MA 201 (Part II), September-November, 2020 session
Partial Differential Equations
Session September-November,2020
Solutions to PDE Tutorial Problems - 1, Date of Discussion: October 20th October, 2020

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First Order ODE, Second Order ODE, Derivation of PDEs, Method of Characteristics

1. Let $y(x)$ be the solution of the differential equation
$$y' = (y-1)(y-3)$$
satisfying $y(0) = 2$. Then, which of the following is/are true

(a) $y$ is not bounded

(b) $y$ is bounded

(c) $\lim_{x\to\infty} y(x) = 1$

(d) $\lim_{x\to-\infty} y(x) = 3$

**Solution:** The ODE can be written as
$$\left\{\frac{1}{y-3} - \frac{1}{y-1}\right\} = 2dx.$$
Which gives
$$(y-3) = C(y-1)e^{2x}.$$
Using the condition $y(0) = 2$, we obtain
$$(y-3) = (1-y)e^{2x}.$$
Therefore, we obtain
$$y = \frac{3+e^{2x}}{1+e^{2x}} = 1 + \frac{2}{1+e^{2x}}.$$
Clearly, $0 < y < 3$ and hence it is bounded. Further,
$$\lim_{x\to\infty} y(x) = 1 \ \ \& \ \ \lim_{x\to-\infty} y(x) = 3.$$

2. Let $y_1$ and $y_2$ be solutions of $y' + y = 17,\ x \in \mathbb{R}$, with $y_1(0) = 0$ and $y_2(0) = 1$. Then (a) $y_1(x)$ and $y_2(x)$ meets exactly one point $x = 17$.

(b) $y_1(x)$ and $y_2(x)$ do not intersect for any values of $x \in \mathbb{R}$.

(c) $y_1(x)$ and $y_2(x)$ intersect at infinitely many points.

(d) $y_1(x)$ and $y_2(x)$ intersect at two points $x = 0$ and $x = 17$.

**Solution:** Clearly, $w = y_2 - y_1$ satisfies following IVP
$$w' + w = 0 \ \ \& \ \ w(0) = 1.$$
Which gives
$$w(x) = e^{-x} \neq 0 \ \ \forall x \in \mathbb{R}.$$
Hence,
$$y_2(x) - y_1(x) \neq 0 \ \ \forall x \in \mathbb{R}.$$
Therefore, $y_1$ and $y_2$ do not intersect for any values of $x \in \mathbb{R}$.

3. Find a solution for the IVP

$$y' = y^{1/3}, \ \ y(0) = 0.$$

Whether the solution is unique?

**Solution:** Clearly the given equation is nonlinear and the given function $f(x, y) = y^{1/3}$ is continuous everywhere. However, continuity of $f$ does not assure the existence of unique solution. To understand the situation more clearly, we must actually solve the problem, which is easy to do since the differential equation is separable. Thus, we have

$$y^{-1/3}dy = dx,$$

so

$$\frac{3}{2}y^{2/3} = x + C$$

and

$$y = \left[\frac{2}{3}(x + C)\right]^{3/2}.$$

The initial condition is satisfied if $C = 0$, so

$$y = y_1(x) = \left(\frac{2}{3}x\right)^{3/2}, \ \ x \geq 0$$

is a solution. On the other hand, the function

$$y = y_2(x) = -\left(\frac{2}{3}x\right)^{3/2}, \ \ x \geq 0$$

is also a solution of the initial value problem. Thus the problem has more than one solution.

4. Solve the IVP

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \ \ y(0) = -1.$$

Whether the solution is unique?

**Solution:** Clearly the given equation is nonlinear and the given function

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y - 1)} \ \ \& \ \ \frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y - 1)^2}$$

are continuous in a neighborhood containing the initial point $(0, -1)$. To solve the problem, we rewrite the equation in the separable form

$$2(y - 1)dy = -(3x^2 + 4x + 2)dx$$

so that

$$(y - 1)^2 = -x^3 - 2x^2 - 2x + C.$$

Then applying the initial condition, we have $C = 4$. Finally, by solving for $y$, we obtain

$$y = 1 \pm \sqrt{-x^3 - 2x^2 - 2x + 4}.$$

This provides two functions namely

$$y = y_1(x) = 1 + \sqrt{-x^3 - 2x^2 - 2x + 4}, \ \ y = y_2(x) = 1 - \sqrt{-x^3 - 2x^2 - 2x + 4}.$$

Since $y_1(x)$ does not satisfy the given condition, thus it can not be a solution. Hence, the problem has a unique solution in region containing $(0, -1)$. In this example, **continuity of $f$ and $\partial f/\partial y$ ensure the existence of unique solution**.

5. Consider the following IVP

$$\frac{dy}{dx} = y^2, \;\; y(0) = 2.$$

Then

(a) Check the existence of unique solution.

