MA 201, Mathematics III, July-November 2022, Fourier Transform (Contd.)

Lecture 19

Fourier Transform

For a given function f, the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(\tau) e^{-\mathrm{i}\sigma\tau} \, d\tau \right] d\sigma = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\sigma t} g(\sigma) d\sigma$$

is called Fourier integral of f.

If f is continuous at t, we have

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g(\sigma) d\sigma.$$

The function g is called *Fourier transform* of a function f and it is denoted by $\mathcal{F}\{f(t)\}$

$$\mathcal{F}\{f(t)\} = g(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau) \, d\tau. \tag{1}$$

The *inverse Fourier transform* is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{\mathrm{i}\sigma t} \, d\sigma. \tag{2}$$

Fourier cosine transform

Suppose f is defined in $(0, \infty)$.

For an even extension f_{e} defined in $(-\infty, \infty)$, the Fourier integral is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\sigma t} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f_{\mathsf{e}}(\tau) e^{-\mathrm{i}\sigma\tau}\, d\tau\right] d\sigma \quad (\text{Basic definition}) \\
&= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f_{\mathsf{e}}(\tau) \cos\sigma\tau \cos\sigma t\, d\tau\, d\sigma \quad (\text{Special case}) \\
&= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \cos\sigma t \left[\int_0^{\infty} \sqrt{\frac{2}{\pi}} f(\tau) \cos\sigma\tau d\tau\right] d\sigma \\
&= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \cos\sigma t g_c(\sigma) d\sigma
\end{aligned}$$

We can define the *Fourier cosine transform* of f as

$$\begin{aligned}
\mathcal{F}_c\{f(t)\} = g_c(\sigma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f_e(\tau)\, d\tau \quad (\text{If we use extension}) \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\tau) \cos\sigma\tau\, d\tau. \quad (\text{From the special case})
\end{aligned}$$

The *inverse Fourier cosine transform* is defined as

$$\begin{aligned}
\mathcal{F}_c^{-1}\{g_c(\sigma)\} = f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\sigma) \cos\sigma t\, d\sigma. \quad (\text{From the special case}) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\sigma t} g_c(\sigma) d\sigma. \quad (\text{If we use extension})
\end{aligned}$$

Fourier sine transform

For an odd extension f_o defined in $(-\infty, \infty)$, the Fourier integral is

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\sigma t} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f_o(\tau) e^{-\mathrm{i}\sigma\tau}\, d\tau \right] d\sigma \quad \text{(Basic definition)}$$
$$= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f_o(\tau) \sin\sigma\tau \sin\sigma t\, d\tau\, d\sigma \quad \text{(From the special case)}$$
$$= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \sin\sigma t \left[\int_0^{\infty} \sqrt{\frac{2}{\pi}} f(\tau) \sin\sigma\tau d\tau \right] d\sigma$$
$$= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \sin\sigma t g_s(\sigma) d\sigma$$

We can define the *Fourier sine transform* of f as

$$\mathcal{F}_s\{f(t)\} = g_s(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f_o(\tau)\, d\tau \quad \text{(If we use extension)}$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\tau) \sin\sigma\tau\, d\tau. \quad \text{(From the special case)}$$

The *inverse Fourier sine transform* is defined as

$$\mathcal{F}_s^{-1}\{g_s(\sigma)\} = f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\sigma) \sin\sigma t\, d\sigma. \quad \text{(From the special case)}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\sigma t} g_s(\sigma) d\sigma. \quad \text{(If we use extension)}$$

Example

Consider $f(t) = e^{-t},\ t > 0$.

Justify that

(a) f has Fourier sine and cosine transforms.

(b) Find Fourier cosine transform of f.

(c) Use part (b) to evaluate the following improper integral:

$$\int_0^\infty \frac{\cos \sigma t}{\sigma^2 + 1} d\sigma, \quad t > 0.$$

(a) Use even and odd extensions of f in $(-\infty, \infty)$.

(b) Suppose f_{e} is the even extension of f in $(-\infty, \infty)$. Then

$$\begin{aligned}\mathcal{F}_c\{f_{\mathsf{e}}(t)\} = g_c(\sigma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\mathrm{i}\sigma\tau} f_{\mathsf{e}}(\tau)\, d\tau \quad (\text{From Basic definition}) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{1+\sigma^2}\end{aligned}$$

Example (Contd.)

