MA201, Lecture 10
Fourier Series

Convergence Problem

- Till now we have been using the notation

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

  - with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \text{and}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots$$

- Can we use $=$ instead of $\approx$? That is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]?$$

- This leads to the **convergence problem of Fourier series**.

Convergence of Fourier Series for Periodic Functions

Example

Recall that Fourier Series of the function $f(x) = x$ for $-L \leq x \leq L$ is given by

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\frac{n\pi x}{L}.$$

- Observe that value of the Fourier series at $x = \pm L$ is zero. But, the value of $f(x)$ at $x = \pm L$ is $\pm L$.
- Thus, the Fourier series of $f(x)$ is not $f(x)$ at $x = \pm L$.

Remark

Under various assumptions on a function $f(x)$, one can prove that the Fourier series of $f(x)$ does converge to $f(x)$ for all x in $[-L, L]$ or $(-L, L)$.

Convergence of Fourier Series for Periodic Functions

- Periodic Function: A function f is periodic of period $2L$ if

$$f(x+2L) = f(x) \text{ for all } x \text{ in the domain of } f.$$

- It is easy to verify that if the functions $f_1, \ldots, f_n$ are periodic of period $2L$, then any linear combination $c_1 f_1(x) + \ldots + c_n f_n(x)$ is also periodic with period $2L$.

- Theorem: Let $f \in C^2[-\pi, \pi]$ with period 2π. If

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0,\ 1,\ 2, \ldots \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1,\ 2, \ldots,$$

then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos nx + b_n \sin nx\right\}$$

converges to $f(x)$ for each x in $[-\pi, \pi]$.

Convergence of Fourier Series for Periodic Functions

▶ Theorem: Let $f \in C^2[-L, L]$ with $f(-L) = f(L)$ and $f'(-L) = f'(L)$. If

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,\ 1,\ 2, \ldots \quad \text{and}$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,\ 2, \ldots,$$

then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}$$

converges to $f(x)$.

Remark

Can we apply this result to comment on the convergence of Fourier series for $f(x) = x,\ -L \leq x \leq L$?

Convergence of FS for piecewise continuous functions

Theorem

Let $f : [-L, L] \to \mathbb{R}$ be a piecewise C^1 function such that $f(-L) = f(L)$. Then the Fourier series of $f(x)$ converges to $f(x)$ on $[-L, L]$.

- Can we apply above result for the cinvergence of Fourier series to

$$f(x) = x, \ -L \leq x \leq L?$$

- Define an **adjusted function** $g : [-L, L] \to \mathbb{R}$ such that

$$g(\pm L) = \frac{1}{2}\{f(L^-)+f(-L^+)\}, \quad g(x) = \frac{1}{2}\{f(x^+)+f(x^-)\}, \ -L < x < L$$

for a given piecewise C^1 function $f : [-L, L] \to \mathbb{R}$.

- Clearly, g is a piecewise C^1 function such that $g(-L) = g(L)$.
- Fourier series of g converges to g in $[-L, L]$. **What about the given function f?**

Convergence of Fourier series for piecewise continuous functions

- Recall that Fourier Series of the function $f(x) = x$ for $-L \leq x \leq L$ is given by

$$\frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin \frac{n\pi x}{L}.$$

- Note that $g(x) = f(x), \ \ -L < x < L, \ g$ is the adjusted function.
- Fourier series of g agrees with the Fourier series of f in $(-L, L)$.
- Hence, for $x \in (-L, L)$, we obtain

$$\frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin \frac{n\pi x}{L} = g(x) = f(x) = x.$$

Remark

- What about the convergence of the Fourier series of f at end points $x = \pm L$?

