Introduction to Optimization

Prakash Kotecha
Debasis Maharana & Remya Kommadath
Department of Chemical Engineering
Indian Institute of Technology Guwahati

Course Outline

- Introduction to Optimization
- Linear and Non-Linear Regression
- Metaheuristic Techniques & their implementation
  - Teaching Learning Based Optimization
  - Particle Swarm Optimziation
  - Differential Evolution
  - Genetic Algorithm (Binary & Real Coded)
  - Artificial Bee Colony Optimization
- Constraint Handling (Penalty based, Correction Approaches)
- Mathematical Programming Techniques
  - Linear Programming (LP)
  - Mixed Integer Linear Programming (MILP)
- Case Study: Production Planning
  - Modelling and Solution as MILP
  - Modelling and Solution using Metaheuristic Techniques

State-of-the-art optimization solvers

➢ Optimization Toolbox of MATLAB

- Inbuilt mathematical programming techniques functions for LP, NLP, MILP
- Inbuilt metaheuristic techniques for solving NLP, MINLP
- Will be used for the implementation of metaheuristic techniques
- Made accessible as part of this course, details would be provided during the course

➢ IBM ILOG CPLEX Optimization Studio

- Easy to use modelling framework
- Full version is available as part of IBM Academic Initiative
- Demo or Evaluation version for Non-Academic Users
- Can be used to solve LP, MILP, QP, MIQP, MIQCP, Constraint Programming.
- https://ur-a2mt-prod.mybluemix.net/a2mt/email-auth

➢ General Algebraic Modelling System (GAMS)

- Modelling language https://www.gams.com/download/
- Will be used to solve LP, MILP, NLP, MINLP

➢ NEOS Optimization Solver

- Free online solver https://neos-server.org/neos/solvers/index.html
- Accepts input from a wide range of modelling platforms

Applications

- Time tabling
- Site selection for an industry
- Production planning, controlling and scheduling
- Optimal tariff design
- Design of civil engineering structures for minimum cost
- Design of aircraft and aerospace structures for minimum weight
- Design of electrical machinery such as motors, generators and transformers
- Design of pumps, turbines and heat exchangers for maximum efficiency
- Design of pipeline networks for process industries
- Determining the trajectories for space vehicles

Engineering optimization: Theory and practice, S. S Rao
Operations research: An introduction, H. A. Taha

Applications

GAMS

DOWNLOAD SUPPORT SALES

News & Events | Products | Documentation | Resources | Community

Documentation | Model Libraries

The GAMS Model Library

The models in the GAMS Model Library have been selected because they represent interesting and sometimes classic problems. Examples of problems included in the library are production and shipment by firms, investm plants, macroeconomics stabilization, applied general equilibrium, international trade in aluminum and in copper, water distribution networks, and many more.

Another criterion for including a model in the library is that it illustrates the modeling capabilities GAMS offers. For example, the mathematical specification of cropping patterns can be represented handily in GAMS. A furt point in the search for the optimal solution of dynamic nonlinear optimization problems.

Show All entries

Seq	Name	Description	Type	Author	Subject
64	abel	Linear Quadratic Control Problem	NLP	Kendrick, D	Macro Economics
208	absmip	Discontinous functions abs() min() max() sign() as MIPs	MIP	GAMS Development Corporation	Mathematics
88	agreste	Agricultural Farm Level Model of NE Brazil	LP	Kutcher, G P	Agricultural Economics
8	aircraft	Aircraft Allocation Under Uncertain Demand	LP	Dantzig, G B	Management Science and OR
189	airsp	Aircraft Allocation	LP	Dantzig, G B	Stochastic Programming
196	airsp2	Aircraft Allocation - stochastic optimization with DECIS	DECIS	Dantzig, G B	Stochastic Programming
60	ajax	Ajax Paper Company Production Schedule	LP	CDC	Management Science and OR
124	alan	A Quadratic Programming Model for Portfolio Analysis	MINLP	Manne, A S	Finance
165	alkyl	Simplified Alkylation Process	NLP	Berna, T J	Chemical Engineering
396	allbases	Enumerate all Feasible Basic Solutions of the Transportation Problem	MIP	Dantzig, G B	Micro Economics
170	alphamet	Alphametics - a Mathematical Puzzle	MIP	de Wetering, A V	Recreational Models

https://www.gams.com/latest/gamslib_ml/libhtml/index.html

Applications

BuildingIQ Develops Proactive Algorithms for HVAC Energy Optimization in Large-Scale Buildings

Large-scale commercial buildings can reduce energy costs by 10–25% with BuildingIQ's energy optimization system.

Challenge
Develop a real-time system to minimize HVAC energy costs in large-scale commercial buildings via proactive, predictive optimization

Solution
Use MATLAB to analyze and visualize big data sets, implement advanced optimization algorithms, and run the algorithms in a production cloud environment

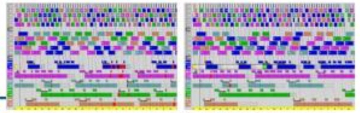
Results
- Gigabytes of data analyzed and visualized
- Algorithm development speed increased tenfold
- Best algorithmic approaches quickly identified

"MATLAB has helped accelerate our R&D and deployment with its robust numerical algorithms, extensive visualization and analytics tools, reliable optimization routines, support for object-oriented programming, and ability to run in the cloud with our production Java applications."

Borislav Savkovic
BuildingIQ

Link to user story

HKM Optimizes Just-in-Time Steel Manufacturing Schedule



Manually reviewed plant schedule (left) and plant schedule automatically optimized with MATLAB genetic algorithms (right). The optimized schedule minimizes schedule conflicts (in red), meets delivery dates, and achieves the target utilization rate.

Challenge
Optimize a steel production process to enable consistent, just-in-time delivery

Solution
Use MATLAB, global optimization, and parallel computing to maximize throughput of more than 5 million tonnes of steel annually

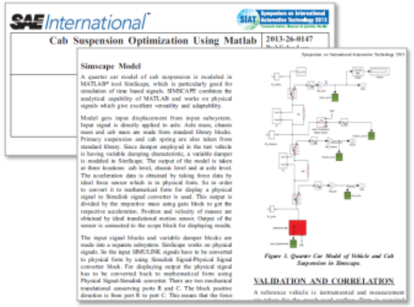
Results
- Algorithm development accelerated by a factor of 10
- Optimization time cut from 1 hour to 5 minutes
- Customer satisfaction increased

"C++, Java, or third-party optimization solutions would have required us to spend significantly more time in development or to simplify our constraints. Only MATLAB provided the flexibility, scalability, development speed, and level of optimization that we required."

Alexey Nagaytsev
Hüttenwerke Krupp Mannesmann

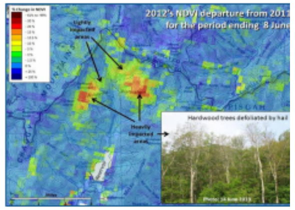
Ashok Leyland Improves Ride Comfort with Simscape Model of Cab Suspension

- Challenge:
  - Reduce driver seat vibrations per ISO 2631 standard
- Solution
  - Optimize parameters in Simscape suspension model to minimize cab acceleration
- Results
  - Simulation results validated against test data
  - 35% reduction in vertical acceleration of cab
  - Process established to validate requirements in initial design phase



with base configuration. An improvement of 35% with the base configuration is achieved. Thus we establish a process to create an analytical lumped parameter model for optimization of ride comfort in initial design phase of cab suspension.

NASA Develops Early Warning System for Detecting Forest Disturbances



U.S. Forest Change Assessment Viewer map showing damage to the Asheville, North Carolina watershed following a 2012 hail storm. Image courtesy ForWarn.

Challenge
Develop a system that uses satellite imagery to quickly detect forest disturbance threats from insects, drought, storms, blights, wildfires, and other events

Solution
Use MATLAB to process multispectral satellite images, construct multidimensional time-series data baselines, and analyze terabytes of data to help detect regionally evident forest disturbances

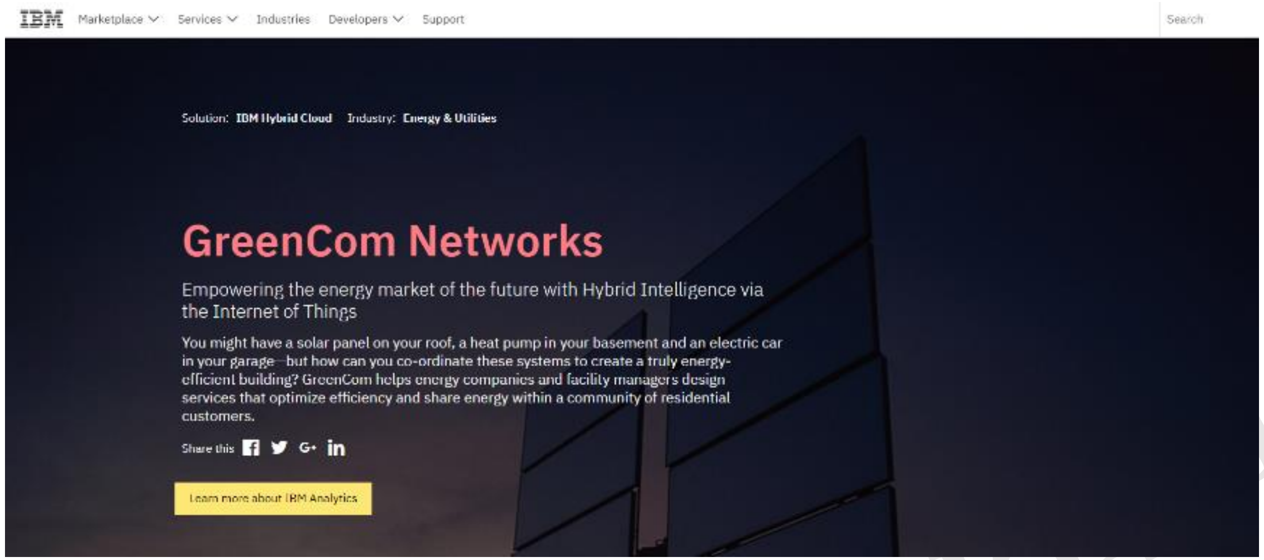
Results
- New ideas implemented and tested in hours
- Years of development time saved
- Reusable production software delivered to growing user community

