Constraint Optimization Problems

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Prakash Kotecha, Associate Professor
Debasis Maharana, Teaching Assistant &
Remya Kommadath, Teaching Assistant
Department of Chemical Engineering
Indian Institute of Technology Guwahati

Example 1

A farm uses at least 800 kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions.

Feedstuff	kg per kg of feedstuff		Cost ($ per kg)
	Protein	Fiber	
Corn	0.09	0.02	0.3
Soyabean meal	0.6	0.06	0.9

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost of feed mix.

Operations research: An introduction, H. A. Taha

Example 1

Let x_1 = kg of corn in the daily mix

x_2 = kg of soybean meal in the daily mix

Feedstuff	kg per kg of feedstuff		Cost ($ per kg)
	Protein	fiber	
Corn	0.09	0.02	0.3
Soyabean meal	0.6	0.06	0.9
Dietary requirements	At least 30%	At most 5%	

farm uses at least 800 lb of special feed daily

Objective: determine minimum-cost feed mix.

Minimize $z = 0.3\,x_1 + 0.9\,x_2$

Subject to,

$x_1 + x_2 \geq 800$ **Daily demand**

$0.09\,x_1 + 0.6\,x_2 \geq 0.3(x_1 + x_2)$ **Protein requirement in total mix**

$-0.21x_1 + 0.3\,x_2 \geq 0$

$0.02\,x_1 + 0.06\,x_2 \leq 0.05(x_1 + x_2)$ **Fiber requirement in total mix**

$-0.03\,x_1 + 0.01\,x_2 \leq 0$

$x_1, x_2 \geq 0$ **Lower bounds of the decision variables**

Operations research: An introduction, H. A. Taha

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Example 2

- Reddy Mikks company produces interior and exterior paints from raw materials, M1 and M2.
- Daily demand for interior paint cannot exceed that for exterior paint by more than 1 unit.
- Maximum daily demand for the interior paint is 2 units.
- Determine optimum quantity of interior and exterior paints that maximizes total daily profit.

	Exterior paint	Interior paint
M1	6	4
M2	1	2

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Operations research an introduction, H.A. Taha

Example 2

Let x_1 = Units of exterior paint produced daily

x_2 = Units of interior paints produced daily

Decision variables

Maximize Profit, $Z = 5x_1 + 4x_2$

Objective function

Subject to

$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$

Raw material constraints

$$x_2 \leq x_1 + 1$$

Demand constraint

Constraints

$$x_1, x_2 \geq 0$$
$$x_2 \leq 2$$

Bound constraints

	Ext. paint	Int. paint	Availability
M1	6	4	24
M2	1	2	6
Profit	5	4	

Daily demand for interior paint cannot exceed that for exterior paint by more than 1 Unit.

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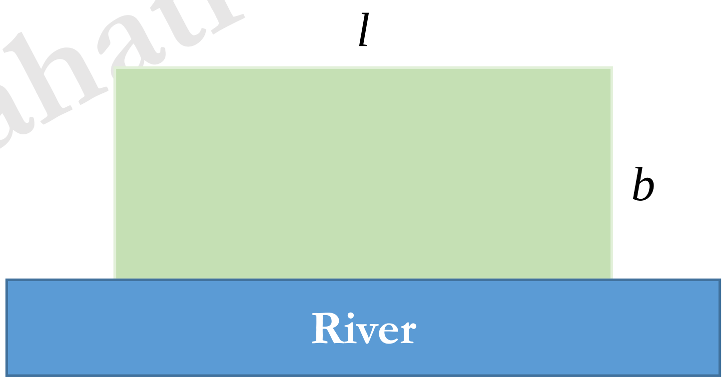
Example 3

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. No fence is required along the river. What are the dimensions of the field that has the largest area?



Area of a rectangular region = lb

Objective: Maximize the area, $\max f = lb$

Constraint: Perimeter should not exceed 2400m

$$l + 2b \leq 2400$$

Bounds: $l, b > 0$

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https://studentsuccess.asu.edu/sites/default/files/optimization.pdf

Constraints

- Inequality constraints (usually resource/requirement constraints)
  - In general denoted by $g(x) \leq 0$
  - Conversion form one form to the other by multiplying by -1
- Equality Constraints (usually material and energy balances)
  - In general denoted by $g(x) = 0$
- Feasible solution: Satisfies all the constraints
- Infeasible solution: Does not satisfy at least one constraint

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Constraints

- Any feasible solution is preferred to any infeasible solution
- Among two feasible solutions, the one having a better objective function value is preferred
- Among two infeasible solutions, the one having a smaller constraint violation is preferred

