MA 201
COMPLEX ANALYSIS
ASSIGNMENT–4 SOLUTIONS

(1) For the following functions, locate and classify all the singular points in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(i) $\sin\left(\dfrac{1}{z}\right)$ (ii) $\dfrac{1}{\sin\left(\frac{1}{z}\right)}$ (iii) $\cot z - (2/z)$ (iv) $\dfrac{z\exp(1/(z-1))}{\exp(z)-1}$

Answer:

(i) $\sin\left(\dfrac{1}{z}\right)$:

Since $\sin\left(\dfrac{1}{z}\right)$ has infinite terms in the principal part in the Laurent series expansion, $z = 0$ is an essential singularity.

Nature of point at infinity:

Consider $g(w) = f(1/w) = \sin(w)$. Then, the function $g(w)$ is analytic at $w = 0$ and hence the function f is analytic at $z = \infty$.

(ii) $\dfrac{1}{\sin\left(\frac{1}{z}\right)}$:

Then, $f(z)$ has simple poles at $z_n = \frac{1}{n\pi}$ for $n \in \mathbb{Z}$ with $n \neq 0$. The point $z = 0$ is the limit point of the poles z_n of f. Therefore, the point $z = 0$ is a non-isolated singularity of f.

Nature of point at infinity:

Consider $g(w) = f(1/w) = \dfrac{1}{\sin(w)}$. Then, $w = 0$ is a simple pole of g. Therefore, the point $z = \infty$ is a simple pole of f.

(iii) $f(z) = \cot z - \left(\dfrac{2}{z}\right) = \dfrac{z\ \cos(z)\ -\ 2\sin(z)}{z\ \sin(z)}$:

If $z_n = n\pi$ for $n \in \mathbb{Z}$ with $n \neq 0$ then z_n is a simple zero of $z\sin z$. Note that $z_n\cos(z_n) - 2\sin(z_n) \neq 0$. Hence f has simple poles at z_n. The point $z = 0$ is a zero of order 1 for the numerator function $z\cos z - 2\sin z$ and is a zero of order 2 for the denominator function $z\sin z$. Therefore, f has a simple pole at $z = 0$. Thus, the function f has simple poles at $z_n = n\pi$ for $n \in \mathbb{Z}$.

Nature of point at infinity:

Consider $g(w) = f(1/w) = \dfrac{\cos(1/w)}{\sin(1/w)} - 2w$. Then, $g(w)$ has simple poles at

$w_n = \frac{1}{n\pi}$ for $n \in \mathbb{Z}$ with $n \neq 0$. The point $w = 0$ is the limit point of the poles w_n of g. Therefore, $w = 0$ is a non-isolated singularity of $g(w)$ and hence $z = \infty$ is a non-isolated singularity of $f(z)$.

(iv) $f(z) = \dfrac{z\exp(1/(z-1))}{\exp(z) - 1}$:
Observe that $z = 2n\pi i$ where $n \in \mathbb{Z}$ is a simple zero for the function $e^z - 1$. Therefore, f has simple poles at $z = 2n\pi i$ where $n \in \mathbb{Z}$ and $n \neq 0$. Further, f has a removable singularity at $z = 0$.
Since $\lim_{z\to 1} \exp(1/(z-1))$ does not exist, the function f has an essential singularity at $z = 1$.
Nature of point at infinity:
Consider $g(w) = f(1/w) = \dfrac{\exp(w/(1-w))}{w(\exp(1/w) - 1)}$. Then, $g(w)$ has simple poles at $w_n = \frac{1}{2n\pi i}$ for $n \in \mathbb{Z}$ with $n \neq 0$. The point $w = 0$ is the limit point of the poles w_n of g. Therefore, $w = 0$ is a non-isolated singularity of $g(w)$ and hence $z = \infty$ is a non-isolated singularity of $f(z)$.

(2) Using Rouché's theorem prove Fundamental Theorem of Algebra.
Answer: If $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ then write $f(z) = z^n$ and $g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$. Now choose $R > 0$ such that $|f(z)| > |g(z)|$ on the circle $|z| = R$. Note that

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right| \to 0$$

as $z \to \infty$. i.e. for $\epsilon = 1 > 0$ there exists $M > 0$ such that

$$\left|\frac{g(z)}{f(z)}\right| < 1 \text{ whenever } |z| > M.$$

Take $R = M + 1$ and apply Rouché's theorem.

