MA 201, Mathematics III, July-November 2022,

Part II: Partial Differential Equations

PDEs in Different Coordinate Systems

Lecture 15

Solutions in different types of domains

For a two-dimensional problem in $u(x,y)$, the Laplacian:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

For a three-dimensional problem in $u(x,y,z)$, the Laplacian:

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

For a two-dimensional problem in $u(r,\theta)$, that is, in polar coordinates, the Laplacian:

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Solution in different types of domains

For a three-dimensional problem in $u(r, \theta, z)$, that is, in cylindrical coordinates, the Laplacian:

$$\nabla^2 u(r, \theta, z) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

For a three-dimensional problem in $u(r, \theta, \phi)$, that is, in spherical coordinates, the Laplacian:

$$\nabla^2 u(r, \theta, \phi) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

Laplace's equation in polar coordinates

For a problem involving a circular disk, polar coordinates are more appropriate than rectangular coordinates.

Let us formulate the steady-state heat flow problem in polar coordinates r, θ,

where $x = r\cos\theta, y = r\sin\theta$.

A circular plate of radius a can be simply represented by

$r < a$ with $0 \leq \theta \leq 2\pi$ with $r = a$ as its boundary.

The unknown temperature inside the plate is now $u = u(r, \theta)$.

The given temperature on the boundary of the plate is considered as

$u(a, \theta) = f(\theta)$, where f is a known function.

Laplace's equation in polar coordinates

Now we have the following boundary value problem:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \;\; 0 \le r \le a, \; 0 \le \theta \le 2\pi, \tag{1}$$

$$u(a,\theta) = f(\theta), \; 0 \le \theta \le 2\pi. \tag{2}$$

There is a periodic boundary condition which is implicit in nature:

$$u(r,\theta) = u(r,\theta + 2\pi). \tag{3}$$

Using the separation of variables method, assume a solution:

$$u(r,\theta) = R(r)T(\theta).$$

Using this in equation (1),

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{T''}{T} = 0. \tag{4}$$

Laplace's equation in polar coordinates

Separating the variables

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{T''}{T}=k, \tag{5}$$

which gives rise to the following ODEs:

$$r^2R''+rR'-kR=0, \tag{6}$$

$$T''+kT=0. \tag{7}$$

We cannot consider negative values of k, since if k is negative,

then the ODE in $T(\theta)$ has exponential solutions, and exponential solutions cannot satisfy periodicity conditions.

Since we are looking for a periodic solution in θ,

we must take $k=\lambda^2$.

But we should keep in mind that $\lambda=0$, corresponding to $k=0$, is also an eigenvalue with corresponding eigenfunction $u_0(r,\theta)=$ constant.

Laplace's equation in polar coordinates

Hence the equations reduce to

$$r^2R'' + rR' - \lambda^2 R = 0, \tag{8}$$

$$T'' + \lambda^2 T = 0. \tag{9}$$

(9) has the general solution

$$T(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \tag{10}$$

The Dirichlet periodic boundary condition (3) will give us

$$\cos(2\pi\lambda) = 1,$$

i.e., $\lambda_n = n$.

Laplace's equation in polar coordinates

$T_n(\theta)$:

$$T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta). \tag{11}$$

Equation (8) takes the form

$$r^2 R'' + rR' - n^2 R = 0, \tag{12}$$

which is of Cauchy-Euler form and the solution is

$$R_n = C_n r^{-n} + D_n r^n. \tag{13}$$

We set $C_n = 0$ since we are seeking a bounded solution in $0 \leq r \leq a$,

and r^{-n} is not bounded when $r \to 0$.

Laplace's equation in polar coordinates

Solution to the BVP:

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta). \quad (14)$$

Using the given boundary condition (2),

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)]. \quad (15)$$

The coefficients are given by

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta,\ n = 0, 1, 2, 3, \ldots, \quad (16a)$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta,\ n = 1, 2, 3, \ldots \quad (16b)$$

(14) is the solution of Laplace's equation with the coefficients given by (16a) and (16b).

Laplace's equation in polar coordinates

What we have just solved is called

an Interior Dirichlet problem for a circle since we have used Dirichlet condition in the region $r \leq a$.

If we change the boundary condition to Neumann,

we have what is called an Interior Neumann problem for a circle.

Now if we change the region to $r > a$ with Dirichlet condition

we have what is called an Exterior Dirichlet problem for a circle.

