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Linear Programming: Simplex Method

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Simplex Method for LP: https://www.youtube.com/watch?v=VsyFFhzQVZM
Branch & Bound Method for MILP: https://www.youtube.com/watch?v=g1Xtmd94zns
Additional resources: tinyurl.com/sksopti, tinyurl.com/sksoptivid

Linear programming (LP)

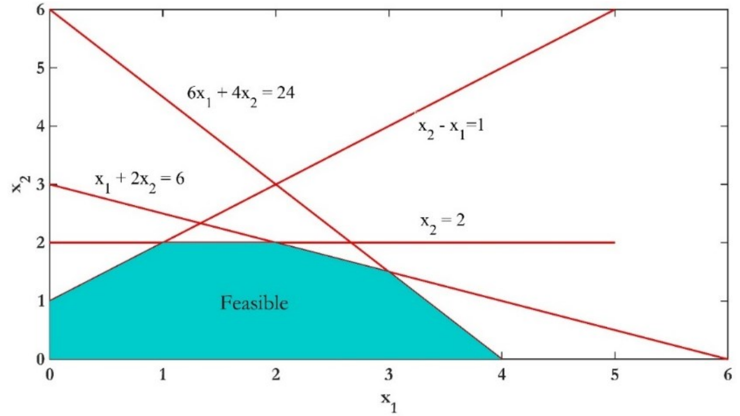
$$
\begin{aligned}
\textit{Minimize / Maximize} \quad & z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\
\textit{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m
\end{aligned}
$$

$$
\begin{aligned}
\textit{Minimize / Maximize} \ & z = c^T X \\
\textit{subject to} \quad & AX \leq b \\
& A_{eq}X = b_{eq}
\end{aligned}
$$

Linear objective function

Linear equality constraints
Linear inequality constraints

Number of decision variables: n
Number of constraints: m



- Applications in resource allocation, production scheduling, workforce planning, transportation, etc.
- At least one optimal solution lies on one of the vertices of the feasible region.
- Algorithms: Simplex method and interior point method

Simplex method

- Developed by George Dantzig in 1947 for solving optimal resource allocation problem.
- Can handle large number of decision variables and constraints.
- Solves for optimum by visiting the vertices (or corner points) of the feasible region.
- To apply the Simplex method
  - Objective function should be maximized
  - All constraints should be expressed as less-than-or-equal-to constraints i.e., $Ax \leq b$.
  - All the decision variables should be non-negative, i.e., $x_i \geq 0$.
  - Right hand side of the constraints should be non-negative, i.e., $b_i \geq 0$.

$$
\begin{aligned}
\textit{Maximize} \quad & z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\textit{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
& x_i \geq 0 \quad i = 1, 2, \ldots
\end{aligned}
$$

Converting general LP into standard LP

- Objective function should be maximization
- All inequality constraints should be converted into equations
- RHS of all the equations must be non-negative
- All variables should be non-negative

Conversion of minimization problem to maximization

Minimize Z → Maximize $(-Z)$

Let *Minimize* $Z = 5x_1 + 4x_2$ be the objective function

Minimize	**Maximize**
$Z = 5x_1 + 4x_2$	$Z = -(5x_1 + 4x_2)$

Converting general LP into standard LP

"$\leq$" inequality constraint can be converted to an equality constraint by introducing a *slack* variable.

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \Rightarrow a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + s_i = b_i$$

s_i : slack variable

$s_i \geq 0$

$$x_1 + 2x_2 \leq 10 \Rightarrow x_1 + 2x_2 + s_1 = 10$$

"$\geq$" inequality constraint can be converted to an equality constraint by introducing a *surplus* variable.

