MA 201: Partial Differential Equations
Lecture - 4

Solution technique for quasi-linear equations: *Method of Characteristics*

- Recall a first-order quasi-linear PDE:

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u). \tag{1}$$

- Let us assume that an integral surface $u = u(x, y)$ of (1) can be found. Write this integral surface in implicit form as

$$F(x, y, u) = u(x, y) - u = 0.$$

- Equation (1) may be written as

$$au_x + bu_y - c = \langle a, b, c\rangle \cdot \langle u_x, u_y, -1\rangle = 0. \tag{2}$$

- This shows that the vector $\langle a, b, c\rangle$ and the gradient vector ∇F are orthogonal at the point (x, y, u).

- In other words, at each point (x, y, u) of the integral surface, the tangent vector to the integral surface is given by the vector $\langle a, b, c\rangle$.
- Recall that a tangent vector to the integral surface at $(x, y, u) = (x(t), y(t), u(t))$ is given by

$$\mathbf{r}'(t) = \langle x'(t), y'(t), u'(t)\rangle, \tag{3}$$

for some t belonging to an interval I.

Then, we must have

$$\frac{dx}{dt} = a(t), \quad \frac{dy}{dt} = b(t), \quad \frac{du}{dt} = c(t). \tag{4}$$

The solutions of (4) are called the characteristic curves for the quasi-linear PDE.

Example

Find an integral surface of the quasi-linear PDE

$$x(y^2+u)p - y(x^2+u)q = (x^2-y^2)u, \quad p = \frac{\partial u}{\partial x} \;\&\; q = \frac{\partial u}{\partial y}$$

which contains the straight line $x+y=0,\ u=1$.

Remarks:

- Characteristic equations are

$$\begin{aligned}
\frac{dx}{dt} &= x(y^2+u) = f_1(x,y,u) \\
\frac{dy}{dt} &= -y(x^2+u) = f_2(x,y,u) \\
\frac{du}{dt} &= (x^2-y^2)u = f_3(x,y,u).
\end{aligned}$$

- These are nonlinear equations in x and y, and we do not go ahead to solve them at this moment. Instead, we next focus on a familiar method due to Lagrange.

Solution Technique: Method of *Lagrange*

Background:

Theorem

If $\phi = \phi(x, y, u)$ and $\psi = \psi(x, y, u)$ are two given functions of $x,\ y$ and u, and if $G(\phi, \psi) = 0$, where G is an arbitrary function of ϕ and ψ, then $u = u(x, y)$ satisfies a first-order PDE

$$\frac{\partial u}{\partial x}\frac{\partial(\phi,\psi)}{\partial(y,u)} + \frac{\partial u}{\partial y}\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}, \tag{5}$$

where

$$\frac{\partial(\phi,\psi)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}.$$

How to establish it?

What is given? A surface $G(\phi, \psi) = 0$. **How to get a PDE out of it?**

- Differentiate $G(\phi, \psi) = 0$ with respect to x and y, respectively:

$$G_\phi(\phi_x + u_x\phi_u) + G_\psi(\psi_x + u_x\psi_u) = 0, \tag{6}$$

$$G_\phi(\phi_y + u_y\phi_u) + G_\psi(\psi_y + u_y\psi_u) = 0. \tag{7}$$

- Nontrivial solutions for $\frac{\partial G}{\partial \phi}(= G_\phi)$ and $\frac{\partial G}{\partial \psi}(= G_\psi)$ can be found when

$$\begin{vmatrix} \phi_x + u_x\phi_u & \psi_x + u_x\psi_u \\ \phi_y + u_y\phi_u & \psi_y + u_y\psi_u \end{vmatrix} = 0.$$

- Expanding this determinant obviously gives a first-order quasi-linear PDE.

- Since surface $G(\phi, \psi) = 0$ leads to equation

$$\frac{\partial u}{\partial x}\frac{\partial(\phi, \psi)}{\partial(y, u)} + \frac{\partial u}{\partial y}\frac{\partial(\phi, \psi)}{\partial(u, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \tag{8}$$

therefore, $G(\phi, \psi) = 0$ is a **solution** for the equation. Since G is arbitrary, this solution is called the **general solution**.

