8. Consider the following initial boundary value problem for $u(x,t)$ governed by one-dimensional wave equation for a flexible string of unit length:

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0;$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \geq 0;$$
$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, \quad 0 < x < 1.$$

Find the displacement of the string at location $x = 1/2$ and time $t = 2$. [3]

3 9. Consider solving the one-dimensional heat conduction equation $u_t = u_{xx}$ for a thin metal rod of length π with an initial temperature distribution $u(x,0) = \sin^2 x$, $0 < x < \pi$. Further, homogeneous Neumann conditions are prescribed at the ends $x = 0, \pi$ for $t \geq 0$. Obtain the solution of the corresponding IBVP. [4]

10. By considering the one-dimensional wave equation for finite spatial domain along with suitable boundary conditions and initial conditions, discuss the basic idea behind Duhamel's principle. [2]

11. Suppose it is required to solve the steady-state heat conduction equation (without source) in a rectangular plate $0 \leq x \leq a, 0 \leq y \leq b$ subject to two homogeneous Dirichlet boundary conditions along $x = 0, x = a$ and two non-homogeneous Dirichlet conditions along $y = 0, y = b$. Write the resulting boundary value problem clearly. Show how you would restructure the problem so that the method of separation of variables can be appropriately used. [1+2]
(You need not solve.)

12. Consider a circular planar disk of unit radius. Solve the corresponding BVP outside the disk with the given boundary condition: [3]

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r > 1, \quad -\pi < \theta < \pi,$$
$$u(1,\theta) = \theta, \quad -\pi < \theta < \pi.$$

13. Obtain the Fourier series expansion of the function f given by [3]

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x \leq 0, \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi. \end{cases}$$

14. (a) By considering an appropriate Fourier transform of the function $f(t) = e^{-\beta t}$, $t > 0$, where $\beta > 0$ is a constant, evaluate the improper integral $\int_0^\infty \frac{\sigma \sin(\sigma t)}{\beta^2 + \sigma^2}\, d\sigma$. [3]
(b) Consider the heat conduction in an infinite thin metal rod having thermal diffusivity α with an initial temperature distribution $\phi(x)$. By using Fourier transform, find the temperature distribution $u(x,t)$ at any point of the rod at any subsequent time $t > 0$. How would you approach this problem if the rod is considered to be semi-infinite? [2+1]

15. (a) Find the Laplace transform of $f(t) = 2H(\sin \pi t) - 1$ where H is Heaviside unit step function. [2]

(b) By using Laplace transform, solve the second-order ODE $\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), t > 0$ subject to the initial conditions $x(0) = a$, $\frac{dx}{dt}(0) = b$ for the case $q - p^2 > 0$, where p, q, a and b are constants, and f is piecewise continuous and of exponential order. [3]

1. Use residue theorem to evaluate the following integrals:

(a) $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,d\theta$, (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{x^2+2x+5}\,dx$. [3+4]

2. Find the image of the lower half plane $\mathrm{Im}(z) < 0$ under the map $w = \dfrac{iz-1}{i-z}$. [3]

3. Find a first-order partial differential equation (PDE) whose characteristic curves are represented by a one-parameter family of circles $x^2+y^2=a^2$. [1]

W 4. Obtain the surface that is orthogonal to the one-parameter family of surfaces $u = cxy\left(x^2+y^2\right)$ (c is a non-zero parameter) and passes through the hyperbola $x^2-y^2=a^2$, $u=b$, $(a,b>0)$. [4]

5. Find a second-order PDE arising from the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{u^2}{c^2}=1$. Here, x and y are the independent variables, $u=u(x,y)$ and a,b,c are arbitrary non-zero constants. [2]

W 6. Write the general form of a second-order linear homogeneous PDE in two dimensions involving one dependent variable only. Considering the PDE as a hyperbolic one, reduce it to a normal form which will be free from mixed derivatives. [1+3]

7. Find the general solution of the PDE $y^2u_{xx}-2yu_{xy}+u_{yy}=u_x+6y$ by reducing to its canonical form. [3]

In questions 8, 9, 10 and 12 on next page, writing/finding the IBVP/BVP and the corresponding solution for the general problem will not carry any marks. Solutions to the specific problem as asked will only carry marks.