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \Rightarrow a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - s_i = b_i$$

s_i : surplus variable

$s_i \geq 0$

$$3x_1 + 2x_2 \geq 11 \Rightarrow 3x_1 + 2x_2 - s_1 = 11$$

Converting general LP into standard LP

Converting negative RHS value to non-negative

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = -b_i \quad \Rightarrow \quad -\left(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n\right) = b_i$$

$$3x_1 + x_2 = -11 \quad \Rightarrow \quad -(3x_1 + x_2) = 11$$

Converting negative/unrestricted variable to non-negative

Let x_i be an unrestricted variable

$$x_i = x_i' - x_i''$$

$3x_1 + x_2 \leq 9$, where x_1 is an unrestricted variable and $x_2 \geq 0$

$$3\left(x_1' - x_1''\right) + x_2 + S_1 = 9$$

$$x_1' \geq 0, x_1'' \geq 0, x_2 \geq 0, S_1 \geq 0$$

Converting general LP into standard LP

Objective function	Minimize Z $Minimize\ Z = 5x_1 + 4x_2$	Maximize (-Z) $Maximize\ Z = -(5x_1 + 4x_2)$
" ≤" inequality constraint can be converted to an equality constraint by introducing a *slack* variable	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq b_i$ $x_1 + 2x_2 \leq 10$	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + s_i = b_i$ s_i is slack variable, $s_i \geq 0$ $x_1 + 2x_2 + s_1 = 10$
" ≥ " inequality constraint can be converted to an equality constraint by introducing a *surplus* variable	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \geq b_i$ $3x_1 + 2x_2 \geq 11$	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n - s_i = b_i$ s_i is surplus variable, $s_i \geq 0$ $3x_1 + 2x_2 - s_1 = 11$
Converting negative RHS value to non-negative	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = -b_i$ $3x_1 + x_2 = -11$	$-(a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n) = b_i$ $-(3x_1 + x_2) = 11$
Converting negative/unrestricted variable to non-negative	Let x_i be an unrestricted variable $3x_1 + x_2 \leq 9$, where x_1 is an unrestricted variable and $x_2 \geq 0$	$x_i = x_i^{'} - x_i^{"}$ $3(x_1^{'} - x_1^{"}) + x_2 + s_1 = 9$ $x_1^{'} \geq 0, x_1^{"} \geq 0, x_2 \geq 0, s_1 \geq 0$

Linear Programming

- Reddy Mikks company produces interior and exterior paints from raw materials, M1 and M2.
- Daily demand for interior paint cannot exceed that for exterior paint by more than 1 unit.
- Maximum daily demand for the interior paint is 2 units.
- Determine optimum quantity of interior and exterior paints that maximizes total daily profit.

	Exterior paint	**Interior paint**	**Availability**
M1	6	4	24
M2	1	2	6
Profit	5	4	

Linear Programming

Let x_1 = Units of exterior paint produced daily

x_2 = Units of interior paints produced daily

Decision variables

Maximize Profit, $Z = 5x_1 + 4x_2$

Objective function

Subject to

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

Raw material constraints

$x_2 \leq x_1 + 1$

Demand constraint

Constraints

$x_1, x_2 \geq 0$

$x_2 \leq 2$

Bound constraints

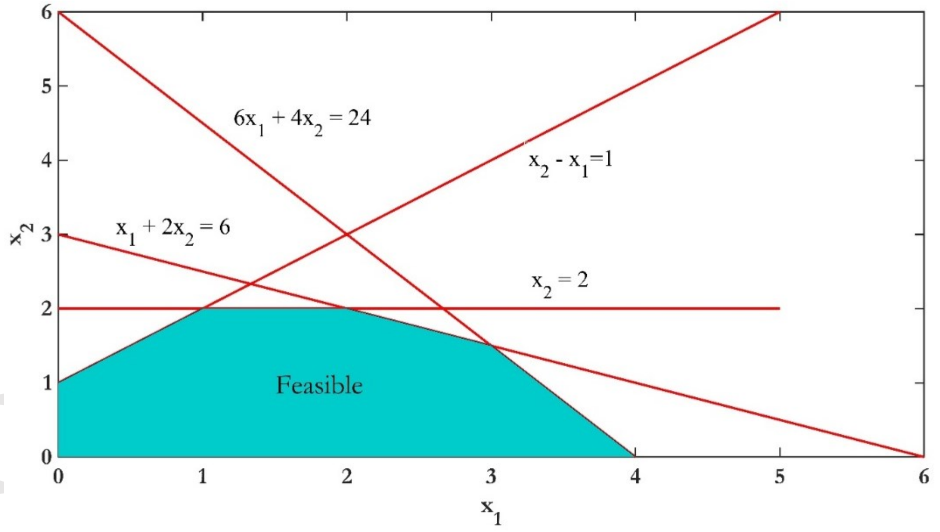
	Ext. paint	Int. paint	Availability
M1	6	4	24
M2	1	2	6
Profit	5	4	

Daily demand for interior paint cannot exceed that for exterior paint by more than 1 Unit.