- This may give an idea to find the general solution for the quasi-linear equation

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u). \tag{9}$$

- **How?** Both equations (8) and (9) will be identical if we select $\phi = \phi(x, y, u)$ and $\psi = \psi(x, y, u)$ in such a way that

$$a = \lambda\frac{\partial(\phi, \psi)}{\partial(y, u)}, \quad b = \lambda\frac{\partial(\phi, \psi)}{\partial(u, x)}, \quad c = \lambda\frac{\partial(\phi, \psi)}{\partial(x, y)},$$

for some λ.

Theorem (The method of Lagrange)

The general solution of the quasi-linear PDE

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \tag{10}$$

is given by

$$G(\phi, \psi) = 0, \tag{11}$$

where G *is an arbitrary function of* ϕ *and* ψ *with* $\phi(x, y, u) = c_1$ *and* $\psi(x, y, u) = c_2$ *being the solutions of the equations*

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{12}$$

Proof. Observe that

$$d\phi = \phi_x dx + \phi_y dy + \phi_u du = 0, \tag{13}$$

$$d\psi = \psi_x dx + \psi_y dy + \psi_u du = 0. \tag{14}$$

On the integral surface, we have (through characteristic equations)

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{15}$$

Then, we must have (comparing with (13) and (14):

$$a\phi_x + b\phi_y + c\phi_u = 0, \tag{16}$$

$$a\psi_x + b\psi_y + c\psi_u = 0. \tag{17}$$

Solving these equations for a, b and c, we obtain

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}}. \tag{18}$$

This completes the rest of the proof.

Remark

Any expression involving such ϕ and ψ is an integral surface for (1).

Example

Find the general integral/general solution of $xu_x + yu_y = u$.

Solution. The associated system of equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

From the first two relations, we have

$$\frac{dx}{x} = \frac{dy}{y} \Longrightarrow \ln x = \ln y + \ln c_1 \Longrightarrow \frac{x}{y} = c_1.$$

Similarly,

$$\frac{du}{u} = \frac{dy}{y} \Longrightarrow \frac{u}{y} = c_2.$$

Take $\phi = \frac{x}{y}$ and $\psi = \frac{u}{y}$. The general integral is given by

$$G\left(\frac{x}{y}, \frac{u}{y}\right) = 0,$$

where G is an arbitrary function of ϕ and ψ.

Example

Find the general integral for $u(x+y)u_x + u(x-y)u_y = x^2 + y^2$.

Solution. The characteristic equations are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2}. \tag{19}$$

Each of these ratios is equivalent to

$$y\frac{dx}{dt} + x\frac{dy}{dt} - u\frac{du}{dt} = 0 = x\frac{dx}{dt} - y\frac{dy}{dt} - u\frac{du}{dt}.$$

Consequently, we have

$$d\left\{xy - \frac{u^2}{2}\right\} = 0 \quad \text{and} \quad d\left\{\frac{1}{2}(x^2 - y^2 - u^2)\right\} = 0.$$

Integrating the above, we obtain

$$2xy - u^2 = c_1 = \phi \quad \text{and} \quad x^2 - y^2 - u^2 = c_2 = \psi.$$

Thus, the general solution is $G(2xy - u^2, x^2 - y^2 - u^2) = 0$, where G is an arbitrary function.

Alternative method of solution. From (19), we can choose

$$\frac{ydx + xdy}{uy(x+y) + ux(x-y)} = \frac{du}{x^2+y^2},$$

which gives us

$$ydx + xdy = udu$$

leading to

$$xy - \frac{u^2}{2} = c_1.$$

In a similar manner, from (19), we can choose

$$\frac{xdx - ydy}{ux(x+y) - uy(x-y)} = \frac{du}{x^2+y^2},$$

which gives us

$$xdx - ydy = udu$$

leading to

$$x^2 - y^2 - u^2 = c_2.$$

Hence, we obtain

$$2xy - u^2 = c_1 = \phi \quad \text{and} \quad x^2 - y^2 - u^2 = c_2 = \psi.$$

Thus, the general solution is $G(2xy - u^2, x^2 - y^2 - u^2) = 0$, where G is an arbitrary function.