MA 201 (PART II), JULY-NOVEMBER, 2022 SESSION
PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS TO TUTORIAL SHEET 3, DATE OF DISCUSSION: OCTOBER 28, 2022

Fourier series

Lectures 9 and 10

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1. Find the Fourier series of the following functions:

$$(a) \quad f(x) = \begin{cases} -x, & -\pi \le x \le 0, \\ x, & 0 \le x \le \pi. \end{cases}$$

$$(b) \quad f(x) = |\sin x|, \quad -\pi < x < \pi.$$

Solution: (a) This is an even periodic function with period 2π. Thus, $B_n = 0 \ \forall n$.

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi}\int_0^{\pi} x dx = \pi, \\ A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{2}{\pi n^2}\Big| \cos nx \Big|_0^{\pi} = \frac{2}{\pi n^2}\{(-1)^n - 1\} \\ &= \begin{cases} 0 & \text{if } n \text{ is even}, \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

$[-L, L]$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

Thus, the Fourier series is given by

$$\begin{aligned} f(x) &\approx \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1} \frac{\cos(2n-1)x}{(2n-1)^2} \\ &= \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}. \end{aligned}$$

(b) Since $|\sin x|$ is an even function, we have $B_n = 0$ for $n = 1, 2, \ldots,$. Further,

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} \sin x\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x]dx \\ &= \frac{2}{\pi}\frac{[1+(-1)^n]}{(1-n^2)} \quad \text{for } n \ne 1 \\ &= \begin{cases} \frac{4}{\pi}\frac{1}{1-4n^2}, & \text{when } n \text{ is even}, \\ 0, & \text{when } n \text{ is odd}. \end{cases} \end{aligned}$$

Hence, the Fourier series of $f(x) = |\sin x|$ is

$$f(x) \approx \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{(1-4n^2)}.$$

2. Find the Fourier series expansion for the function $f(x)$ as given:

(a) $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$

(b) $f(x) = \begin{cases} -\pi/2, & -\pi < x < 0, \\ \pi/2, & 0 < x < \pi. \end{cases}$

(c) $f(x)$ is given by the line joining $(-\pi, 0)$ and $(0, 2)$ in $(-\pi, 0)$ and given by the line $f(x) = 2$ in $(0, \pi)$.

Solution:

(a) $A_0 = \pi/2,\ A_n = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\cos nx dx = \dfrac{(-1)^n - 1}{\pi n^2},$

$B_n = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\sin nx dx = \dfrac{(-1)^{n+1}}{n}.$

Hence $f(x) \approx \pi/4 + \displaystyle\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 1}{\pi n^2}\cos nx + \dfrac{(-1)^{n+1}}{n}\sin nx\right].$

(b) It is an odd function.

Hence $A_n = 0,\ B_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \dfrac{\pi}{2}\sin nx dx = \dfrac{2(1-\cos n\pi)}{n} = \begin{cases} \frac{2}{n}, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases}$

Hence, $f(x) \approx 2\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x}{2n-1}.$

(c) To find $f(x)$ for the interval $(-\pi, 0)$, use the fact that it is a straight line through $(-\pi, 0)$ and $(0, 2)$ to get

$$f(x) = \begin{cases} \frac{2}{\pi}(x+\pi), & -\pi < x < 0, \\ 2, & 0 < x < \pi. \end{cases}$$

$A_0 = 3,\ A_n = \dfrac{2}{(n\pi)^2}\{1 - (-1)^n\},\ B_n = \dfrac{2(-1)^{n+1}}{n\pi}.$

Hence, $f(x) \approx 3/2 + 2\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{\{1-(-1)^n\}}{(n\pi)^2}\cos nx + \dfrac{(-1)^{n+1}}{n\pi}\sin nx\right].$

3. For the following functions, find the Fourier cosine series and the Fourier sine series on the interval $0 < x < \pi$:

(a) $f(x) = 1$, (b) $f(x) = \pi - x$, (c) $f(x) = x^2$.

Solution:

Make an odd extension to get a sine series and an even extension to get a cosine series.

(a) cosine series: $f(x) = 1$, sine series: $f(x) \approx \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x}{2n-1}.$

(b) cosine series: $f(x) \approx \dfrac{\pi}{2} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2}$, sine series: $f(x) \approx 2\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin nx}{n}.$

(c) cosine series: $f(x) \approx \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n\cos nx}{n^2},$

sine series: $f(x) \approx 2\pi^2\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{(-1)^{n+1}}{n\pi} - 2\dfrac{(1-(-1)^n)}{(n\pi)^3}\right]\sin nx.$

4. Given the Fourier series for the function $f(x) = x^4$, $-\pi < x < \pi$, as

$$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos nx,$$

find the Fourier series for $f(x) = x^5$, $-\pi < x < \pi$.

Solution: Integrate the given series w.r.t. x to get

$$\frac{x^5}{5} = \frac{\pi^4}{5}x + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\frac{\sin nx}{n} + A.$$

Put $x = 0$ to obtain $A = 0$.

Hence, the series becomes

$$\frac{x^5}{5} = \frac{\pi^4}{5}x + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^5}(\pi^2 n^2 - 6)\sin nx.$$

Recall that the Fourier series for a function $g(x) = x$, $-\pi < x < \pi$ is given by

$$g(x) \approx 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Utilizing this, we get

$$x^5 = 2\pi^4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx + 40\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}(n^2\pi^2 - 6)\sin nx.$$

5. Deduce the Fourier series for the function $f(x) = e^{ax}$, $-\pi < x < \pi$, a a real number. Hence find the values of the four series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}$, (b) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a^2+n^2}$, (d) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2}$.

Solution:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{a\pi - \sinh a\pi}{2a^2 \sinh a\pi}$.