(3) Find the isolated singularities and compute the residue of the functions

$$a)\ \frac{e^z}{z^2 - 1}, \quad b)\ \frac{3z}{z^2 + iz + 2}, \quad c)\ \cot \pi z, \quad d)\ \frac{\pi \cot \pi z}{(z + \frac{1}{2})^2}.$$

Answer:

a) Singularities at ± 1. Residue at $1 = \lim_{z\to 1}(z-1)\dfrac{e^z}{z^2 - 1} = \dfrac{e}{2}$

Residue at $-1 = \lim_{z\to -1}(z+1)\dfrac{e^z}{z^2 - 1} = -\dfrac{1}{2e}$.

b) Singularities at i and $-2i$.

Residue at $i = \lim_{z\to i}(z-i)\dfrac{3z}{z^2 + iz + 2} = 1$

Residue at $-2i = \lim_{z \to -2i} (z+2i)\dfrac{3z}{z^2+iz+2} = 2.$

c) Singularities at $\pm n$.

Residue at $n = \lim_{z \to n} (z-n)\cot \pi z = \lim_{z \to n} (z-n)\dfrac{(-1)^n \cos \pi z}{\sin(z-n)\pi} = \dfrac{1}{\pi}.$

d) Singularities at $\pm n$ and $-\frac{1}{2}$.

Residue at $\pm n = \lim_{z \to \pm n} (z \mp n)\dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2} = \dfrac{1}{(\pm n + \frac{1}{2})^2}.$

Residue at $-\frac{1}{2} = \lim_{z \to -\frac{1}{2}} \dfrac{1}{2}\dfrac{d}{dz}\left[\left(z+\dfrac{1}{2}\right)^2 \dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2}\right] = -\dfrac{\pi^2}{2}.$

(4) Find the residues of the function $\dfrac{1}{z^3 - z^5}$ at all isolated singular points in $\mathbb{C}$.

Answer:

The function $f(z) = \dfrac{1}{z^3 - z^5}$ has:

a pole of order 3 at $z = 0$

a simple pole at $z = 1$

a simple pole at $z = -1$

We know that $\sum_{n=0}^{\infty} z^{2n} = \dfrac{1}{1-z^2}$ for $|z| < 1$. Therefore, $\dfrac{1}{z^3 - z^5} = \sum_{n=0}^{\infty} z^{2n-3}$ for $0 < |z| < 1$.

The residue of f at $z = 0$ is $a_{-1} = \operatorname{Res}(f,\ 0) = 1.$

$$\operatorname{Res}(f,\ 1) = \lim_{z \to 1} \frac{(z-1)}{z^3(1-z)(1+z)} = \frac{-1}{2}\ .$$

$$\operatorname{Res}(f,\ -1) = \lim_{z \to -1} \frac{(z+1)}{z^3(1-z)(1+z)} = \frac{-1}{2}\ .$$

(5) Find the residues of $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$, $(m,\ a$ real) at its singularities in $\mathbb{C}$.

Answer:

The function $f(z) = \dfrac{e^{imz}}{z^2+a^2}$ has:

a simple pole at $z = ia$

a simple pole at $z = -ia$

Set $g(z) = e^{imz}$ and $h(z) = z^2 + a^2$. Then, $h'(z) = 2z$.

$$\operatorname{Res}(f,\ ia) = \frac{g(ia)}{h'(ia)} = \frac{e^{im \times ia}}{2ia} = \frac{-i\ e^{-ma}}{2a}\ .$$

$$\operatorname{Res}(f,\ -ia) = \frac{g(-ia)}{h'(-ia)} = \frac{e^{im\times(-ia)}}{-2ia} = \frac{i\ e^{ma}}{2a}\ .$$

(6) Show that the residue at the point at infinity for the function $f(z) = \left(\dfrac{z^4}{2z^2-1}\right)\sin\left(\dfrac{1}{z}\right)$ is equal to $(-1/6)$.