Algebraic form

$$\text{Maximize } z = 5x_1 + 4x_2$$

$$\textit{subject to} \quad 6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$x_2 \le x_1 + 1$$
$$x_1 \ge 0,\ x_2 \le 2$$



Algebraic form

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$$

Number of non-basic variables = Number of variables – Number of constraints

$= 6 - 4 = 2$

Algebraic form

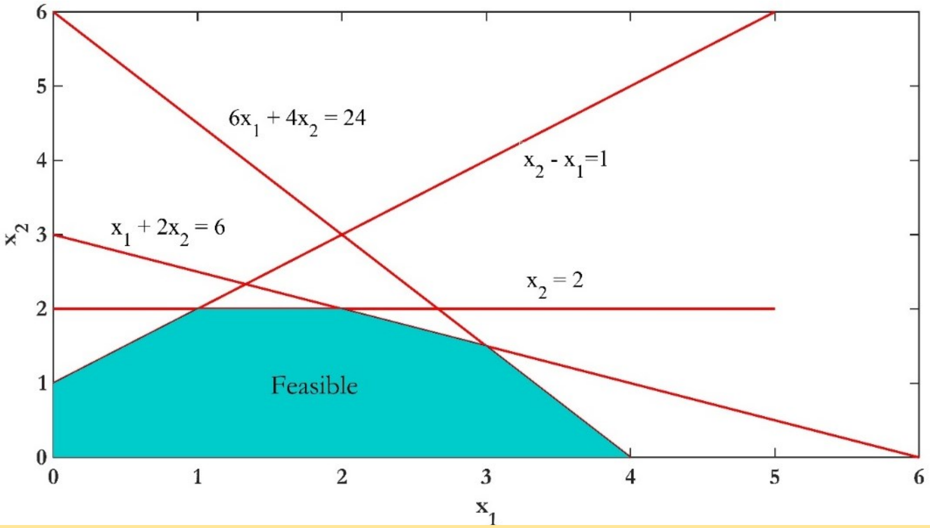
Maximize $z = 5x_1 + 4x_2$

subject to $6x_1 + 4x_2 \le 24$

$x_1 + 2x_2 \le 6$

$x_2 \le x_1 + 1$

$x_1 \ge 0,\ \ x_2 \le 2$



Algebraic form

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$$

For a system of equations $\{\mathrm{A}x = \mathrm{b}, x \ge 0\}$ with A being a $\mathrm{M} \times \mathrm{N}$ matrix of rank M and b an $\mathrm{M} \times 1$ vector

- A point $\mathrm{x} = [x_\mathrm{B}\ x_\mathrm{N}]$ is called a basic solution of the system, $x_\mathrm{B} = \mathrm{B}^{-1}\mathrm{b}$ and $x_\mathrm{N} = 0$.
- Elements of x_B and x_N are called basic and non-basic variables respectively.

