MA 201: Partial Differential Equations
One-dimensional Wave Equation (Contd.)
Lecture - 12

Finite Vibrating String with no External Force

- Recall the finite string problem in a computational domain $(x,t) \in [0,L] \times [0,\infty)$ (Lecture 11)
  - The governing equation
$$u_{tt} = c^2 u_{xx}, \quad (x,t) \in (0,L) \times (0,\infty). \tag{1}$$
  - The boundary conditions for all $t > 0$:
$$u(0,t) = 0, \quad u(L,t) = 0. \tag{2}$$
  - The initial conditions for $0 \leq x \leq L$:
$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \tag{3}$$

Finite Vibrating String with no External Force

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \tag{4}$$

with

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,\ \ n = 1,2,3,\ldots \\ B_n &= \frac{2}{n\pi c}\int_0^L \psi(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,\ \ n = 1,2,3,\ldots. \end{aligned}$$

Example:

For a string of length L stretched between the points $x = 0$ and $x = L$, find the vibration in the string subject to the following initial conditions:

$$u(x,0) = \sin(\pi x/L) + 1/2\sin(3\pi x/L), \quad u_t(x,0) = 0.$$

Solution of the Finite Vibrating String Problem: Example

Solution:

Here, initial conditions are

$$\phi(x) = \sin(\pi x/L) + 1/2\sin(3\pi x/L), \;\; \psi(x) = 0.$$

Therefore, $B_n = 0$.

and

$$A_n = \frac{2}{L}\int_0^L \left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)dx.$$

Due to the orthogonality of the set $\{\sin\left(\frac{n\pi x}{L}\right) : n = 1,\ 2, \ldots\}$,

only A_1 and A_3 are non-zero, and

$$A_1 = 1, A_3 = 1/2.$$

Therefore, the solution is (since $A_n = 0 \quad \text{for} \quad n \neq 1, 3$)

$$u(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right).$$

Finite Vibrating String with Gravity

We now consider an external force due to the gravitational acceleration g only (consider a string oriented horizontally). Then the one-dimensional wave equation becomes

$$u_{tt} = c^2 u_{xx} - g, \quad 0 < x < L, \quad t > 0. \tag{5}$$

We seek to find

the displacement of the string at any position and at any time subject to the following boundary conditions (for $t > 0$) and initial conditions ($0 \leq x \leq L$):

$$u(0, t) = 0, \tag{6a}$$

$$u(L, t) = 0, \tag{6b}$$

and

$$u(x, 0) = \phi(x), \tag{7a}$$

$$u_t(x, 0) = \psi(x). \tag{7b}$$

Finite Vibrating String with Gravity (Contd.)

Due to the presence of the term g in equation (5), which has made the equation non-homogeneous,

the direct application of the method of separation of variables will not work.

Now seek a solution in the form:

$$u(x,t) = v(x,t) + h(x). \tag{8}$$

We intend to split the given problem into two problems – one IBVP in $v(x,t)$ and another BVP in $h(x)$.

The idea is to have

- $v(x,t)$ giving the solution for a string without the external force.
- $h(x)$ taking care of the non-homogeneous term in the equation.

Now, substituting (8) in equation (5), we obtain

$$v_{tt} = c^2[v_{xx} + h''(x)] - g. \tag{9}$$

Finite Vibrating String with Gravity (Contd.)

We choose $h(x)$ such that

$$c^2 h''(x) - g = 0, \tag{10}$$

in order to allow $v(x, t)$ to satisfy the homogeneous wave equation

$$v_{tt} = c^2 v_{xx}, \;\; 0 < x < L, \;\; t > 0. \tag{11}$$

Both functions $v(x, t)$ and $h(x)$ together satisfy the boundary conditions

$$v(0, t) + h(0) = 0, \; t > 0, \tag{12a}$$
$$v(L, t) + h(L) = 0, \; t > 0, \tag{12b}$$

and also the initial conditions

$$v(x, 0) + h(x) = \phi(x), \; 0 < x < L, \tag{13a}$$
$$v_t(x, 0) = \psi(x), \; 0 < x < L. \tag{13b}$$

Finite Vibrating String with Gravity (Contd.)