Convergence of FS for piecewise continuous functions

- What about the convergence of Fourier Series for the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

- Recall that the Fourier Series is

$$\frac{4}{\pi}\Big[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\Big].$$

- What about the series

$$\frac{4}{\pi}\Big[\sin\frac{1}{2} + \frac{1}{3}\sin\frac{3}{2} + \frac{1}{5}\sin\frac{5}{2} + \dots\Big] =?$$

Convergence of Fourier series for piecewise continuous functions

- Note that given function f is not defined at $x = 0,\ \pi,\ -\pi$.
- Define adjusted function $g : [-\pi, \pi] \to \mathbb{R}$ as

$$g(\pm\pi) = \frac{1}{2}\{f(\pi^-) + f(-\pi^+)\} = 0$$
$$g(x) = \frac{1}{2}\{f(x^-) + f(x^+)\}, \text{ otherwise.}$$

- Clearly g is well defined and piece-wise C^1 function in $[-\pi, \pi]$. Further, $g(-\pi) = g(\pi)$ and $g(x) = f(x)$ in $(-\pi, 0) \cup (0, \pi)$.

Convergence of Fourier series for piecewise continuous functions

- Hence, Fourier series of g converges to g in $[-\pi, \pi]$.
- Therefore, Fourier series of f converges to f in $(-\pi, 0) \cup (0, \pi)$.
- Thus,

$$\frac{4}{\pi}\Big[\sin\frac{1}{2} + \frac{1}{3}\sin\frac{3}{2} + \frac{1}{5}\sin\frac{5}{2} + \dots\Big] = f\left(\frac{1}{2}\right) = 1.$$

- What about at the end points and at the point of discontinuity?

Half-range Series and its Periodic Extension

- Suppose f is a defined and integrable function in $(0, L)$.
- Fourier series of f is a defined via an even extension or odd extension.
- Define the even extension $f_e : [-L, L] \to \mathbb{R}$ of f by

$$\begin{aligned} f_e(x) &= f(x) \text{ if } 0 < x < L, \\ f_e(x) &= f(-x) \text{ if } -L < x < 0. \end{aligned}$$

- Define the odd extension of f by

$$\begin{aligned} f_O(x) &= f(x) \text{ if } 0 < x < L, \\ f_O(x) &= -f(-x) \text{ if } -L < x < 0. \end{aligned}$$

Half-range Series

Example

Check the convergence of Fourier cosine series expansion for

$$f(x) = x,\ 0 < x < 1.$$

- Suppose $f_e : [-1, 1] \to \mathbb{R}$ is the even periodic extension of f. Then

$$f_e(x) = \begin{cases} x, & 0 < x < 1, \\ -x, & -1 < x < 0. \end{cases}$$

- Let g be the adjusted function of f_e.
- Thus, $g(x) = f_e(x)$ where the function f_e is continuous.
- Thus, Fourier series of $f_e(x)$ converges to $f_e(x)$ in $(-1, 0) \cup (0, 1)$.
- Again Fourier cosine series of f is the Fourier series of $f_e(x)$ in $(0, 1)$. So, Fourier cosine series of f converges in $(0, 1)$.

Convergence of FS for piecewise continuous functions

Theorem

Let $f : [-L, L] \to \mathbb{R}$ be a piecewise C^1 function and let g be the adjusted function. Then Fourier series of f converges to g in $[-L, L]$.

Note:

- The existence of Fourier series depends on the evaluation of Fourier coefficients.
- On the other hand, convergence of the Fourier series is done via an adjusted function g.
- This is a much stronger result in the sense that above result takes care the convergence of Fourier series of a function f at the end points as well as at the discontinuous points.
- A discontinuous function is approximated by trigonometric functions.
- **The bottom line is:**
  A Fourier series converges to a given function $f(x)$ for all points at which $f(x)$ is continuous. At a point of discontinuity x, the series converges to $\frac{1}{2}\left\{f(x^+) + f(x^-)\right\}$.

Series in terms of π: Consequence of Fourier Series

Consider Fourier cosine series of $f(x) = x,\ 0 < x < 1$, so we carry out Fourier series of f_e so that

- $a_0 = 2\int_0^1 x dx = 1,\ a_n = 2\int_0^1 x \cos n\pi x dx = \frac{2}{n^2\pi^2}(\cos n\pi - 1).$
  a_n vanishes for even n. Thus, Fourier series of $f_e(x)$ is given by

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2} = g(x) \ \ \forall x \in [-1, 1].$$

- Setting $x = 0$, we obtain

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 0$$

giving

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Applications of Fourier Series

Square wave-high frequencies

One application of Fourier series, the analysis of a square wave in terms of its Fourier components, occurs in electronic circuits designed to handle sharply rising pulses.

