

MA 201 (PART II), SEPTEMBER-NOVEMBER, 2020 SESSION
 PARTIAL DIFFERENTIAL EQUATIONS
 SOLUTIONS TO PROBLEMS IN PDE TUTORIAL 5
 (DATE OF DISCUSSION: NOVEMBER 24, 2020)

Fourier Integral and Transform: Properties and applications

Laplace Transform: Properties and applications

1. Find the Fourier integral representation of the following non-periodic function:

$$f(t) = \begin{cases} \sin t, & t^2 < \pi^2, \\ 0, & t^2 > \pi^2. \end{cases}$$

Solution: The Fourier transform of $f(t)$:

$$\begin{aligned} \mathcal{F}\{f(t)\} &= g(\sigma) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\sigma t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin t e^{-i\sigma t} dt \\ &= -\frac{1}{\sqrt{2\pi}} \frac{2i \sin \sigma \pi}{1 - \sigma^2}. \end{aligned}$$

Taking the inverse

$$\begin{aligned} f(t) = \mathcal{F}^{-1}\{g(\sigma)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{i\sigma t} d\sigma \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{2i \sin \sigma \pi}{1 - \sigma^2} \right] e^{i\sigma t} d\sigma \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin \sigma \pi \sin \sigma t}{1 - \sigma^2} d\sigma. \end{aligned}$$

2. Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} 0, & -\infty < t < -1, \\ -1, & -1 < t < 0, \\ 1, & 0 < t < 1, \\ 0, & 1 < t < \infty. \end{cases}$$

Solution: Realize that the given function is an odd function.

We can directly use the sine integral formula:

$$\begin{aligned} \mathcal{F}_s\{f(t)\} &= g_s(\sigma) \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \sigma t dt \\ &= \sqrt{\frac{2}{\pi}} |(-\cos \sigma t)/\sigma|_0^1 \\ &= \sqrt{\frac{2}{\pi}} \frac{1 - \cos \sigma}{\sigma}. \end{aligned}$$

Taking the inverse

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos \sigma)}{\sigma} \sin \sigma t d\sigma.$$

3. Express

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi, \\ 0, & x > \pi, \end{cases}$$

as Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1 - \cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma.$$

Solution:

From the convergence of Fourier integral, we obtain

$$f_o(\pi) = \frac{1}{2}(f_o(\pi^+) + f_o(\pi^-)) = \frac{1}{2} = \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) g_s(\sigma) d\sigma.$$

Here, $g_s(\sigma)$ is the Fourier sine transform of f_o (odd extension of f). By definition, $g_s(\sigma)$ is given by

$$\begin{aligned} g_s(\sigma) &= \int_0^\infty \sqrt{\frac{2}{\pi}} f_o(\tau) \sin \sigma \tau d\tau \\ &= \int_0^\pi \sqrt{\frac{2}{\pi}} \sin \sigma \tau d\tau = -\sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \cos \sigma \tau \Big|_0^\pi \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} (1 - \cos \sigma \pi) \end{aligned}$$

Finally, we have

$$\begin{aligned} \frac{1}{2} &= \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) g_s(\sigma) d\sigma \\ &= \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} (1 - \cos \sigma \pi) d\sigma \\ &= \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma. \end{aligned}$$

Hence,

$$\int_0^\infty \frac{1 - \cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma = \frac{\pi}{4}.$$

4. If $U(x, t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time by solving the following boundary value problem:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0 \\ \frac{\partial U}{\partial x}(0, t) &= 0, \quad U(x, 0) = f(x). \end{aligned}$$

Solution:

Taking Fourier cosine transform

$$\frac{d}{dt} \overline{U}_c(\sigma, t) + \alpha \sigma^2 \overline{U}_c = 0,$$

which gives us

$$\overline{U}_c(\sigma, t) = Ae^{-\alpha\sigma^2 t}$$

Using the initial condition,

$$\overline{U}_c(\sigma, t) = \overline{f}_c(\sigma)e^{-\alpha\sigma^2 t},$$

where $\overline{f}_c(\sigma)$ is the Fourier cosine transform of $f(x)$. Taking inverse, we get the solution

$$U(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f}_c(\sigma) e^{-\alpha\sigma^2 t} \cos \sigma x \, d\sigma.$$

5. Find the Laplace transforms of

$$(i) \, te^{3t} \cos 4t, (ii) \, t \int_0^t e^{-3t} \sin 2t \, dt, (iii) \, \int_0^t \frac{e^{-3t} \sin 2t}{t} \, dt.$$

Solution:

