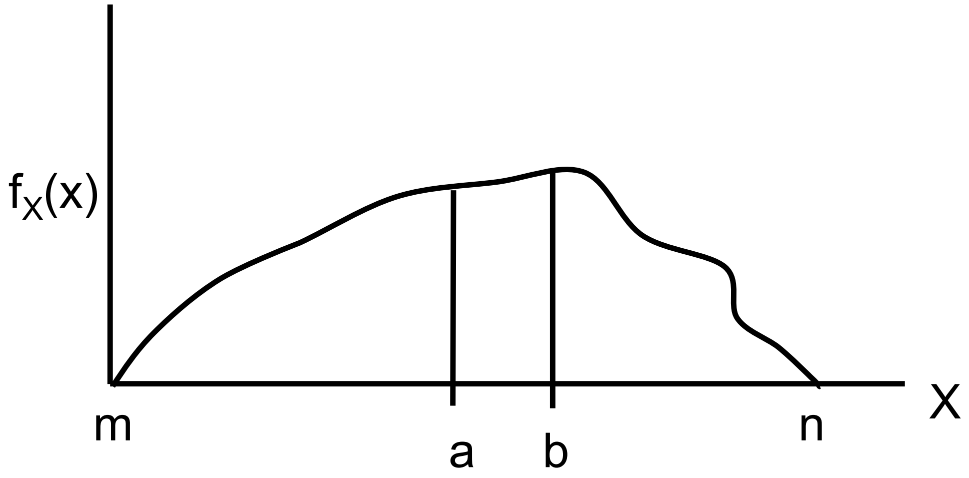
where $f_X(x)$ satisfies

$$f_X(x) \geq 0 \qquad \int_m^n f_X(x)\, \mathrm{d}x = 1$$

Continuous Probability Distribution

Area under curve

$$p(a \leq X \leq b) = \int_{a}^{b} f_X(x)\,\mathrm{d}x$$



It is to be noted that the value $f_X(x)$ is not a probability. It is probability density function. In fact, $f_X(x)$ is probability/interval range. The area under the curve gives the probability of a random variable between a and b.

