MA 201: Partial Differential Equations
Lecture - 2

How do first-order PDEs occur?

- First-order PDEs mainly connect to geometry.
- **Two-parameter family of surfaces:** Let

$$f(x, y, u, a, b) = 0 \tag{1}$$

represent two-parameter family of surfaces in $\mathbb{R}^3$, where a and b are arbitrary constants.

Differentiating (1) with respect to x and y yields the relations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{2}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0. \tag{3}$$

Eliminating a and b from (1), (2) and (3), we get a relation of the form

$$F(x, y, u, p, q) = 0, \tag{4}$$

which is a first-order PDE for the unknown function u of two independent variables x and y.

Example

The equation

$$x^2 + y^2 + (u - c)^2 = r^2, \tag{5}$$

where r and c are arbitrary constants, represents the set of all spheres whose centers lie on the u-axis.
Differentiating (5) with respect to x, we obtain

$$x + (u - c)\frac{\partial u}{\partial x} = 0. \tag{6}$$

Differentiate (5) with respect to y to have

$$y + (u - c)\frac{\partial u}{\partial y} = 0. \tag{7}$$

Eliminating the arbitrary constant c from (6) and (7), we obtain a first-order PDE:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \qquad \text{-linear PDE.} \tag{8}$$

Example

The equation

$$x^2 + y^2 = (u - c)^2 \tan^2 \alpha, \tag{9}$$

where c and α are arbitrary constants, represents a family of all right circular cones having the u-axis as their axes.
Differentiating (9) with respect to x, we obtain

$$\frac{\partial u}{\partial x}(u - c) \tan^2 \alpha = x. \tag{10}$$

Differentiate (9) with respect to y to have

$$\frac{\partial u}{\partial y}(u - c) \tan^2 \alpha = y. \tag{11}$$

Eliminating the arbitrary constants c and α from (10) and (11), we obtain a first-order PDE:

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0, \tag{12}$$

which is interestingly the same as (8).

- **Unknown function of known functions**
  - *Unknown function of a single known function*

    Let
$$u = f(g), \tag{13}$$
    where f is an unknown function and g is a known function of two independent variables x and y.

    Differentiating (13) with respect to x and y, respectively, yields the equations
$$u_x = f'(g)g_x \tag{14}$$
    and
$$u_y = f'(g)g_y, \tag{15}$$
    where $f'(g) = \frac{df}{dg}$.

    Eliminating $f'(g)$ from (18) and (19), we obtain
$$g_y u_x - g_x u_y = 0,$$
    which is a first-order PDE for u.

Example

Consider the surfaces described by an equation of the form

$$u = f(x^2 + y^2), \tag{16}$$

where f is an arbitrary function of a known function $g(x, y) = x^2 + y^2$. Differentiating (16) with respect to x and y, it follows that

$$u_x = 2xf'(g), \qquad u_y = 2yf'(g),$$

where $f'(g) = \frac{df}{dg}$.
Eliminating $f'(g)$ from the above two equations, we obtain a first-order PDE

$$yu_x - xu_y = 0. \tag{17}$$

It is clear that (17) is the same PDE as (8) and (12). This is because the forms of (5) and (9) allow for both equations to possess the similar kind of solutions. Both (5) and (9) have the similar properties as mentioned, i.e., $u = f(x^2 + y^2)$.

- *Unknown functions of two known functions*

Let
$$u = f(x - ay) + g(x + ay), \tag{18}$$
where $a > 0$ is a constant.
With $v(x, y) = x - ay$ and $w(x, y) = x + ay$, we can write (18) as
$$u = f(v) + g(w). \tag{19}$$

Differentiating (19) w. r. t. x and y, respectively, yields
$$\begin{aligned} p = u_x &= f'(x - ay) + g'(x + ay), \\ q = u_y &= -af'(x - ay) + ag'(x + ay). \end{aligned}$$

Eliminating $f'(v)$ and $g'(w)$, we get
$$q_y = a^2 p_x.$$

In terms of u, the above PDE is the well-known one-dimensional wave equation
$$u_{yy} = a^2 u_{xx}.$$

Example (Geometrical problem)

All functions $u(x, y)$ such that the tangent plane to the graph $u = u(x, y)$ at any arbitrary point $(x_0, y_0, u(x_0, y_0))$ passes through the origin is characterized by the PDE $xu_x + yu_y - u = 0$.

