MA 201
COMPLEX ANALYSIS
ASSIGNMENT–3

(1) Show that $\displaystyle\int_\gamma \frac{e^{az}}{z^2+1}dz = 2\pi i \sin a$, where $\gamma(t) = 2e^{it},\ t \in [0, 2\pi]$.

Answer:

$$\int_\gamma \frac{e^{az}}{z^2+1}dz = \int_\gamma e^{az}\frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz = \frac{1}{2i}\left[\int_\gamma \frac{e^{az}}{z-i}dz - \int_\gamma \frac{e^{az}}{z+i}dz\right]$$

By Cauchy integral formula,

$$\frac{1}{2i}\left[\int_\gamma \frac{e^{az}}{z-i}dz - \int_\gamma \frac{e^{az}}{z+i}dz\right] = 2\pi i \times \frac{1}{2i}[e^{ia} - e^{-ia}] = 2\pi i \sin a.$$

(2) Evaluate $\displaystyle\int_0^{2\pi} e^{e^{i\theta}} d\theta$.

Answer: Put $e^{i\theta} = z$. Then $d\theta = \frac{dz}{iz}$. So

$$\int_0^{2\pi} e^{e^{i\theta}} d\theta = \int_{|z|=1} e^z \frac{dz}{iz} = 2\pi \text{ (by Cauchy integral formula)}.$$

(3) Let f be an entire function such that $\displaystyle\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = 0$. Show that f is constant.

Answer: Given that $\displaystyle\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = 0$. for every $\epsilon > 0$ there exists a $M > 0$ such that $\left|\frac{f(z)}{z}\right| < \epsilon$ whenever $|z| > M$. i.e

$$|f(z)| < \epsilon|z| \text{ whenever } |z| > M \Longrightarrow f(z) = az + b$$

for some $a, b \in \mathbb{C}$. But

$$\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = \lim_{z\to\infty}\left|\frac{az+b}{z}\right| = \lim_{z\to\infty}\left|a + \frac{b}{z}\right| = 0.$$

So $a = 0$ and hence f is constant.

(4) Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\mathbb{C} \setminus \{0\}$ and bounded on $B(0, \frac{1}{2})$. Show that $\int_{|z|=R} f(z)dz = 0$ for all $R > 0$.

Answer: By deformation theorem, $\int_{|z|=R} f(z)dz = \int_{|z|=r} f(z)dz$ for every $r > 0$. Take $0 < r < \frac{1}{2}$. Given that f is bounded (by M say) on $B(0, \frac{1}{2})$, then by ML inequality

$$\left|\int_{|z|=r} f(z)dz\right| \leq M(2\pi r) \to 0 \text{ as } r \to 0.$$

(5) Show that an entire function satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$ is a constant.

Answer: It follows from the hypothesis that

$$f(z) = f(z+n) = f(z+im), \quad \text{for all} \quad z \in \mathbb{C}, \quad \text{and for all} \quad n, m \in \mathbb{Z}.$$

Let S be the rectangle with vertices $0, 1, 1+i$ and i. For any $z = x+iy \in \mathbb{C}$, there exits integers n and m and $z_0 = x_0 + iy_0 \in S$ such that,

$$z = x + iy = x_0 + n + i(y_0 + m) = z_0 + n + im.$$

This implies that $f(z) = f(z_0)$. In particular $f(\mathbb{C}) = f(S)$. Since S is a compact set and f is a continuous function then $f(S)$ must be a bounded set. All together implies that f is a bounded entire function. By Liouville's theorem we get f is a constant function.

(6) Let $g(z)$ be an analytic in $B(0,2)$. Compute $\int_{|z|=1} f(z)dz$ if

$$f(z) = \frac{a_k}{z^k} + \cdots + \frac{a_1}{z} + a_0 + g(z)$$

where a_i's are complex constants.

Answer: Since $a_0 + g(z)$ is analytic by cauchy's theorem $\int_{|z|=1} [a_0 + g(z)]dz = 0$. Again $\frac{a_k}{z^k}$ has an antiderivative for $k \neq 1$ therefore $\int_{|z|=1} \left[\frac{a_k}{z^k} + \cdots + \frac{a_2}{z^2}\right] dz = 0$. Therefore

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{a_1}{z}dz = 2\pi i \times a_1.$$

(7) Let f be an entire function such that $|f(0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Then either $f(0) = 0$ or f is constant.