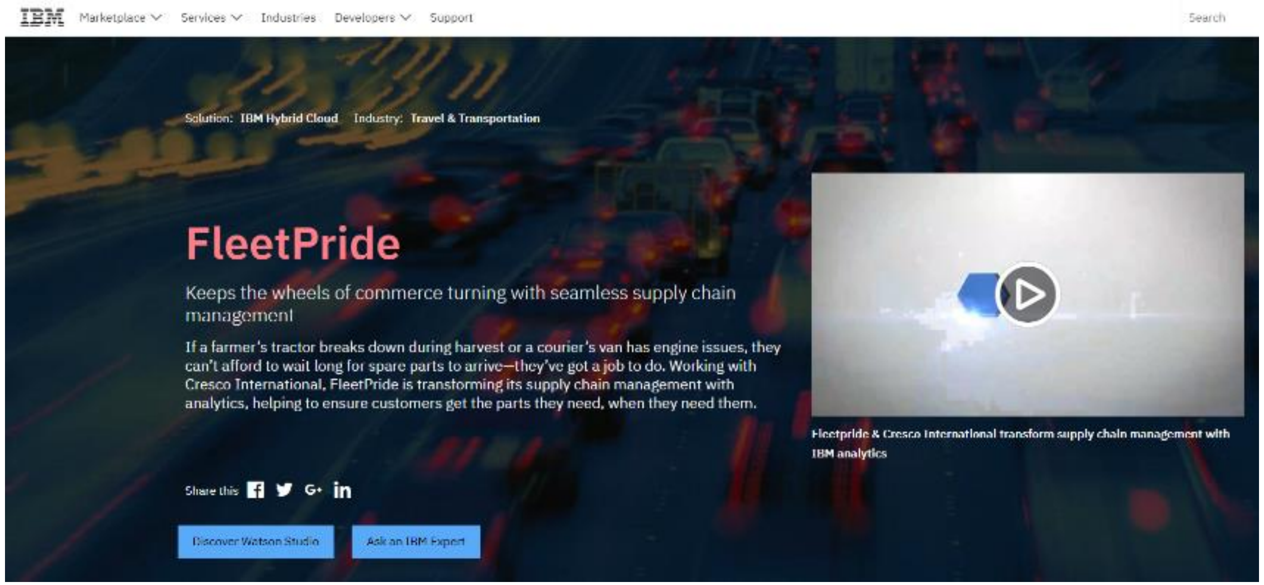
"Soon after ForWarn moved into production, it detected previously unnoticed hail damage that posed a threat to a watershed. We would not have been able to do this work as efficiently without MATLAB."
– Duane Armstrong, NASA Stennis Space Center