Physically, square wave contains many **high-frequency components**.

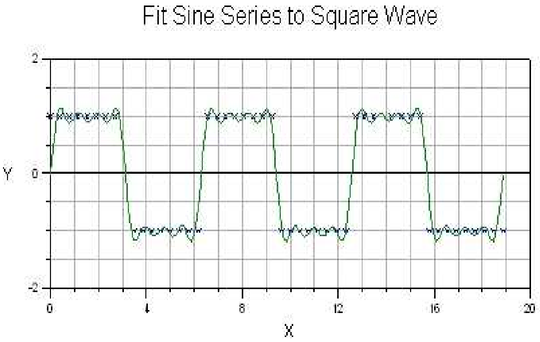

Figure : Square wave

Applications of Fourier Series

Let us try to find the FS of the following function:

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ h, & 0 < x < \pi. \end{cases} \tag{1}$$

We can easily calculate the Fourier coefficients to be

$$a_0 = h, \quad a_n = 0, \quad n = 1, 2, 3, \dots, \tag{2}$$

$$b_n = \begin{cases} \frac{2h}{n\pi}, & n \;\; \text{odd}, \\ 0, & n \;\; \text{even}. \end{cases} \tag{3}$$

So, the resulting series is

$$\frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{4}$$

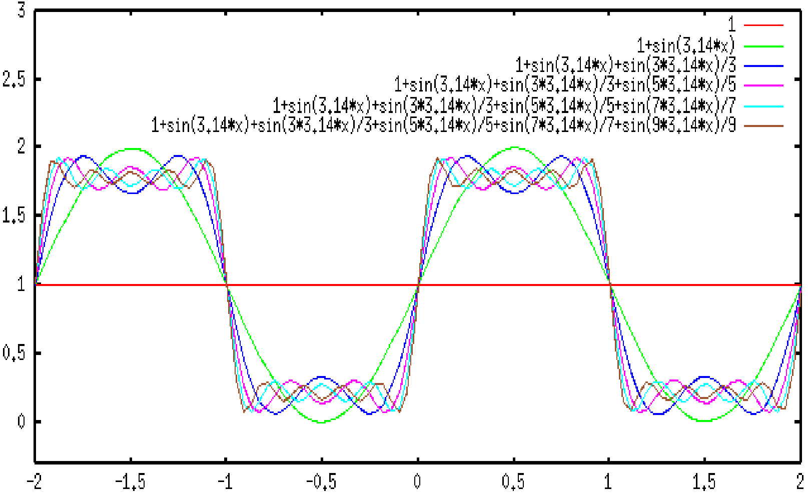

Figure : Square wave

Parseval's Identity for Fourier Series

Theorem

If $f(t)$ and $g(t)$ are continuous in $(-L, L)$, and provided $\int_{-L}^{L} |f(t)|^2 dt < \infty$ and $\int_{-L}^{L} |g(t)|^2 dt < \infty$, and if A_n, B_n are Fourier coefficients of $f(t)$ and C_n, D_n are Fourier coefficients of $g(t)$, then

$$\int_{-L}^{L} f(t)g(t)dt = \frac{L}{2}A_0 C_0 + L\sum_{n=1}^{\infty}(A_n C_n + B_n D_n).$$

- **Proof:** We can express $f(t)$ and $g(t)$ in terms of Fourier series as

$$\begin{aligned} f(t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi t}{L} + B_n \sin\frac{n\pi t}{L}\right], \\ g(t) &= \frac{C_0}{2} + \sum_{n=1}^{\infty}\left[C_n \cos\frac{n\pi t}{L} + D_n \sin\frac{n\pi t}{L}\right]. \end{aligned}$$

Parseval's Identity for Fourier Series

- Taking product of $f(t)$ with $g(t)$ we obtain

$$f(t)g(t) = \frac{A_0}{2}g(t) + \sum_{n=1}^{\infty}\left[A_n\, g(t)\cos\frac{n\pi t}{L} + B_n\, g(t)\sin\frac{n\pi t}{L}\right].$$