Answer: If $f(0) = 0$ then the proof is trivial. If $f(0) \neq 0$ then $\left|\frac{1}{f(z)}\right| \leq \left|\frac{1}{f(0)}\right|$. So $\frac{1}{f(z)}$ is entire and bounded and by Liouville's theorem f is constant.

(8) Find the radius of convergence of the following power series:

(a) $\sum_{n\geq 0} z^{n!}$ (**R=1**)

(b) $\sum_{n\geq 0} 2^{n^2} z^n$ (**R=0**)

(c) $\sum_{n\geq 0} \frac{(-1)^n}{n} z^{n(n+1)}$ (**R=1**)

(d) $\sum_{n\geq 0} a_n z^n$ where $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 3^n & \text{if } n \text{ is even.} \end{cases}$ $(\mathbf{R} = \mathbf{\tfrac{1}{3}})$

(9) Find the power series expansion of the following functions about the point $z_0 = 0$ and find its radius of convergence
(i) $f(z) = \cos^2 z$ (ii) $f(z) = \sinh^2 z$ (iii) $f(z) = \mathrm{Log}\,(1+z)$
(iv) $f(z) = \sqrt{z+2i}$
(v) $f(z) = \displaystyle\int_0^z \exp(w^2)\,dw$

Answer:

(i) We know that $f(z) = \cos^2 z = (1+\cos 2z)/2$ and $\cos z = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{2n!}$ for $z \in \mathbb{C}$. Therefore,

$$f(z) = \cos^2 z = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n}}{2n!} \qquad \text{for } z \in \mathbb{C}\ .$$

The radius of convergence of this series is $R = \infty$.

(ii) We know that $f(z) = \sinh^2 z = (\cosh(2z) - 1)/2$ and $\cosh(z) = \dfrac{e^z + e^{-z}}{2} = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}$ for $z \in \mathbb{C}$. Therefore,

$$f(z) = \sinh^2 z = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2z)^{2n}}{2n!} \qquad \text{for } z \in \mathbb{C}\ .$$

The radius of convergence of this series is $R = \infty$.

(iii) $f(z) = \mathrm{Log}\,(1+z)$

$$\begin{aligned}
f(z) &= \mathrm{Log}\,(1+z) &&\implies f(0) = 0\\
f'(z) &= \frac{1}{(1+z)} &&\implies f'(0) = 1\\
f''(z) &= \frac{(-1)}{(1+z)^2} &&\implies f''(0) = -1\\
f^{(n)}(z) &= \frac{(-1)^{n-1}(n-1)!}{(1+z)^n} &&\implies f^{(n)}(0) = (-1)^{n-1}(n-1)!\\
a_n &= \frac{f^{(n)}(0)}{n!} &&\implies a_n = \frac{(-1)^{n-1}}{n} \quad \text{for } n \geq 1\ .
\end{aligned}$$

This gives that

$$f(z) = \mathrm{Log}\,(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, z^n \qquad \text{for } |z| < 1\ .$$

The radius of convergence of this series is $R = 1$.

(iv) We know that $f(z) = \sqrt{z+2i} = \sqrt{2i}\ \left(1+\dfrac{z}{2i}\right)^{\frac{1}{2}}$.
Using the binomial series, for $|z/2i| < 1$, we can expand

$$\left(1+\frac{z}{2i}\right)^{\frac{1}{2}} = 1+\frac{1}{2}\left(\frac{z}{2i}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}\left(\frac{z}{2i}\right)^2+\cdots+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{(3-2n)}{2}\right)}{n!}\left(\frac{z}{2i}\right)^n+\cdots$$

$$\begin{aligned}
\sqrt{2i}\ \left(1+\frac{z}{2i}\right)^{\frac{1}{2}} &= \sqrt{2i}\ \left[1+\frac{1}{2}\left(\frac{z}{2i}\right)+\sum_{n=2}^{\infty}\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\left(\frac{(3-2n)}{2}\right)}{n!}\left(\frac{z}{2i}\right)^n\right] \\
&= \sqrt{2i}\ \left[1+\frac{1}{2}\left(\frac{z}{2i}\right)+\sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{2}\right)^n\frac{1\cdot 3\cdots(2n-3)}{n!}\left(\frac{z}{2i}\right)^n\right] \\
&= \sqrt{2i}\ \left[1+\frac{1}{2}\left(\frac{z}{2i}\right)+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n)}\left(\frac{z}{2i}\right)^n\right]
\end{aligned}$$

converges for $\left|\dfrac{z}{2i}\right| < 1$. That is, the series converges for $|z| < 2$. The radius of convergence of this series is $R = 2$.