Number of non-basic variables = Number of variables – Number of constraints

$$= 6 - 4 = 2$$

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

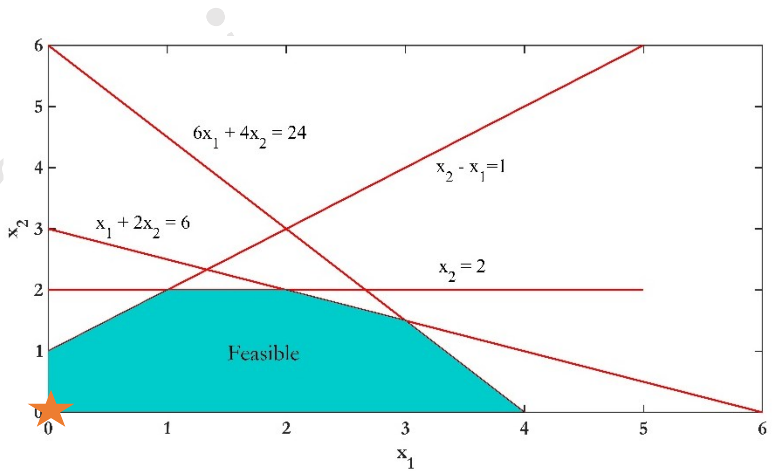
Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

$$z - 5x_1 - 4x_2 = 0$$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative



Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	$\boxed{\mathbf{-5}}$	$\boxed{-4}$	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the ‘most negative’ coefficient in the first row

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	**-5**	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	**-5**	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	**-5**	-4	0	0	0	0	0	
s_1	0	6	4	1	0	0	0	24	24/6 = 4
s_2	0	1	2	0	1	0	0	6	6/1 = 6
s_3	0	-1	1	0	0	1	0	1	1/-1 = -1
s_4	0	0	1	0	0	0	1	2	2/0= ∞

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column
- **Pivot row:** row with the minimum positive value for the ratio

Entering variable

Basic	z	x_1								
z	1	**-5**	-4	0	0	0	0	0		
s_1	0	6	4	1	0	0	0	24	24/6 = 4	Minimum
s_2	0	1	2	0	1	0	0	6	6/1 = 6	
s_3	0	-1	1	0	0	1	0	1	1/-1 = -1	Ignore
s_4	0	0	1	0	0	0	1	2	2/0= ∞	Ignore

Simplex method: Iteration 1

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column
- **Pivot row:** row with the minimum positive value for the ratio
- **Leaving basic variable:** basic variable of the pivot row
- **Pivot element:** element at the intersection of the pivot column and pivot row

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	**-5**	-4	0	0	0	0	0	
s_1	0	6	4	1	0	0	0	24	4
s_2	0	1	2	0	1	0	0	6	6
s_3	0	-1	1	0	0	1	0	1	-1
s_4	0	0	1	0	0	0	1	2	∞

Leaving variable

Minimum

Simplex method: Iteration 1

Entering variable

Leaving variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	**-5**	-4	0	0	0	0	0	
s_1	0	6	4	1	0	0	0	24	4
s_2	0	1	2	0	1	0	0	6	6
s_3	0	-1	1	0	0	1	0	1	-1
s_4	0	0	1	0	0	0	1	2	∞

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	**-5**	-4	0	0	0	0	0
x_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 1

Determine the next basic feasible solution by dividing pivot row by pivot element

New pivot row = Current pivot /pivot element

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
x_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
x_1	0	1	4/6	1/6	0	0	0	24/6 = 4
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Row (x_1) = Row (x_1) /6

Simplex method: Iteration 1

- Determine the next basic feasible solution by dividing pivot row by pivot element
- Perform row operations to make all element in pivot column except pivot element to zero

New row = Current row – current row pivot column coefficient x New pivot row

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	-5	-4	0	0	0	0	0
x_1	0	1	4/6	1/6	0	0	0	4
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
s_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Row z=Row z – (– 5) x Row x_1

Row z=[1 -5 -4 0 0 0 0 0]– (– 5) x [0 1 4/6 1/6 0 0 0 4]

Row s_2=Row s_2 – 1 x Row x_1

Row s_2=[0 1 2 0 1 0 0 6]– 1 x [0 1 4/6 1/6 0 0 0 4]

