MA 201, Mathematics III, July-November 2022, Fourier Integral and Fourier Transform

Lecture 18

Fourier Integral

We already know that Fourier series and orthogonal expansions can be used as tools to solve boundary value problems over bounded regions such as intervals, rectangles, disks and cylinders.

But we can understand that the modelling of certain physical phenomena will give rise naturally to boundary or initial value problems over unbounded regions.

For example:

To describe the temperature distribution in a very long insulated wire, we can suppose that the length of the wire is infinite, which gives rise to an initial value problem over an infinite line.

To solve this type of problems, we can generalize the notion of Fourier series by developing Fourier transform.

Fourier Integral

Recall that for a piece-wise C^1 and continuous function $f_p(t)$, its Fourier series in the interval $(-L, L)$ satisfies (f_p is periodic with period $2L$)

$$f_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right) \right], \tag{1}$$

where

$$A_n = \frac{1}{L} \int_{-L}^{L} f_p(t) \cos\left(\frac{n\pi t}{L}\right)\, dt, \quad n = 0, 1, 2, 3, \ldots,$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f_p(t) \sin\left(\frac{n\pi t}{L}\right)\, dt, \quad n = 1, 2, 3, \ldots.$$

We know

$$\cos\left(\frac{n\pi t}{L}\right) = \frac{1}{2}[\exp(\mathrm{i}n\pi t/L) + \exp(-\mathrm{i}n\pi t/L)],$$

$$\sin\left(\frac{n\pi t}{L}\right) = \frac{1}{2\mathrm{i}}[\exp(\mathrm{i}n\pi t/L) - \exp(-\mathrm{i}n\pi t/L)].$$

Fourier Integral

Putting these expressions in (1),

$$\begin{aligned} f_p(t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \left(\frac{1}{2}\{\exp(\mathrm{i}n\pi t/L) + \exp(-\mathrm{i}n\pi t/L)\} \right) \right. \\ &+ \left. B_n \left(\frac{1}{2\mathrm{i}}\{\exp(\mathrm{i}n\pi t/L) - \exp(-\mathrm{i}n\pi t/L)\} \right) \right]. \end{aligned}$$

From above, A_n and B_n can be absorbed in C_n as

$$C_n = \frac{1}{2}\left(A_n - \mathrm{i}B_n\right), \quad \overline{C_n} = \frac{1}{2}\left(A_n + \mathrm{i}B_n\right) = C_{-n}, \quad n = 0,\ 1,\ 2, \ldots$$

Subsequently, we obtain Fourier series for $f_p(t)$ in complex form as

$$f_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{\mathrm{i}n\pi t/L}, \tag{2}$$

where

$$C_n = \frac{1}{2L}\int_{-L}^{L} f_p(\tau) e^{-\mathrm{i}n\pi\tau/L}\, d\tau, \quad n = 0, \pm 1, \pm 2, \ldots \tag{3}$$

Fourier Integral

Substituting (3) into (2):

$$f_p(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-\mathrm{i}n\pi\tau/L} \, d\tau \right] e^{\mathrm{i}n\pi t/L} \left(\frac{\pi}{L} \right). \tag{4}$$

Define the frequency of the general term by

$$\sigma_n = \frac{n\pi}{L}, \tag{5}$$

and the difference in frequencies between successive terms by

$$\varDelta\sigma = \sigma_{n+1} - \sigma_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}. \tag{6}$$

Then (4) reduces to

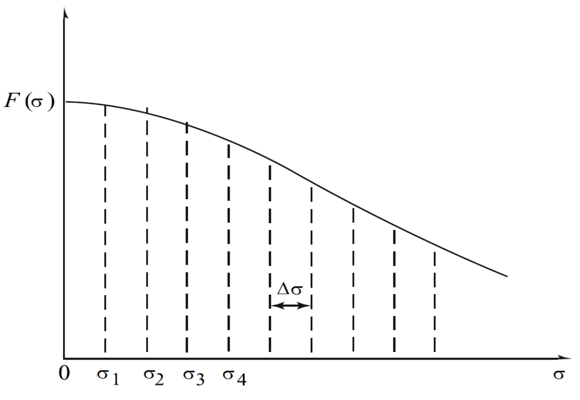
$$\begin{aligned} f_p(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{e^{\mathrm{i}\sigma_n t}}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-\mathrm{i}\sigma_n \tau} \, d\tau \right] \varDelta\sigma \\ &= \sum_{n=-\infty}^{\infty} F(\sigma_n) \varDelta\sigma, \quad \text{with} \quad F(\sigma) = \left[\frac{e^{\mathrm{i}\sigma t}}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-\mathrm{i}\sigma\tau} \, d\tau \right]. \end{aligned} \tag{7}$$

