MA 201 (PART II), JULY-NOVEMBER, 2022 SESSION
PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS TO TUTORIAL SHEET - 1, DATE OF DISCUSSION: OCTOBER 14, 2022

Derivation of PDEs, General integrals, Cauchy problems,
Integral surface through given curves, Orthogonal surfaces

Lectures 1–5

1. Find the partial differential equation arising from each of the following surfaces:
   (*i*) $u = f(x - y)$, (*ii*) $2u = (ax + y)^2 + b$, (*iii*) $f(x^2 + y^2, x^2 - u^2) = 0$.

   (x, y: independent variables, u: dependent variable, a, b: arbitrary constants, f: arbitrary function.)

   Solution: (*i*) $u_x + u_y = 0$.
   (Differentiating w.r.t. x gives $u_x = f'(x - y)$ and then w.r.t. y gives $u_y = -f'(x - y)$. Eliminating f' gives the PDE)
   (*ii*) $xu_x + yu_y = (u_y)^2$.
   (Differentiating w.r.t. x gives $u_x = a(ax + y)$ and then w.r.t. y gives $u_y = ax + y$.
   $\Rightarrow a = u_x/u_y \Rightarrow u_y = (u_x/u_y)x + y$ which gives the PDE.)

   (*iii*) $u_x y - u_y x = \dfrac{xy}{u}$.
   (Take $x^2 + y^2 = u_1, x^2 - u^2 = u_2$). With $u_x = p$ and $u_y = q$, we obtain

$$\begin{aligned}
\frac{\partial f}{\partial u_1} \cdot \left\{ \frac{\partial u_1}{\partial x} + p\frac{\partial u_1}{\partial u} \right\} + \frac{\partial f}{\partial u_2} \left\{ \frac{\partial u_2}{\partial x} + p\frac{\partial u_2}{\partial u} \right\} &= 0, \\
\frac{\partial f}{\partial u_1} \left\{ \frac{\partial u_1}{\partial y} + q\frac{\partial u_1}{\partial u} \right\} + \frac{\partial f}{\partial u_2} \left\{ \frac{\partial u_2}{\partial y} + q\frac{\partial u_2}{\partial u} \right\} &= 0.
\end{aligned}$$

   (Find the partial derivatives as:
   $\frac{\partial u_1}{\partial x} = 2x$, $\frac{\partial u_1}{\partial y} = 2y$, $\frac{\partial u_2}{\partial x} = 2x$, $\frac{\partial u_2}{\partial y} = 0$, $\frac{\partial u_1}{\partial u} = 0$, $\frac{\partial u_2}{\partial u} = -2u$.
   Eliminate the partial derivatives of f by considering

$$\begin{vmatrix} 2x & (2x - 2up) \\ 2y & -2uq \end{vmatrix} = 0$$

   which gives the PDE.)

2. Is it must that a well-posed initial value problem possesses a unique solution?

   Solution: Yes. A well-posed initial value problem must possess a unique solution. It is needed in the context of real word applications.

   In general, this is important not only for real world applications; uniqueness hugely simplifies the problems. Say, that you have to solve for your abstract mathematics problem: a complex PDE. If you can "guess" the solution, and it works, then it works! You are done - that is the only solution, and you can go forward. If there were multiple solutions, you would probably need to check if the solution you found is "the right one" for the problem at hand. It is useful in all sorts of context.

3. What is a characteristic curve? How does a PDE with two independent variables become an ODE along the characteristic curve.

Solution: Characteristic curve of a first-order linear PDE is an integral curve in the solution space for which the direction of any tangent to this curve is parallel to the direction of the derivative of the dependent variable when it is expressed in the form of directional derivative.

If the PDE has two independent variables, then based on these variables, one can find characteristic curves that span the solution space. Therefore, a new single variable can be formed as a function of the independent variables that represent the characteristic curve. Subsequently, the whole solution space can be reproduced by the new single variable and the PDE can also be rewritten in terms of this new variable only. This converts the PDE into an ODE of this new variable and along the characteristic curve, the new variable remains constant.

4. Find the general integral of the following partial differential equations, where $u_x = p$ and $u_y = q$.
(*i*) $x^2p + y^2q + u^2 = 0$, (*ii*) $x^2(y-u)p + y^2(u-x)q = u^2(x-y)$.

Solution:
(*i*) $F(1/x - 1/y, 1/y + 1/u) = 0$.
(Write the auxiliary equations as $dx/x^2 = dy/y^2 = du/(-u^2)$. Take the first two fractions to get $1/x - 1/y = c_1$ and take the second and third fractions to get $1/y + 1/u = c_2$. [You can also take the first and third fractions.])
(*ii*) $F(1/x + 1/y + 1/u, xyu) = 0$.
(**Step 1:** Write the auxiliary equations as $\dfrac{dx/x^2}{y-u} = \dfrac{dy/y^2}{u-x} = \dfrac{du/u^2}{x-y}$. Then rewrite them as $\dfrac{(dx/x^2)+(dy/y^2)}{y-x} = \dfrac{du/u^2}{x-y}$ which gives $\dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{du}{u^2} = 0 \Rightarrow 1/x + 1/y + 1/u = c_1$.
Step 2: Rewrite the auxiliary equations as $\dfrac{dx/x}{x(y-u)} = \dfrac{dy/y}{y(u-x)} = \dfrac{du/u}{u(x-y)}$. Then $\dfrac{(dx/x)+(dy/y)}{u(y-x)} = \dfrac{du/u}{u(x-y)}$ which gives $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{du}{u} = 0 \Rightarrow \log xyu = \log c_2$.)
These steps give us the solution.