(v) We know that $e^{w^2} = \displaystyle\sum_{n=0}^{\infty}\frac{w^{2n}}{n!}$ for all $w \in \mathbb{C}$ and this series converges uniformly in the closed disk $\{z \in \mathbb{C} : |z| \leq r\}$ for every $r > 0$. Therefore, the term by term integration is valid and it gives that

$$\begin{aligned}
f(z) = \int_0^z \exp(w^2)\,dw &= \int_0^z\left(\sum_{n=0}^{\infty}\frac{w^{2n}}{n!}\right)dw \\
&= \sum_{n=0}^{\infty}\left(\int_0^z\frac{w^{2n}}{n!}\,dw\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{(n!)\ (2n+1)}\,z^{2n+1} \qquad \text{for all } z \in \mathbb{C}
\end{aligned}$$

The radius of convergence of this series is $R = \infty$.

(10) Find the Taylor series for the function $\dfrac{1}{z}$ about the point $z_0 = 2$. Then, by differentiating that series term by term, show that $\dfrac{1}{z^2} = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}(-1)^n\,(n+1)\left(\frac{z-2}{2}\right)^n$ for $|z-2| < 2$.

Answer:

We know that $f(z) = 1/z$ is analytic in $\mathbb{C}\setminus\{0\}$. Therefore, we can expand f

into a power series about the point $z_0 = 2$ using the Taylors theorem.

$$
\begin{aligned}
f(z) &= \frac{1}{z} &\Longrightarrow\quad & f(2) = \frac{1}{2} \\
f'(z) &= \frac{-1}{(z)^2} &\Longrightarrow\quad & f'(2) = \frac{(-1)}{2^2} \\
f''(z) &= \frac{(-1)^2 1\cdot 2}{(z)^3} &\Longrightarrow\quad & f''(2) = \frac{2}{2^3} \\
f^{(n)}(z) &= \frac{(-1)^n (n)!}{(z)^{n+1}} &\Longrightarrow\quad & f^{(n)}(2) = \frac{(-1)^n (n)!}{2^{n+1}} \\
a_n &= \frac{f^{(n)}(2)}{n!} &\Longrightarrow\quad & a_n = \frac{(-1)^n}{2^{n+1}} \quad \text{for } n \geq 0 \; .
\end{aligned}
$$

This gives that

$$
f(z) = \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^n \qquad \text{for } |z-2| < 2 \; .
$$

Differentiating the above series term by term, we get

$$
\begin{aligned}
f'(z) = \frac{-1}{z^2} \quad &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} n(z-2)^{n-1} \qquad \text{for } |z-2| < 2 \\
\frac{1}{z^2} \quad &= \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n+1}\, n \left(\frac{z-2}{2}\right)^{n-1} \qquad \text{for } |z-2| < 2
\end{aligned}
$$

Put $n = k + 1$ to make the index to start with $k = 0$,

$$
\begin{aligned}
\frac{1}{z^2} &= \frac{1}{4} \sum_{k=0}^{\infty} (-1)^{k+2} (k+1) \left(\frac{z-2}{2}\right)^{k} \qquad \text{for } |z-2| < 2 \\
&= \frac{1}{4} \sum_{k=0}^{\infty} (-1)^{k} (k+1) \left(\frac{z-2}{2}\right)^{k} \qquad \text{for } |z-2| < 2
\end{aligned}
$$

(11) Expand $f(z) = \dfrac{1}{1-z}$ in a power series about the point $z_0 = 2i$.

