MA201, Lecture 9
Introduction to Fourier Series

IBVP for Vibrating string with no external forces

- We consider the one-dimensional wave equation problem in a computational domain

$$(x, t) \in [0, L] \times [0, \infty).$$

- The IBVP under consideration consists of the following:
- The governing equation:

$$u_{tt} = c^2 u_{xx}, \quad (x, t) \in (0, L) \times (0, \infty). \tag{1}$$

▶ The boundary conditions for all $t > 0$:

$$u(0, t) = 0, \qquad u(L, t) = 0. \tag{2}$$

▶ The initial conditions for $0 \leq x \leq L$ are

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x). \tag{3}$$

HISTORICAL REMARKS on Fourier Series

- The theory of Fourier series has its historical origin in the middle of the eighteenth century, when several mathematicians were studying the vibrations of stretched strings and some other problems involving steady-state heat conduction.
- For the case of a string stretched between the points $x = 0$ and $x = \pi$, Daniel Bernoulli (in 1753) gave the solution of (1) as a series of the form

$$u(x,t) = b_1 \sin x \cos ct + b_2 \sin 2x \cos 2ct + \cdots . \tag{4}$$

▶ Observe that:

- A typical term of this series, $b_n \sin nx \cos nct$, is a solution of (1).
- Further, every finite sum of such terms is a solution.
- The series (4) will also be a solution if term-by-term differentiation of the series is justified.

HISTORICAL REMARKS on Fourier Series (contd.)

- When $t = 0$, the series (4) reduces to

$$u(x,0) = b_1 \sin x + b_2 \sin 2x + \ldots \tag{5}$$

- This should give the initial shape of the string, that is, the curve

$$u = u(x,0) = \phi(x).$$

- Thus, we should have

$$\phi(x) = b_1 \sin x + b_2 \sin 2x + \ldots \tag{6}$$

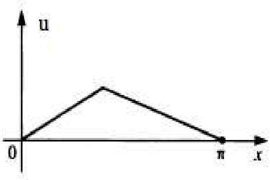
- It is clear on physical grounds that there is a great amount of freedom in the way the string can be constrained in its initial position.

HISTORICAL REMARKS on Fourier Series (contd.)

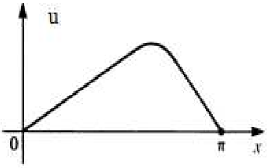
▶ For Example:

(a) If the string is plucked aside at a single point, then the shape will be a broken line,

(b) If it is pushed aside by using a circular object of some kind, then the shape will be partly a straight line, partly an arc of a circle, and partly another straight line.



(a)



(b)

- It is reasonable to ask whether the single analytic expression (6) could represent a straight line on part of the interval $[0, \pi]$, a circle on another part, and a second straight line on still another part?

Fourier Series *on its way*

- Therefore, as a result of mathematically analyzing this physical problem, Bernoulli arrived at an idea that has had very far-reaching influence on the history of mathematics and physical science, namely, the possibility that any function can be expanded in a trigonometric series of the form (6).
- However, D'Alembert and Euler rejected Bernoulli's idea - questioning the authenticity of such an (absurd) idea.
- The controversy bubbled on for many years, and in the absence of mathematical proofs, no one could convince anyone else to his way of thinking.

Fourier Series *on its way*

- In 1807, the French physicist-mathematician Fourier announced in this connection that an arbitrary function f(x) can be represented in the form (6).
- He supplied no proofs, but instead heaped up the evidence of many solved problems and many convincing specific expansions – in fact, so many that the mathematicians of the time began to spend more effort on proving, rather than disproving, his conjecture.
- The first major result of this shift in the winds of opinion was the classical paper of Dirichlet in 1829, in which he proved with full mathematical rigour that such a series actually exists.

Let us know Fourier

Jean Baptiste Joseph Fourier (March 21, 1768 – May 16, 1830) was a French mathematician and physicist who is best known for initiating the investigation of Fourier series and their application to problems of heat flow. He is also known for Fourier transform and Fourier's heat law.

Some Facts:

Alma Mater: Ecole Normale Suprieure

Academic Advisor: Joseph-Louis Lagrange

Notable students:
Peter Gustav Lejeune Dirichlet
Claude-Louis Navier
Giovanni Plana

Fourier Series: *Orthogonal Sets*

▶ We begin our treatment with some observations:
for $m, n = 1, 2, 3, \ldots,$

- $$\int_{-L}^{L} \cos \frac{n\pi x}{L} dx = 0 \;\; \& \;\; \int_{-L}^{L} \sin \frac{n\pi x}{L} dx = 0, \tag{7}$$
- $$\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0, \tag{8}$$
- $$\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} 0, & m \neq n, \\ L, & m = n, \end{cases} \tag{9}$$
- $$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases} \tag{10}$$

● Orthogonal Functions: A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be an orthogonal set on the interval $[a, b]$ if

$$\int_{a}^{b} f_n(x) f_m(x) dx = 0, \quad m \neq n. \tag{11}$$

Fourier Series: *Orthogonal Sets*

- For an orthogonal set $\{f_n(x)\}_{n=1}^{\infty}$ in $[a, b]$ and a given function $f(x)$, if we have

$$\begin{aligned} f(x) &= c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \cdots, \quad \text{then} \\ c_n &= \frac{\int_a^b f(x) f_n(x) dx}{\int_a^b (f_n(x))^2 dx}, \quad n = 1, 2, \ldots. \end{aligned}$$

  - We already know that the set of trigonometric functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

    is orthogonal on $[-\pi, \pi]$.