The equation of the tangent plane at $(x_0, y_0, u(x_0, y_0))$ is

$$u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) - (u - u(x_0, y_0)) = 0.$$

Since this plane passes through the origin $(0, 0, 0)$, we have

$$-u_x(x_0, y_0)x_0 - u_y(x_0, y_0)y_0 + u(x_0, y_0) = 0. \tag{20}$$

For equation (20) to hold for all (x_0, y_0) in the domain of u, it must be that u satisfies

$$xu_x + yu_y - u = 0,$$

which is a first-order linear PDE.

Cauchy's problem or IVP for first-order PDEs:
Let Γ be a given curve in $\mathbb{R}^2$ described parametrically by the equations

$$x = x_0(s), \qquad y = y_0(s); \quad s \in I, \tag{21}$$

where $x_0(s)$, $y_0(s)$ are in $C^1(I)$.
The IVP or Cauchy's problem for first-order PDE

$$F(x, y, u, p, q) = 0 \tag{22}$$

is to find a function $\phi = \phi(x, y)$ with the following properties:

- $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous in a region Ω of $\mathbb{R}^2$ containing the curve Γ.
- $\phi = \phi(x, y)$ is a solution of (22) in Ω, i.e.,

$$F(x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)) = 0 \quad \text{in } \Omega.$$

- On the curve Γ, we have

$$\phi(x_0(s), y_0(s)) = u_0(s), \quad s \in I. \tag{23}$$

The curve Γ is called the initial curve of the problem and the function $u_0(s)$ is called the initial data. Equation (23) is called the initial condition (or side condition) of the problem.

- **Well-posed Problem** (In the sense of Hadamard)

The Cauchy's problem (PDE + side condition) is said to be well-posed if it satisfies the following criteria:

1. The solution exists.
2. The solution is unique.
3. The solution depends continuously on the initial and/or boundary data.

If one or more of the above conditions does not/do not hold, we say that the problem is ill-posed.

Linear First-Order PDEs

The most general first-order linear PDE has the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y), \tag{24}$$

where a, b, c, and d are given functions of x and y. These functions are assumed to be continuously differentiable.
Observe the left hand side of (24):

$$a(x,y)u_x + b(x,y)u_y = \nabla u \cdot (a,b),$$

which is (essentially) a directional derivative of $u(x,y)$ in the direction of the vector (a,b), where (a,b) is defined and is non-zero.

Remark: When a and b are constants, the vector (a,b) has a fixed direction and magnitude, but now it is seen that the vector (a,b) can change as its base point (x,y) varies. Thus, (a,b) is a vector field on the plane.

The equations

$$\boxed{\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y),} \tag{25}$$

determine a family of curves $x = x(t)$, $y = y(t)$ whose tangent vector $(\frac{dx}{dt}, \frac{dy}{dt})$ coincides with the direction of the vector (a, b). Here t is considered as a parameter, not time as usually considered.

Therefore,

$$\begin{aligned}
\frac{d}{dt}u\{(x(t), y(t))\} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\
&= u_x(x(t), y(t))a(x(t), y(t)) \\
&\quad + u_y(x(t), y(t))b(x(t), y(t)) \\
&= c(x(t), y(t))u(x(t), y(t)) + d(x(t), y(t)) \\
&= c(t)u(t) + d(t),
\end{aligned}$$

where we have used the chain rule and (24).

Thus, along these curves, $u(t) = u(x(t), y(t))$ satisfies the ODE

$$u'(t) - c(t)u(t) = d(t). \tag{26}$$

Let $\mu(t) = \exp\left[-\int_0^t c(\tau)d\tau\right]$ be an integrating factor for (26). Then, the solution is given by

$$u(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(\tau)d(\tau)d\tau + u(0)\right]. \tag{27}$$

The approach described above is called the method of characteristics. It is based on the geometric interpretation of the partial differential equation (24).

Remarks.

- The system of ODEs (25) is known as the characteristic equation for the PDE (24). The solution curves of the characteristic equation are the characteristic curves for (24).
- The values $u(t)$ of the solution u along the entire characteristics curve are completely determined, once the value $u(0) = u(x(0), y(0))$ is prescribed.
- Assuming certain smoothness conditions on the functions a, b, c, and d, the existence and uniqueness theory for ODEs guarantees a unique solution curve $(x(t), y(t), u(t))$ of (25) and (26).