MA 201 (Part II), September-November, 2020 session

Partial Differential Equations

Solutions to PDE Tutorial, Date of Discussion: November 17, 2020

Separation of variables: one-dimensional wave equation

One-dimensional diffusion equation, two-dimensional steady-state equation

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1. A string is fixed at $x = 0$ and $x = L$ and lies initially along the $x-$axis. If it is set in motion by giving all points $0 < x < L$ a constant transverse velocity $\frac{\partial u}{\partial t} = u_0$ at $t = 0$, then find the subsequent motion of the string.
   Solution: Due to the string lying initially along x-axis, there is zero initial displacement implying $A_n = 0$. Hence the solution is given by
   $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$$
   where
   $$B_n = \begin{cases} 0, & \text{n even} \\ \frac{4u_0 L}{n^2\pi^2 c}, & \text{n odd.} \end{cases}$$
   Therefore
   $$u(x,t) = \frac{4u_0 L}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{L}\sin\frac{(2n-1)\pi ct}{L}.$$

2. A guitar string of length $L = 1$, is pulled in the middle so that it reaches a height h. Assuming the initial position of the string as
   $$u(x,0) = \begin{cases} 2hx, & 0 < x < 1/2, \\ 2h(1-x), & 1/2 \le x < 1, \end{cases}$$
   what is the subsequent motion of the string if it is suddenly released?
   Solution: Solution procedure is same as for the above problem.
   $$u(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)\pi x\ \cos(2n-1)\pi ct.$$

3. A metal bar of length 100 metre has its ends $x = 0$ and $x = 100$ kept at 0 degree Celsius. Initially half of the bar is at 60 degrees while the other half is at 40 degrees. Assuming a thermal diffusivity of 0.16 cgs units and that the surface of the bar is insulated, find the temperature everywhere in the bar at time t.
   Solution:
   $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{100} e^{-0.16\frac{n^2\pi^2 t}{10^4}}$$
   where
   $$B_n = \frac{1}{n\pi}(120 - 80\cos n\pi - 40\cos\frac{n\pi}{2}).$$

4. Solve the following IBVP with non-homogeneous BCs:

$$\begin{aligned}
&u_t = u_{xx},\ 0 < x < 1,\\
&u(0,t) = 0,\\
&u(1,t) = 1,\\
&u(x,0) = x^2.
\end{aligned}$$

Solution:

$$u(x,t) = x - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)\pi x\ \ e^{-(2n-1)^2\pi^2 t}.$$

5. Using Duhamel's principle, solve

$$\begin{aligned}
u_{tt} - u_{xx} &= x - t,\ -\infty < x < \infty, && (1)\\
u(x,0) &= 0, \quad u_t(x,0) = 0. && (2)
\end{aligned}$$

Solution: We first solve the related problem for $v(x,t,s)$.

$$\begin{aligned}
v_{tt} - v_{xx} &= 0, \quad -\infty < x < \infty, && (3)\\
v(x,0,s) &= 0, \quad v_t(x,0,s) = F(x,s) = x - s, \quad -\infty < x < \infty,\ s > 0. && (4)
\end{aligned}$$

For fixed $s > 0$, D'Alembert's solution is given by

$$\begin{aligned}
v(x,t,s) &= \frac{1}{2}\int_{x-t}^{x+t} F(\tau,s)d\tau = \frac{1}{2}\int_{x-t}^{x+t}(\tau - s)d\tau && (5)\\
&= \frac{1}{2}\Big[\frac{\tau^2}{2} - s\tau\Big]_{x-t}^{x+t} = xt - ts = t(x-s). && (6)
\end{aligned}$$

Thus

$$v(x, t-\tau, \tau) = (t-\tau)(x-\tau). \qquad (7)$$

Due to Duhamel's principle

$$\begin{aligned}
u(x,t) &= \int_0^t v(x, t-\tau, \tau)d\tau && (8)\\
&= \int_0^t (t-\tau)(x-\tau)d\tau = -\frac{t^3}{6} + \frac{t^2 x}{2}. && (9)
\end{aligned}$$

6. Find a solution $u(x,y)$ of the following steady-state heat conduction problem

$$\begin{aligned}
u_{xx} + u_{yy} &= 0, \quad 0 < x < \pi,\ 0 < y < 1,\\
u(x,0) &= 0, \quad 0 < x < \pi,\ \ u(x,1) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases},\\
u(0,y) &= u(\pi,y) = 0, 0 < y < 1.
\end{aligned}$$

Solution:

$$u(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\frac{\sinh(2n-1)y}{\sin h(2n-1)}\sin(2n-1)x.$$

7. Consider transient heat conduction in a circular region of radius a. Considering that heat conduction takes place only radially, find the solution of the transient heat conduction in the circular disk at any point r at any time $t > 0$
(i) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = 100$,
(ii) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = r$.
Solution:
Governing equation:

$$u_t = \alpha(u_{rr} + \frac{1}{r}u_r).$$

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a}r\right) e^{-\alpha\frac{\nu_n^2}{a^2}t},$$

where ν_n are zeros of $J_0(\lambda a)$, and

$$A_n = \frac{\int_0^a rf(r)J_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$

(i) $f(r) = 100$:

$$A_n = 100\frac{\int_0^a rJ_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$

(ii) $f(r) = r$:

$$A_n = \frac{\int_0^a r^2J_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$

8. Solve the following boundary value problem in a circular disk:

$$\begin{aligned} \nabla^2 u &= 0, r < a,\ 0 \le \theta < 2\pi, \\ u(a,\theta) &= 4 + 3\sin\theta, 0 \le \theta < 2\pi. \end{aligned}$$

Solution:

$$u(r,\theta) = 4 + \frac{3}{a}r\sin\theta.$$

(All coefficients for $\cos n\theta, n \neq 0$ will vanish due to the given BC and $a_0 = 8$. Also for the coefficients of $\sin\theta$, all will be zero for $n \neq 1$ and $b_1 = 3/a$.)
Recall the solution:

$$u(r,\theta) = a_0/2 + \sum_{n=1}^{\infty}[a_n\cos n\theta + b_n\sin n\theta]r^n,$$

with

$$\begin{aligned} a_n &= \frac{1}{\pi a^n}\int_0^{2\pi} u(a,\theta)\cos n\theta d\theta,\ n = 0, 1, 2, 3, \ldots \\ b_n &= \frac{1}{\pi a^n}\int_0^{2\pi} u(a,\theta)\sin n\theta d\theta,\ n = 1, 2, 3, \ldots. \end{aligned}$$