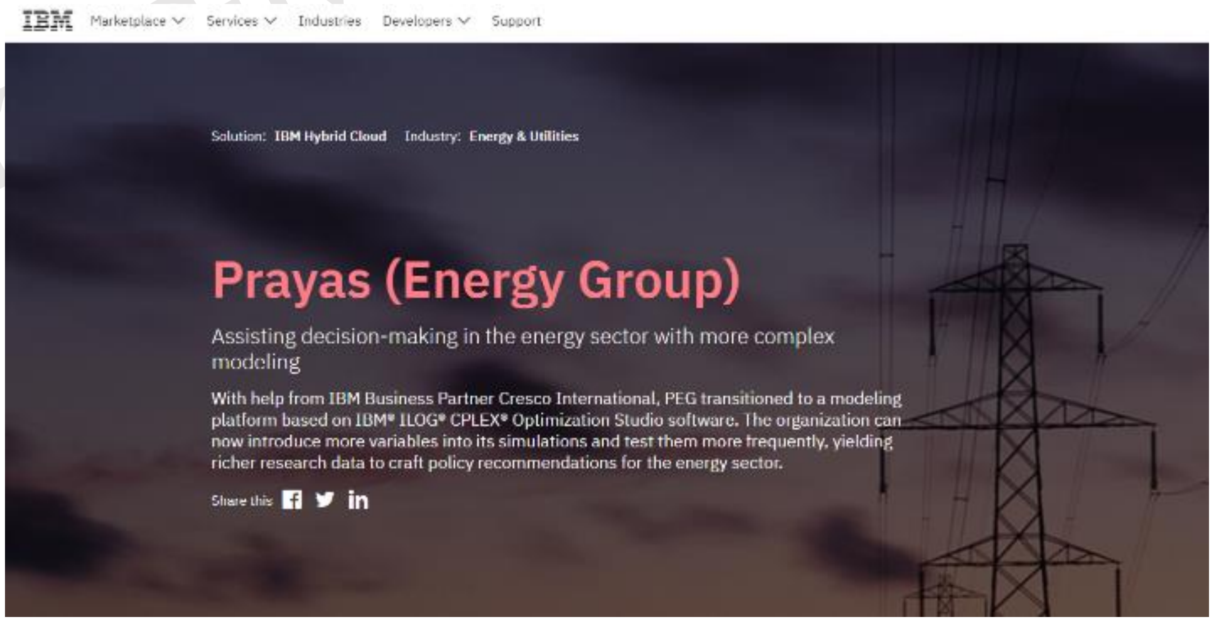
Link to user story

https://in.mathworks.com/company/user_stories

Applications

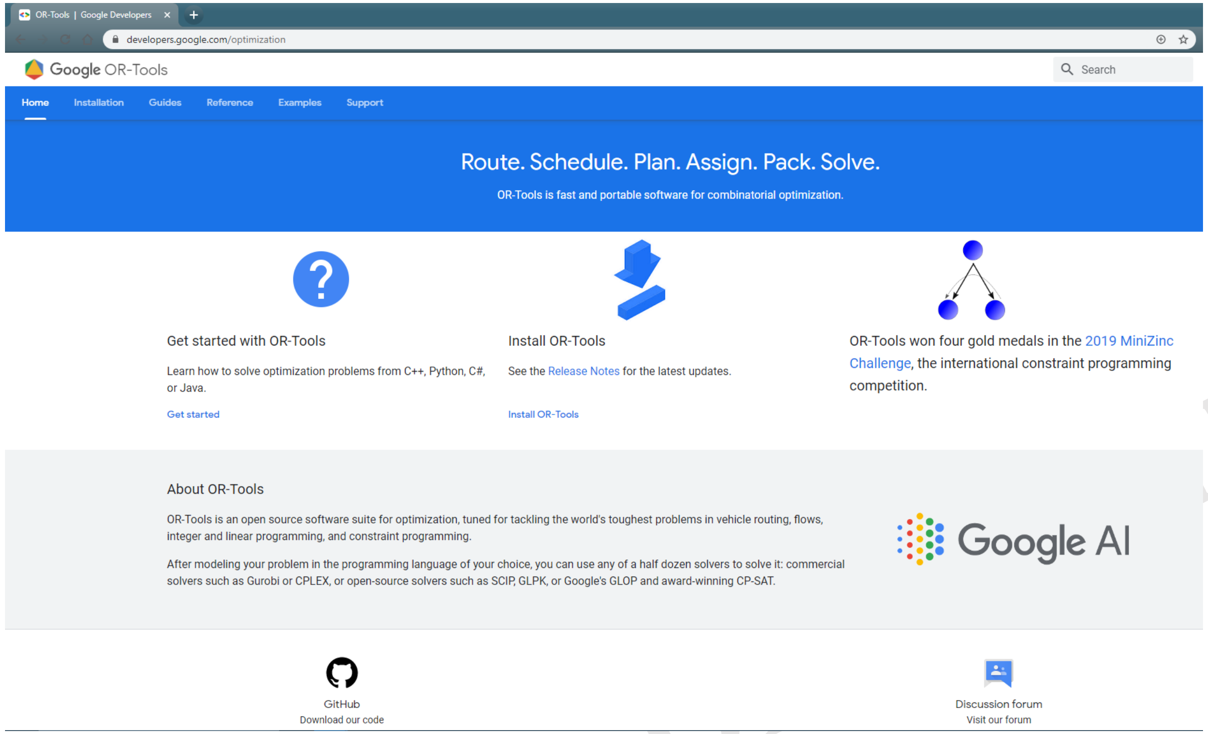


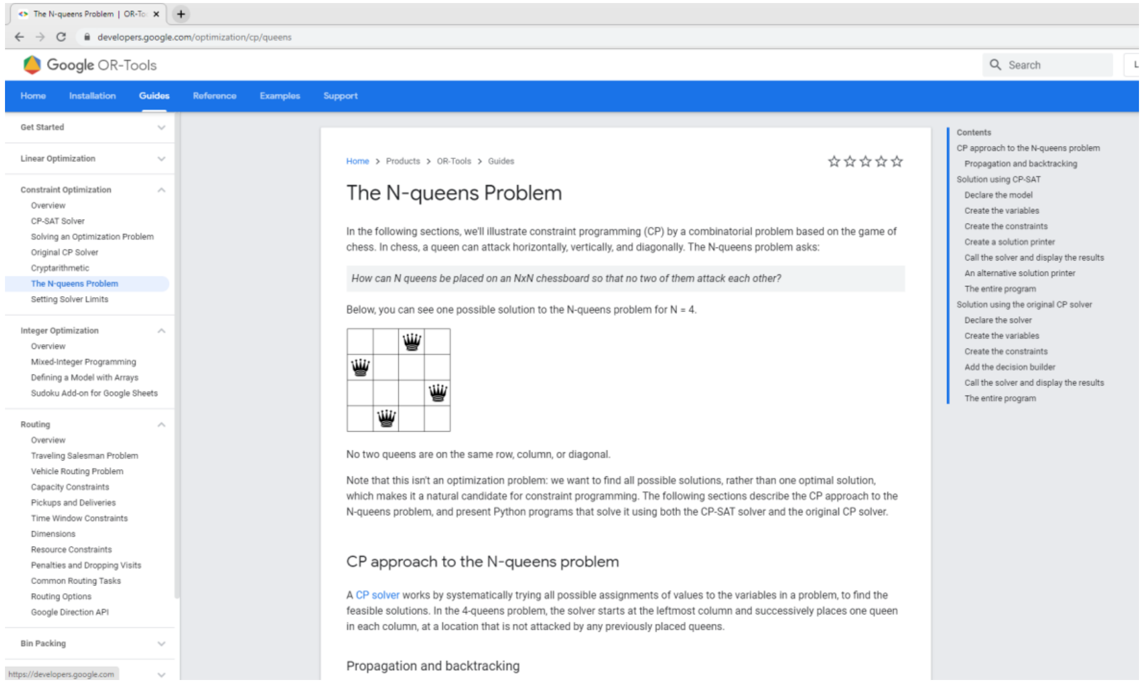




https://www.ibm.com/case-studies/

Applications



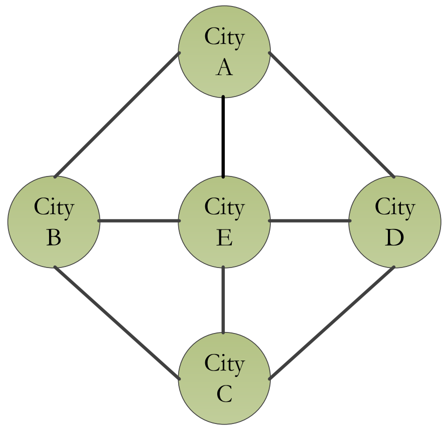


https://developers.google.com/optimization

Optimization problems

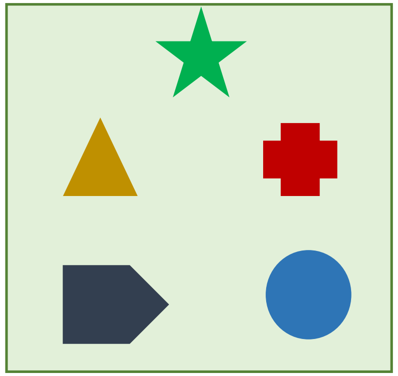
➢Travelling salesman

- A sale person wants to find the minimum cost tour in an n city situation.
- Each city is visited exactly once before returning to the starting point.



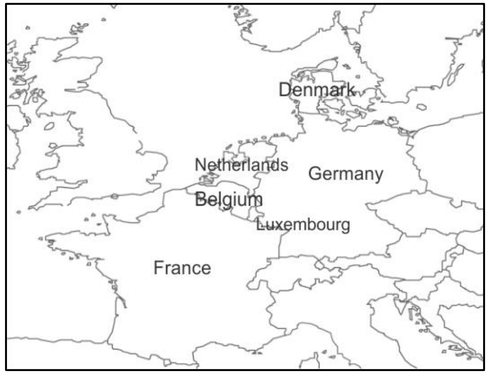
➢Knapsack problem

- A hiker wants to fill the knapsack to a maximum value.
- Objective is to maximize total value of the knapsack satisfying the weight constraint.



➢Map coloring (Feasibility problem)

- Four sets of colours (blue, white, yellow, green) are used to colour six countries.
- No pair of neighbouring countries can have the same colour.



Operations research: An introduction, H. A. Taha
IBM Knowledge Centre; https://www.ibm.com/support/knowledgecenter/SSSA5P_12.7.0/ilog.odms.cpo.help/CP_Optimizer/Getting_started/topics/color_describe.html

Sudoku

- ➤Puzzle played in a 9 x 9 partially filled matrix (standard Sudoku)
- ➤Fill a 9 x 9 matrix with integers from 1 to 9
- ➤Each integer appears only once, across row, column and in 3 x 3 major region

	7				1	9		
9				6			5	
6	4	1						
	2			7			9	6
		6				7		3
	5			2				
								9
1						6		
			4	3	6			

3	7	5	2	8	1	9	6	4
9	8	2	3	6	4	1	5	7
6	4	1	9	5	7	3	8	2
4	2	3	1	7	8	5	9	6
8	1	6	5	4	9	7	2	3
7	5	9	6	2	3	4	1	8
2	6	4	7	1	5	8	3	9
1	3	7	8	9	2	6	4	5
5	9	8	4	3	6	2	7	1

3	7	5	2	8	1	9	6	4
9	8	2	3	6	4	1	5	7
6	4	1	9	5	7	3	8	2
4	2	3	1	7	8	5	9	6
8	1	6	5	4	9	7	2	3
7	5	9	6	2	3	4	1	8
2	6	4	7	1	5	8	3	9
5	3	7	8	9	2	6	4	1
1	9	8	4	3	6	2	7	5

3	7	5	2	8	1	9	6	4
9	8	2	3	6	4	1	5	7
6	4	1	9	5	7	8	3	2
4	2	3	1	7	8	5	9	6
8	1	6	5	4	9	7	2	3
7	5	9	6	2	3	4	1	8
2	6	4	7	1	5	3	8	9
5	3	7	8	9	2	6	4	1
1	9	8	4	3	6	2	7	5

$$\sum_{k=1}^{9} x_{ijk} = 1 \quad \forall i, j \in \{1,2,\ldots,9\}$$

$$\sum_{i=1}^{9} x_{ijk} = 1 \quad \forall j, k \in \{1,2,\ldots,9\}$$

$$\sum_{j=1}^{9} x_{ijk} = 1 \quad \forall i, k \in \{1,2,\ldots,9\}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_{ijk} = 1 \quad \forall i, j \in \{1,2,3\}, k \in \{1,2,\ldots,9\}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_{(i+U)(j+V)k} = 1 \quad where\, U, V \in \{0,3,6\}, k \in \{1,2,\ldots,9\}$$

```
using CP;
range d = 0..8;
int input[d, d] = ...;
dvar int sudoku[d, d] in 1..9 ;
constraints {
    forall (i in d) allDifferent (all (j in d) sudoku[i,j]);
    forall (j in d) allDifferent (all(i in d) sudoku [i,j]);

    forall ( block_row , block_column in 0..2)
      allDifferent (all(l, m in 0..2) sudoku[3 * block_row + l , 3 * block_column + m]);

    forall (i, j in d: input[i, j] != 0) sudoku[i, j] == input[i, j];
};
```

https://in.mathworks.com/help/optim/examples/solve-sudoku-puzzles-via-integer-programming.html
https://www.ibm.com/developerworks/community/blogs/jfp/entry/solving_the_hardest_sudoku?page=0&lang=en

