MA 201 (Part II), September-November, 2020 session
Partial Differential Equations
Session September-November,2020
Solutions to PDE Tutorial Problems - 2, Date of Discussion: October 27th October, 2020

Cauchy Problem, Method of Characteristics, General Solution, Lagrange Method

1. Find the general integral of the following partial differential equations, where $u_x = p$ and $u_y = q$.

   (i) $x^2p + y^2q + u^2 = 0$, (ii) $x^2(y - u)p + y^2(u - x)q = u^2(x - y)$.

   Solution:
   (i) $F(1/x - 1/y, 1/y + 1/u) = 0$.
   (Write the auxiliary equations as $dx/x^2 = dy/y^2 = du/(-u^2)$. Take the first two fractions to get $1/x - 1/y = c_1$ and take the second and third fractions to get $1/y + 1/u = c_2$. [You can also take the first and third fractions.])

   (ii) $F(1/x + 1/y + 1/u, xyu) = 0$.

   (**Step 1:** Write the auxiliary equations as $\dfrac{dx/x^2}{y-u} = \dfrac{dy/y^2}{u-x} = \dfrac{du/u^2}{x-y}$. Then rewrite them as $\dfrac{(dx/x^2) + (dy/y^2)}{y-x} = \dfrac{du/u^2}{x-y}$ which gives $\dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{du}{u^2} = 0 \ \Rightarrow 1/x + 1/y + 1/u = c_1$.

   Step 2: Rewrite the auxiliary equations as $\dfrac{dx/x}{x(y-u)} = \dfrac{dy/y}{y(u-x)} = \dfrac{du/u}{u(x-y)}$. Then $\dfrac{(dx/x) + (dy/y)}{u(y-x)} = \dfrac{du/u}{u(x-y)}$ which gives $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{du}{u} = 0 \ \Rightarrow \log xyu = \log c_2$.)
   These steps give us the solution.

2. Find an integral surface of $xp+yq = z$ passing through the curve $x+y = 1$ and $x^2+y^2+z^2 = 25$.

   Solution: $x^2 + y^2 + z^2 = 25(x + y)^2$.
   (Find the curves as $x/y = c_1$ and $z/y = c_2$. Using the given condition, relate the constants by $C_1^2 + C_2^2 + 1 = 25(1 + C_1)^2$.)

3. Show that the integral surface of the equation $2y(u - 3)p + (2x - u)q = y(2x - 3)$ that passes through the circle $x^2 + y^2 = 2x$, $u = 0$ is $x^2 + y^2 - u^2 - 2x + 4u = 0$.

   Solution: Take the first and third fractions to get the first curve as $x^2 - u^2 - 3x + 6u = C_1$. Then use multipliers $1, 2y, -2$ to write $\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} - 2\dfrac{du}{dt} = 0$ (each fraction is equal to this). This gives $d\{x + y^2 - u\} = 0 \ \Rightarrow x + y^2 - 2u = c_2$ to get the second curve. Using the given conditions, $x^2 - 3x = c_1$, $x + y^2 = c_2$ adding which we get $x^2 + y^2 - 2x = c_1 + c_2$ and finally the relation $c_1 + c_2 = 0$ which will give the required surface.

4. Find the solution of the following Cauchy problems:

   (i) $u_x + u_y = 2$, $u(x, 0) = x^2$; (ii) $5u_x + 2u_y = 0$, $u(x, 0) = \sin x$.

   Solution:(i) The characteristics equations are: $\frac{dx(t,\tau)}{dt} = 1$, $\frac{dy(t,\tau)}{dt} = 1$, $\frac{du(t,s)}{dt} = 2$, whose solutions are given by

$$x(t, \tau) = t + C_1(\tau), \quad y(t, \tau) = t + C_2(\tau), \quad u(t, \tau) = 2t + C_3(\tau).$$

Using the parametric initial conditions $x(0,\tau)=\tau,\ \ y(0,\tau)=0,\ \ u(0,\tau)=\tau^2$, we obtain

$$x(t,\tau)=t+\tau,\quad y(t,\tau)=t,\quad u(t,\tau)=2t+\tau^2.$$

Now, writing (t,τ) as functions of (x,y), we have $t=y \quad \tau=x-y$. Thus, the integral surface is given by

$$U(x,y)=u(t(x,y),\tau(x,y))=2y+(x-y)^2.$$

(ii) Proceed as in Q.4 (i) to get the solution. The integral surface is given by $U(x,y)=\sin(x-\frac{5}{2}y)$.

5. Consider the PDE $xu_x+yu_y=4u$, where $x,y\in\mathbb{R}$. Find the characteristic curves for the equation. an explicit solution that satisfies $u=1$ on the circle $x^2+y^2=1$.