(c) Finally, use inverse Fourier cosine transform to have

$$f_e(t) = \mathcal{F}_c^{-1}\{g_c(\sigma)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(\sigma)\cos\sigma t\, d\sigma. \quad \text{(From Special case)}$$
$$= \int_0^\infty \frac{2}{\pi}\frac{\cos\sigma t}{\sigma^2+1}d\sigma. \quad \text{(From part (b))}$$

The required integral has the form

$$\int_0^\infty \frac{\cos\sigma t}{\sigma^2+1}d\sigma = \frac{e^{-t}\pi}{2}$$

What about the following integrals?

$$(a)\ \int_0^\infty \frac{\sigma^3\sin\sigma t}{\sigma^4+4}d\sigma, \quad t>0.$$
$$(b)\ \int_0^\infty \frac{\sin\sigma t}{\sigma}d\sigma, \quad t>0.$$

Here, f is not given explicitly – for (a), we can take $f(t) = e^{-t}\cos t,\ t>0.$

Application of Fourier Transform to PDEs

Fourier transform:

The Fourier transform of a function $U(x,t)$ with respect to x is defined as

$$\mathcal{F}\{U(x,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma x} U(x,t)\, dx = \overline{U}(\sigma,t). \tag{3}$$

Inverse Fourier transform:

The inverse Fourier transform $U(x,t)$ of $\overline{U}(\sigma,t)$ is defined as

$$U(x,t) = \mathcal{F}^{-1}\{\overline{U}(\sigma,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma x} \overline{U}(\sigma,t)\, d\sigma.$$

Application of Fourier Transform to PDEs

Under the assumption that U and $\frac{\partial U}{\partial x}$ vanish as $x \to \pm\infty$, we obtain the following results of transforms of the partial derivatives:

$$\mathcal{F}\left\{\frac{\partial}{\partial x}U(x,t)\right\} = \mathrm{i}\sigma\overline{U}(\sigma,t),$$

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}U(x,t)\right\} = (\mathrm{i}\sigma)^2\overline{U}(\sigma,t),$$

$$\mathcal{F}\left\{\frac{\partial}{\partial t}U(x,t)\right\} = \frac{d}{dt}\overline{U}(\sigma,t),$$

$$\mathcal{F}\left\{\frac{\partial^2}{\partial t^2}U(x,t)\right\} = \frac{d^2}{dt^2}\overline{U}(\sigma,t).$$

Application of Fourier Transform to PDEs

Result I:

$$\mathcal{F}\left\{\frac{\partial}{\partial x}U(x,t)\right\} = \mathrm{i}\sigma\overline{U}(\sigma,t). \tag{4}$$

Proof:

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial}{\partial x}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}dx \\
&= \frac{1}{\sqrt{2\pi}}\left\{[e^{-\mathrm{i}\sigma x}U(x,t)]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx\right\} \\
&= \mathrm{i}\sigma\overline{U}(\sigma,t).
\end{aligned}$$

□

Application of Fourier Transform to PDEs

Result II:

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}U(x,t)\right\} = (\mathrm{i}\sigma)^2\overline{U}(\sigma,t). \tag{5}$$

Proof:

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial^2 U}{\partial x^2}dx\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left\{[e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}dx\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left\{\mathrm{i}\sigma([e^{-\mathrm{i}\sigma x}U(x,t)]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx)\right\} \\
&= (\mathrm{i}\sigma)^2\overline{U}(\sigma,t).
\end{aligned}$$

□

Application of Fourier Transform to PDEs

Result III:

$$\mathcal{F}\left\{\frac{\partial}{\partial t}U(x,t)\right\} = \frac{d}{dt}\overline{U}(\sigma,t). \tag{6}$$

Proof:

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial}{\partial t}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial t}dx \\
&= \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\
&= \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\
&= \frac{d}{dt}\overline{U}(\sigma,t).
\end{aligned}$$

□

Proceeding in a similar manner as above, Result 4 can be obtained as follows:

$$\mathcal{F}\left\{\frac{\partial^2}{\partial t^2}U(x,t)\right\} = \frac{d^2}{dt^2}\overline{U}(\sigma,t). \tag{7}$$

□

Application of Fourier Transform to PDEs

The use of Fourier transform to solve partial differential equations is best described by examples.

Example A:

An infinitely long string extending in $-\infty < x < \infty$ under uniform tension is displaced into the curve $y = f(x)$ and let go from rest with velocity $g(x)$. To find the displacement $U(x,t)$ at any point at any subsequent time.