Optimization & its components

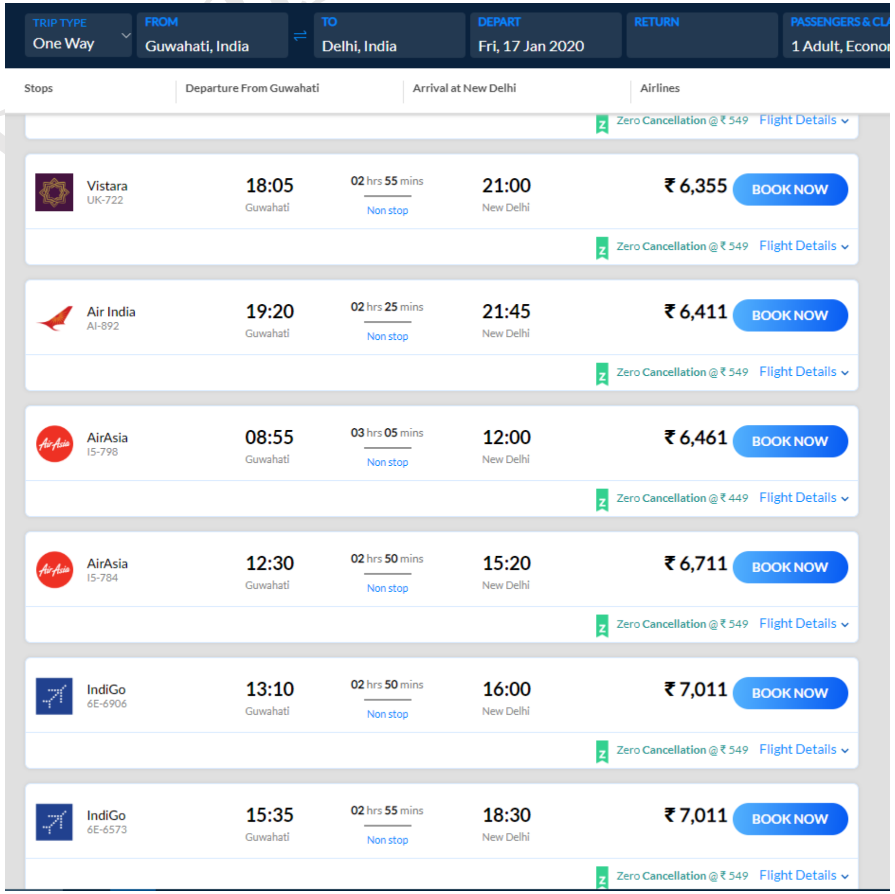
➢ Selection of best choice (based on some criteria) from a set of available alternatives.

➢ Decision variables

➢ Objective function: Relation of decision variables

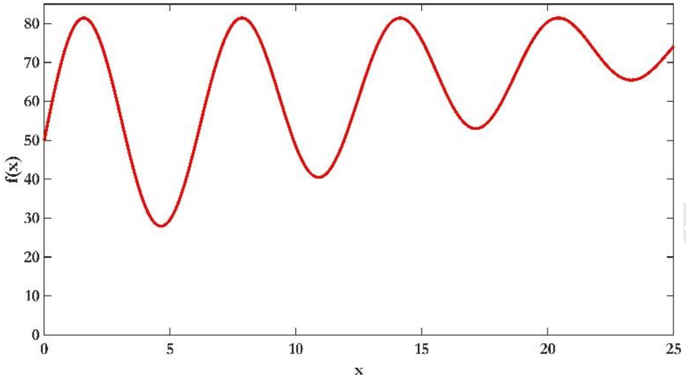
➢ Constraints: Restrictions on the decision variables

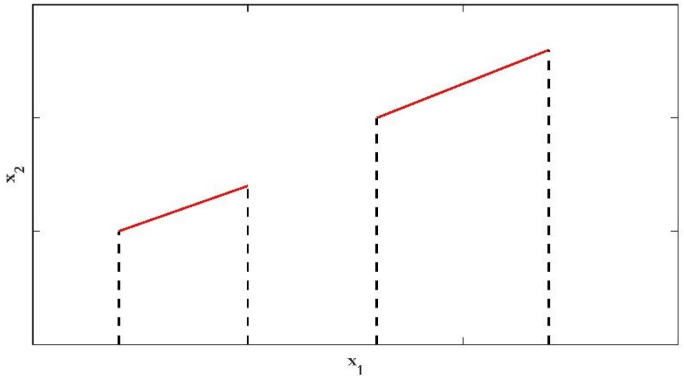
➢ Helps in the classification of problems and techniques.

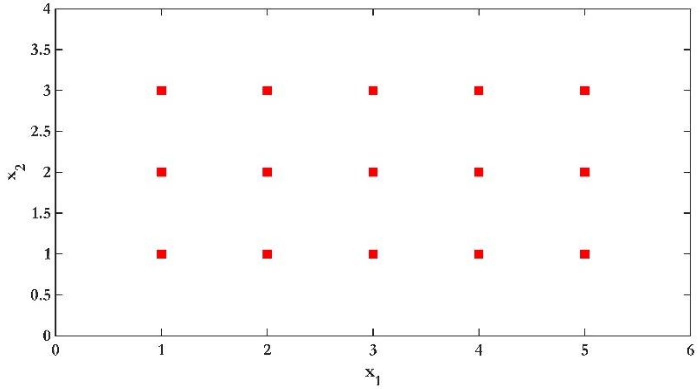


Decision variables

- Formulation of an optimization problem begins with identifying the decision variables.
- Relates the objective function and constraints.
- Can be continuous, semi-continuous, discrete or sets.





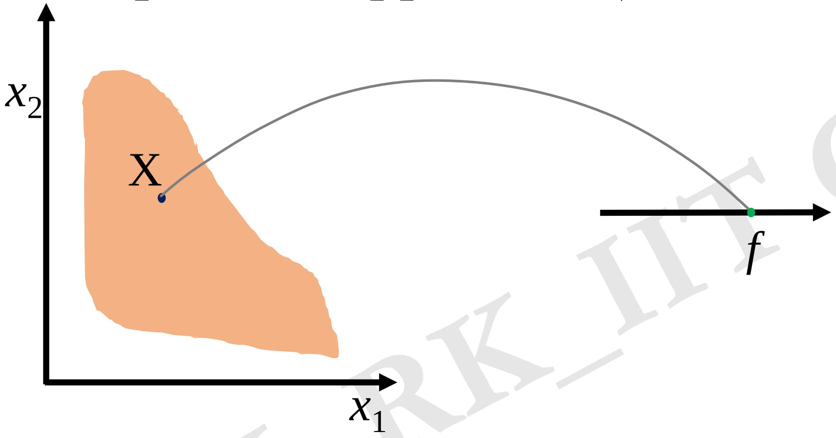


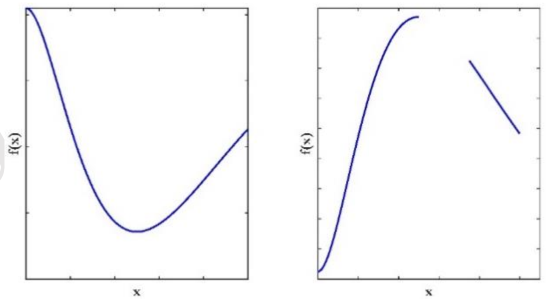
Choice of paint color is a decision variable (s).
Possible value of the decision variable: $s \in \{\text{'Red'}, \text{'Blue'}, \text{'Green'}, \text{'Yellow'}\}$

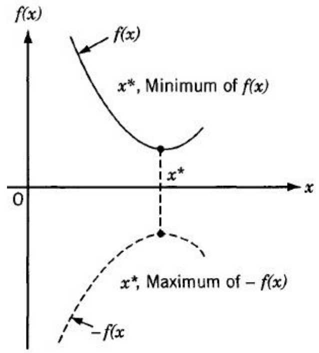
- Can be bounded or unbounded.

Objective function

- Criteria with respect to which the decision variables are to be optimized.
- Every solution in variable space is mapped to objective.







- Can be continuous or semi-continuous.
- Maximization problem to a minimization problem: multiple by –1.

$$\text{Max } f(x) \equiv \text{Min } (-f(x))$$

- Can be bounded or unbounded.
- Absence of objective function (in the presence of constraints) leads to feasibility problems.

3	7	5	2	8	1	9	6	4
9	8	2	3	6	4	1	5	7
6	4	1	9	5	7	3	8	2
4	2	3	1	7	8	5	9	6
8	1	6	5	4	9	7	2	3
7	5	9	6	2	3	4	1	8
2	6	4	7	1	5	8	3	9
1	3	7	8	9	2	6	4	5
5	9	8	4	3	6	2	7	1

Engineering optimization, S S Rao
Optimization of chemical processes, Himmelblau

Constraints

- Inequality constraints (usually resource constraints)
  - In general, denoted by $g(x) \leq 0$
  - Conversion from one form to the other by multiplying with -1

$$x_1 + x_2 \leq 10$$
$$-(x_1 + x_2) \geq -10$$

- Equality Constraints (usually balances)
  - In general, denoted by $h(x) = 0$

$$x_1 + x_2 = 3$$
$$xe^{-x^2-y^2} = 5$$

- Feasible solution: Satisfies all the constraints
- Infeasible solution: Does not satisfy at least one constraint
- Hard Constraints: Must be satisfied in order to accept a solution
- Soft Constraints: Allowed to relax to some extent to accept a solution

Feasibility of a solution

$$\text{Minimize} \quad f(x) = c\left(\frac{\pi x_1^2}{2} + \pi x_1 x_2\right)$$

c = 0.065

$$\text{subject to} \quad \frac{\pi x_1^2 x_2}{4} \geq 300 \leftarrow g(x) \geq 300$$

