MA201 Partial Differential Equations

Lecture 8

Vibrating string and the wave equation

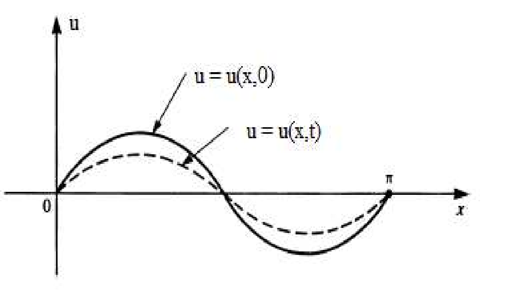

Figure : String problem

Vibrating string and the wave equation

Consider a stretched string of length π with the ends fastened to the ends $x = 0$ and $x = \pi$.

Suppose that the string is set to vibrate by displacing it from its equilibrium position.

Let $u(x, t)$ denote the transverse displacement at time $t \geq 0$ of the point on the string at position x.

That is, we assume that each point of the string moves only in the vertical direction.

In particular, $u(x, 0)$ denotes the initial shape of the string and $u_t(x, 0)$ denotes the initial velocity.

That is, the string is set to vibrate by supplying an initial velocity $u_t(x, 0)$ to it from its equilibrium position $u(x, 0)$ and then releasing it.

Vibrating string and the wave equation (Contd.)

Under some physical assumptions, we arrive at the following equation known as the **one-dimensional wave equation**:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \tag{1}$$

which governs the entire process. Here c represents a physical quantity.

The above equation corresponds to the situation of without consideration of any external force acting on the string.

Suppose that we do wish to include such an external force on the string (due to its weight or other impressed external forces (like gravity or pressure)), referred to as a **load**, given by

$$f(x,t) = \text{vertical force per unit length at point } x, \text{ at time } t.$$

Vibrating string and the wave equation (Contd.)

Then, the result will be the nonhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = F(x,t), \tag{2}$$

where $F(x,t) = \frac{1}{\rho} f(x,t)$, with ρ as the mass per unit length of the string.

Two special cases of (2) particularly generate interest:

- When the external force is due to the gravitational acceleration g only, the equation becomes

$$u_{tt} = c^2 u_{xx} - g. \tag{3}$$

- When the external force is due to the resistance of the medium (say, a string vibrating in a fluid), the equation becomes

$$u_{tt} - c^2 u_{xx} = -k u_t, \quad k \text{ is a positive constant,} \tag{4}$$

known as the damped wave equation.

Wave Equation

It is to be noted that the use of string problem to demonstrate the wave equation is a matter of convenience. There are more applications in physics and engineering.

For instance,

$$u(x,t) = \sin(x \pm ct)$$

represents sinusoidal waves traveling with speed c in the positive and negative directions, respectively, without change of shape.

Figure : Water Wave

The D'Alembert's solution of the wave equation

Method of characteristics is very useful for hyperbolic equations.

Please note

- Two families of characteristics of hyperbolic equations, being real and distinct, are of considerable practical value.
- In one-dimensional progressive wave propagation, consideration of characteristics can give us a good deal of information about the propagation of the wave fronts.
- This solution of one-dimensional wave equation, known as D'Alembert's solution, was discovered by a French mathematician named Jean Le Rond D'Alembert.

The D'Alembert solution of the wave equation (Contd.)

Consider the one-dimensional wave equation:

$$u_{tt} = c^2 u_{xx}, \;\; -\infty < x < \infty, \; t > 0. \tag{5}$$

Here $B^2 - 4AC = 4c^2 > 0$.

The characteristics are given by

$$\xi = x - ct, \; \eta = x + ct.$$

Under this transformation

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \\
u_t &= u_\xi \xi_t + u_\eta \eta_t = -c u_\xi + c u_\eta, \\
u_{xx} &= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x) + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) \\
&= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\
u_{tt} &= -c(u_{\xi\xi}\xi_t + u_{\xi\eta}\eta_t) + c(u_{\eta\xi}\xi_t + u_{\eta\eta}\eta_t), \\
&= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).
\end{aligned}$$

The D'Alembert solution of the wave equation (Contd.)