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Constraints

- Any feasible solution is preferred to any infeasible solution
- Among two feasible solutions, the one having a better objective function value is preferred
- Among two infeasible solutions, the one having a smaller constraint violation is preferred
- A solution $x^{(i)}$ is said to be *constrain – dominate* a solution $x^{(j)}$:
  - Solution $x^{(i)}$ is feasible and solution $x^{(j)}$ is not.
  - Solutions $x^{(i)}$ and $x^{(j)}$ are both infeasible, but solution $x^{(i)}$ has a smaller constraint violation.
  - Solutions $x^{(i)}$ and $x^{(j)}$ are feasible and solution $x^{(i)}$ dominates solution $x^{(j)}$ with respect to objective function.

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Constraints

- Any feasible solution is preferred to any infeasible solution
- Among two feasible solutions, the one having a better objective function value is preferred
- Among two infeasible solutions, the one having a smaller constraint violation is preferred
- A solution $x^{(i)}$ is said to be *constrain – dominate* a solution $x^{(j)}$:
  - Solution $x^{(i)}$ is feasible and solution $x^{(j)}$ is not.
    - (S1 constrain-dominate S2)
  - Solutions $x^{(i)}$ and $x^{(j)}$ are both infeasible, but solution $x^{(i)}$ has a smaller constraint violation.
    - (S3 constrain-dominate S2)
  - Solutions $x^{(i)}$ and $x^{(j)}$ are feasible and solution $x^{(i)}$ dominates solution $x^{(j)}$ with respect to objective function.
    - (S4 constrain-dominate S1)

	S1	S2	S3	S4
x	0.7	0.4	0.65	0.8
f	0.35	0.2	0.325	0.4
violation	0	0.29	0.0275	0

$$\text{Min} \quad f = \frac{x}{2}$$
$$s.t. \quad x^2 \geq 0.45, \quad 0 < x < 1$$

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Preserving feasibility of solutions

$$\begin{aligned} \text{Minimize} \quad & 3x_1 + 2x_2 + x_3 \\ s.t. \quad & 2x_1 + x_2 + x_3 \geq 5 \\ & x_1 + x_2 + 0.5x_3 \leq 4 \\ & x_1 - x_2 = 2 \\ & x_1, x_2, x_3 \in \mathbb{R} \end{aligned}$$

$$\Rightarrow \quad \boxed{x_2 = x_1 - 2} \quad \Rightarrow$$

$$\begin{aligned} \text{Minimize} \quad & 5x_1 + x_3 - 4 \\ s.t. \quad & 3x_1 + x_3 \geq 7 \\ & 2x_1 + 0.5x_3 \leq 6 \\ & x_1, x_3 \in \mathbb{R} \end{aligned}$$

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Preserving feasibility of solutions

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$$\Rightarrow \boxed{x_2 = x_1 - 2} \Rightarrow$$

$$\begin{aligned} \textit{Minimize} \quad & 5x_1 + x_3 - 4 \\ s.t. \quad & 3x_1 + x_3 \geq 7 \\ & 2x_1 + 0.5x_3 \leq 6 \\ & x_1, x_3 \in \mathbb{R} \end{aligned}$$

Algorithm

1) Generate randomly a single solution/a set of solutions – Partial decision variables

2) Based on the fitness of the current solution/set of solutions, suggest other solutions

Partial set of decision variables →

← **Fitness**

Problem Statement

1) Generate the remaining set of decision variables

2) Evaluate Objective function

3) Check for Constraint Satisfaction

4) Evaluation of fitness (based on 1,2 & 3)

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Constraint Handling

- Case 1: Unconstrained optimization problem
  - Fitness function = Objective function

Solution	Objective function	Fitness function
S_1	5	5
S_2	7	7

- Case 2: Constrained optimization problem (Minimization case)
  - Fitness function = Objective function + λ (Penalty)

$$\text{Min } f(x)$$
$$s.t.\ \ g(x) \geq b$$

Solution	Objective function	Feasibility
S_1	5	Yes
S_2	7	No

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Constraint Handling

➢ Case 1: Unconstrained optimization problem

- Fitness function = Objective function

Solution	Objective function	Fitness function
S_1	5	5
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➢ Case 2: Constrained optimization problem (Minimization case)

- Fitness function = Objective function + λ (Penalty)

$$\text{Min } f(x)$$
$$s.t.\ \ g(x) \geq b$$

Solution	Objective function	Feasibility	Violation	Fitness function
S_1	5	Yes	0	5
S_2	7	No	2	47