- Integrating this series from $-L$ to L gives

$$\int_{-L}^{L} f(t)g(t)dt = \frac{A_0}{2}\int_{-L}^{L} g(t)dt + \sum_{n=1}^{\infty}\left[A_n\int_{-L}^{L} g(t)\cos\frac{n\pi t}{L}dt + B_n\int_{-L}^{L} g(t)\sin\frac{n\pi t}{L}dt\right]$$

- Putting back the values of the Fourier coefficients C_n and D_n:

$$\frac{1}{L}\int_{-L}^{L} f(t)g(t)dt = \frac{A_0C_0}{2} + \sum_{n=1}^{\infty}[A_nC_n + B_nD_n].$$

- As a consequence, we have

$$\frac{1}{L}\int_{-L}^{L}[f(t)]^2dt = \frac{A_0^2}{2} + \sum_{n=1}^{\infty}[A_n^2 + B_n^2].$$

Parseval's Identity for Fourier Series

- Parseval's identity can be used **to determine the power (P) delivered** by an electric current, $I(t)$, flowing under a voltage, $E(t)$, through a resistor of resistance R.
- Then, P, I, E and R are related by

$$P = EI = RI^2.$$

$$\begin{aligned}
\text{Average Power} = P_{av} &= \frac{1}{2L}\int_{-L}^{L} RI^2(t)\, dt \\
&= \frac{R}{2L}\int_{-L}^{L} I^2(t)\, dt \\
&= R\left[\frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)\right],
\end{aligned}$$

- Here we have made use of the Fourier expansion of $I(t)$:

$$I(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi t}{L} + B_n \sin\frac{n\pi t}{L}\right].$$

Parseval's Identity for Fourier Series

- Mean square of the current is:

$$I_{av} = \frac{1}{2L}\int_{-L}^{L} I^2(t)dt = \frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2).$$

- Root mean square of the current is:

$$I_{rms} = \sqrt{\frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)}.$$

Application of Parseval's Identity

- **Example:** Given the Fourier series $t^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nt, \ -\pi < t < \pi,$ deduce that $\sum_{n=1}^{\infty} \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$.
- Here $A_0 = \dfrac{2\pi^2}{3}, \ \ A_n = \dfrac{4(-1)^n}{n^2}, \ \ B_n = 0.$
- Use Parseval's identity

$$\frac{1}{L}\int_{-L}^{L} [f(t)]^2 dt = \frac{A_0^2}{2} + \sum_{n=1}^{\infty} [A_n^2 + B_n^2].$$

to have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} t^4 dt = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

giving us

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

Gibbs' Phenomenon

- This is about *Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity*. It is named after the American mathematical physicist J. W Gibbs (1899).
- Consider the following periodic function whose definition in one period is:

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 2. \end{cases} \tag{5}$$

- This function can be represented as

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}. \tag{6}$$

- To see how well this infinite series represents the function, let us truncate the series after N terms.
- Let the sum of these first N terms of the infinite series be denoted by S_N:

$$S_N = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin(2n-1)\pi x}{2n-1}. \tag{7}$$

Gibbs' Phenomenon

- Graphs of equation (7) are shown for $N = 2, 4, 8, 16$ and 32.
- In the figures, the overshoot (*overshoot is the occurrence of a signal or function exceeding its target*) at $x = 1^-$ and the undershoot at $x = 1^+$ are characteristics of Fourier series at the points of discontinuity.
- and is known as Gibbs' phenomenon
- This phenomenon persists even though a large number of terms are considered in the partial sum.
- In the approximation of functions, overshoot/undershoot is one term describing quality of approximation. Here, the convergence is in *point-wise sense*.