Substituting these into the given equation, we get

$$u_{\xi\eta} = 0. \tag{6}$$

Integrating partially with respect to η:

$$u_{\xi} = f'(\xi).$$

Integrating partially w.r.t. ξ:

$$u = f(\xi) + g(\eta).$$

The solution in physical variables:

$$u(x,t) = f(x-ct) + g(x+ct) \tag{7}$$

where f and g are arbitrary functions.

The D'Alembert solution of the wave equation (Contd.)

The physical interpretation of these functions is quite interesting.

The functions f and g represent two progressive waves travelling in opposite directions with the speed c.

To see this, let us first consider the solution $u = f(x - ct)$.

At $t = 0$, it defines the curve $u = f(x)$, and after time $t = t_1$, it defines the curve $u = f(x - ct_1)$.

But these curves are identical except that the latter is translated to the right a distance equal to ct_1.

Method of Characteristics (Contd.)

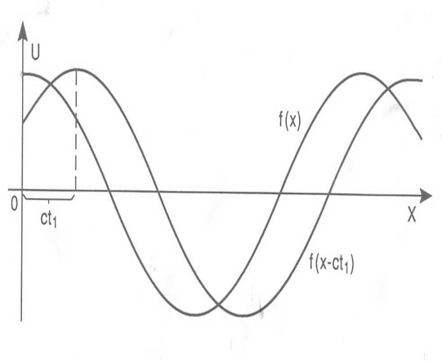

Figure : A Progressive Wave

The D'Alembert solution of the wave equation (Contd.)

Thus the entire configuration moves along the positive direction of the x-axis a distance of ct_1 in time t_1.

The velocity with which the wave is propagated is, therefore,

$$v = \frac{ct_1}{t_1} = c$$

Similarly, the function $g(x + ct)$ defines a wave progressing in the negative direction of the x-axis with constant velocity c.

The total solution is, therefore, the algebraic sum of these two travelling waves.

Solution (7) is a very convenient representation for progressive waves

which travel large distances through a uniform medium.

The D'Alembert solution of the wave equation (Contd.)

Let us consider the following two initial conditions for a uniform medium over $-\infty < x < \infty$.

$$\text{Displacement: } u(x,0) = \phi(x), \tag{8}$$

$$\text{Velocity: } u_t(x,0) = \psi(x). \tag{9}$$

That is,

we consider the vibration of a thin string of infinite length with some given initial displacement and initial velocity.

From solution (7), by utilizing the conditions (8) and (9), we find that

$$f(x) + g(x) = \phi(x), \tag{10}$$

$$-cf'(x) + c\,g'(x) = \psi(x), \tag{11}$$

for all values of x.

The D'Alembert solution of the wave equation (Contd.)

Integrating the second equation with respect to x:

$$-f(x) + g(x) = \frac{1}{c}\int_{x_0}^{x} \psi(\tau)d\tau + A, \tag{12}$$

where A is an integration constant and τ is a dummy variable.

From (10) and (12), we get

$$f(x) = \frac{1}{2}\left[\phi(x) - \frac{1}{c}\int_{x_0}^{x} \psi(\tau)d\tau\right] - A/2, \tag{13}$$

$$g(x) = \frac{1}{2}\left[\phi(x) + \frac{1}{c}\int_{x_0}^{x} \psi(\tau)d\tau\right] + A/2, \tag{14}$$

Substituting these expressions into (7):

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)d\tau. \tag{15}$$

This is $\boxed{\textit{D'Alembert's solution}}$ for one-dimensional wave equation.

The D'Alembert solution of the wave equation (Contd.)

Thus, for a given initial displacement and velocity in the vertical direction,

the wave equation (for an infinite string) is completely solved and this solution is usually called the progressive wave solution.

It is to be noted that

the use of string problem to demonstrate the solution of the wave problem is a matter of convenience. However, any variables satisfying the wave equation possess the same mathematical properties developed for the string.

The D'Alembert solution of the wave equation (Contd.)

It is clear that the wave equation can be handled very easily

by introducing the characteristic variables (ξ, η).

The relationship between the physical plane and the characteristic plane for this particular example can be demonstrated graphically.

Equation (7) represents the solution as the sum of two progressive waves: one going to the right and the other to the left.

The wave celerity is c.

The D'Alembert solution of the wave equation (Contd.)