Solution: The characteristic equations $x'(t)=x,\quad y'(t)=y,\quad u'(t)=4u$ yield the solutions

$$x(t)=C_1e^t,\quad y(t)=C_2e^t,\quad u(t)=C_3e^{4t}\ \ t\in I.$$

The characteristics curves are given by $\frac{x}{y}=C$. With $x_0(\tau)=\tau,\ \ y_0(\tau)=(1-\tau^2)^{1/2},\ \ u_0(\tau)=1$, it now follows that $U(x,y)=u(t(x,y),\tau(x,y))=(x^2+y^2)^2$, which is the required integral surface.

6. Find a function $u(x,y)$ that solves the Cauchy problem

$$x^2u_x+y^2u_y=u^2,\quad u(x,2x)=x^2,\ \ x\in\mathbb{R}.$$

Is the solution defined for all x and y? Check whether the transversality condition holds.

Solution: Solving the characteristic equations: $\frac{dx(t,\tau)}{dt}=x^2,\quad \frac{dy(t,\tau)}{dt}=y^2,\quad \frac{du(t,\tau)}{dt}=u^2$, we get

$$-\frac{1}{x}=t+C_1(\tau),\quad -\frac{1}{y}=t+C_2(\tau),\quad -\frac{1}{u}=t+C_3(\tau).$$

Using the parametric initial conditions $x(0,\tau)=\tau,\ \ y(0,\tau)=2\tau,\ \ u(0,\tau)=\tau^2$, we obtain

$$\frac{1}{x}=\frac{1}{\tau}-t,\quad \frac{1}{y}=\frac{1}{2\tau}-t,\quad \frac{1}{u}=\frac{1}{\tau^2}-t.$$

Now, writing (t,τ) as functions of (x,y), we obtain $\tau=\frac{xy}{2(y-x)} \quad t=\frac{y-2x}{xy}$. Thus, the integral surface is given by

$$U(x,y)=u(t(x,y),\tau(x,y))=\frac{\tau^2}{1-t\tau^2}=\frac{x^2y^2}{\{4(y-x)^2-xy(y-2x)\}}.$$

The solution is not defined on the curve $4(y-x)^2=xy(y-2x)$ that passes through the origin.

7. Find a function $u(x,y)$ that satisfies the PDE $-yu_x+xu_y=0$ subject to the side condition $u(x,x^2)=x^3,\quad (x>0)$.

Solution

$$\text{PDE:}\qquad -yu_x+xu_y=0 \tag{1}$$
$$\text{Side Condition:}\qquad u(x,x^2)=x^3,\quad (x>0). \tag{2}$$

Step 1. (Finding characteristic curves $(x(t,\tau),y(t,\tau),u(t,\tau))$)

Solve

$$\frac{d}{dt}x(t,\tau)=-y(t,\tau),\quad \frac{d}{dt}y(t,\tau)=x(t,\tau),\quad \frac{d}{dt}u(t,\tau)=0.$$

with initial conditions $x(0,\tau) = \tau, \quad y(0,\tau) = \tau^2, \quad u(0,\tau) = \tau^3$. The general solution is

$$x(t,\tau) = c_1(\tau)\cos(t) + c_2(\tau)\sin(t), \;\; y(t,\tau) = c_1(\tau)\sin(t) - c_2(\tau)\cos(t).$$

Step 2. (Applying IC)
Using ICs, we find that

$$c_1(\tau) = \tau, \quad c_2(\tau) = -\tau^2,$$

and hence

$$x(t,\tau) = \tau\cos(t) - \tau^2\sin(t) \quad \text{and} \;\; y(t,\tau) = \tau\sin(t) + \tau^2\cos(t).$$

Step 3. (Writing the parametric form of the solution)
Note that $c(x,y) = 0$ and $d(x,y) = 0$. Therefore, it follows that

$$d(t,\tau) = 0, \quad \mu(t,\tau) = 1.$$

In view of the given initial curve and $u = u(t,\tau)$, we obtain

$$u(x(0,\tau), y(0,\tau)) = u(\tau,\tau^2) = g(\tau) = \tau^3, \quad u(t,\tau) = \tau^3.$$

Thus, the parametric form of the solution of the problem is given by

$$x(t,\tau) = \tau\cos(t) - \tau^2\sin(t), \quad y(t,\tau) = \tau\sin(t) + \tau^2\cos(t), \quad u(t,\tau) = \tau^3.$$

Step 4. (Expressing $u(\tau,t)$ in terms of $U(x,y)$) It is left as an exercise to show that

$$U(x,y) = \frac{1}{\sqrt{8}}\left[-1 + \sqrt{1 + 4(x^2+y^2)}\right]^{3/2}.$$

8. Find a solution $u = u(x,y)$ to the partial differential equation

$$x\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -xy, \; x > 0$$

and subject to $u = 5$ on $xy = 1$. Further, justify that solution exists when $xy \leq 19$.

