This is known as the *convolution integral*, a general result that applies to any linear dynamic system.

Specializing Eq. (4.2.2) for the SDF system by substituting Eq. (4.1.7) for the unit impulse response function gives *Duhamel's integral*:

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau) e^{-\zeta\omega_n(t-\tau)} \sin[\omega_D(t-\tau)]\, d\tau \tag{4.2.3}$$

For an undamped system this result simplifies to

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin[\omega_n(t-\tau)]\, d\tau \tag{4.2.4}$$

Implicit in this result are "at rest" initial conditions, $u(0) = 0$ and $\dot{u}(0) = 0$. If the initial displacement and velocity are $u(0)$ and $\dot{u}(0)$, the resulting free vibration response given by Eqs. (2.2.4) and (2.1.3) should be added to Eqs. (4.2.3) and (4.2.4), respectively. Recall that we had used Eq. (4.2.4) in Section 1.10.2, where four methods for solving the equation of motion were introduced.

Duhamel's integral provides a general result for evaluating the response of a linear SDF system to arbitrary force. This result is restricted to linear systems because it is based on the principle of superposition. Thus it does not apply to structures deforming beyond their linearly elastic limit. If $p(\tau)$ is a simple function, closed-form evaluation of the integral is possible and Duhamel's integral is an alternative to the classical method for solving differential equations (Section 1.10.1). If $p(\tau)$ is a complicated function that is described numerically, evaluation of the integral requires numerical methods. These will not be presented in this book, however, because they are not particularly efficient. More effective methods for numerical solution of the equation of motion are presented in Chapter 5.

PART B: RESPONSE TO STEP AND RAMP FORCES

4.3 STEP FORCE

A *step force* jumps suddenly from zero to p_o and stays constant at that value (Fig. 4.3.1b). It is desired to determine the response of an undamped SDF system (Fig. 4.3.1a) starting at rest to the step force:

$$p(t) = p_o \tag{4.3.1}$$

The equation of motion has been solved (Section 1.10.2) using Duhamel's integral to obtain

$$u(t) = (u_{\text{st}})_o(1 - \cos\omega_n t) = (u_{\text{st}})_o\left(1 - \cos\frac{2\pi t}{T_n}\right) \tag{4.3.2}$$

where $(u_{\text{st}})_o = p_o/k$, the static deformation due to force p_o.

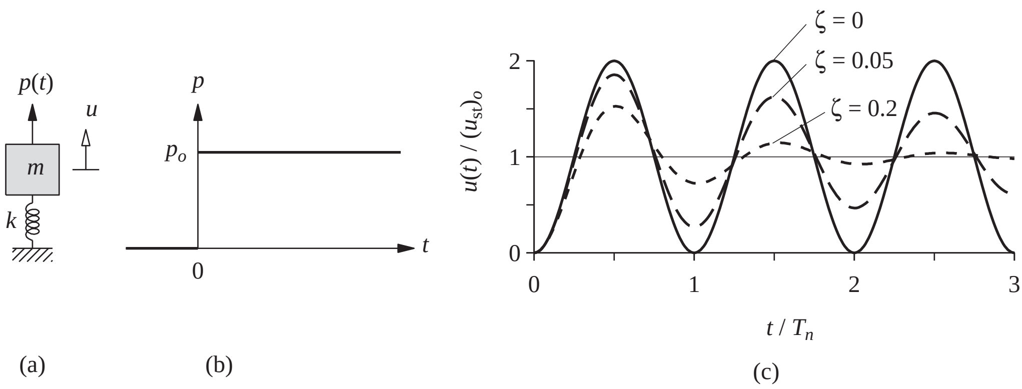

Figure 4.3.1 (a) SDF system; (b) step force; (c) dynamic response.

The normalized deformation or displacement, $u(t)/(u_{\text{st}})_o$, is plotted against normalized time, t/T_n, in Fig. 4.3.1c. It is seen that the system oscillates at its natural period about a new equilibrium position, which is displaced through $(u_{\text{st}})_o$ from the original equilibrium position of $u = 0$. The maximum displacement can be determined by differentiating Eq. (4.3.2) and setting $\dot{u}(t)$ to zero, which gives $\omega_n \sin \omega_n t = 0$. The values t_o of t that satisfy this condition are

$$\omega_n t_o = j\pi \qquad \text{or} \qquad t_o = \frac{j}{2} T_n \tag{4.3.3}$$

where j is an odd integer; even integers correspond to minimum values of $u(t)$. The maximum value u_o of $u(t)$ is given by Eq. (4.3.2) evaluated at $t = t_o$; these maxima are all the same:

$$u_o = 2(u_{\text{st}})_o \tag{4.3.4}$$

Thus a suddenly applied force produces twice the deformation it would have caused as a slowly applied force.

The response of a system with damping can be determined by substituting Eq. (4.3.1) in Eq. (4.2.3) and evaluating Duhamel's integral to obtain

$$u(t) = (u_{\text{st}})_o \left[1 - e^{-\zeta\omega_n t} \left(\cos \omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_D t \right) \right] \tag{4.3.5}$$

For analysis of damped systems the classical method (Section 1.10.1) may be easier, however, than evaluating Duhamel's integral. The differential equation to be solved is

$$m\ddot{u} + c\dot{u} + ku = p_o \tag{4.3.6}$$

Its complementary solution is given by Eq. (f) of Derivation 2.2, the particular solution is $u_p = p_o/k$, and the complete solution is

$$u(t) = e^{-\zeta\omega_n t}(A \cos \omega_D t + B \sin \omega_D t) + \frac{p_o}{k} \tag{4.3.7}$$

where the constants A and B are to be determined from initial conditions. For a system starting from rest, $u(0) = \dot{u}(0) = 0$ and

$$A = -\frac{p_o}{k} \qquad B = -\frac{p_o}{k}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Substituting these constants in Eq. (4.3.7) gives the same result as Eq. (4.3.5). When specialized for undamped systems this result reduces to Eq. (4.3.2), already presented in Fig. 4.3.1c.

