MA201: Partial Differential Equations
Lecture - 6

- A second-order PDE in two independent variables x and y in its most general form is given by

$$F(x, y, u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}) = 0. \tag{1}$$

- The linear form of a second-order PDE in two independent variables is the following when the unknown function $u(x, y)$ satisfies

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0. \tag{2}$$

where A, B, C, D, E, F and G are, in general, functions of x and y.

Facts:

- The expression $Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}$, containing the second-order partial derivatives, is called the Principal part of equation (2).

- Classification of such PDEs is based on this principal part. We will discuss it later.
- Second-order PDEs have immense importance in all branches of science and engineering.

- Consider the second-order linear equation in two independent variables x and y given by (2) in the following form:

$$(Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu)(x, y) = -G(x, y). \quad (3)$$

- In operator notation,

$$(T(u))(x, y) = -G(x, y) = f(x, y) \text{ (say)}, \quad (4)$$

with

$$T(u) = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu.$$

- Since T is linear, we have

$$T(u_1 + u_2) = T(u_1) + T(u_2) \;\&\; T(cu) = cT(u) \,\forall c \in \mathbb{R}. \quad (5)$$

Remark:

- Equation (4) is called homogeneous, if $f \equiv 0$, otherwise it is called non-homogeneous.

The Principle of Superposition

Theorem

Suppose u_1 solves linear PDE $T(u) = f_1$ and u_2 solves $T(u) = f_2$, then $u = c_1 u_1 + c_2 u_2$ solves $T(u) = c_1 f_1 + c_2 f_2$.

In particular, if u_1 and u_2 both solve the same homogeneous linear PDE $T(u) = 0$, so does $u = c_1 u_1 + c_2 u_2$.

Remark:

- Any linear combination of solutions of a linear homogeneous PDE is also a solution.
- A solution $u = u(x, y)$ to a homogeneous equation $T(u) = 0$ is called the *general solution* if it contains two arbitrary functions.
- If u is a general solution to homogeneous PDE $T(u) = 0$ and u_p is a particular solution to non-homogeneous PDE $T(w) = f$, then $u + u_p$ is also a solution to the non-homogeneous equation and it is called the general solution to the PDE $T(w) = f$.

▶With the notations $D = \partial/\partial x$ and $D' = \partial/\partial y$, a PDE with constant coefficients can be written as

$$F(D, D')u = f. \tag{6}$$

▶We classify PDE (6) into two main types (with respect to the appearance of the operators):

- **Reducible:** Equation (6) is called reducible if it can be written as the product of linear factors of the form $aD + bD' + c$, with constants a, b, c. For example, consider the equation

  $$u_{xx} - u_{yy} = 0.$$

  In this case

  $$F(D, D') = D^2 - (D')^2 = (D + D')(D - D').$$

- **Irreducible:** Equation (6) is called irreducible if it is not reducible. For example, when we consider $F(D, D') = D^2 - D'$.

- An n-th order reducible PDE can be written as

$$F(D, D')u = \Big(\prod_{r=1}^{n}(a_r D + b_r D' + c_r)\Big)u = f. \tag{7}$$

Theorem 1

If $(a_r D + b_r D' + c_r)$ is a factor of $F(D, D')$, $a_r \neq 0$, then

$$u_r = \exp\left\{-\frac{c_r x}{a_r}\right\}\phi_r(b_r x - a_r y)$$

is a solution of the equation $F(D, D')u = 0$. Here, ϕ_r is an arbitrary real-valued function.

Theorem 2

If $(b_r D' + c_r)$ is a factor of $F(D, D')$ and ϕ_r is an arbitrary real-valued single variable function, then

$$u_r = \exp\left\{-\frac{c_r y}{b_r}\right\}\phi_r(b_r x)$$

is a solution of the equation $F(D, D')u = 0$.

Linear Equations with Constant Coefficients: Reducible Equation

Theorem 3

If $(aD + bD' + c)^m$ $(m \leq n,\ a \neq 0)$ is a factor of $F(D, D')$ and $\phi_1,\ \phi_2, \ldots, \phi_m$ are arbitrary real-valued functions, then

$$\exp\left\{-\frac{cx}{a}\right\}\sum_{i=1}^{m} x^{i-1}\phi_i(bx - ay)$$

is a solution of the equation $F(D, D')u = 0$.

Theorem 4

If $(bD' + c)^m$ $(m \leq n)$ is a factor of $F(D, D')$ and $\phi_1,\ \phi_2, \ldots, \phi_m$ are real-valued single variable functions, then

$$\exp\left\{-\frac{cy}{b}\right\}\sum_{i=1}^{m} x^{i-1}\phi_i(bx)$$

is a solution of the equation $F(D, D')u = 0$.

NOTE: n is the order of the PDE.

Example

- General solution of
$$u_{xx} - u_{yy} = 0$$
is given by
$$u = \phi_1(x + y) + \phi_2(x - y),$$
ϕ_1 and ϕ_2 are arbitrary real-valued single functions.
- By Theorem 1, $D^2 - D^{'^2} = (D - D')(D + D')$ and $a_1 = 1, b_1 = -1, a_2 = 1, b_2 = 1$ and $c_1 = 0 = c_2$.
- Hence the solution.

Example

- General solution of

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 2\frac{\partial^4 u}{\partial x^2 \partial y^2}$$

is given by

$$u = x\phi_1(x-y) + \phi_2(x-y) + x\psi_1(x+y) + \psi_2(x+y).$$

- We have $D^4 + D^{'4} - 2D^2D^{'2} = (D^2 - D^{'2})^2 = (D+D')^2(D-D')^2$.
- By using Theorem 3, m=2. Also, $n = 2$ for both expressions. For $(D+D')^2$ part, $a = 1, b = 1$ whereas for the $(D-D')^2$ part, $a = 1, b = -1$.
- Hence the solution.

Classification

- Consider

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0. \tag{8}$$

- At a point (x, y), equation (8) is said to be

▶Hyperbolic if $B^2(x,y) - 4A(x,y)C(x,y) > 0$

▶Parabolic if $B^2(x,y) - 4A(x,y)C(x,y) = 0$

▶Elliptic if $B^2(x,y) - 4A(x,y)C(x,y) < 0$

- Each category relates to specific problems such as

1. Wave Equation: $\mathbf{u_{tt} - c^2 u_{xx} = 0}$. (Hyperbolic)
2. Laplace's Equation: $\mathbf{u_{xx} + u_{yy} = 0}$. (Elliptic)
3. Heat (or Diffusion) Equation: $\mathbf{u_t} = \alpha \mathbf{u_{xx}}$. (Parabolic)

Methods and Techniques for Solving PDEs

- Change of coordinates: A PDE can be converted to ODEs or to an easier PDE by changing the coordinates of the problem.
- Separation of variables: A PDE in n independent variables is reduced to n ODEs.
- Integral transforms: A PDE in n independent variables is reduced to a PDE in $(n-1)$ independent variables. Hence, a PDE in two variables gets reduced to an ODE.
- Numerical Methods

Our immediate focus will be on a method in which we will use transformation of the independent variables x and y to another set, say ξ and β.
The aim is to transform the PDE to an easier one so that the solution can be obtained by simple integration only.