Since, $h(x)$ is a user-defined function, we can set

$$h(0) = 0 \;\;\&\;\; h(L) = 0. \tag{14}$$

so that $v(x,t)$ satisfies zero boundary conditions at $x = 0$ and $x = L$ allowing us to use separation of variables method for the IBVP in $v(x,t)$.

Subsequently, the original non-homogeneous IBVP can be conveniently split into the following two problems:

Problem I (BVP):

$$\begin{aligned} &c^2 h''(x) = g, \\ &h(0) = 0 = h(L). \end{aligned}$$

Problem II (IBVP):

$$\begin{aligned} &v_{tt} = c^2 v_{xx}, \\ &v(0,t) = 0 = v(L,t), \\ &v(x,0) = \phi(x) - h(x), \;\; v_t(x,0) = \psi(x). \end{aligned}$$

Finite Vibrating String with Gravity (Contd.)

The solution for Problem I can be easily found by integrating $h''(x)$ twice:

$$h(x) = \frac{gx^2}{2c^2} + Ax + B.$$

Upon using the conditions $h(0) = 0 = h(L)$, we get

$$B = 0 \ \ \& \ \ A = -gL/(2c^2)$$

and hence

$$h(x) = -g\frac{(L-x)x}{2c^2}. \tag{15}$$

Finite Vibrating String with Gravity (Contd.)

The solution of Problem II is known to us, which is

$$v(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \tag{16}$$

where A_n and B_n are given, respectively, by

$$A_n = \frac{2}{L}\int_0^L [\phi(x) - h(x)] \sin\left(\frac{n\pi x}{L}\right)\, dx,\ n = 1, 2, 3, \dots, \tag{17}$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right)\, dx,\ n = 1, 2, 3, \dots \tag{18}$$

Hence the solution $u(x,t)$ for our IBVP is given by the sum of (15) and (16).

Finite Vibrating String with Gravity (Contd.)

In other words,

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] - g\frac{(L-x)x}{2c^2}, \qquad (19)$$

where A_n and B_n are given, respectively, by

$$A_n = \frac{2}{L}\int_0^L \left(\phi(x) - g\frac{(L-x)x}{2c^2}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx,\ \ n = 1, 2, 3, \ldots, \qquad (20)$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,\ \ n = 1, 2, 3, \ldots \qquad (21)$$

Remark: Clearly, this splitting method would be applicable only when the non-homogeneous term in the governing equation is a constant or a function of x.

For problems in which the non-homogeneous term is a function of x and t, Laplace transform method is considered to be the most appropriate one. However, Duhamel's Principle can also be applied to solve non-homogeneous problems.

Duhamel's Principle: Finite String Problem

Assume that $v(x,t,s)$ is the solution of the problem

$$v_{tt} - c^2 v_{xx} = 0, \;\; (x,t) \in (0,L) \times (0,\infty), \tag{22}$$

$$\text{with BCs } \; v(0,t,s) = 0, \;\; v(L,t,s) = 0, \; t > 0, \; s > 0, \tag{23}$$

$$\text{with ICs } \; v(x,0,s) = 0, \;\; v_t(x,0,s) = f(x,s), \;\; s > 0. \tag{24}$$

Then, $u(x,t)$ defined by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau) d\tau \tag{25}$$

is the solution to the non-homogeneous problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \;\; (x,t) \in (0,L) \times (0,\infty), \tag{26}$$

$$\text{with BCs } \; u(0,t) = 0, \;\; u(L,t) = 0, \; t > 0, \tag{27}$$

$$\text{with ICs } \; u(x,0) = 0, \;\; u_t(x,0) = 0. \tag{28}$$

Duhamel's Principle: Finite String Problem

It is to be noted that

the vibration of the string takes place due to one of the following:

- At least one of the initial conditions is non-zero (with no source),
- There is a source term present in the governing equation (when both initial conditions are zero). This condition leads to the consideration of a non-homogeneous governing equation.

