MA 201, Mathematics III, July-November 2022, Laplace Transform (Contd.)

Lecture 17

Application of Laplace transform in solving ODEs

ODEs with constant coefficients

Example ODE1 (First-order ODE):

$$\frac{dx}{dt} + 3x = 0, \; x(0) = 1.$$

By taking Laplace transform on both sides of the equation,

$$\begin{aligned}
&\mathcal{L}\{\frac{dx}{dt}\} + \mathcal{L}\{3x\} = 0 \\
\Rightarrow\ & s\mathcal{L}\{x\} - x(0) + 3\mathcal{L}\{x\} = 0 \\
\Rightarrow\ & (s+3)\mathcal{L}\{x\} = 1 \\
\Rightarrow\ & \mathcal{L}\{x\} = \frac{1}{s+3}.
\end{aligned}$$

Taking inverse transform

$$x = e^{-3t}.$$ ♣

Application of Laplace transform in solving ODEs

ODEs with constant coefficients

Example ODE2 (Second-order ODE):

$$\frac{d^2x}{dt^2} + x = t,\ x(0) = 1, \frac{dx}{dt}(0) = -2.$$

By taking Laplace transform on both sides of the equation,

$$\begin{aligned}
&\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{t\} \\
\Rightarrow\ & s^2\mathcal{L}\{x\} - sx(0) - \dot{x}(0) + \mathcal{L}\{x\} = 1/s^2 \\
\Rightarrow\ & (s^2+1)\mathcal{L}\{x\} = \frac{1}{s^2} + s - 2 \\
\Rightarrow\ & \mathcal{L}\{x\} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}.
\end{aligned}$$

Taking inverse transform

$$x = t + \cos t - 3\sin t.$$ ♣

Application of Laplace transform in solving ODEs

ODEs with variable coefficients

Example ODE3 (Second-order ODE):

$$t\frac{d^2x}{dt^2} + 2(t-1)\frac{dx}{dt} + (t-2)x = 0.$$

By taking Laplace transform on both sides of the equation,

$$\mathcal{L}\left\{t\frac{d^2x}{dt^2}\right\} + 2\mathcal{L}\left\{(t-1)\frac{dx}{dt}\right\} + \mathcal{L}\{(t-2)x\} = 0$$
$$\Rightarrow -\frac{d}{ds}\mathcal{L}\{\ddot{x}\} - 2\frac{d}{ds}\mathcal{L}\{\dot{x}\} - 2\mathcal{L}\{\dot{x}\} - \frac{d}{ds}\mathcal{L}\{x\} - 2\mathcal{L}\{x\} = 0,$$

which will ultimately lead to the first-order ODE:

$$\frac{d}{ds}\mathcal{L}\{x\} + \frac{4s+4}{s^2+2s+1}\mathcal{L}\{x\} = \frac{3x_0}{(s+1)^2}, \quad \text{where } x(0) = x_0.$$

Application of Laplace transform in solving ODEs

The solution (find the integrating factor as $(s+1)^4$) is

$$\mathcal{L}\{x\} = \frac{x_0}{s+1} + \frac{C}{(s+1)^4}.$$

Taking inverse transform,

$$x = x_0\, e^{-t} + C\frac{t^3}{6}\, e^{-t}.$$

♣

Application of Laplace transform in solving ODEs

Simultaneous ODEs

Example ODE4 (First-order system):

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - 3y, \\ \frac{dy}{dt} &= y - 2x, \end{aligned}\right\} \quad x(0) = 8,\ y(0) = 3.$$

Taking Laplace transform on both sides of the first equation,

$$(s-2)\mathcal{L}\{x\} + 3\mathcal{L}\{y\} = 8. \tag{1}$$

Similarly, taking Laplace transform on both sides of the second equation,

$$2\mathcal{L}\{x\} + (s-1)\mathcal{L}\{y\} = 3. \tag{2}$$

By application of Cramer's rule in (1) and (2),

$$\mathcal{L}\{x\} = \frac{5}{s+1} + \frac{3}{s-4},\ \ \mathcal{L}\{y\} = \frac{5}{s+1} - \frac{2}{s-4}.$$

By taking the inverse transform,

$$x(t) = 5e^{-t} + 3e^{4t},\ \ y(t) = 5e^{-t} - 2e^{4t}.$$

♣

Convolution

Example A: Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\}$

We can easily observe that the Laplace transforms of $F(t) = e^{2t}$ and $G(t) = \cos t$, are, respectively, $f(s) = \frac{1}{s-2}$ and $g(s) = \frac{s}{s^2+1}$.

