MA 201 (PART II), JULY-NOVEMBER, 2022 SESSION

PARTIAL DIFFERENTIAL EQUATIONS

PROBLEM SHEET - 2, DATE OF DISCUSSION: OCTOBER 21, 2022

Topics: 2nd order PDEs with constant coefficients, Classification of 2nd order PDEs,
Canonical forms, The wave equation: Infinite string problem (D'Alembert's solution)

Homogeneous PDE: ⟨⟨Factorization approach⟩⟩

Lectures 6-8

$a u_x + b u_y + c u = 0$

sol: $u = e^{-\frac{c}{a}x} \cdot \phi(bx - ay)$

1. Find the general solution of

$F(D, D')u = 0$

$u_i \sim$ sol.

$\Rightarrow \sum_{i=1}^{3} c_i u_i$ is a sol?

(i) $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$,

(ii) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$,

(iii) $\dfrac{\partial^3 u}{\partial x^3} - 2\dfrac{\partial^3 u}{\partial x^2 \partial y} - \dfrac{\partial^3 u}{\partial x \partial y^2} + 2\dfrac{\partial^3 u}{\partial y^3} = 0$.

Solution:

$F(D, D')u = 0 \Rightarrow (3D + D')(D + 3D')u = 0$

(i) Here

$$F(D, D') = 3D^2 + 10DD' + 3(D')^2 = (3D + D')(D + 3D').$$

Take: $(D + 3D')u = 0$

AE: $\frac{dx}{1} = \frac{dy}{3} = \frac{du}{0}$

→ char. curves

→ then find sol.

Hence, general solution is given by

$$u(x, y) = \phi_1(x - 3y) + \phi_2(3x - y).$$

Think like you know solution of simple PDEs, e.g., By multiplying one set higher order PDE.

(ii) Here

$$F(D, D') = D^2 + 4DD' + 4(D')^2 = (D + 2D')^2.$$

Hence, general solution is given by

$$u(x, y) = \phi_1(2x - y) + x\phi_2(2x - y).$$

(iii) Here

$$\begin{aligned} F(D, D') &= D^3 - 2D^2D' - D(D')^2 + 2(D')^3 = (D - 2D')(D^2 - (D')^2) \\ &= (D - 2D')(D + D')(D - D'). \end{aligned}$$

Hence, general solution is given by

$$u(x, y) = \phi_1(2x + y) + \phi_2(x - y) + \phi_3(x + y).$$

2. Why is it so that only the principal part $Au_{xx} + Bu_{xy} + Cu_{yy}$ of the 2nd-order PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$ determines the nature of the PDE?

Solution: The rate of change of a variable to some extent decides the behaviour of the variable. The partial derivatives u_x or u_y influence $u(x, y)$, similarly, the second-order derivatives of u influence the first-order derivatives and in turn u. Therefore, in a second-order PDE, the second-order derivatives decide the behaviour of first-order derivatives and hence the variable itself.

More discussion will take place in the meeting.

Hyperbolic

$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2-4AC}}{2A} = \frac{-4 \pm \sqrt{16-12}}{2}$

$= \frac{-4 \pm 2}{2} = \frac{-6}{2} \text{ or } \frac{-2}{2}$

$= -3 \text{ or } -1$

4.(i)

$\xi = y - 3x$

$\eta = y - x$

$u = u(x,y)$, $u_y = u_\xi \cdot \xi_y + u_\eta \eta_y$

$= u_\xi + u_\eta$

$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x$

$= u_\xi \cdot -3 + u_\eta \cdot -1$

$= -3u_\xi - u_\eta$

$u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$

$u_{xx} = -3(u_{\xi\xi} \cdot -3 + u_{\xi\eta} \cdot -1)$

$-(u_{\eta\eta}(-1) + u_{\xi\eta} \cdot (-3))$

$= +9u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta} + 3u_{\xi\eta}$

$u_{xy} = u_{\xi\xi} \cdot \xi_x + u_{\xi\eta} \cdot \eta_x + u_{\eta\eta} \cdot \eta_x + u_{\eta\xi} \cdot \xi_x$