Further

- Duhamel's principle allows the shifting of the non-homogeneous term (the source) from the original equation to an initial condition for the modified problem.
- Since the source term is connected to some sort of force, therefore it gets shifted to the initial condition connected to the velocity, i.e., $u_t(x, 0)$, not to $u(x, 0)$.

Finite String Problem: Application of Duhamel's Principle

Example: Find $u(x, t)$ such that

$$u_{tt} - u_{xx} = t \sin\left(\frac{\pi x}{L}\right), \quad (x, t) \in (0, L) \times (0, \infty), \tag{29}$$

$$\text{with ICs } \; u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in (0, L), \tag{30}$$

$$\text{with BCs } \; u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0. \tag{31}$$

Solution: Suppose $v(x, t, s)$ is a solution to the user defined problem:

$$v_{tt} - v_{xx} = 0, \quad (x, t) \in (0, L) \times (0, \infty), \tag{32}$$

$$\text{with ICs } \; v(x, s, 0) = 0, \quad v_t(x, s, 0) = s \sin\left(\frac{\pi x}{L}\right), \quad x \in (0, s, L), \; s > 0, \tag{33}$$

$$\text{with BCs } \; v(0, s, t) = 0, \quad v(L, s, t) = 0, \quad t > 0. \tag{34}$$

Finite String Problem: Application of Duhamel's Principle

The solution to IVBP (32)-(34) is given by

$$v(x,t,s) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right] \tag{35}$$

with

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0,\ n = 1,2,3,\ldots,\ \text{(since initial displacement is zero)},\\
B_n &= \frac{2}{n\pi}\int_0^L \psi(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,\ n = 1,2,3,\ldots\\
&= \frac{2}{n\pi}\int_0^L s\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx.
\end{aligned}$$

Finite String Problem: Application of Duhamel's Principle

Thus, $B_1 = \frac{sL}{\pi}$ and $B_n = 0,\ n \neq 1$, and hence

$$v(x,t,s) = \frac{sL}{\pi} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi t}{L}\right).$$

Then solution $u(x,t)$ of the given problem is obtained as

$$\begin{aligned}
u(x,t) &= \int_0^t v(x, t-\tau, \tau) d\tau \\
&= \int_0^t \frac{\tau L}{\pi} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi (t-\tau)}{L}\right) d\tau \\
&= \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right) \int_0^t \tau \sin\left(\frac{\pi (t-\tau)}{L}\right) d\tau.
\end{aligned}$$

Evaluate the integral and find the solution.

Two-dimensional Wave Equation

We will not discuss the solution of two-dimensional wave equation. But knowing the associated IBVP will be helpful in understanding how wave propagates in a rectangular membrane.

Let the four sides of a rectangular membrane ($0 < x < a,\ 0 < y < b$) be fixed along its boundaries. Then, an initial displacement and an initial velocity (or either one of them) will induce deflection $u(x, y, t)$ of the membrane.

The IBVP:

Governing Equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 < x < a,\ 0 < y < b, \tag{36}$$

Boundary Conditions:

$$u(0, y, t) = 0,\ u(a, y, t) = 0,\ u(x, 0, t) = 0,\ u(x, b, t) = 0, \tag{37}$$

Initial Conditions:

$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y, 0) = \psi(x, y). \tag{38}$$

Summary

- By associating appropriate boundary and/or initial conditions, an IBVP or IVP can be formulated and solved which gives the deflection or displacement in a string.
- For an IBVP due to a homogeneous equation, both boundary conditions can be zero but at least one of the initial conditions must be non-zero. Here the deflection is due to the initial condition(s).
- For an IBVP due to a non-homogeneous equation, both boundary conditions and both initial conditions may be zero. Here the deflection is due to the source term present in the governing equation.
- The instantaneous displacements are due to the wave created.
- For an infinite string (recall D'Alembert's solution), it is a progressive wave and hence the solution is known as progressive wave solution.
- For a finite string, it is a standing or stationary wave and hence the solution is known as stationary wave solution since it remains within the bounded domain.