

MA 201 (PART II), JULY-NOVEMBER, 2022 SESSION
 PARTIAL DIFFERENTIAL EQUATIONS
 SOLUTIONS TO TUTORIAL SHEET 9, DATE OF DISCUSSION: NOVEMBER 11, 2022

Separation of variables: one-dimensional wave equation

One-dimensional diffusion equation, two-dimensional steady-state equation

1. A string is fixed at $x = 0$ and $x = L$ and lies initially along the x -axis. If it is set in motion by giving all points $0 < x < L$ a constant transverse velocity $\frac{\partial u}{\partial t} = u_0$ at $t = 0$, then find the subsequent motion of the string.

Solution: Due to the string lying initially along the x -axis, there is zero initial displacement implying $A_n = 0$. Hence the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)$$

where

$$B_n = \begin{cases} 0, & n \text{ even} \\ \frac{4u_0 L}{n^2 \pi^2 c}, & n \text{ odd.} \end{cases}$$

Therefore

$$u(x, t) = \frac{4u_0 L}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{L}\right) \sin\left(\frac{(2n-1)\pi ct}{L}\right).$$

2. A guitar string of length $L = 1$, is pulled in the middle so that it reaches a height h . Assuming the initial position of the string as

$$u(x, 0) = \begin{cases} 2hx, & 0 < x < 1/2, \\ 2h(1-x), & 1/2 \leq x < 1, \end{cases}$$

what is the subsequent motion of the string if it is suddenly released?

Solution: Solution procedure is same as for the above problem.

$$u(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin((2n-1)\pi x) \cos((2n-1)\pi ct).$$

3. A metal bar of length 100 metre has its ends $x = 0$ and $x = 100$ kept at 0 degree Celsius. Initially half of the bar is at 60 degree C while the other half is at 40 degree C. Assuming a thermal diffusivity of 0.16 cgs units and that the surface of the bar is insulated, find the temperature everywhere in the bar at time t .

Solution:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{100}\right) e^{-0.16 \frac{n^2 \pi^2 t}{10^4}}$$

where

$$B_n = \frac{1}{n\pi} (120 - 80 \cos n\pi - 40 \cos \frac{n\pi}{2}).$$

4. Solve the following IBVP with non-homogeneous BCs:

$$\begin{aligned}u_t &= u_{xx}, \quad 0 < x < 1, \\u(0, t) &= 0, \\u(1, t) &= 1, \\u(x, 0) &= x^2.\end{aligned}$$

Solution:

$$u(x, t) = x - \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin((2n-1)\pi x) e^{-(2n-1)^2 \pi^2 t}.$$

5. Using Duhamel's principle, solve

$$u_{tt} - u_{xx} = x - t, \quad -\infty < x < \infty, \quad (1)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (2)$$

Solution: We first solve the related problem for $v(x, t, s)$.

$$v_{tt} - v_{xx} = 0, \quad -\infty < x < \infty, \quad (3)$$

$$v(x, 0, s) = 0, \quad v_t(x, 0, s) = f(x, s) = x - s, \quad -\infty < x < \infty, \quad s > 0. \quad (4)$$

For fixed $s > 0$, D'Alembert's solution is given by

$$v(x, t, s) = \frac{1}{2} \int_{x-t}^{x+t} F(\tau, s) d\tau = \frac{1}{2} \int_{x-t}^{x+t} (\tau - s) d\tau \quad (5)$$

$$= \frac{1}{2} \left[\frac{\tau^2}{2} - s\tau \right]_{x-t}^{x+t} = xt - ts = t(x - s). \quad (6)$$

Thus

$$v(x, t - \tau, \tau) = (t - \tau)(x - \tau). \quad (7)$$

Due to Duhamel's principle

$$u(x, t) = \int_0^t v(x, t - \tau, \tau) d\tau \quad (8)$$

$$= \int_0^t (t - \tau)(x - \tau) d\tau = -\frac{t^3}{6} + \frac{t^2 x}{2}. \quad (9)$$

6. Find a solution $u(x, y)$ of the following steady-state heat conduction problem

$$\begin{aligned}u_{xx} + u_{yy} &= 0, \quad 0 < x < \pi, \quad 0 < y < 1, \\u(x, 0) &= 0, \quad 0 < x < \pi, \quad u(x, 1) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}, \\u(0, y) &= u(\pi, y) = 0, \quad 0 < y < 1.\end{aligned}$$

Solution:

$$u(x, y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \frac{\sinh((2n-1)y)}{\sinh(2n-1)} \sin((2n-1)x).$$

7. Consider transient heat conduction in a circular region of radius a . Considering that heat conduction takes place only radially, find the solution of the transient heat conduction in the circular disk at any point r at any time $t > 0$
- (i) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = 100$,
- (ii) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = r$.

Solution:

Governing equation:

$$u_t = \alpha \left(u_{rr} + \frac{1}{r} u_r \right).$$

$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0 \left(\frac{\nu_n}{a} r \right) e^{-\alpha \frac{\nu_n^2}{a^2} t},$$

where ν_n are zeros of $J_0(\lambda a)$, and

$$A_n = \frac{\int_0^a r f(r) J_0 \left(\frac{\nu_n}{a} r \right) dr}{\int_0^a r \left(J_0 \left(\frac{\nu_n}{a} r \right) \right)^2 dr}.$$

(i) $f(r) = 100$:

$$A_n = 100 \frac{\int_0^a r J_0 \left(\frac{\nu_n}{a} r \right) dr}{\int_0^a r \left(J_0 \left(\frac{\nu_n}{a} r \right) \right)^2 dr}.$$

(ii) $f(r) = r$:

$$A_n = \frac{\int_0^a r^2 J_0 \left(\frac{\nu_n}{a} r \right) dr}{\int_0^a r \left(J_0 \left(\frac{\nu_n}{a} r \right) \right)^2 dr}.$$

8. Solve the following boundary value problem in a circular disk:

$$\begin{aligned} \nabla^2 u &= 0, r < a, 0 \leq \theta < 2\pi, \\ u(a, \theta) &= 4 + 3 \sin \theta, 0 \leq \theta < 2\pi. \end{aligned}$$

Solution:

$$u(r, \theta) = 4 + \frac{3}{a} r \sin \theta.$$

(All coefficients for $\cos(n\theta)$, $n \neq 0$ will vanish due to the given BC and $A_0 = 8$. Also for the coefficients of $\sin \theta$, all will be zero for $n \neq 1$ and $B_1 = 3/a$.)

Recall the solution:

$$u(r, \theta) = A_0/2 + \sum_{n=1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)] r^n,$$

with

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} u(a, \theta) \cos n\theta d\theta, \quad n = 0, 1, 2, 3, \dots$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} u(a, \theta) \sin n\theta d\theta, \quad n = 1, 2, 3, \dots$$

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