Consider two solutions $S_1 = [3 \;\; 10]$ and $S_2 = [8 \;\; 6]$

$S_1 = [3 \;\; 10]$
$f_1 = 7.05$
$g_1 = 70.69$

Violates the constraint
Infeasible solution

Satisfies the constraint
Feasible solution

$S_2 = [8 \;\; 6]$
$f_2 = 16.34$
$g_2 = 301.59$

A feasible solution is preferred to an infeasible one

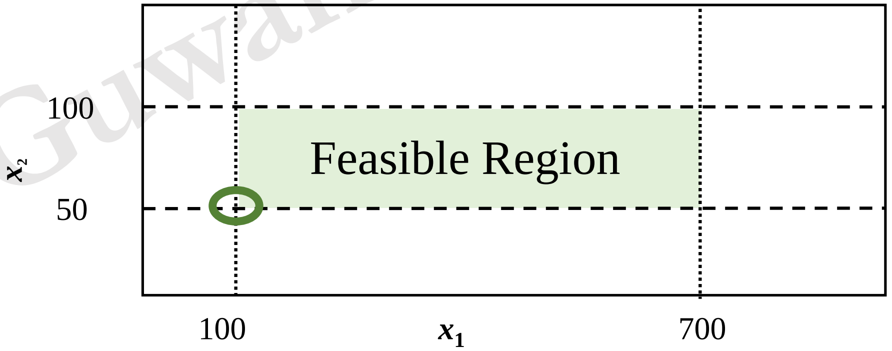
Bounded and unbounded problem

$$\text{Min } f = 3x_1 + 2x_2$$

$$f = -\infty,\ x_1 = -\infty,\ x_2 = -\infty$$

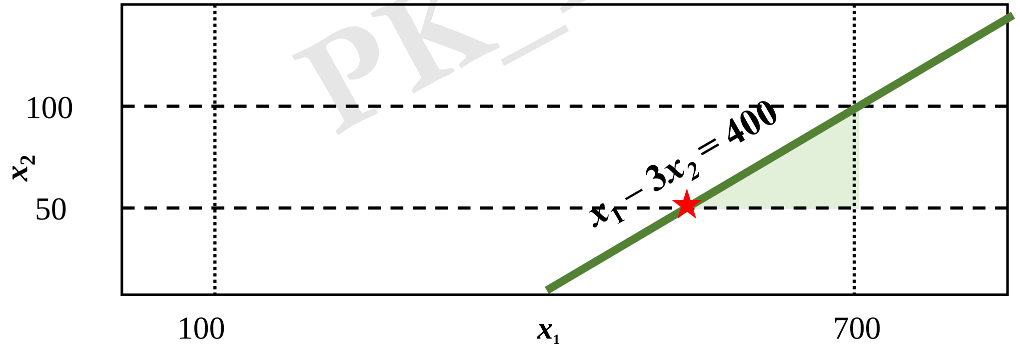
Bounds on variables: $100 \le x_1 \le 700$

$50 \le x_2 \le 100$



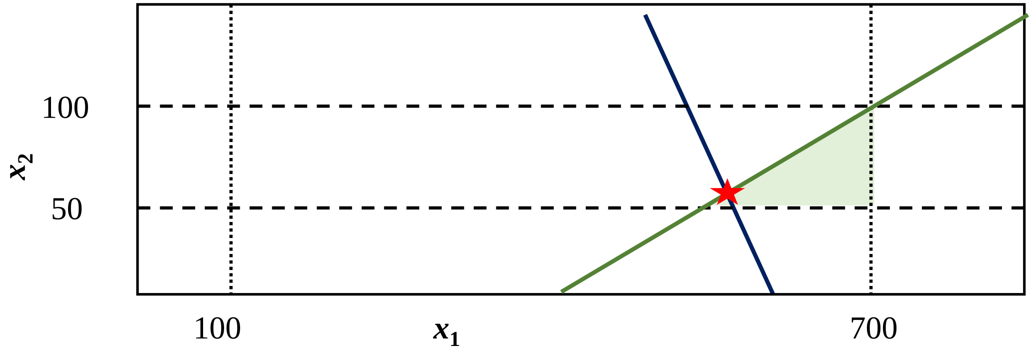
$$f = 400,\ x_1 = 100,\ x_2 = 50$$

Constraint bounding: $x_1 - 3x_2 \ge 400$
Let $x_1 = 0$, then $-3x_2 = 400$ ➔ $x_2 = -400/3 = -133.33$
Let $x_2 = 0$, then $x_1 = 400$
Line corresponding to $x_1 - 3x_2 \ge 400$ passes through (0, 133.33) and (400, 0)



$$f = 1750;\ x_1 = 550;\ x_2 = 50$$

Bounds on Objective: $f \ge 1800$



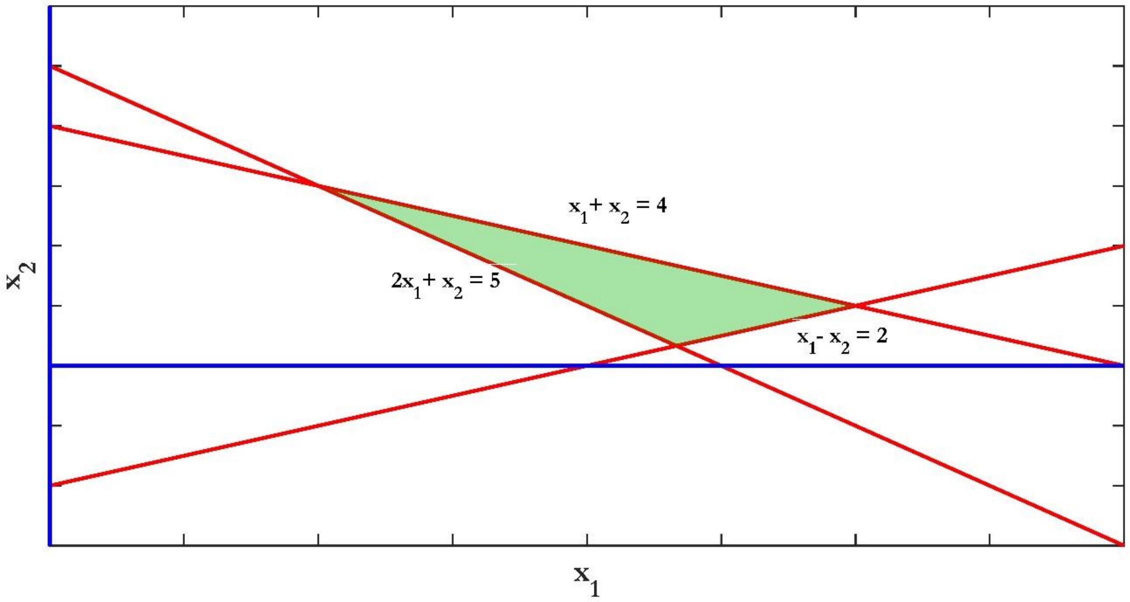
$$f = 1800;\ x_1 = 564;\ x_2 = 54$$

Bounded by only constraints

No bounds on decision variables

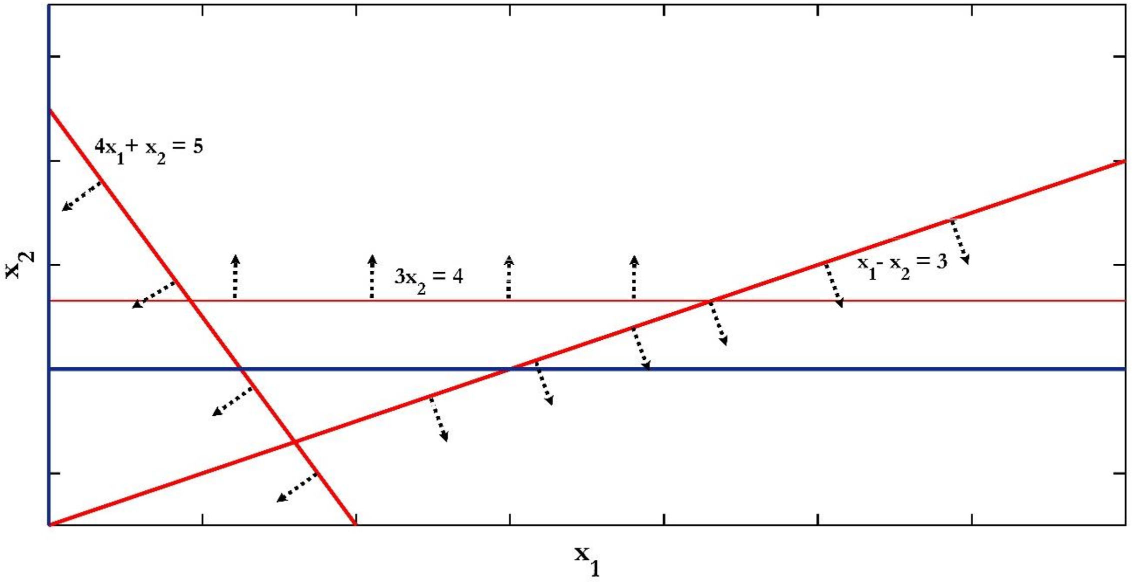
Feasible problem

$$\begin{aligned} \text{Minimize} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \in \mathrm{R} \end{aligned}$$



Infeasible problem – Conflicting constraints

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 5 \\ & 3x_2 \geq 4 \\ & x_1 - x_2 \geq 3 \\ & x_1, x_2 \in \mathrm{R} \end{aligned}$$

