SOLUTION
MA 201: COMPLEX ANALYSIS ASSIGNMENT–1

(1) Prove the following:

(a) Prove that $\max\{|\text{Re }(z)|, |\text{Im}(z)|\} \leq |z| \leq |\text{Re }(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$.

Answer: Observe that, for any two real numbers x and y, we have $(|x| - |y|)^2 \geq 0 \Longrightarrow |x|^2 + |y|^2 \geq 2|x|\,|y|$. Let $z = x + iy$ be any complex number. Now,

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x|\,|y| \leq |x|^2 + |y|^2 + |x|^2 + |y|^2$$
$$= 2\left(|x|^2 + |y|^2\right) = 2\left(x^2 + y^2\right) = 2|z|^2 .$$

This gives that $|\text{Re }(z)| + |\text{Im}(z)| = |x| + |y| \leq \sqrt{2}\,|z|$.

$$|z| = \sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2 + 2|x|\,|y|} = \sqrt{|x|^2 + |y|^2 + 2|x|\,|y|}$$
$$= \sqrt{(|x| + |y|)^2} = |x| + |y| .$$

That is, $|z| \leq |\text{Re }(z)| + |\text{Im}(z)|$.

(b) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds if and only if one is a nonnegative (real) scalar multiple of the other.

Answer: Do yourself.

(c) If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?

Answer: Case I: $|z_1| = 1$ and $|z_2| \neq 1$

$$\left|\frac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right|^2 = \frac{|z_1|^2 + |z_2|^2 - 2\text{Re }(z_1\overline{z_2})}{1 + |\overline{z_1}z_2|^2 - 2\text{Re }(z_1\overline{z_2})}$$
$$= \frac{1 + |z_2|^2 - 2\text{Re }(z_1\overline{z_2})}{1 + |z_2|^2 - 2\text{Re }(z_1\overline{z_2})} = 1 .$$

Observe that the denominator $1 + |\overline{z_1}z_2|^2 - 2\text{Re }(z_1\overline{z_2}) \neq 0$ if $|z_1| = 1$ and $|z_2| \neq 1$.

Case II: $|z_2| = 1$ and $|z_1| \neq 1$. It can be worked out similarly as in the previous case.

Case III: $|z_1| = 1$ and $|z_2| = 1$

Then, $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right|^2 = \dfrac{2 - 2\text{Re }(z_1\overline{z_2})}{2 - 2\text{Re }(z_1\overline{z_2})} = 1$ if the denominator $2 - 2\text{Re }(z_1\overline{z_2}) \neq 0$. That is, $\text{Re }(z_1\overline{z_2}) \neq 1$ if and only if $z_1 \neq z_2$. So, the exception is to be made for the validity of the above equality in this case is $z_1 \neq z_2$.

(2) Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C}\colon\ |z| < 1\}$.

Answer: Suppose α is a solution. So $|\alpha| < 1$ and $\alpha^4 + \alpha = -5$. Then $5 = |\alpha^4 + \alpha| \leq 2$.

(3) If z and w are in $\mathbb{C}$ such that $\text{Im}(z) > 0$ and $\text{Im}(w) > 0$, show that $\left|\frac{z-w}{z-\overline{w}}\right| < 1$.

Answer: $\left|\frac{z-w}{z-\bar{w}}\right|^2 \leq 1 \Longleftrightarrow (z - \bar{z})(w - \bar{w}) < 0$. Clearly $(z - \bar{z})(w - \bar{w}) < 0$ if $\text{Im}(z) > 0$ and $\text{Im}(w) > 0$.

(4) When does $az + b\overline{z} + c = 0$ has exactly one solution?

Answer: Let $z = x + iy, a = a_1 + ia_2, b = b_1 + ib_2$ and $c = c_1 + ic_2$ and put these values in the given equation $az + b\overline{z} + c = 0$. After simplification (please check it carefully!)we have,

$$(a_1x - a_2y) + i(a_2x + a_1y) + (b_1x + b_2y) + i(b_2x - b_1y) + c_1 + ic_2 = 0$$

After equating real and imaginary parts we get the following system of linear equations

$$(a_1 + b_1)x + (b_2 - a_2)y = c_1$$

$$(a_2 + b_2)x + (a_1 - b_1)y = c_2.$$

Therefore given equation has exactly one solution if the above system of linear equations has unique solution. In this case

$$(a_1 + b_1)(a_1 - b_1) - (b_2 - a_2)(a_2 + b_2) \neq 0.$$

In fact the given equation has exactly one solution if $|a| \neq |b|$.

(5) If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that $\Pi_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j$.