Row s_3=Row s_3 – (– 1) x Row x_1

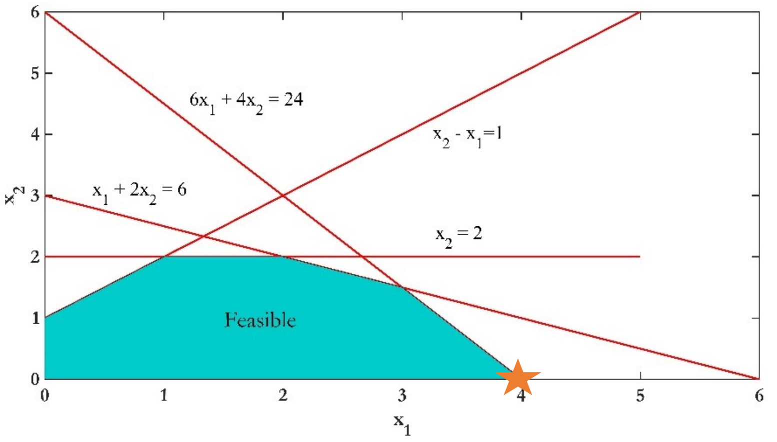
Row s_3=[0 -1 1 0 0 1 0 1]– (– 1) x [0 1 4/6 1/6 0 0 0 4]

Row s_4=Row s_4 – 0 x Row x_1

Row s_4=[0 0 1 0 0 0 1 2] – 0 x [0 1 4/6 1/6 0 0 0 4]

Simplex method: Iteration 1

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	$\boxed{-2/3}$	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
s_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2



- Optimality check:
  - Every coefficient of the non-basic variables in first row should be non-negative.
  - Solution not optimal — **Coefficient of x_2 is negative**
  - Perform next iteration

Simplex method: Iteration 2

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
s_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 2

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x_1	0	1	2/3	1/6	0	0	0	4	6
s_2	0	0	4/3	-1/6	1	0	0	2	3/2
s_3	0	0	5/3	1/6	0	1	0	5	3
s_4	0	0	1	0	0	0	1	2	2

Simplex method: Iteration 2

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the 'most negative' coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column
- **Pivot row:** row with the minimum positive value for the ratio

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x_1	0	1	2/3	1/6	0	0	0	4	6
s_2	0	0	4/3	-1/6	1	0	0	2	3/2
s_3	0	0	5/3	1/6	0	1	0	5	3
s_4	0	0	1	0	0	0	1	2	2

Minimum

Simplex method: Iteration 2

- **Optimality criteria:** coefficient of the non-basic variables in first row should be non-negative
- **Entering basic variable:** variable with the ‘most negative’ coefficient in the first row
- **Pivot column:** column of the entering basic variable
- **Ratio:** divide right side by the corresponding element of the pivot column
- **Pivot row:** row with the minimum positive value for the ratio
- **Leaving basic variable:** basic variable pivot row
- **Pivot element:** element at the intersection of the pivot column and pivot row

Entering variable

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x_1	0	1	2/3	1/6	0	0	0	4	6
s_2	0	0	4/3	-1/6	1	0	0	2	3/2
s_3	0	0	5/3	1/6	0	1	0	5	3
s_4	0	0	1	0	0	0	1	2	2

Leaving variable

Minimum

Simplex method: Iteration 2

Entering variable: x_2

Leaving variable: s_2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x_1	0	1	2/3	1/6	0	0	0	4	6
s_2	0	0	4/3	-1/6	1	0	0	2	3/2
s_3	0	0	5/3	1/6	0	1	0	5	3
s_4	0	0	1	0	0	0	1	2	2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
x_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Simplex method: Iteration 2

Determine the next basic feasible solution by dividing pivot row by pivot element

New pivot row = Current pivot /pivot element

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
x_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Row (x_2) = Row (x_2) /(4/3)

Simplex method: Iteration 2

- Determine the next basic feasible solution by dividing pivot row by pivot element
- Perform row operations to make all element in pivot column except pivot element to zero

New row = Current row – current row pivot column coefficient x New pivot row

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Row z: Row z – (– 2/3) x Row x_2