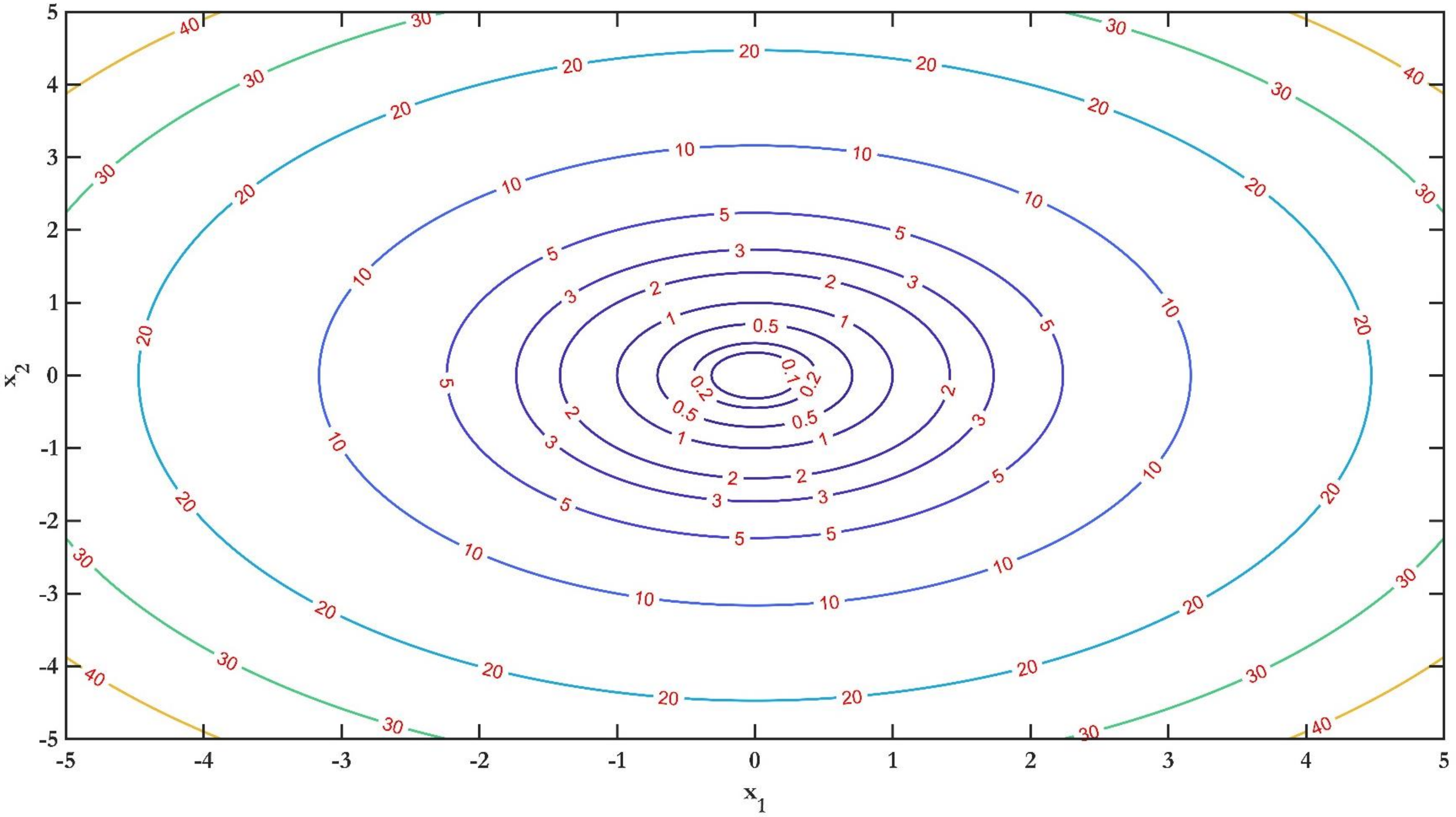


Contour plot

Sphere function

$$f\left(x\right) = x_1^2 + x_2^2$$

x_1	x_2	$f(x)$
2	4	20
4	2	20
1		20
3		20
0		20



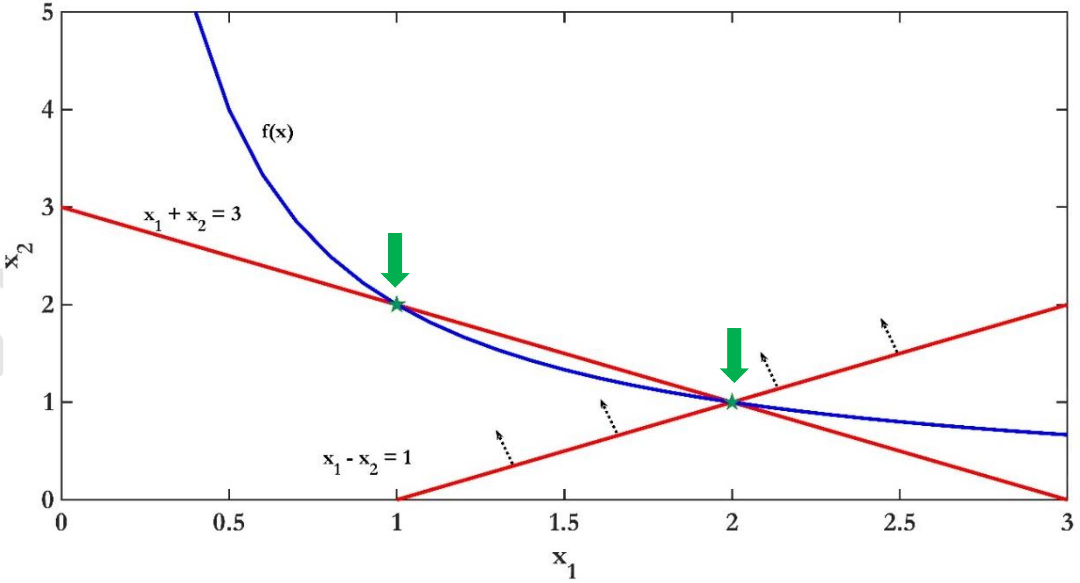
Realizations

➢Two or more solutions with same objective function value

$$\begin{aligned} &Maximize \quad f = x_1 x_2 \\ &subjected\ to\ \ x_1 + x_2 = 3 \\ &\qquad\qquad\qquad x_1 - x_2 \le 1 \\ &\qquad\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

$S1: x_1 = 1, x_2 = 2, f = 2$

$S2: x_1 = 2, x_2 = 1, f = 2$

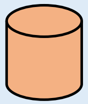


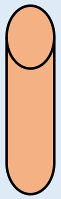
Example: Design a can considering diameter (*d* in cm) and height (*h* in cm) as the two decision variables. The can needs to have a volume of at least 300 ml and the objective is to minimize the material cost of the can.

$$\begin{aligned} &Minimize \quad f(d,h) = c\left(\frac{\pi d^2}{2} + \pi d h\right) \\ &subject\ to \quad \frac{\pi d^2 h}{4} \ge 300 \\ &\qquad 0 \le d \le 31 \\ &\qquad 0 \le h \le 31 \end{aligned}$$

c = 0.065

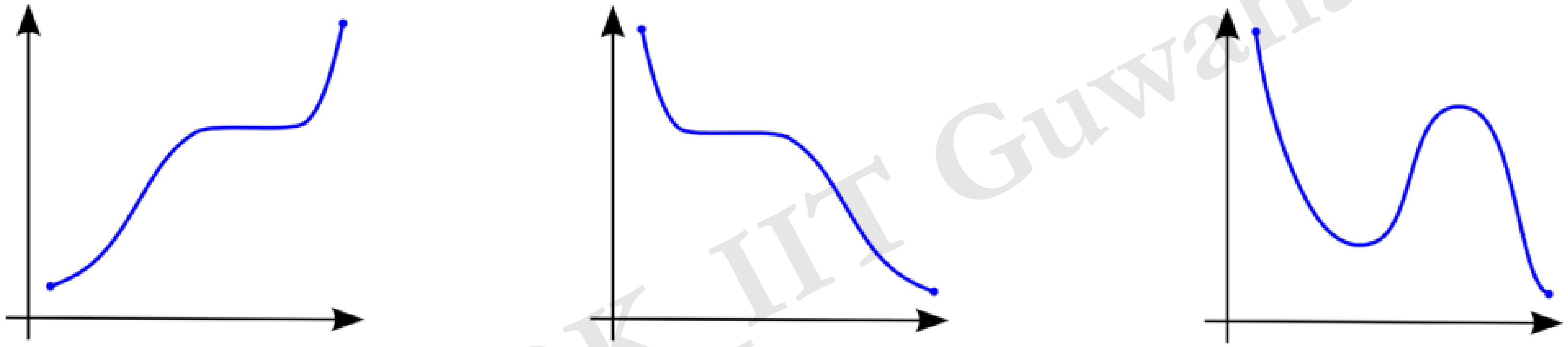
Realizations: $d = 8$, $h = 10$ and $f = 22.87$ $\qquad$ $d = 5$, $h = 19.9$ and $f = 22.87$





Monotonic & convex functions

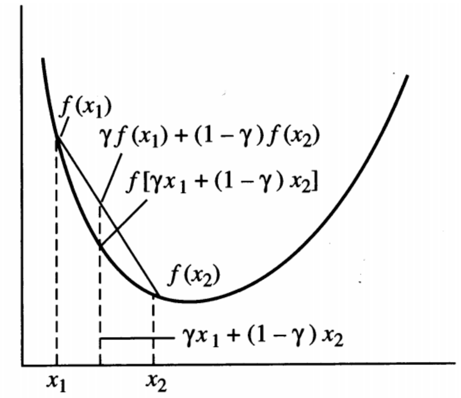
- **Monotonic functions:** Functions are continuously increasing or decreasing.