Solution: The characteristics equations are

$$\frac{dx}{dt} = x, \;\; \frac{dy}{dt} = yu, \;\; \frac{du}{dt} = -xy.$$

Clearly

$$y\frac{dx}{dt} + x\frac{dy}{dt} + u\frac{du}{dt} = xy = -\frac{du}{dt}.$$

Which gives

$$d(xy) + udu + du = 0.$$

Integration leads to

$$xy + \frac{u^2}{2} + u = c_1.$$

Apply initial condition to have

$$1 + \frac{25}{2} + 1 = c_1.$$

Thus, $c_1 = 37/2$ and hence solution is given by

$$xy + \frac{u^2}{2} + u = \frac{37}{2}.$$

After simplification, we obtain

$$(u+1)^2 = 38 - 2xy \;\; \text{or} \;\; u + 1 = \pm\sqrt{38 - 2xy}.$$

Which is valid for $38 - 2xy \geq 0$.

9. Find a integral surface for the initial value problem

$$uu_x + xu_y = 1, \;\; u\Big(\frac{1}{2}x^2 + 1, \frac{1}{6}x^3 + x\Big) = x.$$

Are there other solutions? Explain lack of uniqueness and find at least two solutions.

Solution: We have

$$\frac{dx}{u} = \frac{dy}{x} = \frac{du}{1}$$

along with initial conditions

$$x(0) = \frac{\tau^2}{2} + 1, \; y(0) = \frac{\tau^3}{6} + \tau, \;\; u(0) = \tau.$$

Above differential equation leads to

$$dx - udu = 0$$

and hence

$$x - \frac{u^2}{2} = c_1.$$

Now, apply the given condition to have $c_1 = 1$. Therefore, a integral surface is given by

$$u = \sqrt{2(x-1)}.$$

Again transversality condition yields

$$\begin{aligned} J &= a(0)y_0'(\tau) - b(0)y_0'(\tau) \\ &= u(0)\Big(\frac{\tau^2}{2} + 1\Big) - x(0)\tau = \tau\Big(\frac{\tau^2}{2} + 1\Big) - \Big(\frac{\tau^2}{2} + 1\Big)\tau = 0. \end{aligned}$$

Thus, either solution does not exist or infinitely many solutions exist. As there exists a solution, therefore there are infinitely many solutions.

For other solution, we solve $dy - xdu = 0$ or $dy - (\frac{u^2}{2} + 1)du = 0$ to have

$$y = \frac{u^3}{6} + u + c_2.$$

Initial condition yields $c_2 = 0$ so that another solution surface is given by

$$y = \frac{u^3}{6} + u.$$

10. A river is defined by the domain $D = \{(x, y) : \;\; |y| < 1, -\infty < x < \infty\}$. A factory spills contaminant into the river. The contaminant is further spread and convected by the flow in the river. The velocity field of the fluid is only in the x-direction. The concentration of the contaminant at a point (x, y) in the river and at time τ is denoted by $u(x, y, \tau)$. Conservation of matter and momentum implies that u satisfies the first-order PDE

$$u_\tau - (y^2 - 1)u_x = 0,$$

and the initial condition is given by $u(x, y, 0) = e^y e^{-x^2}$. Find the concentration $u(x, y, \tau)$ for fixed y. Further, conclude with justification whether the solution is unique.

Solution: The Characteristics equations, for fixed y, are given by

$$\frac{dx}{dt} = (y^2 - 1), \;\; \frac{d\tau}{dt} = -1 \;\; \& \;\; \frac{du}{dt} = 0. \tag{3}$$

Solving these equations, we obtain

$$x(t) \quad = \quad (y^2-1)t+C_1, \quad \tau(t) = -t+C_2 \ \ \& \ \ u(t) = C_3. \tag{4}$$

Use initial condition for fixed y to we obtain

$$u(0) = u(x(0), y, \tau(0)) = u(x_0(s), y, \tau_0(s)) = e^y e^{-s^2}$$

with

$$x_0(s) = s \ \ \& \ \ \tau_0(s) = 0$$

Then (4) gives

$$x(t) \quad = \quad (y^2-1)t+s, \quad \tau(t) = -t \ \ \& \ \ u(t) = e^y e^{-s^2}. \tag{5}$$

Therefore,

$$t = -\tau, \quad s = x-(y^2-1)\tau, \ \ \& \ \ u = e^y e^{-[x-(y^2-1)\tau]^2}.$$

For uniqueness, consider the transversality condition

$$J = \begin{vmatrix} y^2-1 & 1 \\ -1 & 0 \end{vmatrix} = 1.$$

Thus, the solution is unique near initial curve.