$= u_{\xi\xi} \cdot (-3) + u_{\xi\eta}(-1) + u_{\eta\eta}(-1) + u_{\eta\xi} \cdot (-3)$

$= -3u_{\xi\xi} - u_{\eta\eta} - 4u_{\xi\eta}$

Reduced PDE : $u_{\xi\eta} = 0$

Integrat^n gives $u_\eta = c_1(\eta)$

again integrat^n yields

$u = \int c_1(\eta) + c_2(\xi)$

$= F(\eta) + G(\xi)$

$= F(y-x) + G(y-3x)$

$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u_x,u_y,u) = 0$

$D = B^2 - 4AC < 0$ (Elliptic), $D = 0$ (Parabolic), $D > 0$ (Hyperbolic)

3. Classify the following second-order partial differential equations:
   (i) $u_{xx} + 4u_{xy} + 4u_{yy} - 12u_y + 7u = x^2 + y^2$; (ii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$;
   (iii) $(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0$; (iv) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$.

   Solution: (i) Parabolic, (ii) Parabolic on the ellipse $\frac{x^2}{4} + y^2 = 1$, hyperbolic inside the ellipse and elliptic outside the ellipse, (iii) Hyperbolic, (iv) Hyperbolic if $x \neq y$, parabolic for $x = y$.

4. Reduce the following equations to canonical form and hence solve them:
   (i) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$; (ii) $4u_{xx} - 12u_{xy} + 9u_{yy} = e^{3x+2y}$,
   (iii) $u_{xx} + 2u_{xy} + u_{yy} = x^2 + 3\sin(x - 4y)$.

① Find nature of PDEs

② Introduce new variables by finding characteristics.

③ Transform into a new PDE

④ Find sol.

   Solution: (i) Equation is hyperbolic. Characteristics are given by $\xi = 3x - y$ and $\eta = x - y$. Canonical form is $u_{\xi\eta} = 0$. The solution is $u = f(3x - y) + g(x - y)$.
   (ii) Equation is parabolic. Characteristics are given by $\xi = x$ and $\eta = 2y + 3x$. Canonical form is $u_{\xi\xi} = \frac{1}{4}e^{\eta}$. The solution is $u = \frac{x^2}{8}e^{3x+2y} + xf(3x + 2y) + g(3x + 2y)$.
   (iii) Equation is parabolic. Characteristics are given by $\xi = y$ and $\eta = y - x$. Canonical form is $u_{\xi\xi} = (\xi - \eta)^2 - 3\sin(3\xi + \eta)$. Solution is $u = \frac{x^4}{12} + \frac{1}{3}\sin(x - 4y) + yf(y - x) + g(y - x)$.

5. Find D'Alembert's solution of one-dimensional wave equation with the following initial conditions:

   $u_{tt} = c^2 u_{xx}$, $u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(\xi)\,d\xi$

   (i) $u(x, 0) = \sin x$, $u_t(x, 0) = 0$, (ii) $u(x, 0) = \sin x$, $u_t(x, 0) = \cos x$.

   $= \varphi(x)$ $\qquad = \psi(x)$

   Solution:
   (i) $u(x, t) = \sin x \cos ct$, (ii) $u(x, t) = \sin x \cos ct + \frac{1}{c}\sin ct \cos x$.

6. A string stretching to infinity in both directions is given the initial displacement

$$\phi(x) = \frac{1}{1 + 4x^2} = u(x,0), \quad u_t(x,0) = 0$$

$\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$

and released from rest. Find its subsequent motion as a function of x and t.
Solution: Recall D'Alembert's solution for one-dimensional wave equation. Here initial displacement $u(x, 0) = \phi(x) = \frac{1}{1 + 4x^2}$ and initial velocity $u_t(x, 0) = v(x) = 0$. The required expression for $u(x, t)$ is

$$\begin{aligned} u(x,t) &= \phi(x + ct) + \phi(x - ct) \\ &= \frac{1}{1 + 4(x + ct)^2} + \frac{1}{1 + 4(x - ct)^2} \\ &= \frac{1 + 4(x^2 + c^2t^2)}{[1 + 4(x + ct)^2][1 + 4(x - ct)^2]} \end{aligned}$$

Equation: $u_{tt} = c^2 u_{xx}$