Answer: Rewrite $f(z)$ by

$$
f(z) = \frac{1}{1-z} = \frac{1}{1-2i-(z-2i)} = \frac{1}{(1-2i)} \left[1 - \left(\frac{z-2i}{1-2i}\right)\right]^{-1} .
$$

Using the geometric series, for $|\frac{z-2i}{1-2i}| < 1$, we expand

$$
\begin{aligned}
\frac{1}{1-z} &= \frac{1}{(1-2i)} \sum_{n=0}^{\infty} \left(\frac{z-2i}{1-2i}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{(1-2i)^{n+1}}\right) (z-2i)^n
\end{aligned}
$$

The above series converges for $|\frac{z-2i}{1-2i}| < 1$. That is, the series converges for $|z-2i| < \sqrt{5}$.

Note: We know that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ in $D_0 : |z| < 1$. But the sum function $\frac{1}{1-z}$ is analytic in $\mathbb{C}$ except at $z = 1$.

(12) If the radius of convergence for the series $\sum_{n=0}^{\infty} a_n z^n$ is R, then find the radius of convergence for the following:

(i) $\displaystyle\sum_{n=0}^{\infty} n^3 a_n z^n$ (ii) $\displaystyle\sum_{n=0}^{\infty} a_n^4 z^n$ (iii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$ (iv) $\displaystyle\sum_{n=0}^{\infty} a_n z^{7+n}$

(v) $\displaystyle\sum_{n=1}^{\infty} n^{-n} a_n z^n$

Answer:

Let $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a given power series with the radius of convergence R.

We know that $\displaystyle\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. That is, $R = \displaystyle\frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$.

It is given that the radius of convergence for the series $\sum_{n=0}^{\infty} a_n z^n$ is R. Then,

(i) $\displaystyle\sum_{n=0}^{\infty} n^3 a_n z^n$ has radius of convergence R.

(ii) $\displaystyle\sum_{n=0}^{\infty} a_n^4 z^n$ has radius of convergence R^4.

(iii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$ has radius of convergence $\sqrt{R}$.

(iv) $\displaystyle\sum_{n=0}^{\infty} a_n z^{7+n}$ has radius of convergence R.

(v) $\displaystyle\sum_{n=1}^{\infty} n^{-n} a_n z^n$ has radius of convergence ∞, if $R > 0$.

(13) Expand each of the following functions about the point $z = 1$ into a power series and find the radius of convergence:

(i) $\displaystyle\frac{z}{z^2-2z+5}$ (ii) $\sin(2z - z^2)$ (iii) $\text{Log}\,(1+z^2)$

Answer:

(i) $\displaystyle\frac{z}{z^2-2z+5}$

$$\begin{aligned}
\frac{z}{z^2-2z+5} &= \frac{z}{(z-1+2i)(z-1-2i)} \\
&= \frac{i}{4}\,\frac{(1-2i)}{((z-1)+2i)} - \frac{i}{4}\,\frac{(1+2i)}{((z-1)-2i)} \\
&= \frac{i}{8i}\,(1-2i)\,\left[1+\left(\frac{z-1}{2i}\right)\right]^{-1} + \frac{i}{8i}\,(1+2i)\,\left[1-\left(\frac{z-1}{2i}\right)\right]^{-1} \\
&= \frac{(1-2i)}{8}\left[1+\left(\frac{z-1}{2i}\right)\right]^{-1} + \frac{(1+2i)}{8}\,\left[1-\left(\frac{z-1}{2i}\right)\right]^{-1} \\
&= \frac{(1-2i)}{8}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2i}\right)^n\right] + \frac{(1+2i)}{8}\,\left[\sum_{n=0}^{\infty}\left(\frac{z-1}{2i}\right)^n\right]
\end{aligned}$$

which converges in $|\frac{z-1}{2i}| < 1$. That is, it converges in $|z-1| < 2$.

(ii) $\sin(2z-z^2)$

$$\begin{aligned}
\sin(2z-z^2) &= \sin\left(1-(z-1)^2\right) \\
&= \sin(1)\ \cos\left((z-1)^2\right)\ -\ \cos(1)\ \sin\left((z-1)^2\right) \\
&= \sin(1)\ \left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{4n}}{(2n)!}\right]\ -\ \cos(1)\ \left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{4n+2}}{(2n+1)!}\right]
\end{aligned}$$

which has radius of convergence $R = \infty$.

