MA 201, Mathematics III, July-November 2022, Partial Differential Equations: 1D heat conduction equation and 2D steady state heat conduction equation

Lecture 14

Example

A thin metal bar of length π is placed in boiling water (temperature 100^0C). After reaching 100^0C throughout, the bar is removed from the boiling water and immediately, at time $t = 0$, the ends are immersed and kept in a medium with constant freezing temperature 0^0C. Taking the thermal diffusivity of the metal to be one, find the temperature $u(x,t)$ for $t > 0$.

Solution: The IBVP will be

$$u_t = u_{xx}, \quad u(0,t) = 0 = u(\pi,t), \quad u(x,0) = 100.$$

We know

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\alpha n^2\pi^2/L\right)t \quad \text{with}$$

$$A_n = \frac{2}{L}\int_0^L u(x,0)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Example

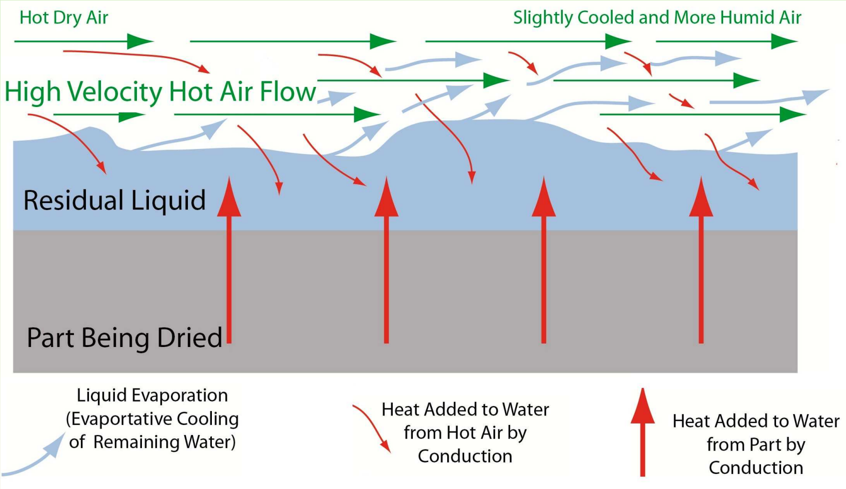
We evaluate A_n as

$$A_n = \frac{2}{\pi}\int_0^{\pi} 100\sin nx\; dx = \frac{200}{n\pi}(1-\cos n\pi).$$

The solution:

$$u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x \quad \exp\{-(2n-1)^2 t\}.$$

Heat conduction problem with flux and radiation conditions



Heat conduction problem with **flux** and radiation conditions

Heat flux is defined as the rate of heat transfer per unit cross-sectional area.

Fourier's heat flow law states flux $= -\alpha u_x$, where α is the conductivity of the metal of the rod under consideration.

The simplest flux condition at boundary $x = 0$ is given by $u_x(0, t) = 0,\ t > 0$.

This is also known as insulation condition, that is, no heat flow takes place across the boundary $x = 0$.

A radiation condition, on the other hand, is a specification of how heat radiates from the end of the rod, say at $x = 0$, into the environment, or how the end absorbs heat from its environment.

Linear, homogeneous radiation condition takes the form

$-\alpha u_x(0, t) + bu(0, t) = 0, t \geq 0$, where b is a constant.

Heat conduction problem with flux and radiation conditions

If $b > 0$, then the flux is negative,

which means that heat is flowing from the rod into its surroundings (radiation).

If $b < 0$, then the flux is positive,

and heat is flowing into the rod (absorption).

A typical problem in heat conduction may have

a combination of Dirichlet, insulation and radiation boundary conditions.

Heat conduction problems with radiation conditions at one end

Consider a problem related to radiation condition.

Take up the following IBVP:

Governing equation

$$u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0. \tag{1}$$

Boundary Conditions

$$u(0,t) = 0,\ t \geq 0, \tag{2a}$$
$$u_x(L,t) + hu(L,t) = 0,\ t \geq 0, \tag{2b}$$

where h is a constant.

Initial condition

$$u(x,0) = \phi(x),\ 0 \leq x \leq L. \tag{3}$$

Heat conduction problems with radiation conditions at one end

Solution can be written as

$$u(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{4}$$

Using boundary condition (2a),

we get $B = 0$.

