DIMENSIONAL ANALYSIS

- ***5.3 Results of dimensional analysis***
- The result of performing dimensional analysis on a physical problem is a single equation. This equation
- relates all of the physical factors involved to one another. This is probably best seen in an example.
- If we want to find the force on a propeller blade we must first decide what might influence this force.
- It would be reasonable to assume that the force, F, depends on the following physical properties:
- diameter, d
- forward velocity of the propeller (velocity of the plane), u
- fluid density, r
- revolutions per second, N
- fluid viscosity, m
- Before we do any analysis we can write this equation:
- F =f *(d, u,* r*, N,* m *)*

Dimensional Analysis

- Dimensional analysis is often used to group the variables in a given physical situation into *dimensionless parameters or numbers* which can be useful in experimentation and correlating data.

- Then a more general procedure is required which is known as the *Buckingham method*.

- In this method the listing of the important variables in the particular physical problem is done first.

- Then we determine the number of dimensionless parameters into which the variables may be combined by using the *Buckingham pi theorem*.

Buckingham theorem

- The ***Buckingham theorem*** states that
  - the functional relationship among ‘q’ quantities or variables whose units may be given in terms of ‘u’ fundamental units or dimensions may be written as (q – u) independent dimensionless groups, often called π’*s*.
  - each π group is a function of ‘u’ governing or repeating variables plus one of the remaining variables.

- Let us consider the following example, to illustrate the use of this method.

- *An incompressible fluid is flowing inside a circular tube of inside diameter D.*

- The significant variables are pressure drop Δp, velocity v, diameter D, tube length L, viscosity μ and density ρ.

- The total number of variables is q = 6.

- The fundamental units or dimensions are u = 3 and are mass M, length L, and time t

- Therefore, the number of dimensionless groups or π's is
- …..q – u = 6 – 3 = 3

Variable	*Units (SI)*	*Dimensions*
Δp	N/m^2	M/Lt^2
D	M	L
v	m/s	L/t
L	M	L
μ	kg/m-s	M/Lt
ρ	kg/m^3	M/L^3

Some rules which should be followed to choose CORE!!!

- From the 2nd theorem number of repeating variables = u

- When combined, these repeating variables variable must contain all of dimensions (M, L, T)
(That is not to say that each must contain M,L and T).

- A combination of the repeating variables must <u>not</u> form a dimensionless group.

- The repeating variables should be chosen to be measurable in an experimental investigation. They should be of major interest to the designer. For example, pipe diameter (dimension L) is more useful and measurable than roughness height (also dimension L).

- $\pi_1 = f(\pi_2, \pi_3)$
- Next, We must select, *a core group of u* (or 3) variables which will appear in each π group and among them contain all the fundamental dimensions.
- Also, no two of the variables selected for the core can have the same dimensions.
- In choosing the core, the variable whose effect one desires to isolate is often excluded (for example, Δp).

- This leaves us with the variables v, D, μ and ρ to be used. (L and D have the same dimensions)
- We will select *D, v and ρ to be the core variables* common to all three groups.
- Then three dimensionless groups are:

$$\pi_1 = D^a v^b \rho^c \Delta p^1$$

$$\pi_2 = D^d v^e \rho^f L^1$$

$$\pi_3 = D^g v^h \rho^i \mu^1$$

- To be dimensionless, the variables must be raised to certain exponents a,b,c etc.,

- First we consider the π_1 group

- $\pi_1 = D^a v^b \rho^c \Delta p^1$

- To evaluate these exponents, we write the above eqn with actual dimensions….

$$\boxed{M^0 L^0 t^0 = 1 = L^a \left(\frac{L}{t}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{Lt^2}\right)}$$