Continuous Probability Distribution

Normal Distribution

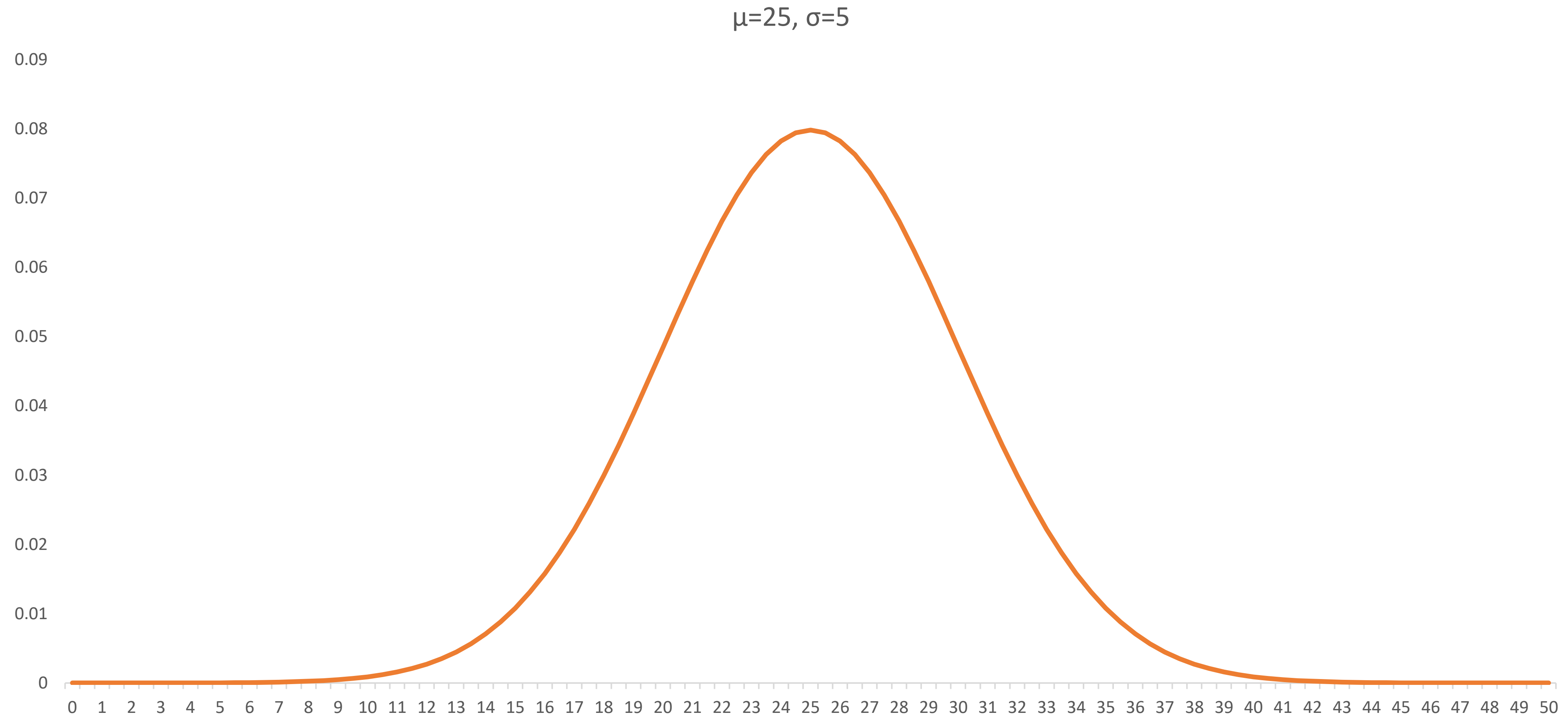
Probability distribution function (PDF) of a random variable X which follows a Normal Distribution depends on two parameters, i.e., X ~ N(μ, σ), where is μ mean, and σ is standard deviation.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \leq x \leq +\infty$$

