MA 201: Partial Differential Equations
Lecture - 7

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$$u_x = u_\xi \, \xi_x + u_\eta \, \eta_x \,, \qquad u_y = u_\xi \, \xi_y + u_\eta \, \eta_y \,,$$

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$$u_{yy} = u_{\xi\xi} \, \xi_y^2 + 2u_{\xi\eta} \, \xi_y \, \eta_y + u_{\eta\eta} \, \eta_y^2 + u_\xi \, \xi_{yy} + u_\eta \, \eta_{yy} \,,$$

$$u_{xy} = u_{\xi\xi} \, \xi_x \, \xi_y + u_{\xi\eta} \left(\xi_x \, \eta_y + \xi_y \, \eta_x \right) + u_{\eta\eta} \, \eta_x \, \eta_y + u_\xi \, \xi_{xy} + u_\eta \, \eta_{xy} \,.$$

- Substitute these values into (1) so as to obtain a new form of the PDE in the variables ξ and η:

$$\widetilde{A} u_{\xi\xi} + \widetilde{B} u_{\xi\eta} + \widetilde{C} u_{\eta\eta} + \widetilde{D} u_{\xi} + \widetilde{E} u_{\eta} + F u = G$$

In above, the new coefficients are as follows:

$$\widetilde{A} = A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2, \quad \widetilde{B} = 2A\,\xi_x\,\eta_x + B(\xi_x\,\eta_y + \xi_y\,\eta_x) + 2C\,\xi_y\,\eta_y,$$

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- Whether the form of the PDE (1) remains invariant even after coordinate transformation?

- It can be observed that

$$\begin{pmatrix} 2\tilde{A} & \tilde{B} \\ \tilde{B} & 2\tilde{C} \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}^t . \quad (2)$$

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- Transformation given by $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ is called canonical transformation or characteristics and the reduced form of the PDE is called the canonical form.

Canonical Transformations: Hyperbolic PDE

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- Canonical form for hyperbolic equation can be written as

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$$\begin{aligned} \xi &= y + \left(\frac{-B + \sqrt{\mathcal{D}}}{2A} \right) x = y - (1/3)x, \\ \eta &= y + \left(\frac{-B - \sqrt{\mathcal{D}}}{2A} \right) x = y - 3x. \end{aligned}$$

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- Integrating the above first with respect to ξ, we get $u_\eta = f'(\eta)$. Integrating now with respect to η, we get

$$u = \mathcal{F}(\eta) + \mathcal{G}(\xi) = \mathcal{F}(y - 3x) + \mathcal{G}(3y - x),$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions.

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Example

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

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with

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- Integrating it twice in succession with respect to η, we will get

$$u = \eta\mathcal{F}(\xi) + \mathcal{G}(\xi) = y\mathcal{F}(y - 2x) + \mathcal{G}(y - 2x),$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions.

Canonical Transformations: Elliptic PDE

- Elliptic PDEs: $B^2 - 4AC < 0$
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  - Leads to complex conjugate canonical transformation ξ and η.
- Since ξ and η are complex, we introduce new real variables

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta),$$

so that

$$\xi = \alpha + i\beta, \quad \eta = \alpha - i\beta.$$

- Under the transformation $(x, y) \to (\alpha, \beta)$, the canonical form is given by

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- For this reason, the method of characteristics is usually not applied to elliptic equations.

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- Integrating partially with respect to ξ:

$$u_{\xi} = \xi\cos\eta + f(\eta).$$

- Again integrating w.r.t. ξ

$$u = \frac{\xi^2}{2}\cos\eta + \xi f(\eta) + g(\eta) = \frac{x^2}{2}\cos(2x+y) + xf(2x+y) + g(2x+y).$$

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- When the roots λ are real complex (appearing as complex conjugates), we get specific characteristics and the equation is called elliptic.
- That is, the classification of such PDEs is simply based on the roots of an algebraic equation - the way we do for conic section.