5. Show that the integral surface of the equation $2y(u-3)p + (2x-u)q = y(2x-3)$ that passes through the circle $x^2 + y^2 = 2x$, $u = 0$ is $x^2 + y^2 - u^2 - 2x + 4u = 0$.

Solution: The auxiliary equations are

$$\frac{dx}{2y(u-3)} = \frac{dy}{2x-u} = \frac{du}{y(2x-3)}.$$

Take the first and third fractions to get the first curve as $x^2 - u^2 - 3x + 6u = C_1$. Then use multipliers $1, 2y, 2$ to write

$$\frac{dx + 2ydy}{2y(u-3) + 2y(2x-u)} = 2\frac{du}{4xy - 6y}.$$

This gives $x + y^2 - 2u = c_2$ as the second curve. Using the given conditions, $x^2 - 3x = c_1$, $x + y^2 = c_2$ adding which we get $x^2 + y^2 - 2x = c_1 + c_2$ and finally the relation $c_1 + c_2 = 0$ which will give the required surface

$$x^2 + y^2 - u^2 - 2x + 4u = 0.$$

6. Find the solution of the following Cauchy problems:

   (*i*) $u_x + u_y = 2, \quad u(x,0) = x^2$; (*ii*) $5u_x + 2u_y = 0, \quad u(x,0) = \sin x$.

   Solution:(*i*) The characteristics equations are: $\frac{dx(t,s)}{dt} = 1, \quad \frac{dy(t,s)}{dt} = 1, \quad \frac{du(t,s)}{dt} = 2$, whose solutions are given by

   $$x(t,s) = t + C_1(s), \quad y(t,s) = t + C_2(s), \quad u(t,s) = 2t + C_3(s).$$

   Using the parametric initial conditions $x(0,s) = s, \ y(0,s) = 0, \ u(0,s) = s^2$, we obtain

   $$x(t,s) = t + s, \quad y(t,s) = t, \quad u(t,s) = 2t + s^2.$$

   Now, writing (t,s) as functions of (x,y), we have $t = y, \quad s = x - y$. Thus, the integral surface is given by

   $$U(x,y) = u(t(x,y), s(x,y)) = 2y + (x-y)^2.$$

   (*ii*) The characteristics equations are: $\frac{dx(t,s)}{dt} = 5, \quad \frac{dy(t,s)}{dt} = 2, \quad \frac{du(t,s)}{dt} = 0$, whose solutions are given by

   $$x(t,s) = 5t + C_1(s), \quad y(t,s) = 2t + C_2(s), \quad u(t,s) = C_3(s).$$

   Using the parametric initial conditions $x(0,s) = s, \ y(0,s) = 0, \ u(0,s) = \sin s$, we obtain

   $$x(t,s) = 5t + s, \quad y(t,s) = 2t, \quad u(t,s) = \sin s.$$

   We get $x = \dfrac{5y}{2} + s \ \Rightarrow s = x - \dfrac{5y}{2}$.
   Hence, the integral surface is given by $U(x,y) = \sin(x - \frac{5}{2}y)$.

7. Show that the Cauchy problem $u_x + u_y = 1, \quad u(x,x) = x$ has infinitely many solutions.

   Solution: Note that the transversality condition is violated, i.e., the Jacobian $J = 0$. Further, the initial curve is a characteristic curve. Therefore, it has infinitely many solutions.

8. Consider the PDE $xu_x + yu_y = 4u$, where $x, y \in \mathbb{R}$. Find the characteristic curves for the equation and determine an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.

   Solution: The characteristic equations $\frac{dx(t,s)}{dt} = x, \quad \frac{dx(t,s)}{dt} = y, \quad \frac{dx(t,s)}{dt} = 4u$ yield the solutions

   $$x(t,s) = C_1(s)e^t, \quad y(t,s) = C_2(s)e^t, \quad u(t,s) = C_3(s)e^{4t}.$$

   The characteristics curves are given by $\frac{x}{y} = C$. With $x_0(s) = s, \ y_0(s) = (1-s^2)^{1/2}, \ u_0(s) = 1$, it now follows that $U(x,y) = u(t(x,y), s(x,y)) = (x^2+y^2)^2$, which is the required integral surface.

9. Find a function $u(x,y)$ that solves the Cauchy problem

   $$x^2 u_x + y^2 u_y = u^2, \quad u(x, 2x) = x^2, \quad x \in \mathbb{R}.$$

   Is the solution defined for all x and y? Check whether the transversality condition holds.

   Solution: Solving the characteristic equations: $\frac{dx(t,s)}{dt} = x^2, \quad \frac{dy(t,s)}{dt} = y^2, \quad \frac{du(t,s)}{dt} = u^2$, we get

   $$-\frac{1}{x} = t + C_1(s), \quad -\frac{1}{y} = t + C_2(s), \quad -\frac{1}{u} = t + C_3(s).$$

   Using the parametric initial conditions $x(0,s) = s, \ y(0,s) = 2s, \ u(0,s) = s^2$, we obtain

   $$\frac{1}{x} = \frac{1}{s} - t, \quad \frac{1}{y} = \frac{1}{2s} - t, \quad \frac{1}{u} = \frac{1}{s^2} - t.$$