MA 201, Mathematics III, July-November 2022, Laplace Transform

Lecture 16

Laplace Transform

Let $F : [0, \infty) \to \mathbb{R}$ satisfy the following conditions:

- $F(t)$ is piecewise continuous on the interval $[0, b]$ for each $b > 0$,
- There exist real constants $M \geq 0$ and real positive number a such that

$$|e^{-at}F(t)| \leq M, \text{ i.e., } |F(t)| \leq Me^{at} \text{ for all } t > 0.$$

(Means F is a function of exponential order.)

Definition (Laplace Transform)

The Laplace transform of $F(t)$ (as defined above), denoted by $\mathcal{L}\{F(t)\}$ or $f(s)$, is defined by

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s), \tag{1}$$

provided this integral exists. Here s is a positive real number or a complex number.

Observe

$$\left|\int_0^\infty F(t)e^{-st}\,dt\right| \leq \int_0^\infty |F(t)|e^{-st}\,dt \leq M\int_0^\infty e^{-(s-a)t}\,dt = \frac{M}{(s-a)}, \quad s > a.$$

The variable s also takes complex values, but for the time being, assume s to be real.

Laplace transform of some elementary functions

1. $\mathcal{L}\{1\} = \dfrac{1}{s},\ s > 0.$
2. $\mathcal{L}\{t\} = \dfrac{1}{s^2},\ s > 0.$
3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ s > 0,\ n$ an integer or

   $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}},\ \ n+1 > 0,\ \ n$ not an integer.
4. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a},\ s > a.$
5. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2},\ \ s > 0.$
6. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2},\ \ s > 0.$
7. $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2},\ \ s > |a|.$
8. $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2},\ \ s > |a|.$

Note: Gamma function $\Gamma(\alpha)$ is defined as an improper definite integral as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$$

Elementary properties of Laplace transform

The Laplace transform has many interesting and useful properties,

the most fundamental of which is linearity.

Theorem

If $F_1(t)$ and $F_2(t)$ are two functions whose Laplace transforms exist, then

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\} \tag{2}$$

where a and b are any real constants.

Proof:

$$\begin{aligned}
\mathcal{L}\{aF_1(t) + bF_2(t)\} &= \int_0^\infty (aF_1 + bF_2)e^{-st}\,dt \\
&= \int_0^\infty (aF_1 e^{-st} + bF_2 e^{-st})\,dt \\
&= a\int_0^\infty F_1 e^{-st}\,dt + b\int_0^\infty F_2 e^{-st}\,dt \\
&= a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}
\end{aligned}$$

Elementary properties of Laplace transform

Theorem

(First Shifting Theorem): *If $F(t)$ is a Laplace transformable function, i.e., $\mathcal{L}\{F(t)\} = f(s)$, then*

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a). \tag{3}$$

Proof:

$$\begin{aligned}
\mathcal{L}\{e^{at}F(t)\} &= \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt \\
&= \int_0^\infty e^{-st}e^{at}F(t)\,dt \\
& \int_0^\infty e^{-(s-a)t}F(t)\,dt \\
&= f(s-a).
\end{aligned}$$

This establishes the theorem. □

Derivative property of Laplace transform

Suppose a differentiable function $F(t)$ has Laplace transform $f(s)$, then

we can obtain the Laplace transform of its derivative $F'(t)$:

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt\ .$$

Similarly if the function $F(t)$ is twice, thrice,..., n-times differentiable,

we can obtain the Laplace transforms of $F''(t), F'''(t), \ldots, F^{(n)}(t)$, respectively.

The following theorems relate to the Laplace transforms of the derivatives of a function.

These results are extremely helpful in solving differential equations of various orders.

Derivative property of Laplace transform

Theorem

If $F(t)$ is a differentiable function of t and $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F'(t)\} = -F(0) + sf(s). \tag{4}$$

Proof:

$$\begin{aligned}
\mathcal{L}\{F'(t)\} &= \int_0^\infty e^{-st} F'(t)\, dt \\
&= \left[\left| e^{-st} F(t) \right|_0^\infty + s \int_0^\infty e^{-st} F(t)\, dt \right] \\
&= -F(0) + sf(s),
\end{aligned}$$

where $F(0)$ is the value of $F(t)$ at $t = 0$.

This is the proof of the theorem. □

Derivative property of Laplace transform

Theorem

If $F(t)$ is a twice differentiable function of t and $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0). \tag{5}$$

Proof:

$$\begin{aligned}
\mathcal{L}\{F''(t)\} &= \int_0^\infty e^{-st} F''(t)\, dt \\
&= \left[\left| e^{-st} F'(t) \right|_0^\infty + s \int_0^\infty e^{-st} F'(t)\, dt \right] \\
&= -F'(0) + \left[\left| s e^{-st} F(t) \right|_0^\infty + s^2 \int_0^\infty e^{-st} F(t)\, dt \right] \\
&= s^2 f(s) - sF(0) - F'(0),
\end{aligned}$$

where $F(0)$ and $F'(0)$ are the values of $F(t)$ and $F'(t)$, respectively, at $t = 0$.