We can draw Normal distribution for various types of data that include,
- Distribution of height of people
- Distribution of errors in any measurement
- Distribution of blood pressure of any patient, etc.

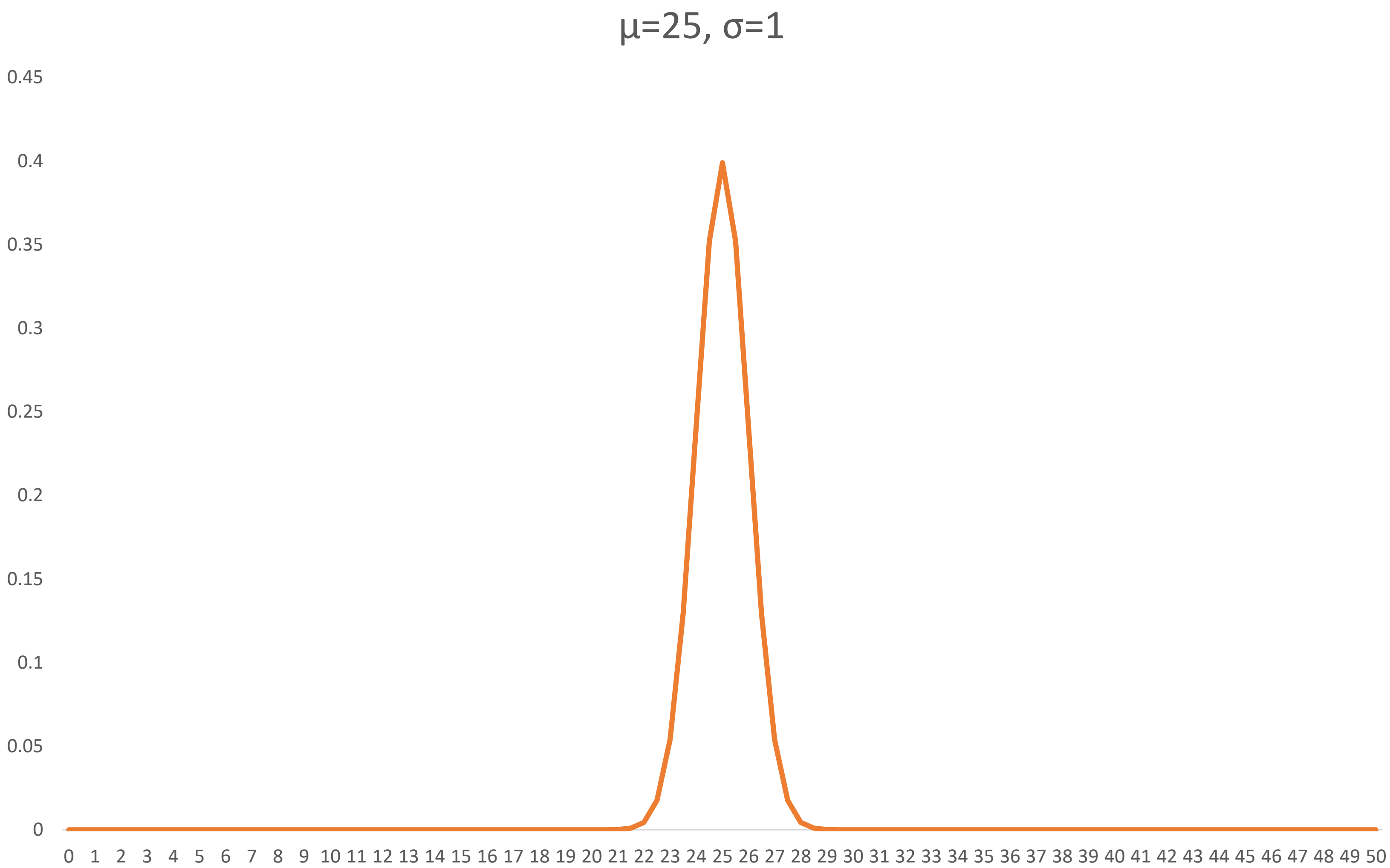
Continuous Probability Distribution

Normal Distribution



Continuous Probability Distribution

Normal Distribution



Continuous Probability Distribution

Normal Distribution

Given the PDF of Normal Distribution (25, 5), how to calculate the probability of a random variable X, e.g.

$$P(20 \leq X \leq 30) = ?$$

$$P(20 \leq X \leq 30) = \int_{20}^{30} f_X(x)\,\mathrm{d}x \qquad \text{where} \qquad f_X(x) = \frac{1}{5\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}$$

Similarly, given the PDF of Normal Distribution (25, 5), how to calculate the probability of a random variable X, e.g.