(b) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{1}{a^2} + \frac{2(a\pi - \sinh a\pi)}{2a^2 \sinh a\pi}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a^2+n^2} = \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{2a^2\sinh(a\pi)}$.

(d) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{1}{a^2} + \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{a^2\sinh(a\pi)}$.

Fourier series is as follows:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos nx + B_n \sin nx],$$

with

$$
\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}dx = \frac{e^{a\pi}-e^{-a\pi}}{a\pi} = \frac{2\sinh(a\pi)}{a\pi},\\
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx dx\\
&= \frac{1}{\pi}\left[\frac{e^{ax}\sin nx}{n}\Big|_{-\pi}^{\pi} - \frac{a}{n}\int_{-\pi}^{\pi} e^{ax}\sin nx dx\right.\\
&= \cdots\\
&= \frac{2a\cos n\pi\sinh(a\pi)}{\pi(a^2+n^2)},\\
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx dx\\
&= \frac{1}{\pi}\left[-\frac{e^{ax}\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{a}{n}\int_{-\pi}^{\pi} e^{ax}\cos nx dx\right.\\
&= \cdots\\
&= -\frac{2n\cos n\pi\sinh(a\pi)}{a^2+n^2}.
\end{aligned}
$$

Hence, Fourier series for the given function is given by

$$\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx)-n\sin(nx)]\right\}. \tag{1}$$

For part (a), put $x=0$, which is a point of continuity, to get the desired result.

For part (b), write

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2} = \sum_{n=-\infty}^{-1}\frac{(-1)^n}{a^2+n^2} + (\text{the term for } n=0) + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2},$$

change n to $-n$ in the first sum on RHS, simplify and use (a) to get the desired result.

Detailed solutions of (c) and (d):

For parts (c) and (d), we have to put $x=\pi$ in the above Fourier series. Observe that we can not directly put $x=\pi$ in above Fourier series due to convergence problem. For this, we define $g:[-\pi,\pi]\to\mathbb{R}$ by

$$g(-\pi)=g(\pi)=\frac{1}{2}\{f(-\pi^+)+f(\pi^-)\}=\frac{e^{-a\pi}+e^{a\pi}}{2}=\cosh(a\pi), g(x)=f(x), x\in(-\pi,\pi).$$

Clearly g is a piece-wise C^1 function in $[-\pi,\pi]$ and $g(-\pi)=g(\pi)$. Thus Fourier series of g converges to g in $[-\pi,\pi]$.

$$g(x)=\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx)-n\sin(nx)]\right\}. \tag{2}$$

Now putting $x=\pi$ we get

$$g(\pi)=\cosh(a\pi)=\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2a\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos(n\pi)\right\}. \tag{3}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{a^2+n^2} = \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{2a^2\sinh(a\pi)}.$$

Similar to part(b), from part (c) we obtain the result for (d) as

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{1}{a^2} + \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{a^2\sinh(a\pi)}.$$

6. Consider $f(x) = \sqrt{1-\cos x}$, $0 < x < 2\pi$.

   (a) Determine Fourier series expansion of f in $(0, 2\pi)$.

   (b) Does the limit of Fourier series exist at $x = 0$?

   (c) Use part (b) to find the series

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots.$$

$1-\cos x$
$= 2\sin^2\frac{x}{2}$

Solution: (a) We have $f(x) = \sqrt{2}\sin\frac{x}{2}$. Fourier series expansion is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}[A_n\cos nx + B_n\sin nx], \quad x \in (0, 2\pi)$$

with $L = \pi$ and

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad n = 0,\ 1,\ 2, \ldots \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx, \quad n = 1,\ 2, \ldots. \end{aligned}$$

We now calculate each of these to have

$$\begin{aligned} A_0 &= \frac{4\sqrt{2}}{\pi} \\ A_n &= -\frac{4\sqrt{2}}{\pi(4n^2-1)} \quad n = 1,\ 2, \ldots, \\ B_n &= 0, \quad n = 1,\ 2, \ldots. \end{aligned}$$

The desired Fourier series is given by

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)}\cos nx.$$

(b) For this part, we define $g : [0, 2\pi] \to \mathbb{R}$ by

$$g(0) = g(2\pi) = \frac{1}{2}\{f(0^+) + f(2\pi^-)\} = 0, \quad g(x) = f(x),\ x \in (0, 2\pi).$$

Clearly g is a piece-wise C^1 function in $[0, 2\pi]$ and $g(0) = g(2\pi)$. Thus Fourier series of g converges to g in $[0, 2\pi]$. Observe that Fourier series of g is given by

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)}\cos nx.$$

Thus, Fourier series of f in part (a) converges to $g(0) = 0$.

(c) Use part (b) to have

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{\pi(4n^2 - 1)} \cos 0 = 0,$$

which gives

$$\frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \frac{2\sqrt{2}}{\pi}.$$

We get

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \cdots = \frac{1}{2}.$$

7. Given the half-range sine series

$$x(\pi - x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}, \quad 0 \le x \le \pi,$$

use Parseval's theorem to deduce the value of the series $\sum_{n=1}^{\infty} 1/(2n-1)^6$.

Solution: $\sum_{n=1}^{\infty} 1/(2n-1)^6 = \frac{\pi^6}{960}$.

Realise that since this is a half-range sine series, we must have the given function as odd in the interval $(-\pi, \pi)$ so that $f(x)$ has the following representation in $(-\pi, \pi)$:

$$f(x) = \begin{cases} x(\pi + x), & -\pi < x < 0, \\ x(\pi - x), & 0 < x < \pi. \end{cases}$$

Parseval's Identity:

$$\frac{1}{L} \int_{-L}^{L} |f|^2 \, dt = \frac{A_0^2}{2} + \sum_{n=1}^{\infty} \left(A_n^2 + B_n^2\right)$$