Fourier Integral

If we plot $F(\sigma)$ against σ, we can clearly see that the sum

$$\sum_{n=1}^{\infty} F(\sigma_n)\Delta\sigma, \quad \Delta\sigma = \frac{\pi}{L} \tag{8}$$

is an approximation to the area under the curve $y = F(\sigma)$.



Fourier Integral

Thus as, $L \to \infty$, (7) can be written as

$$\begin{aligned} \lim_{L\to\infty} f_p(t) = f(t) &= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} F(\sigma_n)\Delta\sigma \\ &= \int_{-\infty}^{\infty} F(\sigma)d\sigma = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{\mathrm{i}\sigma t} \int_{-\infty}^{\infty} f(\tau)e^{-\mathrm{i}\sigma\tau}\, d\tau\right] d\sigma, \end{aligned} \quad (9)$$

which is known as the complex form of Fourier integral for the function f.

A function which has its period going to infinity is called a non-periodic function.

Note that Fourier integral is a valid representation of the non-periodic function $f(t)$, a function whose period goes to infinity provided that

(a) in every finite interval, $f(t)$ is defined and is piecewise continuous,
(b) the improper integral $\int_{-\infty}^{\infty} |f(t)|dt$ exists.

Fourier Integral

Now proceed to obtain the trigonometric representation of the Fourier integral:

Equation (9) can, after writing $e^{-\mathrm{i}\sigma(\tau-t)} = \cos\sigma(\tau-t) - \mathrm{i}\sin\sigma(\tau-t)$, be expressed as

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\tau-t)}d\sigma\right)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} \cos\sigma(\tau-t)d\sigma\right)d\tau. \end{aligned}$$

(Since $\sin\sigma(\tau-t)$ is an odd function of σ, the part containing sine term will vanish.)

Also note that $\cos\sigma(\tau-t)$ is an even function of σ, therefore

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\tau-t)}d\sigma\right)d\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\left(\int_{0}^{\infty} \cos\sigma(\tau-t)d\sigma\right)d\tau. \qquad (10) \end{aligned}$$

Now, the integral has a representation with cosine terms only (with respect to σ).

Fourier Integral

Fourier cosine and sine integrals: Expanding $\cos\sigma(\tau - t)$ in (10),

$$\begin{aligned} f(t) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(\tau)\cos\sigma\tau\cos\sigma t\, d\tau + \int_0^\infty f(\tau)\cos\sigma\tau\cos\sigma t\, d\tau\right] d\sigma \\ &+ \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(\tau)\sin\sigma\tau\sin\sigma t\, d\tau + \int_0^\infty f(\tau)\sin\sigma\tau\sin\sigma t\, d\tau\right] d\sigma. \quad (11) \end{aligned}$$

Changing τ to $-\tau$ in the first and third integrals:

$$\begin{aligned} f(t) &= \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty f(-\tau)\cos\sigma\tau\cos\sigma t\, d\tau + \int_0^\infty f(\tau)\cos\sigma\tau\cos\sigma t\, d\tau\right] d\sigma \\ &+ \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty f(-\tau)(-\sin\sigma\tau)\sin\sigma t\, d\tau + \int_0^\infty f(\tau)\sin\sigma\tau\sin\sigma t\, d\tau\right] d\sigma. \end{aligned} \quad (12)$$

Remark: Above integral representation does not demand that the function f should be defined in $(-\infty, \infty)$.

Therefore, we can now think of defining Fourier integral of a function defined in $(0, \infty)$ only.

Fourier Integral

For a function f defined in $(0, \infty)$, we can use the concept of even or odd functions (or even or odd function extension) to define Fourier integral and subsequently, Fourier transform.