If in above, the condition is replaced by Neumann,

we have what is called an Exterior Neumann problem for a circle.

Time-dependent diffusion in a disk

Consider a circular planar disk of radius a for which

- initial temperature is a function of the radial distance r alone,
- the boundary is held at zero degree Celcius.

Intuition tells that

the temperature u in the disk depends only on time and the distance r from the centre.

To be precise, the initial temperature u will be the same for some $r = r_n$ irrespective of what value of θ is assigned.

That is, if the initial temperature u is, say, u_1 for some $r = r_1$, then it is imperative that the temperature is same for that specific r.

Consider now that the initial temperature u is, say, u_2 for some $r = r_2$, then it is imperative that the temperature is same for that specific r.

This, in turn means that

the heat flow will take place either from $r = r_1$ to $r = r_2$ or from $r = r_2$ to $r = r_1$ depending on which has a higher temperature.

Diffusion in a disk

It then allows us to consider u simply as

$u = u(r,t)$ although the originally equation seemed to contain r, θ, t as the independent variables.

This assumption looks perfectly all right

because there is nothing in the initial condition or boundary condition to cause heat to diffuse in an angular direction.

Heat will flow only along the rays emanating from the origin.

Now it is obvious that

the diffusion equation looks simpler than what it was originally.

Diffusion in a disk

Diffusion equation is now as simple as follows:

$$u_t = \alpha\left(u_{rr} + \frac{1}{r}u_r\right),\ 0 \le r \le a,\ t > 0. \tag{17}$$

Boundary condition:

$$u(a,t) = 0,\ t \ge 0. \tag{18}$$

Initial condition:

$$u(r,0) = f(r),\ 0 \le r < a, \tag{19}$$

where f is a given initial radial temperature distribution.

Diffusion in a disk

Assume a solution in the form:

$$u(r,t) = R(r)T(t).$$

From the given equation,

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{T'}{\alpha T} = k.$$

$k = -\lambda^2$ gives rise to the following pair of ODEs:

$$r^2R'' + rR' + \lambda^2 r^2 R = 0, \tag{20}$$

$$T' + \alpha\lambda^2 T = 0. \tag{21}$$

Equation (21) can be easily solved to write as

$$T(t) = Ce^{-\alpha\lambda^2 t}. \tag{22}$$

Can you recognize equation (20)??

Diffusion in a disk

Equation (20) is Bessel's equation of order 0.

Its solution can be written as

$$R(r) = AJ_0(\lambda r) + BY_0(\lambda r), \tag{23}$$

where

J_0 and Y_0 are, respectively, Bessel's function of first kind and second kind of order zero.

We are looking for a bounded solution as $r \to 0$,

we must take $B = 0$ as $Y_0(\lambda r) \to -\infty$ as $r \to 0$.

Diffusion in a disk

Solution $u(r,t)$ can be written as

$$u(r,t) = AJ_0(\lambda r)e^{-\alpha\lambda^2 t}. \tag{24}$$

Applying boundary condition (18), we get

$0 = AJ_0(\lambda a)$ implying

$$J_0(\lambda a) = 0.$$

Hence $\lambda_n a = \nu_n$,

where ν_n are the zeros of J_0.

Hence the eigenvalues are given by

$$\lambda_n = \frac{\nu_n}{a}. \tag{25}$$

Diffusion in a disk

This gives us

$$u_n(r,t) = A_n J_0\left(\frac{\nu_n}{a} r\right) e^{-\alpha \frac{\nu_n^2}{a^2} t}.$$

The solution $u(r,t)$ is

$$\begin{aligned} u(r,t) &= \sum_{n=1}^{\infty} u_n(r,t) \\ &= \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a} r\right) e^{-\alpha \frac{\nu_n^2}{a^2} t}. \end{aligned} \tag{26}$$

Now using the initial condition (19):

$$f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a} r\right). \tag{27}$$

Diffusion in a disk

Note the difference between the orthogonal properties of sine/cosine functions and Bessel functions.

Bessel functions $\{J_\mu(\lambda_n r)\}$ form an orthogonal set with respect to the weight function r.

For finding the coefficient A_n, we multiply (27) by $rJ_0\left(\frac{\nu_n}{a}r\right)$

and then integrate with respect to r from 0 to a to get

$$A_n = \frac{\int_0^a rf(r)J_0\left(\frac{\nu_n}{a}r\right)\,dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\,dr}. \tag{28}$$

The following orthogonality property is used:

$$\int_0^a rJ_0\left(\frac{\nu_n}{a}r\right)J_0\left(\frac{\nu_m}{a}r\right)\,dr = 0,\quad n \neq m.$$

(26) gives the solution of the given problem with A_n given by (28).