$\lambda = 20$

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Constraint Handling

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S_1	5	Yes	0	5
S_2	7	No	2	47
S_3	2	No	0.1	4

λ = 20

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Constraint Handling

➢Case 1: Unconstrained optimization problem

- Fitness function = Objective function

Solution	Objective function	Fitness function
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➢Case 2: Constrained optimization problem (Minimization case)

- Fitness function = Objective function + λ (Penalty)

$$\text{Min } f(x)$$
$$s.t.\ \ g(x) \geq b$$

Solution	Objective function	Feasibility	Violation	Fitness function $\lambda = 20$	Fitness function $\lambda = 200$
S_1	5	Yes	0	5	5
S_2	7	No	2	47	407
S_3	2	No	0.1	4	22

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Constraint Handling

- Given Problem

$$\begin{aligned} &Min \quad f(x) \\ &s.t. \quad g_j(x) \geq \mathrm{b} \qquad j = 1, 2, \ldots J \\ &\qquad x_l \leq x_i \leq x_u \qquad i = 1, 2, \ldots I \end{aligned}$$

- Normalize Constraints

$$\underline{g}_j(x) \geq 0 \qquad j = 1, 2, \ldots J$$

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Constraint Handling

- Given Problem

$$\begin{aligned} & Min \quad f(x) \\ & s.t. \quad g_j(x) \geq \mathrm{b} \qquad j = 1, 2, \ldots J \\ & \qquad x_l \leq x_i \leq x_u \qquad i = 1, 2, \ldots I \end{aligned}$$

- Normalize Constraints

$$\underline{g}_j(x) \geq 0 \qquad j = 1, 2, \ldots J$$

- Estimate the extent of violation

$$\omega_j(x) = \begin{cases} \left|\underline{g}_j(x)\right|, & \text{if } \underline{g}_j(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

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Constraint Handling

- Given Problem

$$\begin{aligned} &Min \quad f(x) \\ &s.t. \quad g_j(x) \geq \mathrm{b} \qquad j = 1,2,\ldots J \\ &\qquad\quad x_l \leq x_i \leq x_u \qquad i = 1,2,\ldots I \end{aligned}$$

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$$\omega_j(x) = \begin{cases} \left|\underline{g}_j(x)\right|, & \text{if } \underline{g}_j(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

- Estimate the total penalty

$$\Omega(x_i) = \sum_{j=1}^{J} \omega_j(x)$$

- Estimate the fitness function

$$F(x_i) = f(x) + R_m \Omega(x)$$

Penalty factor (R_m): Usually an order of magnitude than the objective function value

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Evaluation of Fitness function

$$\textit{Minimize} \qquad f(x) = \frac{1 + x_2}{x_1}$$

$$\textit{s.t.} \qquad g_1(x) = 9x_1 + x_2 \geq 6$$

$$g_2(x) = 9x_1 - x_2 \geq 1$$

$$0.1 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 5$$

Normalized Constraints

$$\underline{g}_1(x) = \frac{9x_1 + x_2}{6} - 1 \geq 0$$

$$\underline{g}_2(x) = 9x_1 - x_2 - 1 \geq 0$$

$$R_m = 20$$

$$\omega_j(x) = \begin{cases} \left|\underline{g}_j(x)\right|, & \text{if } \underline{g}_j(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Omega(x) = \sum_{j=1}^{J} \omega_j(x)$$

$$F(x) = f(x) + R_m \Omega(x)$$

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Solution	Decision Variables		Objective function (f)
	x_1	x_2	
S1	0.31	0.89	6.10
S2	0.38	2.73	9.82
S3	0.22	0.56	7.09
S4	0.59	3.63	7.85
S5	0.46	2.9	8.48
S6	0.66	4.11	7.74

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Evaluation of Fitness function

Minimize $f(x) = \dfrac{1+x_2}{x_1}$

s.t. $g_1(x) = 9x_1 + x_2 \geq 6$

$g_2(x) = 9x_1 - x_2 \geq 1$

$0.1 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 5$

Normalized Constraints

$$\underline{g}_1(x) = \frac{9x_1 + x_2}{6} - 1 \geq 0$$

$$\underline{g}_2(x) = 9x_1 - x_2 - 1 \geq 0$$

$$R_m = 20$$

$$\omega_j(x) = \begin{cases} \left|\underline{g}_j(x)\right|, & \text{if } \underline{g}_j(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Omega(x) = \sum_{j=1}^{J} \omega_j(x)$$

$$F(x) = f(x) + R_m \Omega(x)$$

Solution	Decision Variables		Objective function (f)	Feasibility	
	x_1	x_2		g_1	g_2
S1	0.31	0.89	6.10	✗	✓
S2	0.38	2.73	9.82	✓	✗
S3	0.22	0.56	7.09	✗	✓
S4	0.59	3.63	7.85	✓	✓
S5	0.46	2.9	8.48	✓	✓
S6	0.66	4.11	7.74	✓	✓