Solution:

The initial value problem is

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0, \tag{8}$$

$$U(x,0) = f(x) \ \text{(initial displacement)}, \tag{9}$$

$$\frac{\partial U}{\partial t}(x,0) = g(x) \ \text{(initial velocity)}. \tag{10}$$

Application of Fourier Transform to PDEs

Taking Fourier transform on both sides of PDE (8),

$$\frac{d^2}{dt^2}\overline{U}(\sigma,t) = -c^2\sigma^2\overline{U}(\sigma,t). \tag{11}$$

(11) can be written in standard form as

$$\frac{d^2}{dt^2}\overline{U}(\sigma,t) + c^2\sigma^2\overline{U}(\sigma,t) = 0. \tag{12}$$

On solving, we get

$$\overline{U}(\sigma,t) = A(\sigma)\cos(c\sigma t) + B(\sigma)\sin(c\sigma t). \tag{13}$$

Taking Fourier transforms on the initial conditions (9) and (10),

$$\overline{U}(\sigma,0) = \overline{f}(\sigma), \tag{14}$$

$$\frac{d}{dt}\overline{U}(\sigma,0) = \overline{g}(\sigma), \tag{15}$$

where $\overline{f}(\sigma)$ and $\overline{g}(\sigma)$, are, respectively the Fourier transform of $f(x)$ and $g(x)$.

Application of Fourier Transform to PDEs

Using the initial conditions, $A(\sigma)$ and $B(\sigma)$ can be obtained as:

$$\overline{U}(\sigma,0) = A(\sigma) = \overline{f}(\sigma),$$
$$\frac{d}{dt}\overline{U}(\sigma,0) = c\sigma B(\sigma) = \overline{g}(\sigma).$$

Now

$$\overline{U}(\sigma,t) = \overline{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\,\sigma}\,\overline{g}(\sigma)\sin(c\sigma t). \tag{16}$$

To get the solution, we use the inverse Fourier transform to obtain

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\overline{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\,\sigma}\,\overline{g}(\sigma)\sin(c\sigma t)\right] e^{i\sigma x}\,d\sigma. \tag{17}$$

Formula (17) gives us the solution of the initial value problem for one-dimensional wave equation (infinite string) in the form of an integral involving the Fourier transform of the initial displacement and velocity.

Application of Fourier Transform to PDEs

Example B:

Consider the heat conduction in an infinite rod with thermal diffusivity α with initial temperature distribution $f(x)$. To find the temperature distribution $U(x,t)$ at any point at any subsequent time.

Solution:

The initial value problem is

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0, \tag{18}$$

$$U(x,0) = f(x) \quad \text{(initial temperature distribution).} \tag{19}$$

Taking Fourier transform on both sides of PDE (18), we observe

$$\overline{U}(\sigma,t) = \overline{f}(\sigma)e^{-\alpha\sigma^2 t}, \tag{20}$$

where $\overline{f}(\sigma)$ is the Fourier transform of $f(x)$.

We use the inverse Fourier transform to obtain

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(\sigma)e^{-\alpha\sigma^2 t}e^{i\sigma x}\, d\sigma. \tag{21}$$

Application of Fourier Transform to PDEs

We summarize the Fourier transform method as follows:

Step 1: Take Fourier transform with respect to x on the given equation in $U(x,t)$ when $-\infty < x < \infty$ and get an ordinary differential equation in $\overline{U}(\sigma,t)$ in the variable t.

Step 2: Solve the ordinary differential equation and find $\overline{U}(\sigma,t)$.

Step 3: Take Fourier transform with respect to x on the given initial condition(s) and use them in the transformed equation in $\overline{U}(\sigma,t)$ to find the coefficients.

Step 4: Take inverse Fourier transform to get $U(x,t)$.

Application of Fourier Transform to PDEs

Example C: Semi-infinite rod with Dirichlet condition

Mathematical Model

The initial boundary value problem is the following:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \;\; x > 0, \; t > 0, \tag{22}$$

$$U(x,0) = f(x), \; t > 0, \;\; U(0,t) = U_0, \; x > 0. \tag{23}$$

Example D: Semi-infinite rod with Neumann condition

Consider the same equation as in the previous example subject to the boundary and initial conditions

$$\frac{\partial U}{\partial x}(0,t) = 0, \;\; U(x,0) = f(x).$$

The Fourier sine and cosine transforms can be employed to solve a partial differential equation when the range of the spatial variable extends from 0 to ∞, not in $-\infty < x < \infty$.