- Next we equate the exponents of L on both sides of this equation, of M, and finally of t….
- (L) 0 = a + b – 3c – 1
- (M) 0 = c + 1
- (t) 0 = – b – 2
- Solving these equations…..
- a = 0
- b = – 2
- c = – 1
- Sub. These values into π_1

$$\boxed{\pi_1 = \frac{\Delta P}{v^2 \rho} = Eu}$$

- Repeating the procedure for π_2 and π_3

$$\boxed{\pi_2 = \frac{L}{D}}$$

$$\boxed{\pi_3 = \frac{\mu}{Dv\rho} = \frac{1}{\mathrm{Re}}}$$

- Then by sub. these expressions…….

$$\boxed{\left(\frac{\Delta P}{v^2\rho}\right) = f\left(\frac{L}{D}, \frac{\mu}{Dv\rho}\right)}$$

$$\boxed{Eu = f\left(\frac{L}{D}, \frac{1}{\mathrm{Re}}\right)}$$

$$we\,know,\; f = \frac{\Delta P D}{2\rho v^2 L}$$

- Combining the *'friction factor'* with the left-hand side of the above eqn. shows that the ***friction factor is a function of the Reynolds number*** (as was shown before in the empirical correlation of friction factor and Reynolds number) and of length/diameter ratio.
- In pipes with L/D >> 1 or pipes with fully developed flow, the friction factor found to be independent of L/D.

- **Wrong choice of physical properties.**
- If, when defining the problem, extra - unimportant - variables are introduced then extra π groups will be formed.
- They will play very little role influencing the physical behaviour of the problem concerned and should be identified during experimental work.
- If an important / influential variable was missed then a π group would be missing.
- Experimental analysis based on these results may miss significant changes.
- <u>*It is therefore, very important that the initial choice of variables is carried out with great care.*</u>

DIMENSIONAL ANALYSIS IN HEAT TRANSFER

- As seen in many of the correlations for fluid flow and heat transfer, many dimensionless groups, such as the Reynolds number and Prandtl number, occur in the correlations.

- Dimensional analysis is often used to group the variables in a given physical situation into dimensionless parameters or numbers which can be useful in experimentation and correlating data.

- By Buckingham method, in which the listing of the significant variables in the particular physical problem is done first.

- Then we determine the number of dimensionless parameters into which the variables may be combined.

Heat transfer inside a pipe

- The Buckingham theorem states that the function relationship among ‘q’ quantities or variables whose units may be given in terms of ‘u’ fundamental units or dimensions may be written as (q – u) dimensionless groups.

- Let us consider a fluid flowing in turbulent flow at velocity ‘v’ inside a pipe of diameter D and undergoing heat transfer to the wall.

- We wish to predict the dimensionless groups relating the heat transfer coefficient ‘h’ to the variables D, ρ, μ, c_p, k, and v. The total number of variables q = 7.

- The fundamental units or dimensions are u = 4 and are mass M, length L, time t and temperature T.

- The units of the variables in terms of these fundamental units are as follows

Variable	Units (SI)	Dimensions
h	W/m^2K	M/t^3T
v	m/s	L/t
D	m	L
μ	kg/m-s	M/Lt
ρ	kg/m^3	M/L^3
Cp	J/kg K	L^2/t^2T
k	W/m K	ML/t^3T

- Therefore, the number of dimensionless groups or π's is $q - u = 7 - 4 = 3$

- We will choose the four variables *D, k, μ, and v* to be common to all the dimensionless groups.

- Then three dimensionless groups are:

$$\pi_1 = D^a k^b \mu^c v^d h$$

$$\pi_2 = D^e k^f \mu^g v^h c_p$$

$$\pi_3 = D^i k^j \mu^k v^l \rho$$

- To be dimensionless, the variables must be raised to certain exponents a,b,c etc.,

- First we consider the $\pi 1$ group

- $\pi_1 = D^a k^b \mu^c v^d h$

- To evaluate these exponents, we write the above eqn with actual dimensions….

$$\boxed{M^0 L^0 t^0 T^0 = 1 = L^a \left(\frac{ML}{t^3 T}\right)^b \left(\frac{M}{Lt}\right)^c \left(\frac{L}{t}\right)^d \left(\frac{M}{Tt^3}\right)}$$