- **Convex Functions:** The line segment between any two points on the graph of a function lies above the graph.

$$f\left[\gamma x_1 + (1-\gamma)x_2\right] \le \gamma f(x_1) + (1-\gamma) f(x_2)$$

where γ is a scalar, $0 \le \gamma \le 1$



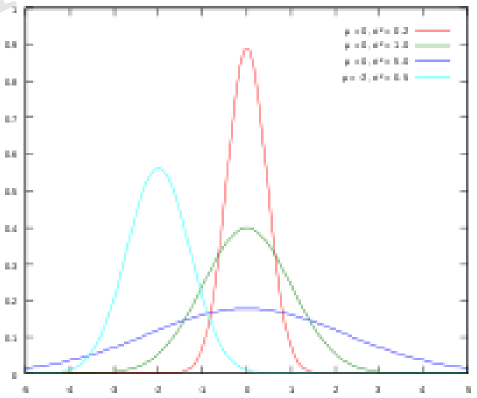
Convex function

Optimization of chemical processes, Himmelblau

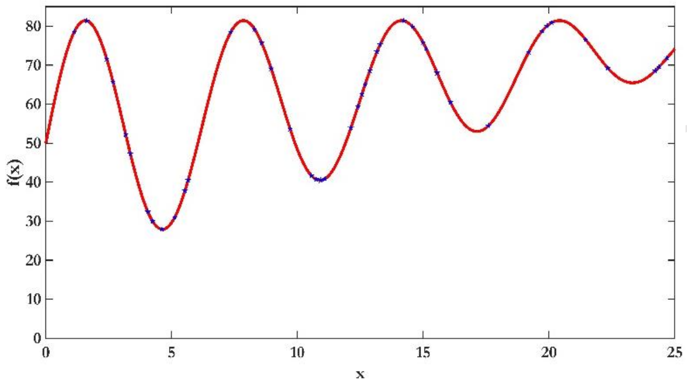
Unimodal and multimodal functions

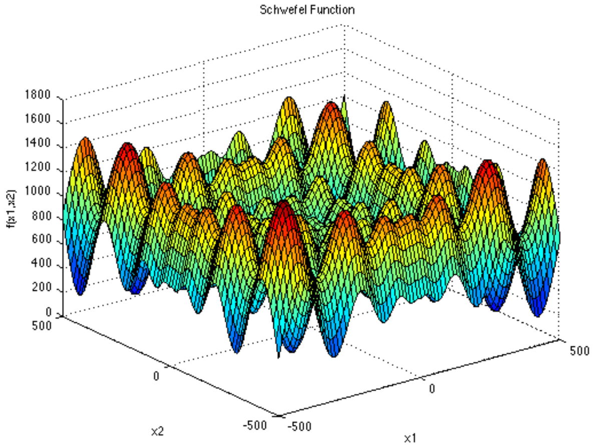
➢ Unimodal functions: for some value m, if the function is monotonically increasing for $x \le m$ and monotonically decreasing for $x \ge m$.

- If so, the maximum value of $f(x)$ is $f(m)$
- No other local maxima.



➢ Multimodal functions: function has multiple global and local optima

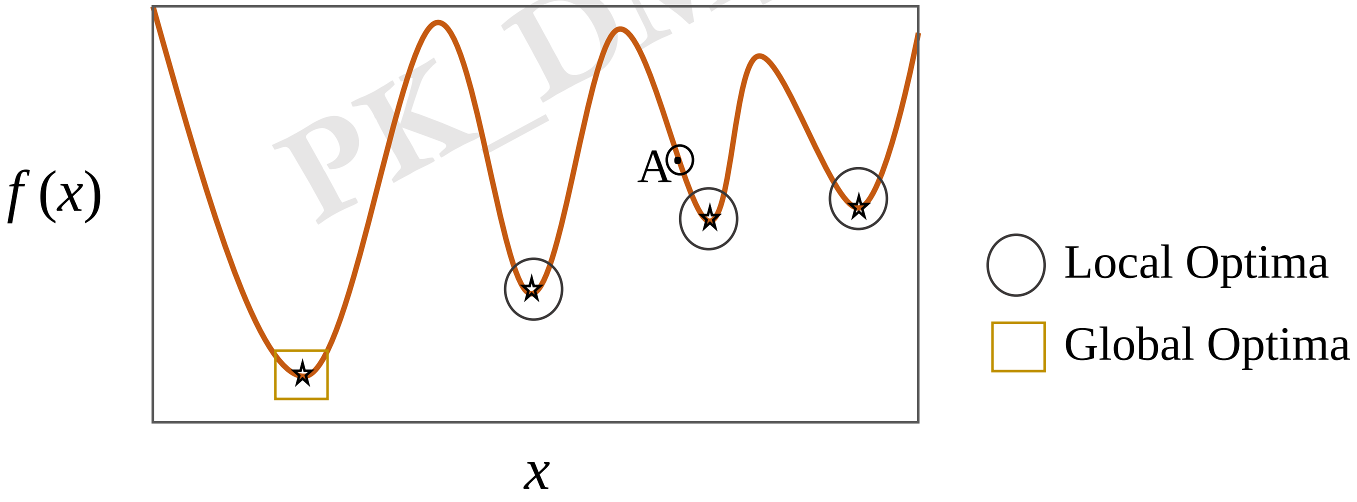




➢ Most real life optimization problems are multimodal

Optimality

- Local Optima (minimization)
  - Smallest function value in its neighborhood.
  - There can be multiple local optima solutions.
- Global Optima (minimization)
  - Smallest function value in the feasible region.
  - If the function is convex, only global optima exists (there will be no local optima).
  - For multimodal functions, most algorithms fail to determine the global optima.



Classification of optimization problems

Linear/Non-linear

$$\begin{cases} Minimize \quad Z = f(x,y) \\ subjected\ to \quad g(x,y) \le 0 \\ h(x,y) = 0 \end{cases}$$

$$x \in X, y \in Y$$

$$X = \{x \mid x \in \mathbb{R}^n, x^L \le x \le x^U\}$$

$$Y = \{y \mid y \in \mathbb{Z}^m, y^L \le y \le y^U\}$$

Category	Variables		Objective function		Constraints	
	Continuous	Discrete	Linear	Non-linear	Linear	Non-linear
Linear Programming	✓	✗	✓	✗	✓	✗
Non Linear Programming	✓	✗	✓	✓	✓	✓
Integer Linear Programming	✗	✓	✓	✗	✓	✗
Mixed Integer Linear Programming	✓	✓	✓	✗	✓	✗
Mixed Integer Non Linear Programming	✓	✓	✓	✓	✓	✓

Classification of Optimization Techniques

Mathematical Programming Techniques

- Based on geometrical properties of the problem
  - Simplex Algorithm, Interior Point Algorithm
  - Steepest Decent, Newton's method, Quasi-Newton
  - Branch & Bound, Cutting Planes

Metaheuristic techniques

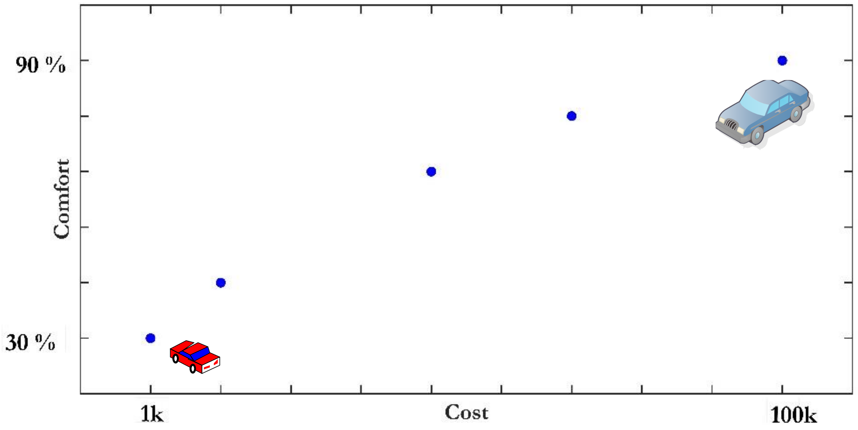
- Genetic Algorithm
- Particle Swarm Optimization
- Differential Evolution
- Teaching Learning Based Optimization
- Artificial Bee Colony
- Grey Wolf Optimization
- Simulated Annealing

Other Techniques

- Nelder Mead/simplex search
- Fibonacci method
- Golden Section method
- Hooke-Jeeves' Pattern Search method
- Powell's Conjugate Direction method