Application of Fourier Transform to PDEs

If the boundary condition is in terms of some value of $U(0,t)$ (i.e., Dirichlet boundary condition), then sine transform is to be used.

When the boundary condition is in terms of some value of $\frac{\partial U}{\partial x}(0,t)$ (i.e., Neumann boundary condition), then cosine transform is to be used.

The *Fourier sine transform* of a function $U(x,t)$ with respect to x is defined as

$$\mathcal{F}_s\{U(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty U(x,t) \sin \sigma x \, dx = \overline{U}_s(\sigma,t). \tag{24}$$

Application of Fourier Transform to PDEs

Under the assumption that U and $\frac{\partial U}{\partial x}$ vanish as $x \to \infty$,

$$\mathcal{F}_s\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\overline{U}_s(\sigma,t), \tag{25}$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\overline{U}_s(\sigma,t), \tag{26}$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}}\sigma U(0,t) - \sigma^2\overline{U}_s(\sigma,t). \tag{27}$$

Application of Fourier Transform to PDEs

The *Fourier cosine transform* of a function $U(x,t)$ with respect to x is defined as

$$\mathcal{F}_c\{U(x,t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty U(x,t)\cos\sigma x \, dx = \overline{U}_c(\sigma,t). \tag{28}$$

Under the assumption that U and $\frac{\partial U}{\partial x}$ vanish as $x \to \infty$,

$$\mathcal{F}_c\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\overline{U}_c(\sigma,t), \tag{29}$$

$$\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\overline{U}_c(\sigma,t), \tag{30}$$

$$\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = -\sqrt{\frac{2}{\pi}}U_x(0,t) - \sigma^2\overline{U}_c(\sigma,t). \tag{31}$$

Application of Fourier Transform to PDEs

Example C:

If $U(x,t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time if the initial temperature distribution is given as $f(x)$ and the boundary is kept at U_0 degrees.

Solution:

The initial boundary value problem is the following:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \;\; x > 0, \; t > 0, \tag{32}$$

$$U(x,0) = f(x), \; t > 0 \;\; U(0,t) = U_0, \; x > 0. \tag{33}$$

The boundary condition suggests that we need to use Fourier sine transform. Taking the transform on (32),

$$\frac{d}{dt}\overline{U}_s(\sigma,t) = \alpha[\sqrt{\frac{2}{\pi}}\sigma U(0,t) - \sigma^2 \overline{U}_s(\sigma,t)]. \tag{34}$$

Application of Fourier Transform to PDEs

Using the boundary condition,

$$\frac{d}{dt}\overline{U}_s(\sigma,t) + \alpha\sigma^2\overline{U}_s(\sigma,t) = \alpha\sqrt{\frac{2}{\pi}}\sigma U_0. \tag{35}$$

On solving,

$$\overline{U}_s(\sigma,t) = A(\sigma)e^{-\alpha\sigma^2 t} + \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}U_0. \tag{36}$$

Using the initial condition,

$$\overline{f_s}(\sigma) = A(\sigma) + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma},$$

where $\overline{f_s}(\sigma)$ is the Fourier sine transform of $f(x)$, that is,

$$A(\sigma) = \overline{f_s}(\sigma) - \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}.$$

Application of Fourier Transform to PDEs

Now $\overline{U}_s(\sigma, t)$ is

$$\begin{aligned}\overline{U}_s(\sigma, t) &= (\overline{f_s}(\sigma) - \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma})e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma} \\ &= \overline{f_s}e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t}). \qquad (37)\end{aligned}$$

The inversion gives

$$U(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\overline{f_s}e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t})\right] \sin\sigma x \, d\sigma. \qquad (38)$$

Application of Fourier Transform to PDEs

Similarly, we can try the same heat conduction problem with a Neumann condition.

Like-wise, we can try the vibration problem in a semi-infinite string $0 < x < \infty$.

In this case, there will be a second-order ordinary differential equation upon using the transform due to the presence of the term $\frac{\partial^2 U}{\partial t^2}$.

The two initial conditions $U(x,0)$ and $\frac{\partial U}{\partial t}(x,0)$ will help us in determining the coefficients of $\overline{U_c}(\sigma,t)$ or $\overline{U_s}(\sigma,t)$ selection of which will depend on the type of boundary condition at $x=0$.