- Thus, if we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} \Big\{ a_n \cos nx + b_n \sin nx \Big\},$$

then all coefficients c_0, a_n and b_n can be determined uniquely.

Fourier Series in $[-L, L]$

Definition:

A function f is called periodic if and only if there exists a positive parameter $2p$ such that for every x in the domain of f, we always have $f(x+2p) = f(x)$. The parameter $2p$ is called a period of f.

Thus, with this definition, we can show that if f is periodic, then

$$f(x) = f(x+2p) = f(x+4p) = f(x+6p) = \cdots = f(x+np) = \cdots,$$

where $n = 1, 2, 3, \ldots$ and hence $2p, 4p, 6p, \ldots, 2np$ are also periods of f. Here $2p$ is the smallest of all the periods and is usually defined as the fundamental period of f.

Definition:

A function f is called sectionally or piecewise continuous over an interval if the interval can be subdivided into a number of finite subintervals in each of which the function is continuous, i.e., a function $f(x)$ must be single-valued but can have a finite number of isolated discontinuities for $x > 0$. These points of discontinuity can be points of jump discontinuity too

Fourier Series in $[-\pi, \pi]$

- An infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big\{ a_n \cos nx + b_n \sin nx \Big\}, \tag{12}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \ldots, \tag{13}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots, \tag{14}$$

is called a Fourier Series of f on the interval $[-\pi, \pi]$.

- a_n and b_n are called Fourier coefficients.
- f and f' are piecewise continuous in $(-\pi, \pi)$.
- f is a periodic function with period 2π.

Fourier Series in $[-L, L]$

- The infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right], \tag{15}$$

▶with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \text{and} \tag{16}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots \tag{17}$$

is called the Fourier series of $f(x)$ on $[-L, L]$.

- a_n and b_n are the Fourier coefficients.
- f and f' are piecewise continuous in $(-L, L)$.
- f is a periodic function with period $2L$.

Fourier Series

Remark

The Fourier series of a function is defined whenever the integrals in (16) and (17) have meaning. This is certainly the case if f is continuous on the interval (open or closed).

- However, the integrals also have meaning when f has jump discontinuities as in the following function:

$$f(x) = \begin{cases} f_1(x), & -L < x < x_1, \\ f_2(x), & x_1 < x < x_2, \\ f_3(x), & x_2 < x < L. \end{cases}$$

- Then the Fourier coefficients a_n and b_n will be given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{x_1} f_1(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_1}^{x_2} f_2(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_2}^{L} f_3(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n &= \frac{1}{L}\int_{-L}^{x_1} f_1(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_1}^{x_2} f_2(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_2}^{L} f_3(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

Example (Contd.)

Example

Find the Fourier Series of the function $f(x) = x$ for $-L \leq x \leq L$.

Solution: We first compute the Fourier coefficients a_n for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx = \frac{x}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx \\ &= 0 + \frac{L}{(n\pi)^2}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} = 0,\ n = 1, 2, 3, \ldots. \end{aligned}$$

For $n = 0$, verify that $a_0 = 0$. Thus, $a_n = 0$, for $n = 0, 1, 2, 3, \ldots$. Next, to compute $b_n,\ n \geq 1$, we have

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx = \frac{-x}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{1}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx \\ &= \frac{-2L}{n\pi}\cos(n\pi) + \frac{L}{(n\pi)^2}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} = \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

Example (Contd.)

Thus, the Fourier Series of $f(x)$ is given by

$$f(x) \approx \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin \frac{n\pi x}{L} = \frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin \frac{n\pi x}{L}.$$

Remark

Here, the symbol $\approx$ means that Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \textit{and}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots$$

is an approximation to $f(x)$.

Example

Find the Fourier Series of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

Solution: Here $L = \pi$. Note that $f(x)$ is an odd function. Since the product of an odd function and an even function is odd, $f(x)\cos nx$ is also an odd function. Hence,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \quad n = 0, 1, 2, \ldots .$$

Since $f(x)\sin nx$ is an even function (as the product of two odd functions), we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} \sin nx dx.$$

Example (Contd.)

Finally, we arrive at

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] \\ &= \begin{cases} 0, & n \text{ even}, \\ \frac{4}{n\pi}, & n \text{ odd}. \end{cases} \end{aligned}$$

Thus, the Fourier series of $f(x)$ is given by

$$f(x) \approx \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

Remark

- *If f is an odd function, then its Fourier series consists only of sine terms.*
- *If f is an even function, then its Fourier series consists only of cosine terms.*

Cosine and sine Fourier expansion

▶Suppose we are given a function f, defined and integrable on the interval $(0, L)$.

▶Define an even extension of f in $(-L, L)$ by

$$\begin{aligned} f_e(x) &= f(x) \text{ if } 0 < x < L, \\ f_e(x) &= f(-x) \text{ if } -L < x < 0. \end{aligned}$$

▶Define an odd extension of f by

$$\begin{aligned} f_O(x) &= f(x) \text{ if } 0 < x < L, \\ f_O(x) &= -f(-x) \text{ if } -L < x < 0. \end{aligned}$$

Remark

- *Then, we define Fourier series of f in $(0, L)$ as the Fourier series of f_e or f_O in $(0, L)$. Fourier expansion in $(0, L)$ is called* **Fourier half range expansion.**
- *Therefore, Fourier half range expansion is either a Fourier cosine or Fourier sine expansion.*

Half-range Series

Example

Find the Fourier cosine series expansion of $f(x) = x,\ 0 < x < 1.$

Solution Hint:

- Construct an even periodic extension of f.
- Suppose, f_e is the even extension of f in $(-1, 1)$.
- Then, evaluate the Fourier cosine expansion of f_e in $(-1, 1)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$