MA 201 (PART II), SEPTEMBER-NOVEMBER, 2020 SESSION
PARTIAL DIFFERENTIAL EQUATIONS
Solutions to PDE Tutorial Problems - 3, Date of Discussion: 3rd of November, 2020

Second Order Linear PDE, Wave Equation, D'Alembert Solution
Duhamel's Principle, Fourier Series

1. Classify the following second-order partial differential equations:
(i) $u_{xx} + 4u_{xy} + 4u_{yy} - 12u_y + 7u = x^2 + y^2$; (ii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$
(iii) $(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0$; (iv) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$.

Solution: (i) Parabolic, (ii) Parabolic on the ellipse $\frac{x^2}{4} + y^2 = 1$, hyperbolic inside the ellipse and elliptic outside the ellipse, (iii) Hyperbolic, (iv) hyperbolic if $x \neq y$, parabolic for $x = y$.

2. Find D'Alembert solution of one-dimensional wave equation with the following initial conditions:
(a) $u(x, 0) = \sin x$, $u_t(x, 0) = 0$, (b) $u(x, 0) = \sin x$, $u_t(x, 0) = \cos x$.

Solution:
(a) $u(x, t) = \sin x \cos ct$, (b) $u(x, t) = \sin x \cos ct + \frac{1}{c} \sin ct \cos x$.

3. A string stretching to infinity in both directions is given the initial displacement

$$\phi(x) = \frac{1}{1 + 4x^2}$$

and released from rest. Find its subsequent motion as a function of x and t.

Solution: Recall D'Alembert solution for one-dimensional wave equation. Here initial displacement $u(x, 0) = \phi(x) = \frac{1}{1 + 4x^2}$ and initial velocity $u_t(x, 0) = \psi(x) = 0$. The required expression for $u(x, t)$ is

$$u(x, t) = \frac{1 + 4(x^2 + c^2t^2)}{[1 + 4(x + ct)^2][1 + 4(x - ct)^2]}$$

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4. Using Duhamel's principle for an infinite string problem, solve

$$\begin{aligned} u_{tt} - u_{xx} &= x - t, \ -\infty < x < \infty, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0. \end{aligned}$$

Solution: We first solve the related problem for $v(x, t, \xi)$.

$$\begin{aligned} v_{tt} - v_{xx} &= 0, \ -\infty < x < \infty, \\ v(x, 0, \xi) &= 0, \ v_t(x, 0, \xi) = F(x, \xi) = x - \xi, \ -\infty < x < \infty, \ \xi > 0. \end{aligned}$$

For fixed $\xi > 0$, D'Alembert's solution is given by

$$\begin{aligned} v(x, t, \xi) &= \frac{1}{2}\int_{x-t}^{x+t} F(\tau, \xi)d\tau = \frac{1}{2}\int_{x-t}^{x+t} (\tau - \xi)d\tau \\ &= \frac{1}{2}\Big[\frac{\tau^2}{2} - \xi\tau\Big]_{x-t}^{x+t} = xt - t\xi = t(x - \xi). \end{aligned}$$

Thus

$$v(x, t-\tau, \tau) = (t-\tau)(x-\tau).$$

Due to Duhamel's principle

$$\begin{aligned} u(x,t) &= \int_0^t v(x, t-\tau, \tau) d\tau \\ &= \int_0^t (t-\tau)(x-\tau) d\tau = -\frac{t^3}{6} + \frac{t^2 x}{2}. \end{aligned}$$

5. Find the Fourier series of the following functions:

$$\begin{aligned} (a) \qquad & f(x) = \begin{cases} -x, & -\pi \le x \le 0, \\ x, & 0 \le x \le \pi. \end{cases} \\ (b) \qquad & f(x) = |\sin x|, \quad -\pi < x < \pi. \end{aligned}$$

Solution: (a) This is an even periodic function with period 2π. Thus, $B_n = 0 \ \forall n$.

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \pi \\ A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{\pi n^2} \left| \cos nx \right|_0^{\pi} = \frac{2}{\pi n^2} \{(-1)^n - 1\} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right\}.$$

(b) Since $|\sin x|$ is an even function, we have $B_n = 0$ for $n = 1, 2, \ldots,$. Further,

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos(nx) dx \\ &= \frac{1}{\pi} \int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] dx \\ &= \frac{2}{\pi} \frac{[1 + (-1)^n]}{(1-n^2)} \quad \text{for } n = 0, 2, 3, \ldots. \end{aligned}$$

For $n = 1$, $A_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x dx = 0$. Hence, the Fourier series of $f(x) = |\sin x|$ is

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{(1-4n^2)}.$$

6. For the following functions, find the Fourier cosine series and the Fourier sine series on the interval $0 < x < \pi$:
(a) $f(x) = 1$, (b) $f(x) = \pi - x$, (c) $f(x) = x^2$.
Solution:

(a) cosine series: $f(x) = 1$, sine series: $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$

(b) cosine series: $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$, sine series: $f(x) = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

(c) cosine series: $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$,

sine series: $f(x) = 2\pi^2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n\pi} - 2\frac{(1-(-1)^n)}{(n\pi)^3}\right]\sin nx$

7. Given the Fourier series for the function $f(t) = t^4$, $-\pi < t < \pi$, as

$$\frac{\pi^4}{5} + \sum_{n=1}^{\infty}\frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos nt.$$

Find the Fourier series for $f(t) = t^5$, $-\pi < t < \pi$.