For each of the two progressive waves, we can also follow the wave motion by observing that in the xt-plane, $\frac{1}{2}f(x-ct)$ is constant along each line $x-ct=$ constant and similarly $\frac{1}{2}g(x+ct)$ is constant along each line $x+ct=$ constant.

Thus, there are two families of parallel lines called the characteristics along which the waves are propagated.

Method of Characteristics (Contd.)

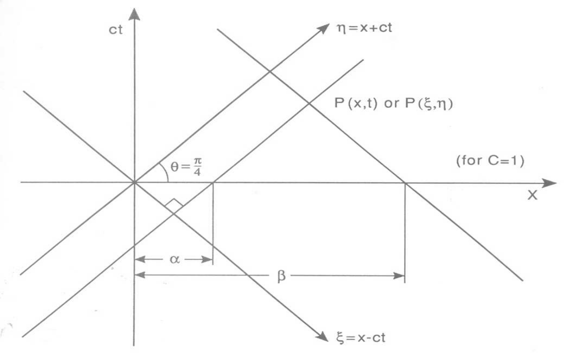

Figure : Relationship between characteristic plane and physical plane

The D'Alembert solution of the wave equation (Contd.)

Furthermore, along the x-axis,

the values of $u(x,0)$ and $u_t(x,0)$ are given as initial conditions of displacement and velocity

and they just suffice to determine the constant values of f and g along the individual characteristic.

The characteristics, therefore, represent the paths in the xt-plane along which disturbances in the medium propagate.

Finally, since the solution of wave equation is $u = f(x - ct) + g(x + ct)$,

the value of u at any point in the xt-plane is the sum of the values of f and g on the respective characteristics which pass through that point.

Special cases of D'Alembert solution

CASE I (*Initial velocity zero*).

Suppose the string has ICs

$$u(x,0) = \phi(x),$$
$$u_t(x,0) = 0.$$

Then, D'Alembert's solution is

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)].}$$

Special cases of D'Alembert solution

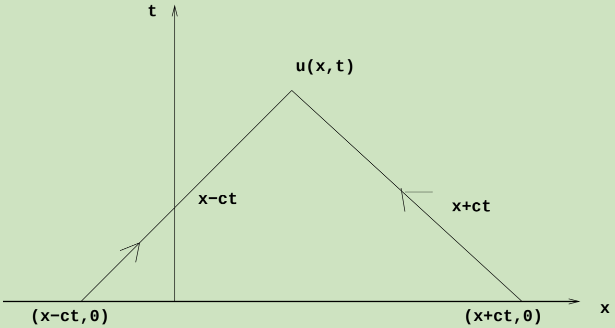

Figure : Geometrical interpretation of $u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)]$

Special cases of D'Alembert solution (Contd.)

CASE 2. (*Initial displacement zero*)

Suppose the string has the following ICs:

$$u(x,0) = 0,$$
$$u_t(x,0) = \psi(x).$$

In this case, the D'Alembert's solution is

$$\boxed{u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\, d\tau.}$$

The solution u at (x,t) may be interpreted as integrating the initial velocity between $x-ct$ and $x+ct$ on the initial line $t=0$.

Special cases of D'Alembert solution (Contd.)

Example (Zero initial velocity)

Solve the IVP:

$$\begin{aligned} \text{PDE:} \quad & u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0 \\ \text{IC:} \quad & u(x,0) = \sin(x), \\ & u_t(x,0) = 0. \end{aligned}$$

Solution: Using D'Alembert's formula with $\phi(x) = \sin(x)$ and $\psi(x) = 0$, we obtain

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right].$$

Special cases of D'Alembert solution (Contd.)

Example (Zero initial displacement)

Consider the IVP:

$$\begin{aligned} &\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0 \\ &\text{I.C.} \quad u(x,0) = 0, \\ &\qquad u_t(x,0) = \sin(x). \end{aligned}$$

Solution:

Here the string is initially straight ($u(x,0) = 0$), but has a variable velocity at $t = 0$ ($u_t(x,0) = \sin(x)$). Thus, using D'Alembert's formula with $\phi(x) = 0$ and $\psi(x) = \sin(x)$, we obtain

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(\tau)d\tau = -\frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right].$$