Answer The points $1 = z_0, z_1, \ldots, z_{n-1}$ are the roots of $z^n - 1 = 0$. So $z^n - 1 = (z-1)\Pi_{j=1}^{n-1}(z - z_j) = (z-1)\sum_{j=0}^{n-1} z^j$. Let $f(z) = \Pi_{j=1}^{n-1}(z - z_j)$ and $g(z) = \sum_{j=0}^{n-1} z^j$. Thus $f(z) = g(z)$ for $z \neq 1$. Since both f, g are continuous it follows that $f(1) = g(1)$ as well.

(6) For each of the following subsets of $\mathbb{C}$, determine whether it is open, closed or neither. Justify your answers.

(a) $S = \{z \in \mathbb{C} : Re(z) = 1 \text{ and } Im(z) \neq 4\}$

Answer: The set $S = \{z = x + iy : x = 1\} \setminus \{1 + 4i\}$ is neither open nor closed. $S^o = \emptyset$ and $\partial S = S' = \overline{S} = \{z = x + iy : x = 1\}$

(b) $\{z \in \mathbb{C} : Re(z) \in (-1, 2) \cup (2, \frac{5}{2}) \text{and } Im(z) = 0\}$

Answer: The $S = \{z \in \mathbb{C} : Re(z) \in (-1, 2) \cup (2, \frac{5}{2}) \text{and } Im(z) = 0\}$ is neither open nor closed. $S^o = \emptyset$ and $\partial S = S' = \overline{S} = \{z = x + iy : x \in [-1, \frac{5}{2}] \text{ and } y = 0\}$.

(c) $\{z = x + iy \in \mathbb{C} : \ xy > 1\}$

Answer: The set $S = \{z = x + iy \in \mathbb{C} : \ xy > 1\}$ is an open set, hence $S = S^o$. $\partial S = \{z = x + iy \in \mathbb{C} : \ xy = 1\}$ and $\overline{S} = S' = \{z = x + iy \in \mathbb{C} : \ xy \geq 1\}$.

(d) $\{z = x + iy \in \mathbb{C} : \ x \in \mathbb{Q} \text{ and } y \in \mathbb{R} \setminus \mathbb{Q}\}$

Answer: The set $S = \{z = x + iy \in \mathbb{C} : \ x \in \mathbb{Q} \text{ and } y \in \mathbb{R} \setminus \mathbb{Q}\}$ is neither open nor closed. $S^o = \emptyset$ and $\partial S = S' = \overline{S} = \mathbb{C}$

(e) $\{\frac{1}{n} + \frac{i}{m} : n, m \in \mathbb{N}\}$

Answer: The set $S = \{\frac{1}{n} + \frac{i}{m} : n, m \in \mathbb{N}\}$ is neither open nor closed. $S^o = \emptyset$, $S' = \{\frac{1}{n} + 0i\} \cup \{0 + \frac{i}{m}\} \cup \{(0, 0)\}$ and $\partial S = \overline{S} = S \cup S'$.

(f) $\{z = re^{i\theta} \in \mathbb{C} : \ 0 < r < 1 \ \text{ and } \theta \in (\frac{\pi}{4}, \frac{\pi}{3})\}$

Answer: The set $S = \{z = re^{i\theta} \in \mathbb{C} : \ 0 < r < 1 \ \text{ and } \theta \in (\frac{\pi}{4}, \frac{\pi}{3})\}$ is an open set, hence $S = S^o$. $\overline{S} = S' = \{z = re^{i\theta} \in \mathbb{C} : \ 0 \leq r \leq 1 \ \text{ and } \theta \in [\frac{\pi}{4}, \frac{\pi}{3}]\}$.
$\partial S = \{z = re^{i\theta} \in \mathbb{C} : \ 0 \leq r \leq 1 \ \text{ and } \theta = \frac{\pi}{4}, \frac{\pi}{3}\} \cup \{z = e^{i\theta} : \ \ \theta \in [\frac{\pi}{4}, \frac{\pi}{3}]\}$.

(g) $\{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$

Answer: The set $\{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$ is neither open nor closed. $S^o = \{z \in \mathbb{C} : Re(z) < 0\}$.
$\partial S = \{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\} \cup \{$ positive real axis $\} \cup \{$ imaginary axis $\}$.
$\overline{S} = S' = \{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\} \cup \{$ positive real axis $\} \cup \{z \in \mathbb{C} : Re(z) \leq 0\}$

(7) Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line $y = 1-x$ joining z_1 and z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$

(Mean value theorem does not extend to complex derivatives).

Answer $\left|\dfrac{f(1) - f(i)}{1 - i}\right| = |\frac{1+i}{1-i}| = 1$. Any point on $[1, i]$ has mod value $\geq \frac{1}{\sqrt{2}}$. . So $|f'(z)| = |3z^2| \geq \frac{3}{2} > 1$.