Solution:

$$t^5 = 2\pi^4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt + 40\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}(n^2\pi^2 - 6)\sin nt.$$

(Integrate the given series. That will give you a sine series along with another t with the constant $\frac{\pi^4}{5}$. Then find the Fourier series for $g(t)$ as

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt.$$

Putting this back into the series for t^5, you get the desired result.)

8. Deduce the Fourier series for the function $f(x) = e^{ax}$, $-\pi < x < \pi$, a a real number. Hence find the values of the four series:

$$\text{(a)} \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}, \quad \text{(b)} \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}, \quad \text{(c)} \sum_{n=1}^{\infty}\frac{1}{a^2+n^2}, \quad \text{(d)} \sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2}.$$

Solution:

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2} = \frac{a\pi - \sinh a\pi}{2a^2\sinh a\pi}$.

(b) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2} = \frac{1}{a^2} + \frac{2(a\pi - \sinh a\pi)}{2a^2\sinh a\pi}$.

(c) $\sum_{n=1}^{\infty}\frac{1}{a^2+n^2} = \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{2a^2\sinh(a\pi)}$.

(d) $\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2} = \frac{1}{a^2} + \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{a^2\sinh(a\pi)}$.

Fourier series of given function $f(x)$ is

$$\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx) - n\sin(nx)]\right\}. \tag{1}$$

For part (a), put $x = 0$, which is a point of continuity, to get the desired result.

For part (b), write

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2} = \sum_{n=-\infty}^{-1}\frac{(-1)^n}{a^2+n^2} + (\text{the term for } n=0) + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2},$$

change n to $-n$ in the first sum on RHS, simplify and use (a) to get the desired result.

Detailed solutions of (c) and (d):

For parts (c) and (d), we have to put $x = \pi$ in the above Fourier series. Observe that we can not directly put $x = \pi$ in above Fourier series due to convergence problem. For this, we define $g : [-\pi, \pi] \to \mathbb{R}$ by

$$g(-\pi) = g(\pi) = \frac{1}{2}\{f(-\pi^+) + f(\pi^-)\} = \frac{e^{-a\pi} + e^{a\pi}}{2} = \cosh(a\pi), g(x) = f(x), x \in (-\pi, \pi).$$

Clearly g is a piece-wise C^1 function in $[-\pi, \pi]$ and $g(-\pi) = g(\pi)$. Thus Fourier series of g converges to g in $[-\pi, \pi]$.

$$g(x) = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx) - n\sin(nx)]\right\}. \tag{2}$$

Now putting $x = \pi$ we get

$$g(\pi) = \cosh(a\pi) = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2a\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos(n\pi)\right\}. \tag{3}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{a^2+n^2} = \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{2a^2\sinh(a\pi)}.$$

Similar to part(b), from part (c) we obtain the result for (d) as

$$\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2} = \frac{1}{a^2} + \frac{a\pi\cosh(a\pi) - \sinh(\pi a)}{a^2\sinh(a\pi)}.$$

9. Consider $f(x) = \sqrt{1-\cos x},\ 0 < x < 2\pi$.

   (a) Determine Fourier series expansion of f in $(0, 2\pi)$.

   (b) Does the limit of Fourier series exist at $x = 0$?

   (c) Use part (b) to find the series

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots.$$

Solution: (a) We have $f(x) = \sqrt{2}\sin\frac{x}{2}$. Fourier series expansion is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\},\quad x \in (0, 2\pi)$$

with $L = \pi$ and

$$\begin{aligned} a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx,\quad n = 0,\ 1,\ 2, \ldots \\ b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx,\quad n = 1,\ 2, \ldots. \end{aligned}$$

We now calculate each of these to have

$$\begin{aligned} a_0 &= \frac{4\sqrt{2}}{\pi} \\ a_n &= -\frac{4\sqrt{2}}{\pi(4n^2-1)}\quad n = 1,\ 2, \ldots, \\ b_n &= 0,\quad n = 1,\ 2, \ldots. \end{aligned}$$

The desired Fourier series is given by

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{\pi(4n^2-1)} \cos nx.$$

(b) For this part, we define $g : [0, 2\pi] \to \mathbb{R}$ by

$$g(0) = g(2\pi) = \frac{1}{2}\{f(0^+) + f(2\pi^-)\} = 0, \;\; g(x) = f(x), \; x \in (0, 2\pi).$$

Clearly g is a piece-wise C^1 function in $[0, 2\pi]$ and $g(0) = g(2\pi)$. Thus Fourier series of g converges to g in $[0, 2\pi]$. Observe that Fourier series of g is given by

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{\pi(4n^2-1)} \cos nx.$$

Thus, Fourier series of f in part (a) converges to $g(0) = 0$.

(c) Use part (b) to have

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{\pi(4n^2-1)} \cos 0 = 0,$$

which gives

$$\frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \frac{2\sqrt{2}}{\pi}.$$