Can we have

$$\mathcal{L}^{-1}\{f(s)g(s)\} = F(t)G(t)?$$

In other sense, whether

$$\mathcal{L}\{F(t)G(t)\} = \{f(s)g(s)\}?$$

This result is true for a special product of functions, known as convolution.

A convolution is an integral that expresses the amount of overlap of one function G as it is shifted over another function F.

It, therefore, "blends" one function with another.

In other words, the output which produces a third function can be viewed as a modified version of one of the original functions.

Convolution

Definition The convolution of two given functions $F(t)$ and $G(t)$ is written as $F * G$ and is defined by the integral

$$F * G = \int_0^t F(\tau)\, G(t-\tau)\, d\tau. \tag{3}$$

Note that the commutative, associative and distributive properties hold true.

Example B:

$$\begin{aligned} t * e^{at} &= \int_0^t \tau e^{a(t-\tau)} d\tau = e^{at} \int_0^t \tau e^{-a\tau} d\tau \\ &= \frac{1}{a^2}(e^{at} - at - 1). \end{aligned}$$

Example C:

$$\begin{aligned} \sin at * \sin at &= \int_0^t \sin a\tau \sin a(t-\tau) d\tau \\ &= \frac{1}{2a}(\sin at - at \cos at). \end{aligned}$$

Convolution

Theorem

If $F(t)$ and $G(t)$ are two functions of exponential order and given $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(\tau)\, G(t-\tau)\, d\tau = F * G, \tag{4}$$

where $*$ is the convolution operator defined in (3) earlier.

Proof: By definition

$$\mathcal{L}\{F(t) * G(t)\} = \int_0^\infty e^{-st} \int_0^t F(\tau)G(t-\tau)\, d\tau\, dt.$$

The domain of this repeated integral takes the form of a wedge in the t, τ-plane.

Write

$$\mathcal{L}\{F(t) * G(t)\} = \int_0^\infty \int_0^t e^{-st} F(\tau)G(t-\tau)\, d\tau\, dt.$$

Convolution

Integrating with respect to t first

$$\begin{aligned}\mathcal{L}\{F(t) * G(t)\} &= \int_0^\infty \int_\tau^\infty e^{-st} F(\tau) G(t-\tau)\, d\tau\, dt \\ &= \int_0^\infty F(\tau) \left\{ \int_\tau^\infty e^{-st} G(t-\tau)\, dt \right\} d\tau.\end{aligned}$$

In the inner integral above, put $u = t - \tau$ so that it can be written as

$$\begin{aligned}\int_\tau^\infty e^{-st} G(t-\tau)\, dt &= \int_0^\infty e^{-s(u+\tau)} G(u)\, du \\ &= e^{-s\tau} g(s).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}\{F(t) * G(t)\} &= \int_0^\infty F(\tau) e^{-s\tau} g(s)\, d\tau \\ &= f(s) g(s),\end{aligned}$$

which gives us the following desired result:

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(\tau)\, G(t-\tau)\, d\tau = F * G.$$

Convolution

Example: Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\}$.

We know

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t), \;\; \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}.$$

Now, use convolution theorem to have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\} = \mathcal{L}^{-1}\{f(s)g(s)\} &= F(t) * G(t) \\
&= e^{2t} * \cos(t) = \int_0^t e^{2\tau}\cos(t-\tau)\,d\tau \\
&= \frac{2}{5}e^{2t} + \frac{1}{5}(\sin t - 2\cos t).
\end{aligned}$$

Inverse Laplace transform by method of residue

If s is considered as complex, then

the inverse Laplace transform $F(t)$ of $f(s)$ is given by

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} f(s) e^{st} ds,$$

where the integration takes place along the vertical line $\mathrm{Re}(s) = a$.

We can use the method of residues to find inverse Laplace transform.

Inverse Laplace transform by method of residue

The following theorem is to be used for evaluating inverse Laplace transform.