(8) If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.

Hint: Use C–R equation.

(9) Let U be an open set and $f \colon U \to \mathbb{C}$ be a differentiable function. Let $\overline{U} := \{\overline{z} \colon z \in U\}$. Show that the function $g \colon \overline{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\overline{z})}$ is differentiable on $\overline{U}$.

Answer:

$$\lim_{w \to w_0 \in \bar{U}} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0 \in U} \frac{g(\bar{z}) - g(\bar{z}_0)}{\bar{z} - \bar{z}_0} = \lim_{z \to z_0 \in U} \overline{\frac{f(z) - f(z_0)}{z - z_0}} = \overline{f'(z_0)}$$

(10) Derive the Cauchy-Riemann equations in polar coordinates.

Answer: Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z_0 = r_0 e^{i\theta_0}$. First we calculate the limit $z \to z_0$ along the ray $\theta = \theta_0$. Then the following limit exists:

$$\begin{aligned} f'(z_o) &= \lim_{r \to r_0} \frac{f(re^{i\theta_0}) - f(r_0 e^{i\theta_0})}{re^{i\theta_0} - r_0 e^{i\theta_0}} \\ &= \frac{1}{e^{i\theta_0}} \lim_{r \to r_0} \frac{u(r, \theta_0) - u(r_0, \theta_0) + i[v(r, \theta_0) - v(r_0, \theta_0)]}{r - r_0} \\ &= \frac{1}{e^{i\theta_0}} \left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right) \qquad (0.1) \end{aligned}$$

Now calculate the limit $z \to z_0$ along the circle $r \to r_0$. In this case we have:

$$\begin{aligned} f'(z_o) &= \lim_{\theta \to \theta_0} \frac{f(r_0 e^{i\theta}) - f(r_0 e^{i\theta_0})}{r_0 e^{i\theta} - r_0 e^{i\theta_0}} \\ &= \frac{1}{r_0} \lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0) + i[v(r_0, \theta) - v(r_0, \theta_0)]}{e^{i\theta} - e^{i\theta_0}} \\ &= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left\{ \left(\frac{u(r_0, \theta) - u(r_0, \theta_0) + i[v(r_0, \theta) - v(r_0, \theta_0)]}{\theta - \theta_0} \right) \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} \right\} \\ &= \frac{1}{ir_0 e^{i\theta_0}} \left(\frac{\partial u}{\partial \theta}(r_0, \theta_0) + i \frac{\partial v}{\partial \theta}(r_0, \theta_0) \right) = \frac{1}{r_0 e^{i\theta_0}} \left(\frac{\partial v}{\partial \theta}(r_0, \theta_0) - i \frac{\partial u}{\partial \theta}(r_0, \theta_0) \right) \qquad (0.2) \end{aligned}$$

To get the C–R equation in polar form , equate the real and imaginary parts of eq.(0.1) and eq.(0.2)

(11) Let Ω be an open connected subset of $\mathbb{C}$ and $f \colon \Omega \to \mathbb{C}$ be a differentiable function. Show that the function $f = u + iv$ is constant if

(a) either of the functions u or v is constant, or

(b) $|f(z)|$ is constant for all $z \in \Omega$, or

(c) if there exists an $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|e^{i\alpha}$ for all $z \in \Omega$.

Hint: Use C–R equations.

(12) Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, $f(z) = f(w)$ whenever $|z| = |w|$. Prove that f is a constant function.

Hint: It is given that $f(z) = f(w)$ if $|z| = |w|$. This means that the function f is independent of argument. (i.e. $f(e^{i\theta} z) = f(z)$ for all θ.)Now use C–R equations in polar coordinates.

(13) Let $f = u + iv$ is an analytic function defined on the whole of $\mathbb{C}$. If $u(x, y) = \phi(x)$ and $v(x, y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a \in \mathbb{C}$, $b \in \mathbb{C}$.

Answer: From C–R equations we have $\phi'(x) = \psi'(y)$ for all $z = x+iy \in \mathbb{C}$. In particular $\phi'(0) = \psi'(y)$ and $\phi'(x) = \psi'(0)$ for all $x, y \in \mathbb{R}$. Also $f'(z) = \phi'(x) = \psi'(y)$ hence $f'(z) = a$ =constant. If we take $g(z) = f(z) - az$, then $g'(z) = 0$. Therefore $g(z) = b$ =constant i.e. $f(z) = az + b$.

(14) Let v be a harmonic conjugate of u. Show that $h = u^2 - v^2$ is a harmonic function.

Answer: Let $f = u+iv$. So by our assumption f is analytic and hence $f^2 = f.f$ is also analytic. So Real part of $f^2 = f.f = u^2 - v^2$ is harmonic.