Equation (4.3.5) is plotted in Fig. 4.3.1c for two additional values of the damping ratio. With damping the overshoot beyond the static equilibrium position is smaller, and the oscillations about this position decay with time. The damping ratio determines the amount of overshoot and the rate at which the oscillations decay. Eventually, the system settles down to the static deformation, which is also the steady-state deformation.

4.4 RAMP OR LINEARLY INCREASING FORCE

In Fig. 4.4.1b, the applied force $p(t)$ increases linearly with time. Naturally, it cannot increase indefinitely, but our interest is confined to the time duration where $p(t)$ is still small enough that the resulting spring force is within the linearly elastic limit of the spring.

While the equation of motion can be solved by any one of several methods, we illustrate use of Duhamel's integral to obtain the solution. The applied force

$$p(t) = p_o \frac{t}{t_r} \tag{4.4.1}$$

is substituted in Eq. (4.2.4) to obtain

$$u(t) = \frac{1}{m\omega_n}\int_0^t \frac{p_o}{t_r}\tau \sin\omega_n(t-\tau)\,d\tau$$

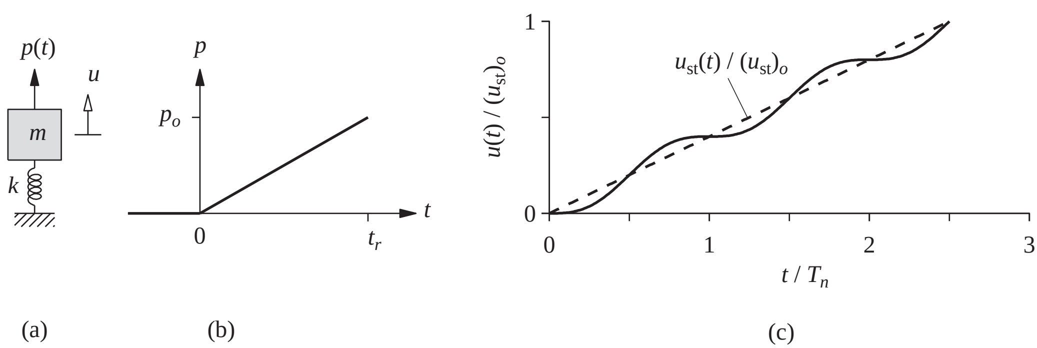

Figure 4.4.1 (a) SDF system; (b) ramp force; (c) dynamic and static responses.

This integral is evaluated and simplified to obtain

$$u(t) = (u_{\mathrm{st}})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) = (u_{\mathrm{st}})_o \left(\frac{t}{T_n} \frac{T_n}{t_r} - \frac{\sin 2\pi t/T_n}{2\pi t_r/T_n} \right) \tag{4.4.2}$$

where $(u_{\mathrm{st}})_o = p_o/k$, the static deformation due to force p_o.

Equation (4.4.2) is plotted in Fig. 4.4.1c for $t_r/T_n = 2.5$, wherein the static deformation at each time instant,

$$u_{\mathrm{st}}(t) = \frac{p(t)}{k} = (u_{\mathrm{st}})_o \frac{t}{t_r} \tag{4.4.3}$$

is also shown; $u_{\mathrm{st}}(t)$ varies with time in the same manner as $p(t)$ and the two differ by the scale factor $1/k$. It is seen that the system oscillates at its natural period T_n about the static solution.

4.5 STEP FORCE WITH FINITE RISE TIME

Since in reality a force can never be applied suddenly, it is of interest to consider a dynamic force that has a finite rise time, t_r, but remains constant thereafter, as shown in Fig. 4.5.1b:

$$p(t) = \begin{cases} p_o(t/t_r) & t \le t_r \\ p_o & t \ge t_r \end{cases} \tag{4.5.1}$$

The excitation has two phases: ramp or rise phase and constant phase.

For a system without damping starting from rest, the response during the ramp phase is given by Eq. (4.4.2), repeated here for convenience:

$$u(t) = (u_{\mathrm{st}})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad t \le t_r \tag{4.5.2}$$

The response during the constant phase can be determined by evaluating Duhamel's integral after substituting Eq. (4.5.1) in Eq. (4.2.4). Alternatively, existing solutions for free vibration and step force could be utilized to express this response as

$$u(t) = u(t_r) \cos \omega_n (t - t_r) + \frac{\dot{u}(t_r)}{\omega_n} \sin \omega_n (t - t_r) + (u_{\mathrm{st}})_o [1 - \cos \omega_n (t - t_r)] \tag{4.5.3}$$

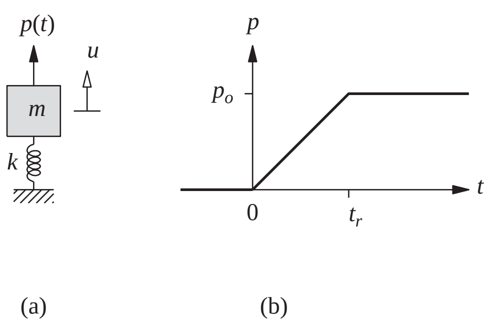

Figure 4.5.1 (a) SDF system; (b) step force with finite rise time.