(i)

$$\begin{aligned} \mathcal{L}\{te^{3t} \cos 4t\} &= -\frac{d}{ds} \mathcal{L}\{e^{3t} \cos 4t\} \\ &= -\frac{d}{ds} \mathcal{L}\{\cos 4t\}_{s \rightarrow s-3} \\ &= -\frac{d}{ds} \left\{ \frac{s-3}{(s-3)^2 + 4^2} \right\} \\ &= \frac{(s-3)^2 - 16}{((s-3)^2 + 16)^2} \end{aligned}$$

(ii)

$$\begin{aligned} \mathcal{L}\{t \int_0^t e^{-3t} \sin 2t\} &= -\frac{d}{ds} \frac{\mathcal{L}\{e^{-3t} \sin 2t\}}{s} \\ &= -\frac{d}{ds} \frac{\mathcal{L}\{\sin 2t\}_{s \rightarrow s+3}}{s} \\ &= -\frac{d}{ds} \frac{2}{s((s+3)^2 + 4)} \\ &= \frac{2(3s^2 + 12s + 13)}{(s^3 + 6s^2 + 13s)^2} \end{aligned}$$

(iii)

$$\begin{aligned} \mathcal{L}\left\{\int_0^t \frac{e^{-3t} \sin 2t}{t} \, dt\right\} &= \frac{1}{s} \int_s^\infty \mathcal{L}\{e^{-3t} \sin 2t\} \, ds \\ &= \frac{1}{s} \int_s^\infty \frac{2}{(s+3)^2 + 4} \, ds \\ &= \frac{1}{s} \left[\tan^{-1} \left(\frac{s+3}{2} \right) \right]_s^\infty \\ &= \frac{1}{s} \left[\tan^{-1} \infty - \tan^{-1} \left(\frac{s+3}{2} \right) \right] \\ &= \frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{s+3}{2} \right) \right] \\ &= \frac{1}{s} \cot^{-1} \left[\frac{s+3}{2} \right] \end{aligned}$$

(Using various properties.)

6. Find the Laplace transform of the following unit step functions:

(i) $2H(\sin \pi t) - 1$, (ii) $H(t^3 - 6t^2 + 11t - 6)$.

Solution: (i) From definition of unit step function

$$H(\sin \pi t) = \begin{cases} 1, & \sin \pi t > 0, \\ 0, & \sin \pi t < 0. \end{cases}$$

This will give the given function as +1 between 0 and 1, 2 and 3, and so on whereas it will be -1 between 1 and 2, 3 and 4, and so on. That is

$$\begin{aligned} \mathcal{L}\{2H(\sin \pi t) - 1\} &= \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt + \int_2^3 e^{-st} dt - \int_3^4 e^{-st} dt + \dots \\ &= \left| \frac{e^{-st}}{-s} \right|_0^1 - \left| \frac{e^{-st}}{-s} \right|_1^2 + \left| \frac{e^{-st}}{-s} \right|_2^3 - \left| \frac{e^{-st}}{-s} \right|_3^4 + \dots \\ &= \frac{1}{s} [(1 - e^{-s}) + (e^{-2s} - e^{-s}) + (e^{-2s} - e^{-3s}) + (e^{-4s} - e^{-3s}) + \dots] \\ &= \frac{1}{s} [1 - 2e^{-s}(1 - e^{-s} + e^{-2s} - e^{-3s} + \dots)] \\ &= \frac{1}{s} \left[1 - 2e^{-s} \frac{1}{1 + e^{-s}} \right] \\ &= \frac{1}{s} \frac{1 - e^{-s}}{1 + e^{-s}} \\ &= \frac{1}{s} \tanh \frac{s}{2} \end{aligned}$$

(ii) $\frac{1}{s} [e^{-s} - e^{-2s} + e^{-3s}]$.

From definition of unit step function

$$H(t^3 - 6t^2 + 11t - 6) = \begin{cases} 1, & (t-1)(t-2)(t-3) > 0, \\ 0, & (t-1)(t-2)(t-3) < 0. \end{cases}$$

This will give the given function as +1 between 1 and 2, and 3 and ∞ .

$$\begin{aligned} \mathcal{L}\{H(t^3 - 6t^2 + 11t - 6)\} &= \int_1^2 e^{-st} dt + \int_3^\infty e^{-st} dt \\ &= \left| \frac{e^{-st}}{-s} \right|_1^2 + \left| \frac{e^{-st}}{-s} \right|_3^\infty \\ &= \frac{1}{s} [e^{-s} - e^{-2s} + e^{-3s}] \end{aligned}$$

7. Find the inverse Laplace transforms:

(i) $\frac{2s+3}{s^2+4s+6}$, (ii) $\frac{2s^2-3s+5}{s^2(s^2+1)}$.