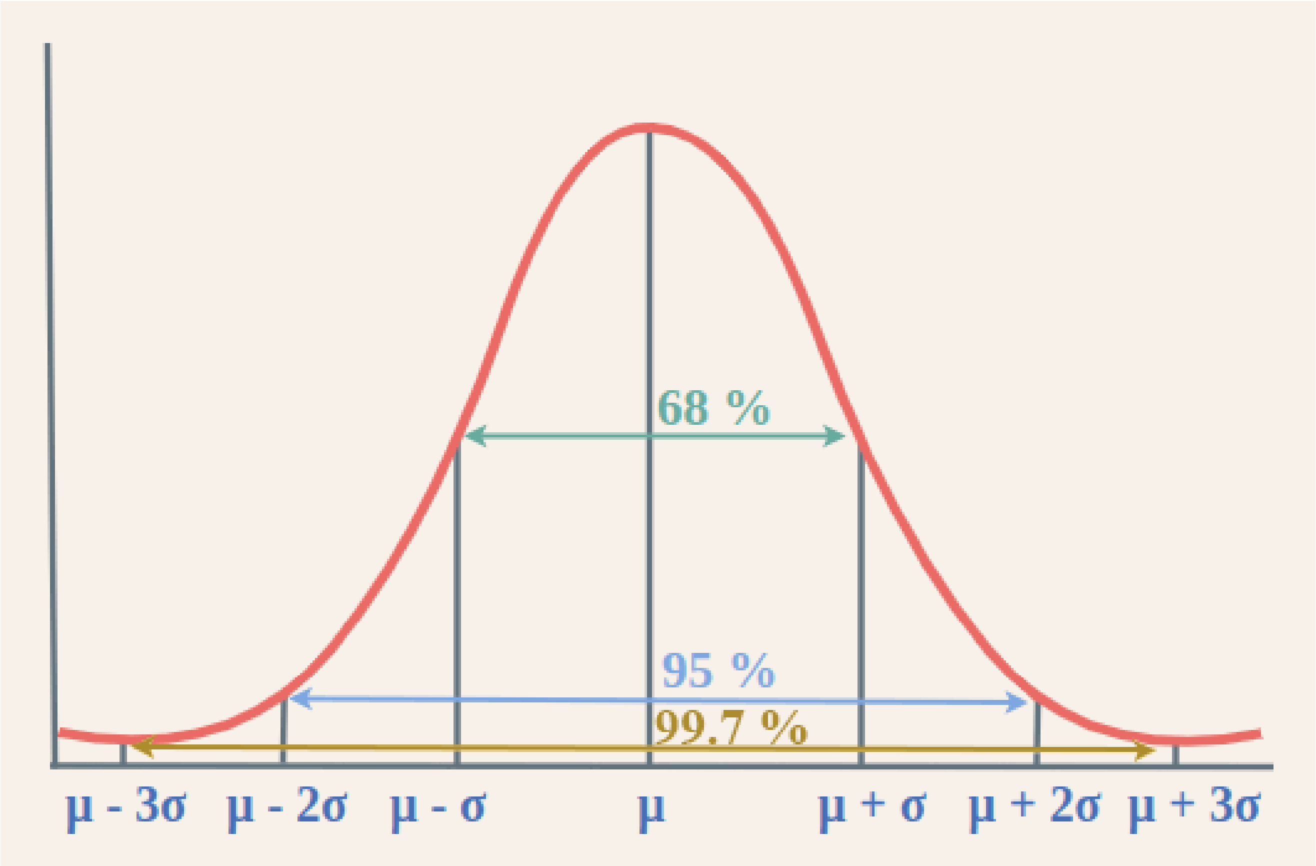
$$P(X \geq 30) = \int_{30}^{+\infty} f_X(x)\,\mathrm{d}x \qquad \text{where} \qquad f_X(x) = \frac{1}{5\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}$$

Continuous Probability Distribution

Normal Distribution

Empirical rule of standard deviation (std.)

- ~68% of the data fall within one std. of the mean, i.e. ($\mu \pm 1\sigma$}
- ~95% of the data fall within two std. of the mean, i.e. ($\mu \pm 2\sigma$}
- ~99.7% of the data fall within a third std. of the mean, i.e. ($\mu \pm 3\sigma$}

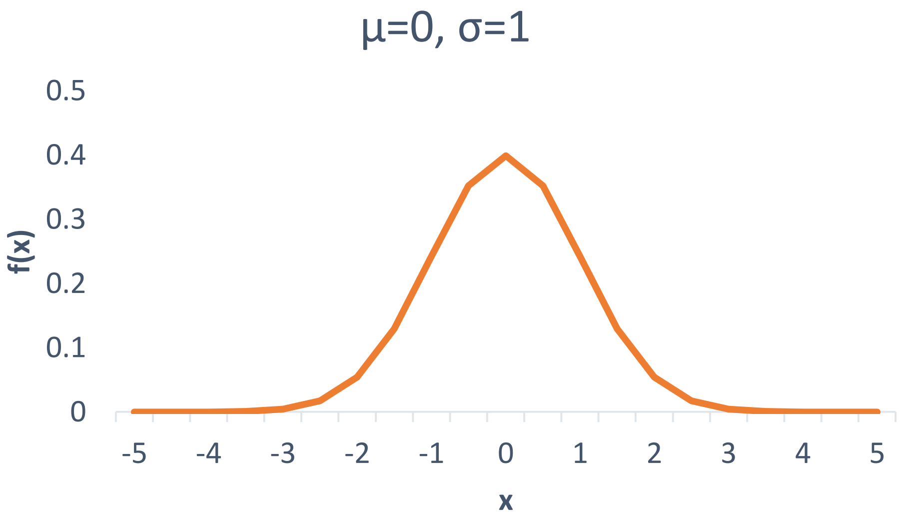


Continuous Probability Distribution

Standard Normal Distribution

Standard Normal Distribution ~N(0, 1). It is also called **z-distribution**.