Using boundary condition (2b),

$$[Ah\sin(\lambda L) + \lambda A\cos(\lambda L)]Ce^{-\alpha\lambda^2 t} = 0.$$

Eigenvalues λ's are given by

$$\lambda/h = -\tan(\lambda L). \tag{5}$$

Equation (5) cannot be solved analytically

but the graphs of the functions λ/h and $-\tan\lambda L$ versus λ (separately) show that the equation has infinitely many positive solutions $\lambda_1, \lambda_2, \ldots$

Heat conduction problems with radiation conditions at one end

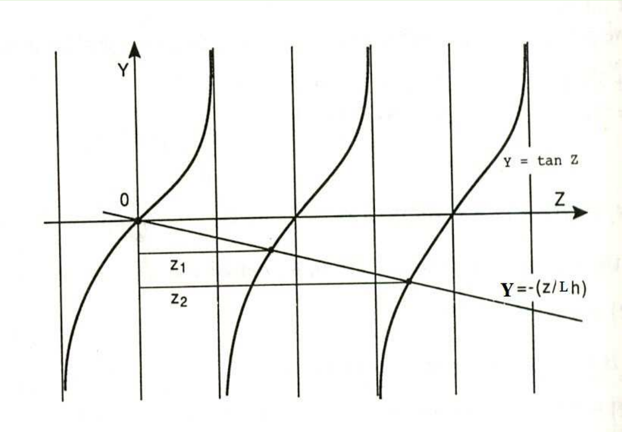


Figure : Graphical representation of the eigenvalues

Heat conduction problems with radiation conditions at one end

Corresponding eigenfunctions are

$$u_n(x,t) = A_n \sin(\lambda_n x) e^{-\alpha \lambda_n^2 t}. \tag{6}$$

Hence the solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\; e^{-\alpha \lambda_n^2 t}. \tag{7}$$

The initial condition will give us A_n as

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin(\lambda_n x)\; dx.$$

Non-Homogeneous Heat Equation: Duhamel's Principle

Find temperature $u(x,t)$ in a thin metal rod such that

$$u_t - \alpha u_{xx} = f(x,t), \;\; (x,t) \in (0,L) \times (0,\infty), \tag{8}$$

$$\text{with BCs } \; u(0,t) = 0, \;\; u(L,t) = 0, \; t \geq 0, \tag{9}$$

$$\text{and IC } \; u(x,0) = 0, \;\; x \in [0,L]. \tag{10}$$

Then the solution u is given by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau) d\tau. \tag{11}$$

What is v?

Duhamel's Principle (Contd.)

Here, v is the solution of the problem

$$v_t - \alpha v_{xx} = 0, \quad (x,t) \in (0,L) \times (0,\infty), \tag{12}$$

$$\text{with BCs } \; v(0,t) = 0, \;\; v(L,t) = 0, \; t \geq 0 \tag{13}$$

$$\text{and IC } \; v(x,0) = f(x,s), \quad s > 0. \tag{14}$$

for some real parameter $s > 0$.

Note that the solution v of the above problem depends on x, t and s.

Thus, $v = v(x,t,s)$. Accordingly, we can modify the BCs and IC as

BCs: $v(0,t,s) = 0, \;\; v(L,t,s) = 0, \; t \geq 0, \; s > 0$ and

IC: $v(x,0,s) = f(x,s), \quad s > 0.$

Example

Solve

$$
\begin{aligned}
&u_t - u_{xx} = t\sin x,\ 0 < x < \pi,\\
&u(0,t) = 0,\ \ u(\pi,t) = 0,\ \ t \geq 0,\\
&u(x,0) = 0.
\end{aligned}
$$

Solution: We first solve the related problem for $v(x,t,s)$:

$$
\begin{aligned}
&v_t - v_{xx} = 0,\ 0 < x < \pi,\\
&v(0,t,s) = 0,\ \ v(\pi,t,s) = 0,\ \ t \geq 0,\ s > 0,\\
&v(x,0,s) = f(x,s) = s\sin x.
\end{aligned}
$$

Example

From method of separation of variables, for fixed s, we obtain

$$v(x,t,s) = \sum_{n=1}^{\infty} A_n(s) \sin nx \, e^{-n^2 t}$$

with A_n

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(x,s) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} s \sin x \sin nx \, dx. \end{aligned}$$

Hence,

$$A_1 = s \quad \& \quad A_n = 0 \ \text{ for } n \neq 1.$$

Example

For any $s > 0$, we obtain

$$v(x,t,s) = se^{-t}\sin x.$$

Therefore

$$v(x,t-\tau,\tau) = \tau e^{-(t-\tau)}\sin x$$

Hence, due to Duhamel's principle, $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t-\tau,\tau)d\tau = \int_0^t \tau e^{-(t-\tau)}\sin x d\tau \\ &= e^{-t}\sin x \int_0^t \tau e^{\tau} d\tau. \end{aligned}$$

Compute the integral

Two-dimensional heat conduction equation

We will not discuss the solution of two-dimensional heat conduction. But knowing the associated IBVP will be helpful in understanding how heat diffuses in a thin rectangular plate.