(b) If possible find the solution.

(c) Determine the interval of definition for the solution.

(d) Determine the largest interval of definition from Picard method.

(e) What is your observation from part (c) and part (d)?

**Solution:** (a) Here

$$f(x, y) = y^2 \;\; \& \;\; \frac{\partial f}{\partial y} = 2y$$

are continuous everywhere. Therefore, the IVP has an unique solution in an interval containing initial point $(0, 2)$.

(b) The solution is given by

$$y = -\frac{1}{x + C}.$$

Which satisfies the given condition for $C = -1/2$ and hence

$$y = \frac{2}{1 - 2x}.$$

(c) It follows from part (b) that the solution is valid for $x \neq 1/2$. Since $x > 1/2$ does not contain $x_0 = 0$, therefore the interval of definition is $x < 1/2$.

(d) Let $a$ and $b$ are parameter, which need to determine, such that the solution is valid in the rectangle

$$R = \{(x, y) : \; |x - 0| \leq a, \; |y - 2| \leq b\}.$$

Then

$$\begin{aligned} M &= \sup\{|f(x, y)| : \; (x, y) \in R\} \\ &= \sup\{y^2 : \; (x, y) \in R\} \\ &= (b + 2)^2. \end{aligned}$$

Thus the unique solution, by Picard result, is valid in the interval $|x| < h$ with

$$h = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{(b+2)^2}\right\} = \min\left\{a, g(b)\right\}, \; g(b) = \frac{b}{(b+2)^2}.$$

For the largest interval, we set $g'(b) = 0$. This gives $b = \pm 2$ and hence the maximum value of $g$ is given by $g(2) = 1/8$. Thus

$$h = \min\left\{a, \frac{1}{8}\right\} = \begin{cases} a, & \text{if } a < 1/8; \\ 1/8, & \text{if } a > 1/8; \\ 1/8, & \text{if } a = 1/8. \end{cases}$$

For all three cases the largest interval is valid given by $|x| < 1/8$.

(e) The largest interval by Picard is much smaller than the interval given by part (c). Thus Picard result gives only local existence of the solution.

6. Let $k, l \in \mathbb{R}$ be such that every solution of

$$y'' + 2ky' + ly = 0$$

satisfies $\lim_{x\to\infty} y(x) = 0$. Then

$$(a)\ 3k^2 + l < 0\ \ \&\ \ k > 0, \quad (b)\ k^2 + l > 0\ \ \&\ \ k < 0$$
$$(c)\ k^2 - l \le 0\ \ \&\ \ k > 0, \quad (d)\ k^2 - l > 0,\ k > 0\ \ \&\ \ l > 0.$$

**Solution:** Roots of the auxillary equation are $m = -k \pm \sqrt{k^2 - l}$, so that two L.I. solutions are

$$y_1 = e^{-kx} e^{\sqrt{(k^2-l)}x}, \quad y_2 = e^{-kx} e^{-\sqrt{(k^2-l)}x}.$$

Due to given condition, we should have

$$\lim_{x\to\infty} y_1(x) = 0\ \ \&\ \ \lim_{x\to\infty} y_2(x) = 0.$$

Option (a) tells $l < 0$ and $k > 0$. For $k = 1$ and $l = -8$, we have $y_1(x) = e^{2x}$ which can not tends to zero as $x \to \infty$. Similarly, we can not have Option (b). For instance, set $k = -1$ and $l = 1$.

For Option (c), write $k^2 - l = -d^2$ so that roots of the auxillary equation are $m = -k \pm id$. Then two L.I. solutions are

$$y_1 = e^{-kx}\cos dx, \quad y_2 = e^{-kx}\sin dx.$$

Now, apply boundedness of sine and cosine functions to conclude

$$\lim_{x\to\infty} y_1(x) = 0\ \ \&\ \ \lim_{x\to\infty} y_2(x) = 0.$$

For Option (d), we have

$$\lim_{x\to\infty} y_2(x) = \lim_{x\to\infty} e^{-kx} e^{-\sqrt{(k^2-l)}x} = 0.$$

For $y_1$, set $d^2 = k^2 - l$. Then observe $d - k < 0$ from $(d + k)(d - k) = -l < 0$ for $d,\ k,\ l > 0$. Hence $y_1(x) = e^{(d-k)x}$ tends to zero as $x \to \infty$. Option (c) and Option (d).

7. Let $y$ be the solution to the differential equation

$$y'' + 5y' + 6y = 0,\ y(0) = 1\ \ y'(0) = -1.$$

Then $y$ attains its maximum value at $x =$ ____.