Answer:

$$f(z) = \left(\frac{z^4}{2z^2-1}\right)\sin\left(\frac{1}{z}\right) = \frac{z^2}{2}\left[1-\left(\frac{1}{2z^2}\right)\right]^{-1}\sin\left(\frac{1}{z}\right)\ .$$

$$f(z) = \frac{z^2}{2}\left[1+\frac{1}{2z^2}+\frac{1}{4z^4}+\frac{1}{8z^6}+\cdots\right]\left[\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z^5}-\frac{1}{7!}\frac{1}{z^7}+\cdots\right] \quad \text{for } |z| > \frac{1}{\sqrt{2}}$$

It yields the Laurent series of f about the point $z=\infty$.
The coefficient a_{-1} of the term $\frac{1}{z}$ in this series is $\frac{-1}{2\times(3!)}+\frac{1}{4}=\frac{1}{6}$.
Therefore, the residue of f at $z=\infty$ is $-a_{-1} = -\frac{1}{6}$.

(7) Evaluate $\displaystyle\int_C \frac{z\,dz}{\cos z}$ where $C:\ \left|z-\dfrac{\pi}{2}\right| = \dfrac{\pi}{2}$.

Answer:

Observe that $\cos(z)$ has simple zeros at $(2n+1)\pi/2$ for $n\in\mathbb{Z}$. Therefore, the function $f(z) = \dfrac{z}{\cos z}$ has a simple poles at $(2n+1)\pi/2$ for $n\in\mathbb{Z}$.
Inside the contour C, the function f has only one pole $z=\frac{\pi}{2}$ which is a simple pole. Then,

$$\operatorname{Res}(f,\pi/2) = \left[\frac{z}{-\sin z}\right]_{z=\pi/2} = \frac{-\pi}{2}\ .$$

By the Cauchy's residue theorem,

$$\int_C \frac{z\,dz}{\cos z} = 2\pi i\,\operatorname{Res}(f,\pi/2) = 2\pi i\ \times\ \frac{(-\pi)}{2} = -\pi^2\, i\ .$$

(8) Using the Cauchy's residue theorem, evaluate $\displaystyle\int_C \frac{(z^2+3z+2)}{(z^3-z^2)}\,dz$ where $C:\ |z|=2$.

Answer:

Let $f(z) = \dfrac{(z^2+3z+2)}{(z^3-z^2)} = \dfrac{(z^2+3z+2)}{z^2(z-1)}$. Then,
the point $z=0$ is a pole of order 2 and the point $z=1$ is a pole of order 1 (simple pole).
Both the poles lie inside the contour $C:|z|=2$.

$$\operatorname{Res}(f,\ 0) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left\{\frac{z^2(z^2+3z+2)}{z^2(z-1)}\right\} = -5\ .$$

$$\mathrm{Res}\,(f,\ 1) = \lim_{z\to 1} \frac{(z-1)(z^2+3z+2)}{z^2(z-1)} = 6\ .$$

By the Cauchy's residue theorem,

$$\int_C \frac{(z^2+3z+2)}{(z^3-z^2)}\,dz \ = \ 2\pi i\,[\mathrm{Res}\,(f,\ 0) + \mathrm{Res}\,(f,\ 1)] = 2\pi i\ [-5+6] = 2\pi i\ .$$

(9) Using the argument principle, evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \cot z\, dz$ where $C : |z| = 7$.

Answer:

Let $f(z) = \sin z$. Then $f'(z) = \cos z$. So, $\dfrac{f'(z)}{f(z)} = \dfrac{\cos z}{\sin z} = \cot z$.

The function $f(z) = \sin z$ has zeros at $n\pi$ where $n \in \mathbb{Z}$.

By the argument principle,

$$\int_C \cot z\, dz \ = \ \int_C \frac{f'(z)}{f(z)}\,dz \ = \ 2\pi i\ (\text{No. of zeros} \ - \ \text{No. of poles}) = (2\pi i)(5{-}0) = 10\pi i\ .$$

(10) Let $f(z) = (z^3+2)/z$. Let $C : z(\theta) = 2e^{i\theta}$, $0 \le \theta \le 2\pi$ be the circle. Let Γ denote the image curve under the mapping $w = f(z)$ as z traverses C once . Determine the change in the argument of $f(z)$ as z describes C once. How many times does Γ wind around the origin in the w-plane and what is the orientation of Γ?