For an even function f or its even extension, Fourier integral of f is defined as (the first and second integrals become the same while the third and fourth integrals cancel each other in (12))

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \cos \sigma\tau \cos \sigma t \; d\tau \; d\sigma. \tag{13}$$

Equation (13) is called the *Fourier cosine integral* of f.

Fourier Integral

For an odd function f or its odd extension, Fourier integral of f is defined as (the first and second integrals cancel each other while the third and fourth integrals become the same in (12))

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(\tau) \sin \sigma\tau \sin \sigma t \, d\tau \, d\sigma. \tag{14}$$

Equation (14) is called the *Fourier sine integral* of f.

With these three definitions of Fourier integrals ((9), (13) and (14)),

we are in a position to define Fourier transforms including Fourier sine and cosine transforms.

Complex Fourier Transform

Recalling equation (9):

$$f(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{\mathrm{i}\sigma t} \int_{-\infty}^{\infty} f(\tau) e^{-\mathrm{i}\sigma\tau} \, d\tau \right] d\sigma,$$

we can now define the complex Fourier transform pair.

The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}\{f(t)\} = g(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau) \, d\tau. \tag{15}$$

The *inverse Fourier transform* is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{\mathrm{i}\sigma t} \, d\sigma. \tag{16}$$

Here $g(\sigma)$ is in the frequency domain and $f(t)$ in the time domain.

Fourier cosine transform

Recalling equation (13):

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \cos \sigma\tau \cos \sigma t \; d\tau \; d\sigma,$$

we can now define the *Fourier cosine transform* of f as

$$\mathcal{F}_c\{f(t)\} = g_c(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \cos \sigma\tau \; d\tau. \tag{17}$$

The *inverse Fourier cosine transform* is defined as

$$f(t) = \mathcal{F}_c^{-1}\{g_c(\sigma)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\sigma) \cos \sigma t \; d\sigma. \tag{18}$$

Fourier sine transform

Recalling equation (14):

$$f(t) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\tau)\sin\sigma\tau \sin\sigma t \, d\tau \, d\sigma,$$

we can now define the *Fourier sine transform* of f as

$$\mathcal{F}_s\{f(t)\} = g_s(\sigma) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\sin\sigma\tau \, d\tau. \tag{19}$$

The *inverse Fourier sine transform* is defined as

$$f(t) = \mathcal{F}_s^{-1}\{g_s(\sigma)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\sigma)\sin\sigma t \, d\sigma. \tag{20}$$

It is to be noted that

any integral transform is always accompanied by its inverse transform.

Some Properties of Fourier transform

Linearity Property:

If $\mathcal{F}\{f_1(t)\} = g_1(\sigma), \mathcal{F}\{f_2(t)\} = g_2(\sigma)$, then

$$\mathcal{F}\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1\mathcal{F}\{f_1(t)\} \pm c_2\mathcal{F}\{f_2(t)\} = c_1 g_1(\sigma) \pm c_2 g_2(\sigma),$$

where c_1 and c_2 are constants.

Theorem I:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(t-a)\} = e^{-\mathrm{i}\sigma a} g(\sigma).$$

Proof:

By definition,

$$\begin{aligned} \mathcal{F}\{f(t-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau - a) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\xi+a)} f(\xi) d\xi, \quad \text{by taking } \tau - a = \xi \\ &= e^{-\mathrm{i}\sigma a} g(\sigma). \end{aligned}$$

This is known as the shifting property.

Some Properties of Fourier transform

Theorem II:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(at)\} = \frac{1}{a} g(\sigma/a).$$

Proof:

By definition,

$$\begin{aligned}
\mathcal{F}\{f(at)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(a\tau) d\tau \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}(\sigma/a)\xi} f(\xi) d\xi/a, \quad \text{by taking } a\tau = \xi \\
&= \frac{1}{a} g(\sigma/a).
\end{aligned}$$

This is known as the scaling property.

Some Properties of Fourier transform

Theorem III:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{e^{\mathrm{i}at} f(t)\} = g(\sigma - a).$$

Proof:

By definition,

$$\begin{aligned} \mathcal{F}\{e^{\mathrm{i}at} f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} e^{\mathrm{i}a\tau} f(\tau) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}(\sigma - a)\tau} f(\tau) d\tau \\ &= g(\sigma - a). \end{aligned}$$

This is known as the translation property.

