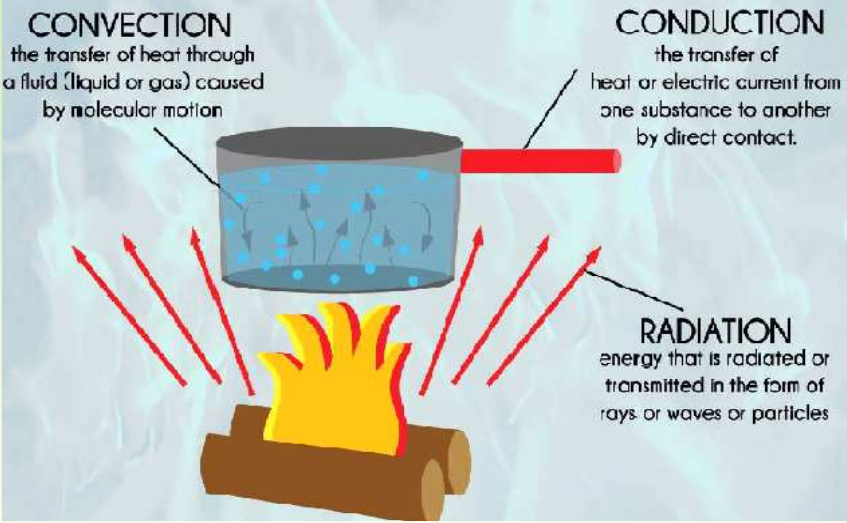
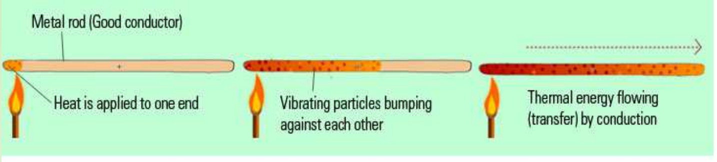
MA 201, Mathematics III, July-November 2022, Partial Differential Equations: One-dimensional heat conduction equation

Lecture 13

Heat Conduction Problem



Heat conduction in a thin rod



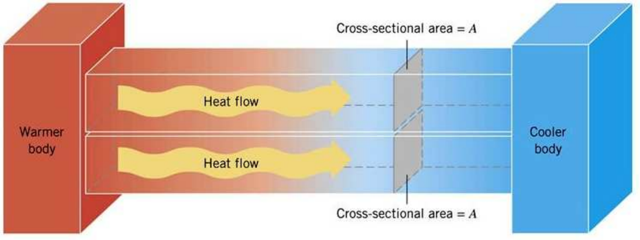
One important and familiar problem in engineering is the heat conduction in a thin metallic rod of finite length.

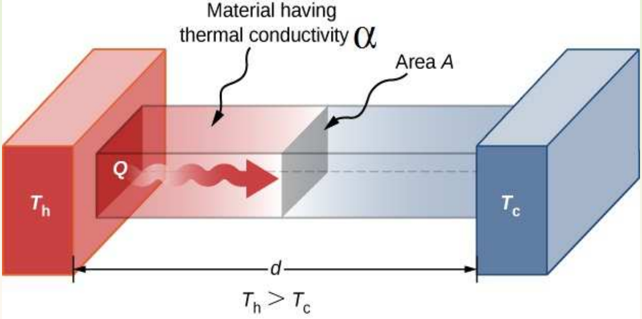
This transient problem represents a class of problem known as diffusion problems.

Just like one-dimensional wave equation, separation of variables can be appropriately applied for this equation (IBVP) too.

An IBVP can be formulated by taking into account the governing equation and the associated conditions.

Heat conduction in a thin rod (Contd.)





Heat conduction in a thin rod (Contd.): Case I Homogeneous Dirichlet boundary condition

The (simple) IBVP under consideration for heat conduction in a thin metallic rod of length L consists of

The governing equation

$$u_t = \alpha u_{xx}, \;\; 0 < x < L, \;\; t > 0, \tag{1}$$

where α is the thermal diffusivity or conductivity.