Theorem

If the Laplace transform $f(s)$ of $F(t)$ is an analytic function of s, except at a finite number of singular points — the poles — each of which lies to the left of the vertical line $Re(s) = a$ and if $sf(s)$ is bounded as s approaches infinity through the half-plane $Re(s) \leq a$, then

$$\begin{aligned} \mathcal{L}^{-1}\{f(s)\} &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)e^{st}\, ds \\ &= \sum \textit{Residues of } f(s)e^{st} \textit{ at all poles} \\ &= R_1 + R_2 + R_3 + \cdots \end{aligned} \tag{5}$$

where

$$R_k = \begin{cases} \lim_{s\to a_k}\{(s-a_k)f(s)e^{st}\}, & k = 1, 2, \ldots, n;\ a_k \ \textit{simple pole}, \\ \lim_{s\to a_k} \frac{1}{r!}\frac{d^r}{ds^r}\{(s-a_k)^{r+1}f(s)e^{st}\}, & a_k \ \textit{multiple pole of order } r+1. \end{cases} \tag{6}$$

Inverse Laplace transform by method of residue

Example D: Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^2-s+3}{s^3+6s^2+11s+6}\right\}.$$

Solution D:

We can see that $\lim_{s\to\infty} sf(s) = 1$ which is bounded. The poles are found to be $s=-1,-2,-3$ which are all simple poles. In order to evaluate the inverse Laplace transform using (6), we need to calculate the residues of the function due to the poles. We get

$$R_1 = (\text{residue at } s=-1) = \frac{5}{2}e^{-t},$$
$$R_2 = (\text{residue at } s=-2) = -9e^{-2t},$$
$$R_3 = (\text{residue at } s=-3) = \frac{15}{2}e^{-3t}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{s^2-s+3}{s^3+6s^2+11s+6}\right\} = \frac{5}{2}e^{-t} - 9e^{-2t} + \frac{15}{2}e^{-3t}.$$

Inverse Laplace transform by method of residue

Example E: Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\}.$$

Solution E:

Here $s=-1$ is a simple pole whereas $s=2$ is a double pole.

$$R_1 = (\text{residue at } s=-1) = \frac{e^{-t}}{9},$$

$$R_2 = (\text{residue at } s=2) = \left(\frac{t}{3}-\frac{1}{9}\right)e^{2t}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{e^{-t}}{9} + \left(\frac{t}{3}-\frac{1}{9}\right)e^{2t}.$$

Application of Laplace transform in solving PDEs

Definition

The Laplace transform of a function $u(x,t)$ with respect to t is defined as

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)\, dt = \overline{u}(x,s). \tag{7}$$

The Laplace transforms of the partial derivatives $\dfrac{\partial u(x,t)}{\partial t}$, $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial^2 u(x,t)}{\partial t^2}$, $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ are as follows:

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0), \tag{8}$$

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{d}{dx}\,\bar{u}(x,s), \tag{9}$$

$$\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2\bar{u}(x,s) - su(x,0) - u_t(x,0), \tag{10}$$

$$\mathcal{L}\left\{\frac{\partial^2 ; u(x,t)}{\partial x^2}\right\} = \frac{d^2}{dx^2}\,\bar{u}(x,s). \tag{11}$$

Application of Laplace transform in solving PDEs

Example PDE1: (First-order)

Find a bounded solution of the following problem:

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{subject to} \quad u(x,0) = 6\, e^{-3x}.$$

Solution PDE1:

Taking Laplace transform on both sides of the given PDE and using the initial condition,

$$\frac{d\bar{u}}{dx} - (2s+1)\bar{u} = -12e^{-3x}.$$

After finding the integrating factor,

$$\bar{u}(x,s) = \frac{6}{s+2}\, e^{-3x} + C\, e^{(2s+1)x}.$$

Application of Laplace transform in solving PDEs

$u(x,t)$ should be bounded when $x \to \infty$.

Hence its Laplace transform $\bar{u}(x,s)$ should also be bounded as $x \to \infty$ and we take $C = 0$.