Steady-state heat conduction in a circular cylinder

Consider a right circular cylinder of radius a and height L having

(a) its convex surface and base in the xy-plane at temperature 0^0C,

(b) the top surface $z = L$ kept at temperature $f(r)^0$C.

To find the steady-state temperature at any point of the cylinder.

The governing equation for this problem will be Laplace's equation in r, θ, z:

$$\nabla^2 u(r, \theta, z) \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

For simplicity, we will consider *Radially Symmetric Solution* for the Laplace's equation.

Radially symmetric solution means that $u(r, \theta, z) = u(r, z)$, that is, the solution does not depend on the polar angle θ.

In other sense, solutions are symmetric under rotation.

Steady-state heat conduction in a circular cylinder

By assuming that the cylinder is symmetrical about its axis, Laplace's equation takes the form:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0,\ 0 \le r \le a,\ 0 \le z \le L. \tag{29}$$

The boundary conditions are

$$\text{(on the surface)}\ \ u(a,z) = 0, 0 \le z \le L, \tag{30a}$$

$$\text{(on the bottom)}\ \ u(r,0) = 0, 0 \le r \le a, \tag{30b}$$

$$\text{(on the top)}\ \ u(r,L) = f(r), 0 \le r \le a. \tag{30c}$$

Assume a solution in the form

$$u(r,z) = R(r)Z(z).$$

Applying it to the governing equation (29):

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0.$$

Steady-state heat conduction in a circular cylinder

By separating the variables:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = k.$$

Observing that only the negative value of the separation constant will give rise to non-trivial solutions,

we get the following ODEs by considering $k = -\lambda^2$:

$$Z'' - \lambda^2 Z = 0, \tag{31}$$

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0. \tag{32}$$

The solutions of the above equations are, respectively, given by

$$Z(z) = A\cosh(\lambda z) + B\sinh(\lambda z), \tag{33}$$

$$R(r) = CJ_0(\lambda r) + DY_0(\lambda r). \tag{34}$$

Steady-state heat conduction in a circular cylinder

The solution $u(r, z)$:

$$u(r, z) = [A\cosh(\lambda z) + B\sinh(\lambda z)][CJ_0(\lambda r) + DY_0(\lambda r)]. \tag{35}$$

We are looking for a bounded solution in $0 \le r \le a$,

we must take $D = 0$ since $Y_0 \to -\infty$ as $r \to 0$.

Equation (35) can be written as

$$u(r, z) = J_0(\lambda r)[A\cosh(\lambda z) + B\sinh(\lambda z)]. \tag{36}$$

Now applying the boundary condition (30a), we get $0 = AJ_0(\lambda a)$

implying

$$J_0(\lambda a) = 0.$$

Steady-state heat conduction in a circular cylinder

Hence

$$\lambda_n a = \nu_n,$$

where ν_n are the zeros of J_0.

The eigenvalues are given by

$$\lambda_n = \frac{\nu_n}{a} \text{ giving us} \tag{37}$$

$$u_n(r,z) = A_n J_0\left(\frac{\nu_n}{a}r\right)\cosh\left(\frac{\nu_n}{a}z\right) + B_n J_0\left(\frac{\nu_n}{a}r\right)\sinh\left(\frac{\nu_n}{a}z\right).$$

Using the boundary condition (30b),

we get $A_n = 0$ thereby reducing the solution to

$$u_n(r,z) = B_n J_0\left(\frac{\nu_n}{a}r\right)\sinh\left(\frac{\nu_n}{a}z\right).$$

Steady-state heat conduction in a circular cylinder

The solution $u(r, z)$ is

$$u(r, z) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh\left(\frac{\nu_n}{a} z\right). \tag{38}$$

The coefficient B_n can be obtained by using the boundary condition (30c):

$$f(r) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\nu_n}{a} r\right) \sinh\left(\frac{\nu_n}{a} L\right) \tag{39}$$

giving us

$$B_n = \frac{\int_0^a r f(r) J_0\left(\frac{z_n}{a} r\right)\, dr}{\sinh\left(\frac{\nu_n}{a} L\right) \int_0^a r \left(J_0\left(\frac{z_n}{a} r\right)\right)^2\, dr}. \tag{40}$$