The boundary conditions for all $t \geq 0$ due to maintaining the temperature at both ends at zero degrees:

$$u(0,t) = 0, \tag{2a}$$

$$u(L,t) = 0. \tag{2b}$$

The initial condition for $0 \leq x \leq L$, i.e., initial temperature distribution:

$$u(x,0) = \phi(x), \;\; 0 < x < L. \tag{3}$$

The solution $u = u(x,t)$ gives the temperature of the rod at any point x for any $t > 0$.

Heat conduction in a thin rod (Contd.)

The assumptions are as follows:

- The rod is very thin so that it can be considered as one-dimensional.
- The curved surface is insulated.
- There is no external source of heat. (The initial temperature controls the process for this case)

We use the separation of variables technique by assuming a solution of the form

$$u(x,t) = X(x)T(t).$$

The governing equation is converted to the following ODEs:

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha T &= 0. \end{aligned}$$

where k is a separation constant. (REFER TO LECTURES 11 AND 12)

Observe that the boundary conditions for this IBVP are the same as those in the one-dimensional wave equation although each equation (IBVP) presents a different physical scenario.

Heat conduction in a thin rod (Contd.)

We can immediately assume that a feasible nontrivial solution exists only for the negative values of $k.$ (BCs for both IBVPs for one-dimensional wave equation and heat conduction equation are the same.)

Above equations $\Rightarrow$

$$X'' + \lambda^2 X = 0, \tag{4}$$

$$T' + \lambda^2 \alpha T = 0. \tag{5}$$

The solutions:

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x), \tag{6}$$

$$T(t) = Ce^{-\alpha\lambda^2 t}. \tag{7}$$

Solution for $u(x,t)$:

$$u(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{8}$$

Using the boundary conditions (2a) and (2b)

$B = 0$ and $\lambda_n = \dfrac{n\pi}{L},\ n = 1, 2, 3, \ldots.$

Heat conduction in a thin rod

Solution corresponding to each n is obtained as

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\ e^{-\alpha \frac{n^2\pi^2}{L^2}\ t}.$$

The general solution:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\ e^{-\alpha \frac{n^2\pi^2}{L^2}\ t}, \end{aligned} \tag{9}$$

where B_n is obtained (refer to the solution of the one-dimensional wave equation) by using the initial condition (3):

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right)\ dx,\ n = 1, 2, 3, \ldots \tag{10}$$

Observe that what we have solved here is a homogeneous equation

subject to homogeneous (zero) conditions. Ideally the boundary conditions are very likely to be non-homogeneous (non-zero) for diffusion problems.

Heat conduction in a thin rod: Case II
Non-homogeneous Dirichlet boundary conditions

Consider the heat conduction in a thin metal rod of length L with insulated sides.

The ends $x = 0$ and $x = L$ are maintained at temperatures $u = {u_0}^0$C and $u = {u_L}^0$C, respectively, for all time $t \geq 0$.

Suppose that the temperature distribution at $t = 0$ is $u(x, 0) = \phi(x),\ 0 < x < L$.

To determine the temperature distribution at any point on the rod at some subsequent time $t > 0$.

Heat conduction in a thin rod

The IBVP under consideration consists of

The governing equation

$$u_t = \alpha u_{xx}, \;\; 0 < x < L, \; t > 0, \tag{11}$$

where α is the coefficient of thermal diffusivity for the specific metal.

The boundary conditions:

$$\begin{aligned} u(0,t) &= u_0, \;\; t \geq 0, & \text{(12a)} \\ u(L,t) &= u_L, \;\; t \geq 0. & \text{(12b)} \end{aligned}$$

The initial condition for $0 \leq x \leq L$:

$$u(x,0) = \phi(x). \tag{13}$$

Heat conduction in a thin rod

Due to the nonhomogeneous boundary conditions (12),

direct application of the method of separation of variables will not work.

We intend to convert the given IBVP into two problems:

one resembling the problem we have solved previously (an IBVP),

the other taking care of the nonhomogeneous terms (a BVP).