(iii) $\text{Log}\ (1+z^2)$

$$\begin{aligned}
f(z) &= \ \text{Log}\ (1+z^2) \implies f(1) = \ln(2) \\
f'(z) = \frac{2z}{1+z^2} &= (z-i)^{-1} + (z+i)^{-1} \implies f'(1) = (1-i)^{-1} + (1+i)^{-1} \\
f''(z) &= (-1)(z-i)^{-2} + (-1)(z+i)^{-2} \implies f''(1) = (-1)(1-i)^{-2} + (-1)(1+i)^{-2}
\end{aligned}$$

$$f^{(n)}(z) = (-1)^{n-1}(n-1)!\left[(z-i)^{-n} + (z+i)^{-n}\right]$$

$$\implies f^{(n)}(1) = (-1)^{n-1}(n-1)!\left[(1-i)^{-n} + (1+i)^{-n}\right] \quad = \quad c_n \text{ (say) for } \ n \geq 1$$

$$a_n = \frac{f^{(n)}(1)}{n!} \implies a_0 = \ln(2) \qquad \text{and} \qquad a_n = \frac{c_n}{n!} \text{ for } n \geq 1\ .$$

This gives that

$$f(z) = \ \text{Log}\ (1+z^2) = \ln(2)\ +\ \sum_{n=1}^{\infty}\frac{c_n}{n!}\,(z-1)^n$$

Finding the radius of convergence of the series.

The radius of convergence of a given power series $\sum a_n(z-z_0)^n = f(z)$ is the distance from the point z_0 to the nearest singularity of f.

Thus, the nearest singular point of $\text{Log}\,(1+z^2)$ from the point $z_0 = 1$ is $\pm i$. Therefore, the radius of convergence of the above series is $R = |1 \pm i| = \sqrt{2}$.

(14) Using the Cauchy product of series, find the first four non-zero terms of the Maclaurin series of $e^z/(1-z)$.

Answer:

We know that

$$\begin{aligned}
e^z &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \text{ for all } z \in \mathbb{C} \\
\frac{1}{1-z} &= 1 + z + z^2 + z^3 + z^4 + \cdots \text{ for } |z| < 1 \\
\frac{e^z}{1-z} &= \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)\left(1 + z + z^2 + z^3 + z^4 + \cdots\right) \text{ for } |z| < 1 \\
&= 1 + (1+1)\,z + \left(1 + 1 + \frac{1}{2}\right) z^2 + \left(1 + 1 + \frac{1}{2} + \frac{1}{6}\right) z^3 + \cdots \text{ for } |z| < 1 \\
&= 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \cdots \text{ for } |z| < 1
\end{aligned}$$

(15) Prove or disprove the existence of an analytic function in a neighborhood of the origin satisfying $|f^{(n)}(0)| \geq (n!)^2$, $n = 1, 2, \cdots$.

Answer:

Suppose f is analytic in a neighborhood of the origin. Then, f has power series representation about $z_0 = 0$ and is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r$ for some $r > 0$ where $a_n = \dfrac{f^{(n)}(0)}{n!}$ for $n = 0,\ 1,\ 2,\ \cdots$.

It is given that $|f^{(n)}(0)| \geq (n!)^2$, $n = 1,\ 2,\ \cdots$. Therefore, $|a_n| \geq (n!)$ for $n \geq 1$. Consequently,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \limsup_{n\to\infty} |n!|^{\frac{1}{n}} = \infty\ .$$

It means that the radius of convergence of the Taylor series of f about $z_0 = 0$ is $R = 0$ which contradicts the fact that $R \geq r > 0$.

Therefore, we conclude that there is **no function** f **exists** such that f is analytic in a neighborhood of the origin and satisfying $|f^{(n)}(0)| \geq (n!)^2$, $n = 1,\ 2,\ \cdots$.

(16) Suppose f is analytic on the open unit disc D and it satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.

Answer: By Cauchy integral formula

$$|f'(0)| \leq \frac{1}{2\pi}\int_{|z|=r}\left|\frac{f(z)}{z^2}\right| dz \leq \frac{1}{2\pi}\frac{1}{r^2} \times 2\pi r = \frac{1}{r}$$

for every $r < 1$. Letting r tending to 1 we get $|f'(0)| \leq 1$.