Gibbs' Phenomenon

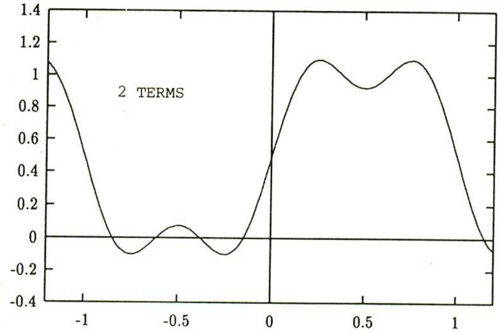

Figure : Gibbs Phenomenon with 2 terms

Gibbs' Phenomenon

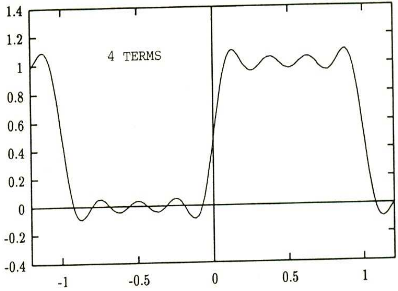

Figure : Gibbs Phenomenon with 4 terms

Gibbs' Phenomenon

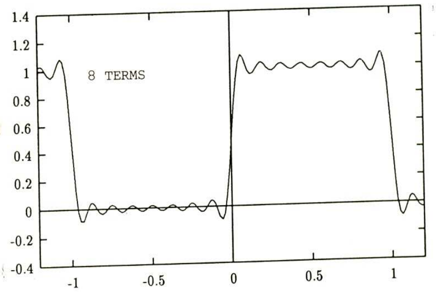

Figure : Gibbs Phenomenon with 8 terms

Gibbs' Phenomenon

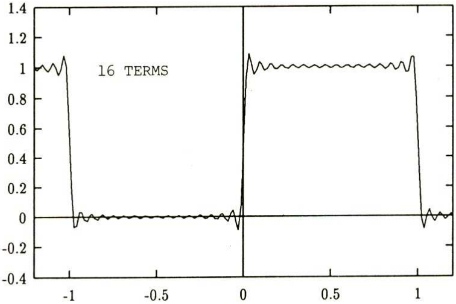

Figure : Gibbs Phenomenon with 16 terms

Gibbs' Phenomenon

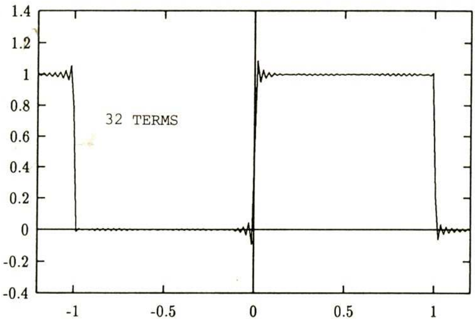

Figure : Gibbs Phenomenon with 32 terms

Differentiation and integration of Fourier series

- The term-by-term differentiation of a Fourier series is not always permissible.

Example

Recall that Fourier series for $f(x) = x, \ -\pi < x < \pi$ is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n},$$

which converges to $f(x)$ for all $x \in (-\pi, \pi)$, that is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

- Term-by-term differentiation leads to

$$1 = \sum_{n=1}^{\infty}(-1)^{n+1}\cos nx, \quad x \in (-\pi, \pi).$$

which fails at $x = 0$. In fact, the RHS series diverges for all x(?)

Differentiation of Fourier series

Theorem (Differentiation of Fourier series)

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous and $f(x+2L) = f(x)$. Let $f'(x)$ and $f''(x)$ be piecewise continuous on $[-L, L]$. Then, The Fourier series of $f'(x)$ can be obtained from the Fourier series for $f(x)$ by termwise differentiation. In particular, if

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right\},$$

then

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} \left\{ -A_n \sin\frac{n\pi x}{L} + B_n \cos\frac{n\pi x}{L} \right\}.$$

Can we comment on the application of above result for Bernoulli's Solution

$$u = b_1 \sin x \cos ct + b_2 \sin 2x \cos 2ct + \dots . \tag{8}$$

Integration of Fourier series

Term-wise integration of a Fourier series is permissible under much weaker conditions.

Theorem (Integration of Fourier series)

Let $f(x) : [-L, L] \to \mathbb{R}$ be a piecewise continuous function with Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right\}.$$

Then, for any $x \in [-L, L]$, we have

$$\int_{-L}^{x} f(\tau) d\tau = \int_{-L}^{x} \frac{A_0}{2} d\tau + \sum_{n=1}^{\infty} \int_{-L}^{x} \left\{ A_n \cos\frac{n\pi \tau}{L} + B_n \sin\frac{n\pi \tau}{L} \right\} d\tau.$$