$$f_X(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}$$



Continuous Probability Distribution

Normal Distribution Z Table: The table shows the area from 0 to Z**.**

Z-Value	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0	0.004	0.008	0.012	0.016	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.091	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.148	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.17	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.195	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.219	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.258	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.291	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.334	0.3365	0.3389
1	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.377	0.379	0.381	0.383
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.398	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.437	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.475	0.4756	0.4761	0.4767
2	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817

Continuous Probability Distribution

Example 1:
Find the probability density function of the normal distribution of the following data. X = 2, μ = 3 and σ = 4.

Given, Variable (X) = 2, Mean = 3, Standard Deviation = 4

Using the formula of the probability density of normal distribution

$$f(x,\mu,\sigma) = \frac{1}{\sqrt[\sigma]{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Simplifying we get the answer as,
f(2, 3, 4) = 0.09666703

Continuous Probability Distribution

Example 2:
If the value of the random variable is 4, the mean is 4 and the standard deviation is 3, then find the probability density function of the Gaussian distribution.

Given, Variable (X) = 4,   Mean = 4,   Standard Deviation = 3

Using the formula of the probability density of normal distribution

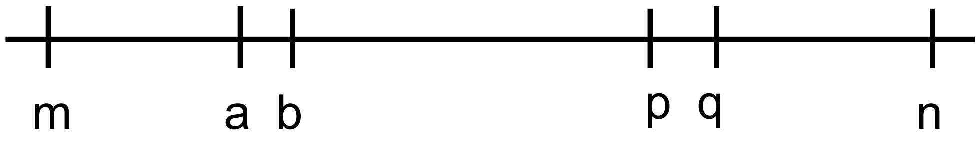
$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Simplifying we get the answer as,
$f(4, 4, 3) = 1/(3\sqrt{2}\pi)e^0$
f(4, 4, 3) = 0.13301

Continuous Probability Distribution

Uniform Distribution

Suppose, we have a number line as follows



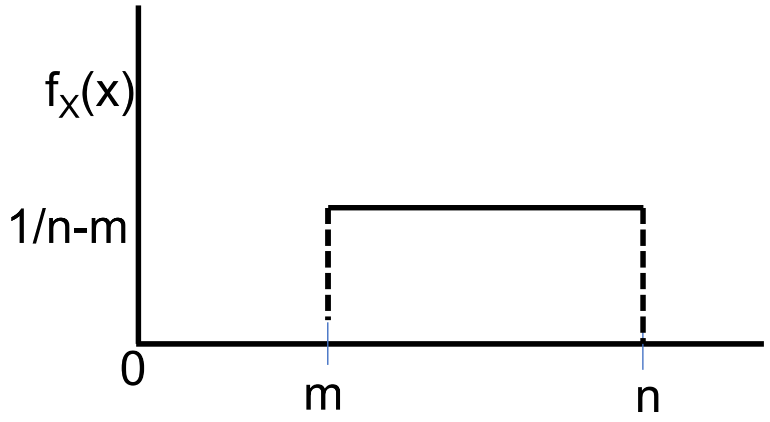
Such that (b - a) = (q – p) = L

Let X be random variable such that m ≤ X ≤ n and if

$$P(a \leq X \leq b) = P(p \leq X \leq q)$$

Then, we say that X ~ U(m, n). The PDF of X is given as

$$f_X(x) = \begin{cases} \dfrac{1}{n-m} & \text{i}f m \leq X \leq n \\ 0 & \text{otherwise} \end{cases}$$



Continuous Probability Distribution

Mean of uniform distribution

$$\text{Mean} = (m + n) / 2$$

Variance of uniform distribution

$$\text{Variance} = (n - m)^2 / 12$$

Median of uniform distribution

$$\text{Median} = (m + n) / 2$$

Continuous Probability Distribution

Example 3:
A random variable X has a uniform distribution over (-2, 2),
(i) find k for which P(X>k) = 1/2 (ii) Evaluate P(X<1) (iii) P[|X-1|<1]