Row x_1: Row x_1 – (2/3) x Row x_2

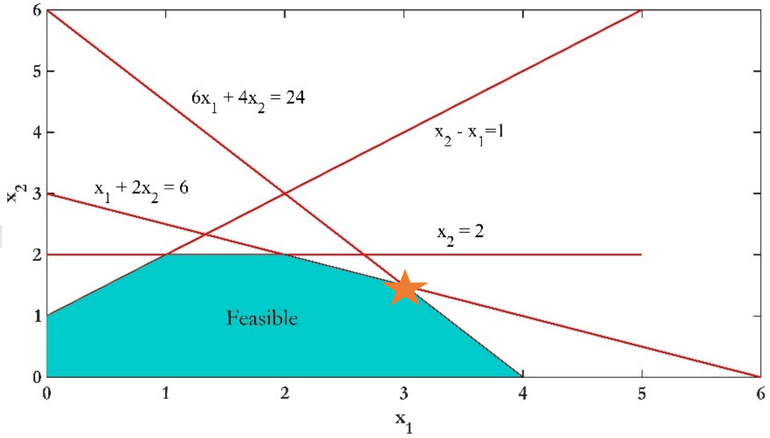
Row s_3: Row s_3 – (5/3) x Row x_2

Row s_4: Row s_4 – 1 x Row x_2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	3/4	1/2	0	0	21
x_1	0	1	0	1/4	-1/2	0	0	3
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	0	1/8	-3/4	0	1	1/2

Simplex method: Iteration 2

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	**3/4**	**1/2**	0	0	21
x_1	0	1	0	1/4	-1/2	0	0	3
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	0	1/8	-3/4	0	1	1/2



- Optimality check:
  - Every coefficient of the non-basic variables in first row should be non-negative.
  - Solution is optimal

Optimal solution: $x_1 = 3$, $x_2 = 1.5$, and objective function value, $z = 21$

Post optimality analysis

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	3/4	1/2	0	0	21
x_1	0	1	0	1/4	-1/2	0	0	3
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	0	1/8	-3/4	0	1	1/2

Max Profit

$$z = 5x_1 + 4x_2$$

$$\begin{aligned} s.t. \quad & 6x_1 + 4x_2 \le 24 \\ & x_1 + 2x_2 \le 6 \\ & x_2 \le x_1 + 1 \\ & x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Max Profit

$$z = 5x_1 + 4x_2$$

$$\begin{aligned} s.t. \quad & 6x_1 + 4x_2 \le 24 \\ & x_1 + 2x_2 \le 6 \\ & -x_1 + x_2 \le 1 \\ & x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Resource	Slack value	Remarks
Raw material M_1	**$s_1 = 0$**	**Complete utilization**
Raw material M_2	**$s_2 = 0$**	**Complete utilization**
Market limit	**$s_3 = 2.5$**	**Abundant**
Demand limit	**$s_4 = 0.5$**	**Abundant**

$$\begin{aligned} & z - 5x_1 - 4x_2 = 0 \\ & 6x_1 + 4x_2 + s_1 = 24 \\ & x_1 + 2x_2 + s_2 = 6 \\ & -x_1 + x_2 + s_3 = 1 \\ & x_2 + s_4 = 2 \\ & x_1, x_2, s_1, s_2, s_3, s_4 \ge 0 \end{aligned}$$

A decrease of 2.5 units in market limit will not change the optima

A decrease of 0.5 units in demand will not change the optima

Other considerations

- Special cases in simplex method
  - Degeneracy: A tie in the minimum ratio test which might lead to cycling.
  - Alternative optima: Objective function is parallel to a non-redundant binding constraint.
  - Unbounded solution: Solution space is unbounded for at least one variable.
  - No feasible solution: Inconsistent constraints.
- Ill conditioned simplex: LPs with (=) and (≥)
  - Use artificial variables.
  - Big M method: Use of Penalty.
  - Two phase method: Solve using two phases.
- Sensitivity analysis: Primal to dual conversion.
- Dual Simplex: Starts with better than optimal but infeasible solution and moves to feasibility.

H. A. Taha, *Operations Research: An Introduction (8th Edition)*: Prentice-Hall, Inc., 2006.

Further Reading

- H. A. Taha, *Operations Research: An Introduction (8th Edition)*: Prentice-Hall, Inc., 2006.

- S. S. Rao, *ENGINEERING OPTIMIZATION Theory and Practice*, Third Enlarged Edition ed.: New Age International Publishers, 2010.

Thank You !!!