Solution: (i)

$$\begin{aligned} f(s) &= \frac{2s+3}{s^2+4s+6} \\ &= \frac{2(s+2) - 1}{(s+2)^2 + (\sqrt{2})^2} \end{aligned}$$

Taking inverse

$$F(t) = e^{-2t} (2 \cos \sqrt{2}t - \frac{1}{\sqrt{2}} \sin \sqrt{2}t)$$

(ii) Take $\frac{2s^2 - 3s + 5}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$. Find $A = -3, B = 5, C = 3, D = -3$ so that

$$\frac{2s^2 - 3s + 5}{s^2(s^2 + 1)} = -\frac{3}{s} + \frac{5}{s^2} + \frac{3s}{s^2 + 1} - \frac{3}{s^2 + 1}.$$

By inverting,

$$F(t) = 5t - 3 - 3(\sin t - \cos t)$$

8. Find the inverse Laplace transforms of the following by the theory of residues:

(i) $\frac{1}{(s+1)(s-2)^2}$, (ii) $\frac{s+2}{(s+1)(s^2+4)}$.

Solution: (i) Poles are $s = -1$ (simple pole) and $s = 2$ (double pole). Residue due to $s = -1$ is $\frac{e^{-t}}{9}$ and due to $s = 2$ is $\frac{te^{2t}}{3} - \frac{e^{2t}}{9}$. Hence the required functions is found as

$$\frac{e^{-t}}{9} + \frac{te^{2t}}{3} - \frac{e^{2t}}{9}.$$

(ii) The poles are $s = -1, s = 2i, s = -2i$. The respective residues are $\frac{e^{-t}}{5}, -\frac{1}{10}(1+3i)e^{2it}, -\frac{1}{10}(1-3i)e^{-2it}$. The required function is obtained as

$$\frac{1}{5}e^{-t} - \frac{1}{5}\cos 2t + \frac{3}{5}\sin 2t.$$

9. Solve the following Initial Value Problems:

(i) $\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin t$, $y(0) = 0$, $\dot{y}(0) = 1$, (ii) $t\ddot{y} + 2\dot{y} + ty = 0$, $y(0) = 1$.

Solution:

(i) After getting the Laplace transform of y as

$$\mathcal{L}\{y(t)\} = \frac{1}{3} \frac{1}{(s+1)^2 + 1} + \frac{2}{3} \frac{1}{(s+1)^2 + 2^2},$$

and then inverting

$$y(t) = \frac{e^{-t}}{3}[\sin t + \sin 2t]$$

(ii)

Taking Laplace transform on the equation,

$$\begin{aligned} -\frac{d}{ds}\mathcal{L}\{\ddot{y}\} + 2\mathcal{L}\{\dot{y}\} - \frac{d}{ds}\mathcal{L}\{y\} &= 0 \\ \Rightarrow -\frac{d}{ds}[s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0)] + 2[s\mathcal{L}\{y\} - y(0)] - \frac{d}{ds}\mathcal{L}\{y\} &= 0, \end{aligned}$$

which gives the ODE

$$\frac{d}{ds}\mathcal{L}\{y\} = -\frac{1}{s^2 + 1},$$

$$\Rightarrow \mathcal{L}\{y\} = -\tan^{-1}(s) + C.$$

Inverting

$$y(t) = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}(\tan^{-1}(s) + C)\right\} = \sin t/t.$$

10. Solve the following IBVP for one dimensional heat conduction equation for a rod of unit length and with unit diffusivity:

$$\begin{aligned}U_t &= U_{xx}, \quad 0 < x < 1, t > 0, \\U(x, 0) &= 3 \sin(2\pi x), \quad 0 < x < 1, \\U(0, t) &= 0 = U(1, t), \quad t > 0.\end{aligned}$$

Solution:

Using $\mathcal{L}\{U(x, t)\} = \bar{u}(x, s)$ and taking transform on the equation, we get

$$\frac{d^2 \bar{u}(x, s)}{dx^2} - s\bar{u}(x, s) = -3 \sin(2\pi x).$$

Solving this non-homogeneous ODE:

$$\bar{u}(x, s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{3 \sin(2\pi x)}{s + 4\pi^2}.$$

Converting and using the boundary conditions, $A(s) = 0 = B(s)$ thereby getting

$$\bar{u}(x, s) = \frac{3 \sin(2\pi x)}{s + 4\pi^2}.$$

Inverting,

$$u(x, t) = 3e^{-4\pi^2 t} \sin(2\pi x).$$