This is the proof of the theorem. □

Derivative property of Laplace transform

We can generalize the earlier results as follows:

Theorem

If $F(t)$ is an n-times differentiable function of t, and $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{(n-2)}(0) - F^{(n-1)}(0). \quad (6)$$

The proof can be completed very easily by induction.

Example ODE1 (First order ODE):

$$\frac{dx}{dt} + 3x = 0,\ x(0) = 1.$$

By taking Laplace transform on both sides of the equation,

$$\begin{aligned}
&\mathcal{L}\{\frac{dx}{dt}\} + \mathcal{L}\{3x\} = 0\\
&\Rightarrow s\mathcal{L}\{x\} - x(0) + 3\mathcal{L}\{x\} = 0\\
&\Rightarrow (s+3)\mathcal{L}\{x\} = 1\\
&\Rightarrow \mathcal{L}\{x\} = \frac{1}{s+3}.
\end{aligned}$$

Elementary properties of Laplace transform

Theorem

(Second Shifting Theorem):

$$\textit{If } \mathcal{L}\{F(t)\} = f(s) \textit{ and } G(t) = \begin{cases} F(t-a), & t \geq a, \\ 0, & t < a, \end{cases}$$

$$\textit{then} \quad \mathcal{L}\{G(t)\} = e^{-as} f(s). \tag{7}$$

Proof:

$$\begin{aligned} \mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} G(t)\, dt \\ &= \int_a^\infty e^{-st} F(t-a)\, dt \\ &= \int_0^\infty e^{-s(a+u)} F(u)\, du, \quad \text{by taking } t-a=u \\ &= e^{-as} \int_0^\infty e^{-su} F(u)\, du \\ &= e^{-as} f(s), \end{aligned}$$

which proves the theorem. □

Some more properties

Recall

Heaviside's unit step function, or simply the unit step function, is defined as

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{8}$$

Since $H(t)$ is precisely the same as 1 for $t \geq 0$,

the Laplace transform of $H(t)$ must be the same as the Laplace transform of 1, i.e., $\dfrac{1}{s}$.

If we shift the origin to t_0,

$$H(t - t_0) = \begin{cases} 0, & t < t_0, \\ 1, & t > t_0. \end{cases} \tag{9}$$

The following results can be obtained very easily:

$$\mathcal{L}\{H(t)\} = \frac{1}{s}. \tag{10}$$

$$\mathcal{L}\{H(t - t_0)\} = \frac{e^{-st_0}}{s}. \tag{11}$$

Elementary properties of Laplace transform

Theorem

(Change of Scale Theorem): *If* $\mathcal{L}\{F(t)\} = f(s)$, *then*

$$\mathcal{L}\{F(at)\} = \frac{1}{a}\, f\left(\frac{s}{a}\right), \quad a > 0. \tag{12}$$

Proof:

$$\begin{aligned}
\mathcal{L}\{F(at)\} &= \int_0^\infty e^{-st} F(at)\, dt \\
&= \frac{1}{a}\int_0^\infty e^{-s\,u/a} F(u)\, du, \quad \text{by taking } at = u \\
&= \frac{1}{a}\int_0^\infty e^{(-s/a)u}\, F(u)\, du \\
&= \frac{1}{a}\, f\left(\frac{s}{a}\right),
\end{aligned}$$

which proves the theorem. □

Elementary properties of Laplace transform

Theorem

If $\mathcal{L}\{F(t)\} = f(s)$, *then*

$$\mathcal{L}\{tF(t)\} = -\frac{d}{ds}f(s) \text{ and in general } \mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s). \tag{13}$$

Proof: We know

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$$

Differentiating w.r.t. s

$$\begin{aligned} \frac{df}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt \\ &= \int_0^\infty -te^{-st} F(t)\, dt. \end{aligned}$$

Hence,

$$\mathcal{L}\{tF(t)\} = -\frac{d}{ds} f(s).$$

Elementary properties of Laplace transform

Now assume that the result holds for n so that

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Differentiating the above w.r.t. s,

$$\begin{aligned} \int_0^\infty -t^{n+1} e^{-st} F(t)\, dt &= (-1)^n \frac{d^{n+1}}{ds^{n+1}} f(s) \\ \implies \int_0^\infty e^{-st} (t^{n+1} F(t))\, dt &= (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s) \\ \implies \mathcal{L}\{t^{n+1} F(t)\} &= (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} f(s), \end{aligned}$$

which shows that it is true for all n and proves the theorem. $\square$

Some more properties

Theorem

If $\mathcal{L}\{F(t)\} = f(s)$, *then*

$$\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}. \tag{14}$$

Proof:

Let $G(t) = \displaystyle\int_0^t F(u)\,du.$

$\Rightarrow \quad G(0) = 0,$ and $G'(t) = F(t).$

$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\mathcal{L}\{G(t)\}$

i.e. $\quad s\mathcal{L}\{G(t)\} = \mathcal{L}\{F(t)\}$

or, $\quad \mathcal{L}\{G(t)\} = \dfrac{f(s)}{s}$

$\Rightarrow \mathcal{L}\left\{\displaystyle\int_0^t F(u)\,du\right\} = \dfrac{f(s)}{s},$

which proves the theorem. □

Some more properties

Theorem

If $\mathcal{L}\{F(t)\} = f(s)$*, then*

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du, \tag{15}$$

assuming that $\mathcal{L}\left\{\frac{F(t)}{t}\right\} \to 0$ *as* $s \to \infty$.