Answer:

The function $f(z) = (z^3+2)/z$ has a simple pole at $z = 0$ and has three zeros in $|z| < 2$.

$$\text{Change in the argument of } f = \Delta_C \arg\ (f(z)) = 2\pi(N-P)$$

where N is the number of zeros and P is the number of poles of f inside C (counting to its multiplicities).

So, $\Delta_C \arg\ (f(z)) = 2\pi(3-1) = 4\pi$.

The image curve Γ winds around the origin two times in the counterclockwise direction in the w-plane.

(11) Using Rouche's theorem, find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lying in $|z| < 1$.

Answer:

Rouche's Theorem: Suppose that (i) two functions f and g are analytic inside and on a simple closed contour C and (ii) $|g(z)| < |f(z)|$ at each point on the contour C. Then the function f and $f+g$ have the same number of zeros, counting multiplicities, inside the contour C.

Set $g(z) = z^9 - 2z^6 + z^2 - 2$, $f(z) = -8z$ and $P(z) = z^9 - 2z^6 + z^2 - 8z - 2$.

Observe that

$$\begin{aligned} |g(z)| = |z^9 - 2z^6 + z^2 - 2| &\leq |z|^9 + 2|z|^6 + |z|^2 + 2 \leq 6 \qquad \text{for } |z| = 1 \\ |f(z)| = |-8z| &= 8|z| = 8 \qquad \text{for } |z| = 1 \\ |g(z)| \leq 6 &< 8 = |f(z)| \qquad \text{on } |z| = 1 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 1$. Since f has only a simple zero at $z = 0$ inside $|z| = 1$, the function $f + g \equiv P$ has only one zero inside $|z| = 1$. Therefore, the equation $P(z) = 0$ has only one root in $|z| < 1$.

(12) How many roots of the equation $z^4 - 5z + 1 = 0$ are situated in the domain $|z| < 1$? In the annulus $1 < |z| < 2$?

Answer:

In the domain $|z| < 1$:

Set $g(z) = z^4 + 1$, $f(z) = -5z$ and $P(z) = z^4 - 5z + 1$.

Observe that

$$\begin{aligned} |g(z)| = |z^4 + 1| &\leq |z|^4 + 1 \leq 2 \qquad \text{for } |z| = 1 \\ |f(z)| = |-5z| &= 5|z| = 5 \qquad \text{for } |z| = 1 \\ |g(z)| \leq 2 &< 5 = |f(z)| \qquad \text{on } |z| = 1 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 1$. Since f has only a simple zero at $z = 0$ inside $|z| = 1$, the function $f + g \equiv P$ has only one zero inside $|z| = 1$. Therefore, the equation $P(z) = 0$ has only one root in $|z| < 1$.

In the domain $|z| < 2$:

Set $g(z) = -5z + 1$, $f(z) = z^4$ and $P(z) = z^4 - 5z + 1$.

Observe that

$$\begin{aligned} |g(z)| = |-5z + 1| &\leq 5|z| + 1 \leq 11 \qquad \text{for } |z| = 2 \\ |f(z)| = |z^4| &= |z|^4 = 16 \qquad \text{for } |z| = 2 \\ |g(z)| \leq 11 &< 16 = |f(z)| \qquad \text{on } |z| = 2 \end{aligned}$$

By the Rouche's theorem, the function f and $f + g \equiv P$ have same number of zeros inside $|z| = 2$. Since f has only a zero of order 4 at $z = 0$ inside $|z| = 2$, the function $f + g \equiv P$ has four zeros inside $|z| = 2$. Therefore, the equation $P(z) = 0$ has four roots in $|z| < 2$.

In the domain $1 < |z| < 2$:

The equation $P(z) = 0$ has 4 roots in $|z| < 2$ and it has 1 root in $|z| < 1$. Therefore, we conclude that the equation $P(z) = 0$ has 3 roots in the domain $1 < |z| < 2$.