$\Rightarrow$

$$\bar{u}(x,s) = \frac{6}{s+2}\, e^{-3x}.$$

Taking the inverse transform

$$u(x,t) = 6e^{-(2t+3x)}.$$

♣

Application of Laplace transform in solving PDEs

Example PDE2: (Second-order)

Consider the following one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ t > 0$$

subject to the following conditions:

$$u(0,t) = 1,\ u(1,t) = 1,\ t \geq 0;\ u(x,0) = 1 + \sin \pi x,\ 0 < x < 1.$$

Solution PDE2:

Taking Laplace transform on both sides and applying the given initial condition,

$$\frac{d^2}{dx^2}\,\bar{u}(x,s) - s\bar{u}(x,s) = -(1 + \sin \pi x).$$

The complementary function and the particular integral of the above equation can be derived as

$$\begin{aligned} \bar{u}_c(x,s) &= Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}, \\ \bar{u}_p(x,s) &= \frac{1}{s} + \frac{\sin \pi x}{s + \pi^2}. \end{aligned}$$

Application of Laplace transform in solving PDEs

$\bar{u}(x,s)$:

$$\bar{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{s+\pi^2}. \tag{12}$$

Convert the boundary conditions in $u(x,t)$ to boundary conditions in $\bar{u}(x,s)$:

$u(0,t) = 1 \Rightarrow \bar{u}(0,s) = \dfrac{1}{s}$,

$u(1,t) = 1 \Rightarrow \bar{u}(1,s) = \dfrac{1}{s}$.

Using these in (12)

$$\frac{1}{s} = A + B + \frac{1}{s} \quad \Rightarrow A + B = 0,$$

$$\frac{1}{s} = Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{s} \quad \Rightarrow Ae^{\sqrt{s}} + Be^{-\sqrt{s}} = 0.$$

Application of Laplace transform in solving PDEs

Both these conditions together imply $A = 0 = B$.

$$\bar{u}(x,s) = \frac{1}{s} + \frac{\sin \pi x}{s + \pi^2}. \tag{13}$$

Solution is obtained by taking the inverse

$$u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x. \tag{14}$$

Application of Laplace transform in solving PDEs

Example PDE3: (Second-order)

$$U_{tt} = c^2 U_{xx} + \sin\left(\frac{\pi x}{L}\right)\sin(\sigma t),\ 0 < x < L,\ t > 0,$$
$$U(0,t) = 0,\ U(L,t) = 0,\ t \geq 0,$$
$$U(x,0) = 0,\ U_t(x,0) = 0,\ 0 < x < L.$$

Taking Laplace transform on the equation, it gets reduced to

$$\frac{d^2}{dx^2}\overline{u}(x,s) - \frac{s^2}{c^2}\overline{u}(x,s) = -\frac{\sigma\sin(\pi x/L)}{c^2(s^2+\sigma^2)},$$

the solution of which can be obtained as

$$\overline{u}(x,s) = A(s)e^{\frac{s}{c}x} + B(s)e^{-\frac{s}{c}x} + \frac{\sigma}{c^2}\frac{\sin(\pi x/L)}{(s^2+\sigma^2)(\frac{s^2}{c^2}+\frac{\pi^2}{L^2})}.$$

It can be simplified to

$$\overline{u}(x,s) = A(s)e^{\frac{s}{c}x} + B(s)e^{-\frac{s}{c}x} + \frac{\sigma\sin(\pi x/L)}{(\frac{c^2\pi^2}{L^2}-\sigma^2)}\left(\frac{1}{s^2+\sigma^2} - \frac{1}{s^2+\frac{c^2\pi^2}{L^2}}\right).$$

Application of Laplace transform in solving PDEs

Taking transforms on the boundary conditions

$$\overline{u}(0,s) = 0,\ \overline{u}(L,s) = 0$$

$\Rightarrow$

$$A(s) = 0 = B(s).$$

This reduces $\overline{u}(x,s)$ simply to

$$\overline{u}(x,s) = \frac{L^2\sigma}{c^2\pi^2 - \sigma^2 L^2}\left(\frac{1}{s^2+\sigma^2} - \frac{1}{s^2 + \frac{c^2\pi^2}{L^2}}\right)\sin(\pi x/L).$$

Inverting

$$U(x,t) = \frac{L^2\sigma}{c^2\pi^2 - \sigma^2 L^2}\left[\frac{1}{\sigma}\sin(\sigma t) - \frac{L}{c\pi}\sin(\pi c t/L)\right]\sin(\pi x/L).$$

That is,

$$U(x,t) = \frac{L^2}{c^2\pi^2 - \sigma^2 L^2}\left[\sin(\sigma t) - \frac{L\sigma}{c\pi}\sin(\pi c t/L)\right]\sin(\pi x/L)$$