# Game Theory

- Strategic interaction among agents or players.
- Strategic interaction means each player or agent consider how the opponent is going to react to their strategy while choosing any strategy.
- Example: chess, tick-tack-toe, bargaining,
- The above two games have two players. Each player has a fixed set of actions. The players take decision or move sequentially- one after another in chess and tick-tack-toe.
- After a few rounds of moves by each player- one of the players win the game.
- The objective of each player is to win the game.

What constitute a Game?

- Players: those who are involved in it.
- Rules: who moves when? What do they know when they move? What can they do?
- Outcome: For each possible set of actions by the players, what is the outcome of the game?
- Payoffs: What are the players preferences (utility function) over the possible outcomes?

- Rules: who moves when? It specifies whether the players take action simultaneously, sequentially.
    - If simultaneously actions are taken, for how long? Like once, twice or for $t$ times.
    - If sequentially, who moves first ? How long the game continues?
- What do they know when they move? It means specification of the information set. If a actions are taken simultaneously, what each player knows? If actions are taken sequentially, how much information is available to the latter movers about the action of their predecessors.
- What can they do? Specify the set of actions a player can choose in each move.

Matching Pennies

- Players: There are two players, 1 and 2.
- Rules: Each player simultaneously put a penny down, either heads up or tails up
- Outcomes: If the two pennies match ( either both head or both tail) player 1 pays 1 rupee to player 2; otherwise, player 2 pays 1 rupee to player 1.
- Payoffs: Amount of money each players gains or loses

# Normal Form Game

- There can be $N$ players.
- Players choose action from a set of actions simultaneously and only once. This is also called simultaneous move single shot game.
- We specify the payoffs for each combinations of actions.
- For two players we can represent them through matrix. It is shown below.

Definition: The normal form representation of a n-player games specifies the players' strategy spaces $S_1, S_2 ... S_n$ and their payoff functions $u_1, u_2 ... u_n$.
A game is denoted as $G = \{S_1, S_2 ... S_n; u_1, u_2 .. u_n\}$.
The payoffs are defined for each combinations of elements belonging to each $S_i$.

In the normal form game $G = \{S_1, S_2 ... S_n; u_1, u_2 .. u_n\}$., let $s_i'$ and $s_i''$ be feasible strategies for player $i$ ( that is $s_i'$ and $s_i''$ are members of $S_i$). Strategy $s_i'$ is strictly dominated by strategy $s_i''$ if for each feasible combination of the other players' strategies, $i's$ payoff from playing from playing $s_i'$ is strictly less than $i's$ payoff from playing $s_i''$:
$u_i(s_1, s_2 .... s_{-i}, s_i', .... s_n) < u_i(s_1, s_2 .... s_{-i}, s_i'', .... s_n)$
for each $(s_1, s_2 .... s_{-i}, .... s_n)$ that can be constructed from the other players' strategy space $S_1, S_2 ... S_{-i}, S_{i+1} ... S_n$.
If the inequality sign is weak inequality for some combinations of $(s_1, s_2 .... s_{-i}, .... s_n)$ and there is atleast combination of $(s_1, s_2 .... s_{-i}, .... s_n)$ the sign must be strict , then it is weakly dominated strategy.

# Problems with Iterated Elimination of dominated strategies

- Common knowledge that players are rational at each step. All of them know that all players are rational. All players know that all the players know that all the players know that all the players are rational and so on.
- Imprecise solution

Nash equilibrium:

Definition: In the n-player normal form game $G = \{S_1, S_2 ... S_n; u_1, u_2 ... u_n\}$, the strategies $(s_1^*, s_2^*, s_3^* ...., s_n^*)$ are a Nash equilibrium if, for each player $i$, $s_i^*$ is ( at least tied for) player i's best response to the strategies specified for the $n-1$ other players, $(s_1^*, s_2^*, .., s_{i-1}^*, s_{i+1}^* ... s_n^*)$:

$u_i(s_1^*, s_2^*, .., s_{i-1}^*, s_i^*, s_{i+1}^* ... s_n^*) > u_i(s_1^*, s_2^*, .., s_{i-1}^*, s_i, s_{i+1}^* ... s_n^*)$,

for every feasible strategy $s_i$ in $S_i$.

$s_i^*$ solves

$\max_{s_i \in S_i} u_i(s_1^*, s_2^* ... s_{i-1}^*, s_i, s_{i+1}^*, .. s_n^*)$. At Nash equilibrium $(s_1^*, s_2^*, s_3^* ...., s_n^*)$ ,

$\max_{s_i \in S_i} u_i(s_1^*, s_2^* ... s_{i-1}^*, s_i^*, s_{i+1}^*, .. s_n^*)$ for each player $i$.

# Results

In the $n$ player normal form game $G = \{S_1 ... S_n; u_1 ... u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, s_2^* ..., s_n^*)$, then these strategies are the unique Nash equilibrium of the game.
In the $n$ player normal form game $G = \{S_1 ... S_n; u_1 ... u_n\}$, if the strategies $(s_1^*, s_2^* ..., s_n^*)$ are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.

Suppose player $i$ has $K$ pure strategies; $S_i = \{s_{i1}, s_{i2}, s_{i3}, ...s_{iK}\}$. Then a mixed strategy for player $i$ is a probability distribution $(p_{i1}, p_{i2}, p_{i3}..p_{iK})$ where $p_{ik}$ is the probability that player $i$ will play strategy $s_{ik}$, for $k = 1, 2, ...K$.

$p_{ik}$ is a probability so $0 \geq p_{ik} \leq 1$ and $\sum\limits_{k=1}^{K} p_{ik} = 1$.

- Suppose the strategy set of player 1 is $S_1 = \{s_{11}, s_{12}, s_{13}, ...s_{1J}\}$. Player 1 has $J$ actions or strategies. Strategy set of player 2 is $S_1 = \{s_{21}, s_{22}, s_{23}, ...s_{2K}\}$, player 2 has $K$ strategies or actions.
- If player 1 believes that player 2 will play the strategies $(s_{21}, s_{21}, ...s_{2k})$ with the probabilities $(p_{21}, p_{22}, ..., p_{2k})$ then player 1's expected payoff from playing $s_{1j}$ is $\sum\limits_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k})$.
- The expected payoff of player 1 from playing mixed strategy $(p_{11}, p_{12}, p_{13}, , , , p_{1J})$ is $E_1(p_i, p_2) = \sum\limits_{j=1}^{J} p_{1j} \left[ \sum\limits_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}) \right] = \sum\limits_{j=1}^{J} \sum\limits_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$.

- Similarly the expected payoff of player 2 from playing mixed strategy $(p_{21}, p_{22}, p_{23}, , , , p_{2K})$ is
$$E_2(p_i, p_2) = \sum_{k=1}^{K} p_{2k}\Big[\sum_{j=1}^{J} p_{1j} u_2(s_{1j}, s_{2k})\Big] = \sum_{k=1}^{K}\sum_{j=1}^{J} p_{1j} p_{2k} u_2(s_{1j}, s_{2k}).$$
- Mixed Strategy Nash equilibrium
In a two player normal form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies $(p_1^*, p_2^*)$ are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy that is the two conditions given below must hold
$E_1(p_1^*, p_2^*) \geq E_1(p_1, p_2^*)$ for every probability distribution $p_1$ over $S_1$.
$E_2(p_1^*, p_2^*) \geq E_2(p_1^*, p_2)$ for every probability distribution $p_2$ over $S_2$.