Seek a solution in the form:

$$u(x,t) = v(x,t) + h(x), \tag{14}$$

where $h(x)$ is an unknown function of x alone to take care of non-homogeneous boundary conditions such that

$$h(0) = u_0 \ \ \& \ \ h(L) = u_L.$$

Heat conduction in a thin rod

Using $u_t - \alpha u_{xx} = 0$, we have

$$v_t = \alpha[v_{xx} + h''(x)] \tag{15}$$

subject to

$$v(0,t) + h(0) = u_0, \;\; t \geq 0, \tag{16a}$$

$$v(L,t) + h(L) = u_L, \;\; t \geq 0, \tag{16b}$$

and

$$v(x,0) + h(x) = \phi(x), \;\; 0 < x < L. \tag{17}$$

Now equations (15)-(17) can be conveniently split into two problems as follows:

Heat conduction in a thin rod

Problem I (BVP):

$$\alpha h''(x) = 0, \;\; 0 < x < L,$$
$$h(0) = u_0, \; h(L) = u_L.$$

Problem II (IBVP):

$$v_t = \alpha v_{xx}, \;\; 0 < x < L, \;\; t > 0,$$
$$v(0,t) = 0 = v(L,t), \;\; t \geq 0,$$
$$v(x,0) = \phi(x) - h(x), \;\; 0 < x < L.$$

The solution of Problem II is known to us from **Case I**, which is

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \; e^{-\alpha \frac{n^2\pi^2}{L^2} \, t}, \tag{18}$$

where

$$A_n = \frac{2}{L}\int_0^L (\phi(x) - h(x)) \sin\left(\frac{n\pi x}{L}\right) \, dx, \; n = 1, 2, 3, \ldots \tag{19}$$

Heat conduction in a thin rod

The solution for Problem I can be easily found in two steps by integrating $h''(x)$ and $h'(x)$:

$$h(x) = Ax + B.$$

Upon using the boundary conditions, we get

$B = u_0$ and $A = (u_L - u_0)/L$.

Hence

$$h(x) = \frac{u_L - u_0}{L}\, x + u_0. \tag{20}$$

Hence the solution $u(x, t)$ for our IBVP is given by

the sum of (18) and (20).

Heat conduction problem in a thin rod

Consider the following heat conduction equation in a thin rod of length L:

$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{21}$$

The initial condition for $0 \leq x \leq L$:

$$u(x, 0) = \phi(x). \tag{22}$$

In practice, temperature u satisfies certain boundary conditions as follows:

(a) Dirichlet Condition: $u(0, t) = u_0, \;\; u(L, t) = u_L, \; t \geq 0,$

(b) Neumann Condition: $u_x(0, t) = u_0, \;\; u_x(L, t) = u_L, \; t \geq 0,$

(c) Both types: $u_x(0, t) = u_0, \;\; u(L, t) = u_L, \; t \geq 0,$ **Or**

$u(0, t) = u_0, \;\; u_x(L, t) = u_L, \; t \geq 0,$

(d) Robin Condition: $u_x(0, t) + a_0 u(0, t) = \alpha, \;\; u_x(L, t) + a_L u(L, t) = \beta.$

Heat conduction problems with different conditions at ends

Problem I

Governing equation: $u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0,$

Boundary conditions: $u(0,t) = 0 = u(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\ e^{-\alpha \frac{n^2\pi^2}{L^2}\ t},$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Heat conduction problems with different conditions at ends

Problem II

Governing equation: $u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0,$

Boundary conditions: $u_x(0,t) = 0 = u(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \operatorname{exp}\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{L^2} t\right] \cos\left(\frac{(2n+1)\pi x}{2L}\right),$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right) dx.$$

Heat conduction problems with different conditions at ends

Problem III

Governing equation: $u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0,$

Boundary conditions: $u(0,t) = 0 = u_x(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

Solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{L^2} t\right] \sin\left(\frac{(2n+1)\pi x}{2L}\right),$$

with

$$A_n = \frac{2}{L}\int_0^l \phi(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right) dx.$$

Heat conduction problems with different conditions at ends

Problem IV

Governing equation: $u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0,$

Boundary conditions: $u_x(0,t) = 0 = u_x(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\ \exp\left[-\alpha \frac{n^2\pi^2}{L^2}\ t\right]$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{n\pi x}{L}\right) dx.$$

TRY

solving problems II-IV yourself.