MA 201
COMPLEX ANALYSIS
ASSIGNMENT–2

(1) Find the values of z such that (a) $e^z \in \mathbb{R}$ and (b) $e^z \in i\mathbb{R}$.

(2) Prove that $\sinh(\mathrm{Im} z) \leq |\sin(z)| \leq \cosh(\mathrm{Im} z)$. Deduce that $|\sin(z)|$ tends to ∞ as $|\mathrm{Im} z| \to \infty$.

(3) Find all the complex numbers which satisfy the following:
(i) $\exp(z) = 1$ (ii) $\exp(z) = i$ (iii) $\exp(z-1) = 1$.

(4) Evaluate the following:
(i) $\log(3-2i)$ (ii) $\mathrm{Log}\, i$ (iii) $(i)^{(-i)}$

(5) If γ is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left| \int_\gamma (e^z - \overline{z})\ dz \right| \leq 60.$

(6) Evaluate $\int_\gamma |z|\, \overline{z}\, dz$ where γ is the circle $|z| = 2$.

(7) Show that $\int_\gamma \frac{e^{az}}{z^2+1} dz = 2\pi i \sin a$, where $\gamma(t) = 2e^{it},\ t \in [0, 2\pi]$.

Answer:

$$\int_\gamma \frac{e^{az}}{z^2+1} dz = \int_\gamma e^{az} \frac{1}{2i} \left[\frac{1}{z-i} - \frac{1}{z+i} \right] dz = \frac{1}{2i} \left[\int_\gamma \frac{e^{az}}{z-i} dz - \int_\gamma \frac{e^{az}}{z+i} dz \right]$$

By Cauchy integral formula,

$$\frac{1}{2i} \left[\int_\gamma \frac{e^{az}}{z-i} dz - \int_\gamma \frac{e^{az}}{z+i} dz \right] = 2\pi i \times \frac{1}{2i} [e^{ia} - e^{-ia}] = 2\pi i \sin a.$$

(8) Evaluate $\int_0^{2\pi} e^{e^{i\theta}} d\theta$.

Answer: Put $e^{i\theta} = z$. Then $d\theta = \frac{dz}{iz}$. So

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta = \int_{|z|=1} e^z \frac{dz}{iz} = 2\pi \text{ (by Cauchy integral formula).}$$