Let the four sides of a thin rectangular plate $(0 \leq x \leq a,\ 0 < y \leq b)$ be maintained at zero degree temperature. Then, an initial temperature distribution will control temperature $u(x, y, t)$ of the plate.

The IBVP:

Governing Equation:

$$u_t = \alpha(u_{xx} + u_{yy}), \quad 0 \leq x \leq a,\ 0 < y \leq b, t \geq 0, \tag{15}$$

Boundary Conditions:

$$\left.\begin{array}{ll} u(0, y, t) = 0,\ u(a, y, t) = 0, & 0 < y \leq b,\ t \geq 0, \\ u(x, 0, t) = 0,\ u(x, b, t) = 0, & 0 \leq x \leq a,\ t \geq 0, \end{array}\right\} \tag{16}$$

Initial Condition:

$$u(x, y, 0) = \phi(x, y),\ 0 \leq x \leq a,\ 0 < y \leq b. \tag{17}$$

The above consideration will also help us in understanding how a steady-state problem arises.

Steady state heat conduction: Laplace's equation

Laplace's equation in two or three dimensions

usually arises in two types of physical problems:

1. As steady state heat conduction.
2. As equation of continuity for incompressible potential flow.

However, here we will emphasize only on the first type.

Steady state solution here means

1. the solution for some large time,
2. the solution does not depend any more on time.

Laplace's equation

Laplace's equation in two dimensions and three dimensions in Cartesian coordinates are, respectively, given by

$$u_{xx} + u_{yy} = 0, \tag{18}$$

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{19}$$

The above equations can be obtained from the two-dimensional and three-dimensional transient heat conduction equations when u does not depend on t.

Hence Laplace's equation models

steady heat flow in a region where the temperature is fixed on the boundary.

Maximum Principle

Theorem: Let $u(x, y)$ satisfy Laplace's equation in D, an open, bounded, connected region in the plane; and let u be continuous on the closed domain $D \cup \partial D$ consisting of D and its boundary. If u is not a constant function, then the maximum and minimum values of u are attained on the boundary of D and nowhere inside D.

. $\square$

This is called maximum principle theorem for Laplace's equation.

Steady state heat conduction in two dimensions

We consider steady state heat conduction

in a two-dimensional rectangular region.

To be specific,

consider the equilibrium temperature inside a rectangle $0 \leq x \leq a, \ 0 < y \leq b$.

Here

the temperature is a prescribed function of position on the boundary.

In general, the Dirichlet BVP will be like

$$u_{xx} + u_{yy} = 0, \ \ 0 \leq x \leq a, \ 0 < y \leq b,$$

$$u(0, y) = g_1(y), \ u(a, y) = g_2(y), \ \ 0 < y \leq b,$$

$$u(x, 0) = f_1(x), \ u(x, b) = f_2(x), \ \ 0 \leq x \leq a.$$

Steady state heat conduction in two dimensions

where

$f_1(x), f_2(x), g_1(y), g_2(y)$ are given functions.

Though the equation is linear and homogeneous,

the BCs are not homogeneous.

Hence

the BVP is needed to be split into four BVPs with each containing one non-homogeneous BC.

Take

$$u = u_1 + u_2 + u_3 + u_4, \;\; 0 \leq x \leq a, \; 0 < y \leq b.$$

Steady state heat conduction in two dimensions

BVP I and BVP II:

$$
\begin{aligned}
u_{1,xx} + u_{1,yy} &= 0; & u_{2,xx} + u_{2,yy} &= 0,\ \ 0 \le x \le a,\ 0 < y \le b;\\
u_1(0,y) &= 0,\ 0 \le y \le b; & u_2(0,y) &= 0,\ 0 < y \le b;\\
u_1(a,y) &= 0,\ 0 \le y \le b; & u_2(a,y) &= 0,\ 0 < y \le b;\\
u_1(x,0) &= f_1(x),\ 0 \le x \le a; & u_2(x,0) &= 0,\ 0 \le x \le a;\\
u_1(x,b) &= 0,\ 0 \le x \le a; & u_2(x,b) &= f_2(x),\ 0 \le x \le a;
\end{aligned}
$$

BVP III and BVP IV:

$$
\begin{aligned}
u_{3,xx} + u_{3,yy} &= 0; & u_{4,xx} + u_{4,yy} &= 0,\ \ 0 \le x \le a,\ 0 < y \le b;\\
u_3(0,y) &= g_1(y),\ 0 \le y \le b; & u_4(0,y) &= 0,\ 0 < y \le b;\\
u_3(a,y) &= 0,\ 0 \le y \le b; & u_4(a,y) &= g_2(y),\ 0 < y \le b;\\
u_3(x,0) &= 0,\ 0 \le x \le a; & u_4(x,0) &= 0,\ 0 \le x \le a;\\
u_3(x,b) &= 0,\ 0 \le x \le a; & u_4(x,b) &= 0,\ 0 \le x \le a.
\end{aligned}
$$

We will consider only one of them.......take $u_1 = u$ for convenience.