Some Properties of Fourier transform

Theorem IV:

If $\mathcal{F}\{f(t)\} = g(\sigma)$ with $f(t)$ as continuously differentiable and $f(t) \to 0$ as $|t| \to \infty$, then

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\sigma g(\sigma).$$

Proof:

By definition,

$$\begin{aligned}
\mathcal{F}\{f'(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f'(\tau) d\tau \\
&= \frac{1}{\sqrt{2\pi}} \left\{ [f(\tau) e^{-\mathrm{i}\sigma\tau}]_{-\infty}^{\infty} + i\sigma \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau) d\tau \right\} \\
&= \mathrm{i}\sigma g(\sigma).
\end{aligned}$$

$\square$

Some Properties of Fourier transform

Theorem V (Extension of Theorem IV):

If $f(t)$ is n-times continuously differentiable and $f^{(k)}(t) \to 0$ as $|t| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the n-th derivative of $f(t)$ is given by

$$\mathcal{F}\{f^{(n)}(t)\} = (\mathrm{i}\sigma)^n \mathcal{F}\{f(t)\} = (\mathrm{i}\sigma)^n g(\sigma).$$

$\square$

Theorem VI (Particular form of Theorem V - important from the point of view of solving differential equations):

If $f(t)$ is twice continuously differentiable; $f(t) \to 0$ as $|t| \to \infty$ and $f^{'}(t) \to 0$ as $|t| \to \infty$, then the Fourier transform of the second derivative of $f(t)$ is given by

$$\mathcal{F}\{f^{''}(t)\} = (\mathrm{i}\sigma)^2 \mathcal{F}\{f(t)\} = (-\sigma^2) g(\sigma).$$

$\square$

Some examples of Fourier integrals

Example: Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} e^{at}, & t < 0, \\ e^{-at}, & t > 0, \end{cases} \quad a > 0.$$

Solution : The Fourier transform of $f(t)$

$$\begin{aligned} \mathcal{F}\{f(t)\} &= g(\sigma) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-\mathrm{i}\sigma\tau} \, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-\mathrm{i}\sigma)\tau} \, d\tau + \int_{0}^{\infty} e^{-(a+\mathrm{i}\sigma)\tau} \, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\sigma^2}. \end{aligned}$$

Taking the inverse

$$\begin{aligned} f(t) = \mathcal{F}^{-1}\{g(\sigma)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{\mathrm{i}\sigma t} \, d\sigma \\ &= \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos \sigma t}{a^2+\sigma^2} \, d\sigma. \end{aligned}$$

Convolution

A convolution is an integral that expresses the amount of overlap of one function f_2 as it is shifted over another function f_1.

It, therefore, "blends" one function with another.

In other words, the output which produces a third function can be viewed as a modified version of one of the original functions.

Definition:

The convolution of two functions $f_1(t)$ and $f_2(t)$, $-\infty < t < \infty$, is defined as

$$f_1(t)*f_2(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\,d\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t-\xi)f_2(\xi)\,d\xi = f_2(t)*f_1(t),$$

provided the integral exists for each t.

This is known as the convolution integral.

Convolution: Some algebraic properties

Property I (Commutative):

$$f_1 * f_2 = f_2 * f_1$$

Property II (Associative):

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$$

Property III (Distributive):

$$(\alpha f_1 + \beta f_2) * f_3 = \alpha(f_1 * f_3) + \beta(f_2 * f_3)$$

Convolution Theorem

If $f_1(t)$ and $f_2(t)$ are Fourier transformable, i.e., we have $\mathcal{F}\{f_1(t)\} = g_1(\sigma)$ and $\mathcal{F}\{f_2(t)\} = g_2(\sigma)$, then

$$\mathcal{F}\{f_1(t) * f_2(t)\} = g_1(\sigma)g_2(\sigma).$$

In other words,

the Fourier transform of the convolution of two functions is equal to the product of the individual Fourier transforms of those functions.

We can also write

$$f_1(t) * f_2(t) = \mathcal{F}^{-1}\{g_1(\sigma)g_2(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau.$$

That is,

the inverse Fourier transform of product of Fourier transforms of two functions is equal to the convolution of these two functions.