**Solution:** Observe the auxillary equation as $m^2+5m+6 = 0$ and their roots $m_1 = -3,\ m_2 = -2$. The general solution is given by

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x},\ \ \&\ \ y'(x) = -3c_1 e^{-3x} - 2c_2 e^{-2x}.$$

Then apply given conditions $y(0) = 1\ \ \&\ \ y'(0) = -1.$ to have

$$c_1 + c_2 = 1, \quad 3c_1 + 2c_2 = 1.$$

So that $c_1 = -1$ and $c_2 = 2$. The solution is given by

$$y(x) = -e^{-3x} + 2e^{-2x},\ \ \&\ \ y'(x) = 3e^{-3x} - 4e^{-2x}.$$

For extremum

$$3e^{-3x} - 4e^{-2x} = 0\ \text{ or }\ e^x = \frac{3}{4}.$$

Hence, $x = \log \frac{3}{4}$.

8. Determine general solution of

$$x^2y'' - 5xy' + 9y = 0, \quad x > 0.$$

**Solution:** The equation is a Cauchy-Euler equation. Set $y = x^r$ to have

$$r(r-1) - 5r + 9 = 0.$$

So that we obtain $r_1 = 3 = r_2$. Hence, general solution is given by

$$y = c_1x^3 + c_2x^3 \log x.$$

9. If $y_1$ and $y_2$ are linearly independent solutions of

$$x^2y'' - 2xy' + (3+x)y = 0$$

and if $\mathcal{W}(y_1, y_2)(1) = 2$, find the value of $\mathcal{W}(y_1, y_2)(5)$.

**Solution:** For any two solutions $y_1$ and $y_2$, we have

$$\mathcal{W}(y_1, y_2)(x) = Ce^{\int -Pdx} = Ce^{\int \frac{2}{x}dx} = Ce^{2\log x} = Cx^2.$$

Now, use the fact that $\mathcal{W}(y_1, y_2)(1) = 2$ to have $C = 2$. Therefore, $\mathcal{W}(y_1, y_2)(x) = 2x^2$ and hence,

$$\mathcal{W}(y_1, y_2)(5) = 2 \times 5^2 = 50.$$

10. Let $y_1$ and $y_2$ be two linearly independent solutions of the differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad x \in [a, b], \ P, \ Q \in C[a, b],$$

and $x_0$ is a point in $(a, b)$. Then

(a) both $y_1$ and $y_2$ can't have local maximum at $x_0$

(b) both $y_1$ and $y_2$ can't have local minimum at $x_0$

(c) $y_1$ can't have a local maximum at $x_0$ and $y_2$ can't have local minimum at $x_0$ simultaneously

(d) both $y_1$ and $y_2$ can't vanish at $x_0$ simultaneously

**Solution:** For any two linearly independent solutions $y_1$ and $y_2$, we have

$$\mathcal{W}(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \ \ \forall x \in [a, b].$$

Therefore, all options are correct.

11. Let $S$ be the collection of all bounded solutions for the differential equation

$$y'' - 2y' - 3y = 0, \quad x > 0. \ \text{Then}$$

(a) $S$ is a real vector space of dimension 2

(b) $S = \text{span}\{xe^{-x} : \ x > 0\}$

(c) $S$ is a real vector space of dimension 1

(d) $S$ contains only trivial solution $y = 0$

**Solution:** The auxillary equation is given by

$$m^2 - 2m - 3 = 0.$$

So that roots are given by $m_1 = 3$ and $m_2 = -1$. Therefore, solution space is generated by the solutions $e^{3x}$ and $e^{-x}$. Since $e^{3x}$ is unbounded for $x > 0$. Thus, the space of bounded solution will be generated by $e^{-x}$ only. Hence, dimension of $S$ is one.

12. Find the partial differential equation arising from each of the following surfaces:

$(i)$ $u = f(x-y)$, $\quad (ii)$ $2u = (ax+y)^2 + b$.

**($x, y$: independent variables, $u$: dependent variable, $a, b$: arbitrary constants, $f$: arbitrary function.)**

**Solution:** $(i)$ $u_x + u_y = 0$.
(Differentiating w.r.t. $x$ gives $u_x = f'(x-y)$ and then w.r.t. $y$ gives $u_y = -f'(x-y)$. Eliminating $f'$ gives the PDE)
$(ii)$ $xu_x + yu_y = (u_y)^2$.
(Differentiating w.r.t. $x$ gives $u_x = a(ax+y)$ and then w.r.t. $y$ gives $u_y = ax+y$.
$\Rightarrow a = u_x/u_y \Rightarrow u_y = (u_x/u_y)x + y$ which gives the PDE.)

13. Consider the PDE $xu_x + yu_y = 4u$, where $x, y \in \mathbb{R}$. Find the characteristic curves for the equation.

**Solution:** The characteristic equations $x'(t) = x, \quad y'(t) = y, \quad u'(t) = 4u$ yield the solutions

$$x(t) = C_1 e^t, \quad y(t) = C_2 e^t, \quad u(t) = C_3 e^{4t} \;\; t \in I.$$