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Evaluation of Fitness function

Minimize $f(x) = \dfrac{1+x_2}{x_1}$

s.t. $g_1(x) = 9x_1 + x_2 \geq 6$

$g_2(x) = 9x_1 - x_2 \geq 1$

$0.1 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 5$

Normalized Constraints

$$\underline{g}_1(x) = \frac{9x_1 + x_2}{6} - 1 \geq 0$$

$$\underline{g}_2(x) = 9x_1 - x_2 - 1 \geq 0$$

$$R_m = 20$$

$$\omega_j(x) = \begin{cases} \left|\underline{g}_j(x)\right|, & \text{if } \underline{g}_j(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Omega(x) = \sum_{j=1}^{J} \omega_j(x)$$

$$F(x) = f(x) + R_m \Omega(x)$$

Solution	Decision Variables		Objective function (f)	Feasibility		Violations		Total penalty Ω	Fitness (F)
	x_1	x_2		g_1	g_2	ω_1	ω_1		
S1	0.31	0.89	6.10	✗	✓	0.39	0.00	0.39	13.90
S2	0.38	2.73	9.82	✓	✗	0	0.31	0.31	16.02
S3	0.22	0.56	7.09	✗	✓	0.58	0.00	0.58	18.69
S4	0.59	3.63	7.85	✓	✓	0	0	0	7.85
S5	0.46	2.9	8.48	✓	✓	0	0	0	8.48
S6	0.66	4.11	7.74	✓	✓	0	0	0	7.74

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Example

$$\textit{Minimize} \qquad f(x) = \frac{1 + x_2}{x_1}$$

$$\textit{s.t.} \qquad g_1(x) = x_2 + 9x_1 \geq 6$$

$$g_2(x) = -x_2 + 9x_1 \geq 1$$

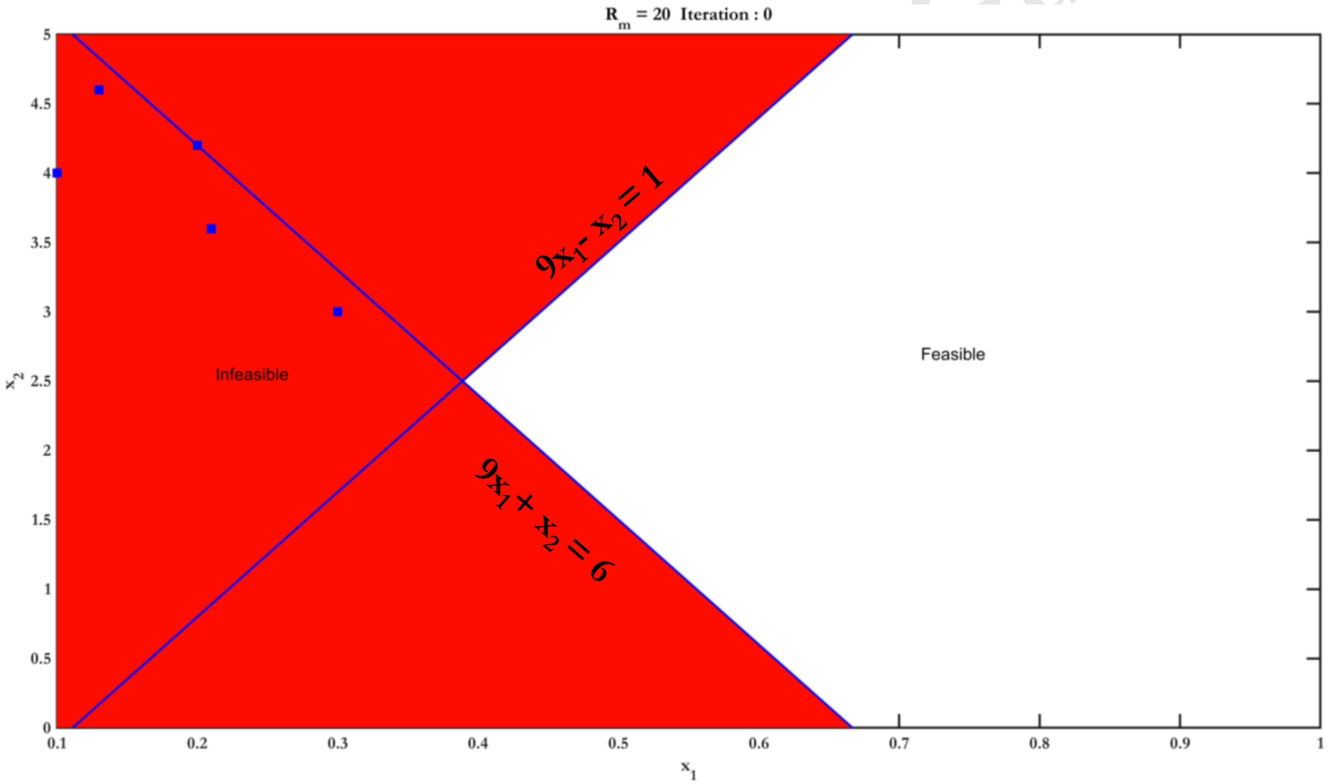
$$0.1 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 5$$