- Next we equate the exponents of L on both sides of this equation, of M, and finally of t….
- (L) 0 = a + b – c + d
- (M) 0 = b + c + 1
- (t) 0 = – 3b – c – d – 3
- (T) 0 = – b – 1
- Solving these equations…..
- a = 1
- b = – 1
- c = 0
- d = 0
- Sub. These values into π_1

$$\boxed{\pi_1 = \frac{hD}{k} = Nu}$$

- Repeating the procedure for π_2 and π_3

$$\boxed{\pi_2 = \frac{c_p \mu}{k}} \qquad \boxed{\pi_3 = \frac{Dv\rho}{\mu} = \mathrm{Re}}$$

- Then by sub. These expressions…….

$$\boxed{\left(\frac{hD}{k}\right) = f\left(\frac{Dv\rho}{\mu}, \frac{c_p \mu}{k},\right)} \qquad \boxed{\Rightarrow Nu = f(\mathrm{Re}, \mathrm{Pr})}$$

- *This is in the form of the familiar equation for heat transfer inside pipes.*

Natural convection heat transfer outside a vertical plane

- In the case of natural convection heat transfer from a vertical plane wall of length L to an adjacent fluid,…..
- The buoyant force due to the difference in density between the cold and heated fluid should be a factor.
- As we know…..buoyancy force depends upon the variables β, g, ρ and ΔT.
- Hence, the list of to be Considered and their fundamental units are as follows:

Variable	**Units**	**Dimensions**
h	W/m^2K	M/t^3T
L	M	L
ΔT	K	T
μ	$kg/m\text{-}s$	M/Lt
ρ	kg/m^3	M/L^3
Cp	$J/kg\ K$	L^2/t^2T
k	$W/m\ K$	ML/t^3T
β	$1/K$	$1/T$
g	m/s^2	L/t^2

- The total number of variables q = 9.
- The fundamental units or dimensions are u = 4 and are mass M, length L, time t and temperature T.
- Since, u = 4…..the number of dimensionless groups or π's = 9 – 4 = 5
- $\pi_1 = f\,(\pi_2, \pi_3, \pi_4, \pi_5)$
- Choose *L, μ, k and g* to be core group

- Then 5 dimensionless groups are:

$$\pi_1 = L^a \mu^b k^c g^d \rho$$

$$\pi_2 = L^e \mu^f k^g g^h c_p$$

$$\pi_3 = L^i \mu^j k^k g^l \beta$$

$$\pi_4 = L^m \mu^n k^o g^p \Delta T$$

$$\pi_5 = L^q \mu^r k^s g^t h$$

- For π1, substituting the dimensions….

$$\boxed{M^0 L^0 t^0 T^0 = 1 = L^a \left(\frac{M}{Lt}\right)^b \left(\frac{ML}{t^3 T}\right)^c \left(\frac{L}{t^2}\right)^d \left(\frac{M}{L^3}\right)}$$

- Solving for the exponents…..
- a = 3/2; b = – 1 ; c = 0 and d = ½
- Then,

$$\boxed{\pi_1 = \frac{L^{3/2} \rho g^{1/2}}{\mu}}$$

$$squaring.........\pi_1 = \frac{L^3 \rho^2 g}{\mu^2}$$

$$\pi_2 = \frac{c_p \mu}{k} = \mathrm{Pr}$$

$$\pi_4 = \frac{k \Delta T}{L \mu g}$$

$$\pi_3 = \frac{L \mu g \beta}{k}$$

$$\pi_5 = \frac{hL}{k} = Nu$$

$$\pi_1\pi_3\pi_4 = \frac{L^{3/2}\rho g^{1/2}}{\mu}\frac{L\mu g\beta}{k}\frac{k\Delta T}{L\mu g} = \frac{L^3\rho^2 g\beta\Delta T}{\mu^2} = Gr$$

$$\boxed{\therefore Nu = f(Gr, \Pr)}$$