(i) $f_X(x) = 1/n\text{-}m = 1/(2+2) = ¼$
$P(X > k) = 1 – P(X \leq k)$
$\Rightarrow 1 - \int_{-2}^{k} f_x(x)\,\mathrm{d}x = 1 – ¼ \int_{-2}^{k} \mathrm{d}x = 1 – ¼\,[k + 2] = 1 – k/4 – ½$
=> for k = 0, P(X > k) = 1/2

(ii) $P(X < 1) = \int_{-2}^{1} f_X(x)\,\mathrm{d}x = ¼\,[1+2] = ¾$

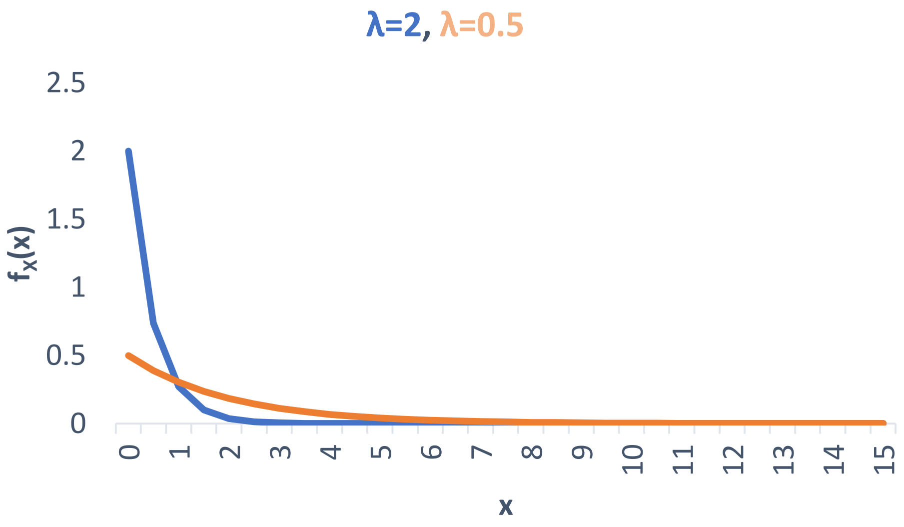
(iii) $P(|X-1| < 1) = P(0 < X < 2) = \int_{0}^{1} f_X(x)\,\mathrm{d}x = ¼[1-0] = 1/4$

Continuous Probability Distribution

Exponential Distribution

If a random variable X follows Exponential distribution, X~E(λ) with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & otherwise \end{cases}$$



Continuous Probability Distribution

Mean of exponential distribution

$$\text{Mean} = 1/\lambda$$

Variance of exponential distribution

$$\text{Variance} = 1/\lambda^2$$

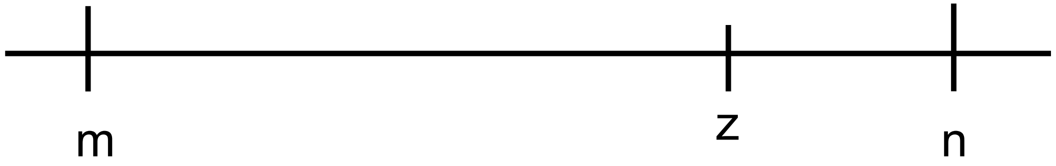
Median of exponential distribution

$$\text{Median} = \ln(2)/\lambda$$

Continuous Probability Distribution

Cumulative Distribution Function (CDF)

If X is a continuous random variable such that m ≤ X ≤ n,



Then, P (X ≤ z) = ?

$$P(X \leq z) = \int_{m}^{z} f_X(x)\,\mathrm{d}x$$

It is called cumulative distribution function (CDF), $F_X(z)$ = P (X ≤ z).

Similarly, cumulative distribution function (CDF) for discrete random variable is $F_X(k)$ = P (X ≤ k),where

$$p(X \leq k) = \sum_{i=a}^{k} P(X = i)$$

Thank You