Steady state heat conduction in two dimensions

Consider the steady state heat conduction in a rectangular region $0 \leq x \leq a,\ 0 < y \leq b$

where three boundaries along $x = 0, x = a, y = b$ are kept at 0^0C

while

the temperature along the boundary $y = 0$ is $f(x)$.

$\boxed{\text{To find}}$ the temperature at any point (x, y).

Steady state heat conduction in two dimensions

BVP will consist of the following:

The governing equation is two-dimensional Laplace's equation:

$$u_{xx} + u_{yy} = 0, \;\; 0 \le x \le a, \; 0 < y \le b. \tag{20}$$

The boundary conditions are:

$$\begin{aligned} u(0,y) &= 0, \; 0 < y \le b, && \text{(21a)} \\ u(a,y) &= 0, \; 0 < y \le b, && \text{(21b)} \\ u(x,0) &= f(x), \; 0 \le x \le a, && \text{(21c)} \\ u(x,b) &= 0, \; 0 \le x \le a. && \text{(21d)} \end{aligned}$$

It being a pure BVP and the solution being a function of x and y,

obviously we will not have any initial conditions.

Hence,

this problem is called a steady-state problem.

Steady state heat conduction in two dimensions

Assume a solution of the form:

$$u(x, y) = X(x)Y(y). \tag{22}$$

Using (22) in (20)

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

On separating the variables x and y,

$$\frac{X''}{X} = -\frac{Y''}{Y} = k(\text{say}).$$

Giving us

$$X'' - kX = 0, \tag{23}$$
$$Y'' + kY = 0. \tag{24}$$

The zero and positive values of k will not give rise to solutions conforming to the boundary conditions. (VERIFY IT.)

Steady state heat conduction in two dimensions

We consider only the negative values of k, say $-\lambda^2$, to write the equations (23) and (24) as

$$\begin{aligned} X'' + \lambda^2 X &= 0, \qquad (25)\\ Y'' - \lambda^2 Y &= 0, \qquad (26)\end{aligned}$$

so that the solution $u(x, y)$ can be written as

$$u(x, y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y). \qquad (27)$$

Using boundary condition (21a), we get

$A = 0$.

Boundary condition (21b) gives

$$\lambda_n = \frac{n\pi}{a},\ n = 1, 2, 3, \ldots$$

$\Rightarrow$

$$u_n(x, y) = \sin\left(\frac{n\pi x}{a}\right)\left[A_n\cosh\left(\frac{n\pi y}{a}\right) + B_n\sinh\left(\frac{n\pi y}{a}\right)\right].$$

Steady state heat conduction in two dimensions

Using boundary condition (21d)

$$B_n = -\frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\, A_n$$

so that the solution $u(x, y)$ can be written as

$$\begin{aligned}
u(x,y) &= \sum_{n=1}^{\infty} u_n(x,y) && (28)\\
&= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\left[\cosh\left(\frac{n\pi y}{a}\right) - \frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sinh\left(\frac{n\pi y}{a}\right)\right]\\
&= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}. && (29)
\end{aligned}$$

Steady state heat conduction in two dimensions

Remaining boundary condition (21c) can be used to evaluate the coefficients A_n:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right).$$

A_n is obtained as

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\, dx. \tag{30}$$

The solution to the BVP described by equations (20)-(21) is given by

(29) with A_n given by (30).

Steady state heat conduction in two dimensions

Similarly we can find the other solutions u_2, u_3 and u_4 and

write the total solution as $u = u_1 + u_2 + u_3 + u_4$.

This problem with Dirichlet conditions

along all boundaries is called a Dirichlet problem for a rectangle.

The problem with Neumann conditions

along all boundaries is called a Neumann problem for a rectangle.

This new problem with all Neumann conditions can be solved by writing the boundary conditions as

$$
\begin{aligned}
u_x(0, y) &= 0, && \text{(31a)}\\
u_x(a, y) &= 0, && \text{(31b)}\\
u_y(x, 0) &= f(x), && \text{(31c)}\\
u_y(x, b) &= 0. && \text{(31d)}
\end{aligned}
$$

TRY to solve it yourself.