Normalized Constraints

$$\underline{g}_1(x) = \frac{9x_1 + x_2}{6} - 1 \geq 0$$

$$\underline{g}_2(x) = 9x_1 - x_2 - 1 \geq 0$$

$$R_m = 20$$



Constraint Handling

- Using hard penalty

$$\text{Min } f(x)$$

$$\left.\begin{aligned} s.t.\ \ g_j(x) &\geq 0 \\ h_k(x) &= 0 \end{aligned}\right\} \forall\ j = 1, 2 \ldots J$$

$$\boxed{F(x) = f(x) + \sum_{j=1}^{J} M_j}$$

$$M_j = \begin{cases} \lambda_j, & \text{if } g_j(x) < 0; \\ 0, & \text{otherwise} \end{cases}$$

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Constraint Handling

➢Using hard penalty

$$F(x) = f(x) + \sum_{j=1}^{J} M_j$$

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➢Using penalty function

$$F(x) = f(x) + \lambda \left[\sum_{j=1}^{J} \langle g_j(x) \rangle + \sum_{k=1}^{K} \left| h_k(x) \right| \right]$$

$$\langle g_j(x) \rangle = \begin{cases} \left| g_j(x) \right|, & \text{if } g_j(x) < 0; \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} g(x) &= -2 \\ \langle g(x) \rangle &= |-2| = 2 \end{aligned}$$

$$\begin{aligned} g(x) &= 2 \\ \langle g(x) \rangle &= 0 \end{aligned}$$

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Constraint Handling

- Using dynamic penalty function

$$F(x) = f(x) + (C \cdot t)^{\alpha} \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^{\beta} + \sum_{k=1}^{K} \left| h_k(x) \right|^{\beta} \right]$$

C, α, β: user defined parameters

t: iteration counter

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Constraint Handling

- Using dynamic penalty function

$$F(x) = f(x) + (C \cdot t)^{\alpha} \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^{\beta} + \sum_{k=1}^{K} |h_k(x)|^{\beta} \right]$$

C, α, β : user defined parameters

t : iteration counter

- Method based on feasible over infeasible solutions.

$$F(x) = f(x) + \lambda \left[\sum_{j=1}^{J} \langle g_j(x) \rangle + \sum_{k=1}^{K} |h_k(x)| \right] + R(t, x)$$

R(t,x): difference between the **best static penalized function value among all infeasible solutions** and **the worst feasible solution**

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Constraint Handling

- Using dynamic penalty function

$$\mathrm{F}(x) = f(x) + (C \cdot t)^{\alpha} \left[\sum_{j=1}^{J} \langle g_j(x) \rangle^{\beta} + \sum_{k=1}^{K} |h_k(x)|^{\beta} \right]$$

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$$\mathrm{F}(x) = f(x) + \lambda \left[\sum_{j=1}^{J} \langle g_j(x) \rangle + \sum_{k=1}^{K} |h_k(x)| \right] + R(t, x)$$

R(t,x): difference between the **best static penalized function value among all infeasible solutions** and **the worst feasible solution**

$$\mathrm{F}(x) = \begin{cases} f(x), & \text{if } x \text{ is feasible} \\ f_{\max} + \sum_{j=1}^{J} \langle g_j(x) \rangle + \sum_{k=1}^{K} |h_k(x)| & \text{otherwise} \end{cases}$$

$f_{\max}$: objective function value of worst feasible solution

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Thank You !!!

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