Proof:

Let $G(t) = \dfrac{F(t)}{t}$ so that $F(t) = tG(t)$.

Using the property $\mathcal{L}\{tG(t)\} = -\dfrac{d}{ds}\mathcal{L}\{G(t)\}$,

$$f(s) = \mathcal{L}\{F(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\} = -\frac{d}{ds}\mathcal{L}\left\{\frac{F(t)}{t}\right\}.$$

Integrating both sides with respect to s from s to ∞,

$$\int_s^\infty f(u)\,du = \left[-\mathcal{L}\left\{\frac{F(t)}{t}\right\}\right]_s^\infty = \mathcal{L}\left\{\frac{F(t)}{t}\right\}\Bigg|_s = \mathcal{L}\left\{\frac{F(t)}{t}\right\},$$

which proves the theorem. $\square$

Some more properties

Theorem

(Third Shifting Theorem) *If $F(t)$ is a function of exponential order in t, then*

$$\mathcal{L}\{H(t-t_0)F(t-t_0)\} = e^{-st_0}\, f(s), \tag{16}$$

where $f(s)$ is the Laplace transform of $F(t)$.

Proof:

$$\begin{aligned}
\mathcal{L}\{H(t-t_0)F(t-t_0)\} &= \int_0^\infty e^{-st} H(t-t_0)F(t-t_0)\, dt \\
&= \int_{t_0}^\infty e^{-st} F(t-t_0)\, dt \\
&= \int_0^\infty e^{-s(t_0+u)} F(u)\, du \\
&= e^{-st_0} \int_0^\infty e^{-su} F(u)\, du \\
&= e^{-st_0}\, f(s),
\end{aligned}$$

which proves the result. □

Some more properties

Example:

Determine the Laplace transform of the sine function switched on at time $t = 3$.

Solution:

The function is

$$S(t) = \begin{cases} \sin t, & t > 3, \\ 0, & t < 3. \end{cases} \tag{17}$$

We can use Heaviside function to write $S(t)$ as

$$S(t) = H(t-3)\sin t.$$

$\sin t = \sin(t-3+3) = \sin(t-3)\cos 3 + \cos(t-3)\sin 3.$

Taking Laplace transform

$$\mathcal{L}\{S(t)\} = \mathcal{L}\{H(t-3)\sin(t-3)\}\cos 3 + \mathcal{L}\{H(t-3)\cos(t-3)\}\sin 3.$$

Then using the third shifting theorem

$$\mathcal{L}\{S(t)\} = \frac{(\cos 3 + s\sin 3)e^{-3s}}{s^2+1}.$$

Inverse Laplace transform

If $F(t)$ has the Laplace transform $f(s)$, i.e., $\mathcal{L}\{F(t)\} = f(s)$, then the inverse Laplace transform is defined by $\mathcal{L}^{-1}\{f(s)\} = F(t)$.

Theorem

(Linearity) *The inverse Laplace transform is linear, i.e.,*

$$\mathcal{L}^{-1}\{a_1 f_1(s) \pm a_2 f_2(s)\} = a_1 \mathcal{L}^{-1}\{f_1(s)\} \pm a_2 \mathcal{L}^{-1}\{f_2(s)\}. \tag{18}$$

Theorem

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t). \tag{19}$$

Theorem

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a), & t \geq a, \\ 0, & t < a. \end{cases} \tag{20}$$

Inverse Laplace transform

Example A: Find the inverse Laplace transform of the following:

(1). $\dfrac{a}{s^2 - a^2}$, (2). $\dfrac{s}{s^2 + a^2}$.

It is very easy to find that the required functions are

$\sinh at$ and $\cos at$ respectively.

Example B:

Determine $\mathcal{L}^{-1}\left\{\dfrac{s-2}{s^2+4s+13}\right\}$.

Solution B:

$$\frac{s-2}{s^2+4s+13} = \frac{s-2}{(s+2)^2+3^2} = \frac{s+2-4}{(s+2)^2+3^2}.$$

Taking inverse

$$F(t) = \frac{e^{-2t}}{3}[3\cos 3t - 4\sin 3t].$$

Inverse Laplace transform

Example C:

Determine $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}$.

Solution C: Use the partial fractions to write

$$\frac{s^2}{(s+3)^3} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3},$$

Equating coefficients of various powers of s, we will ultimately get

$A = 1, B = -6, C = 9$ so that

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}.$$

Taking inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t}.$$