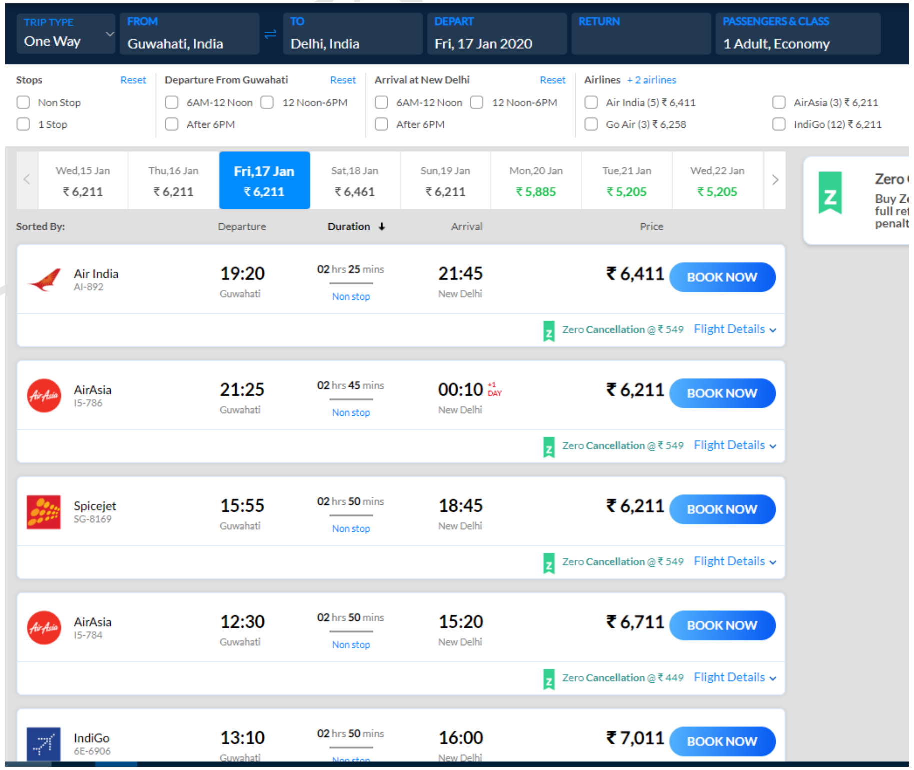
Multi-objective optimization

- ➢Two or more conflicting objectives which can be either minimized or maximized.
- ➢Each objective function corresponds to different optimal solution, no single optima.
- ➢Obtain a set of optimal solutions where a gain in one objective deteriorates the other objectives.



Point 1: Low Comfort and Low Cost

Point 2: High Comfort & High Cost



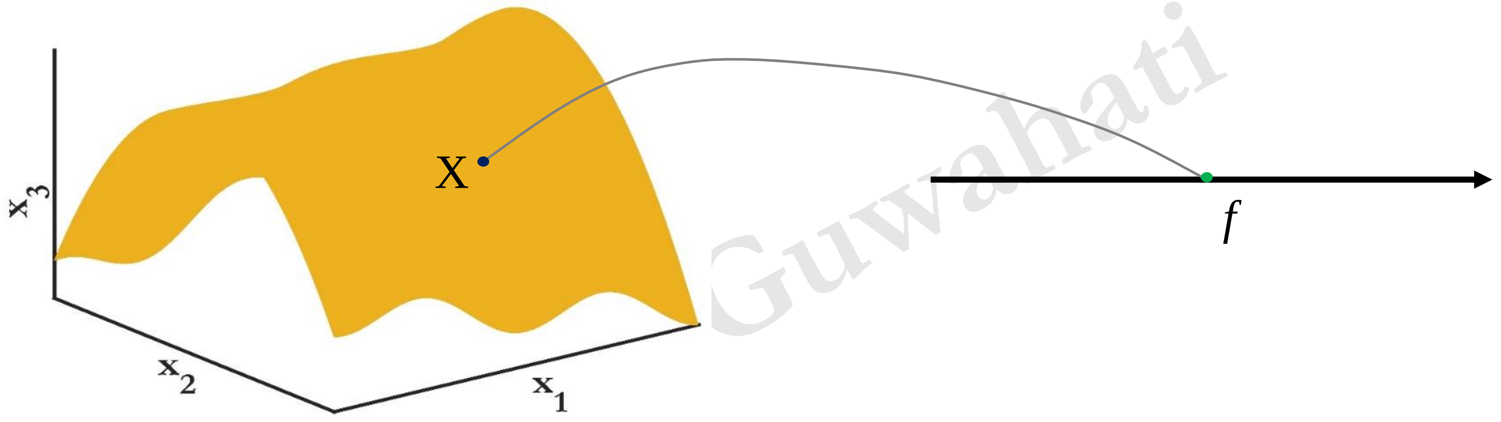
Min f_1 Min f_2

Duration Price

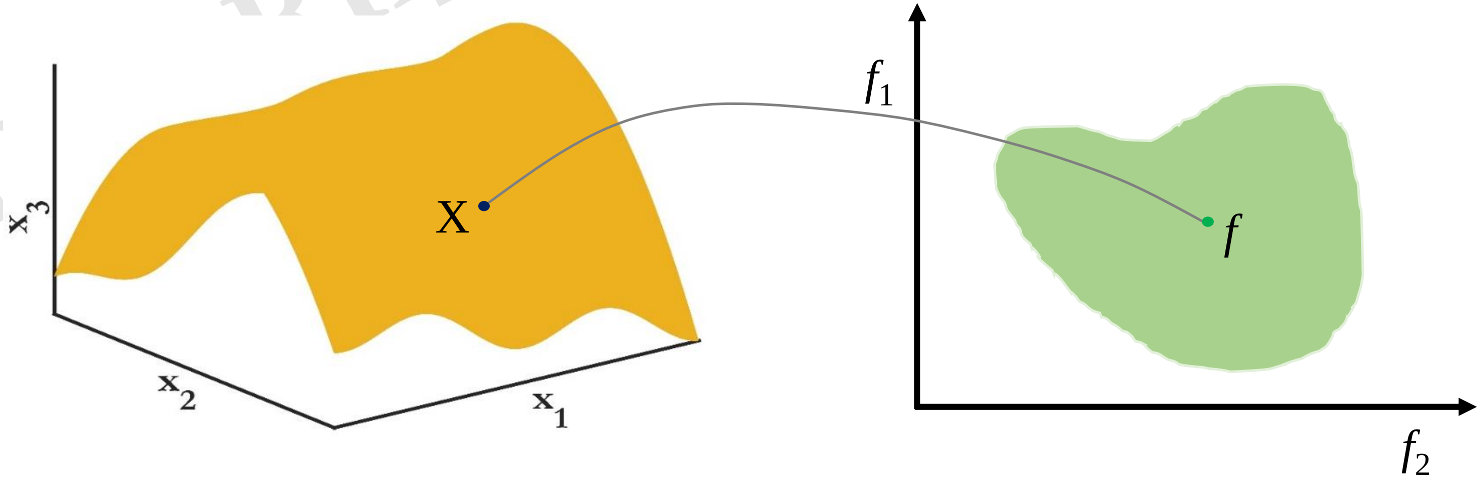
Multi-objective optimization using evolutionary algorithms, K. Deb

Search space

Single objective optimization



Multi-objective optimization



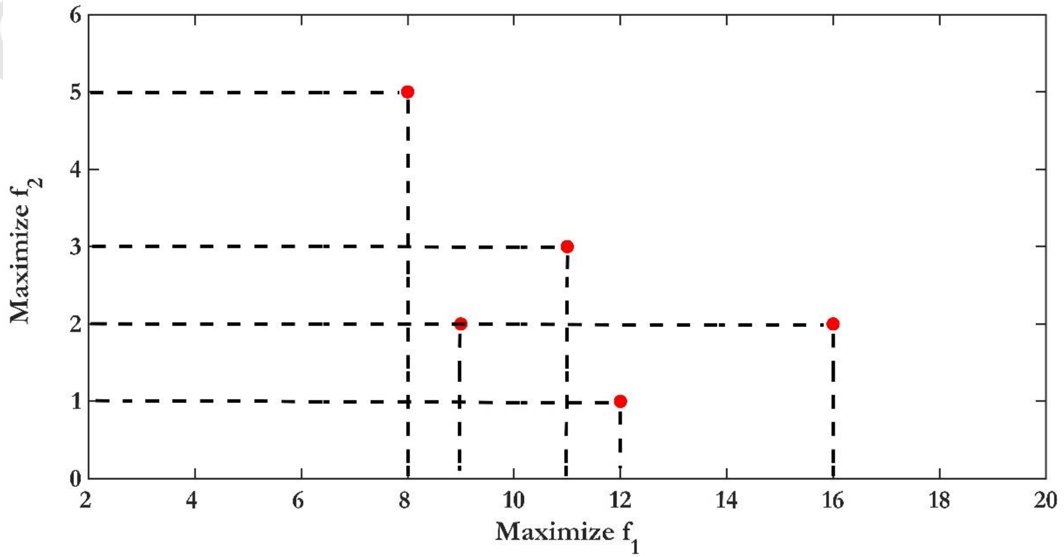
Objective search space in addition to decision variable space; difficult to locate the best solution in the objective space

Pareto solutions

A solution S1 is said to dominate solution S2, if both the following conditions are true

1. S1 is no worse than S2 in all the objectives
2. S1 is strictly better than S2 in at least one objective

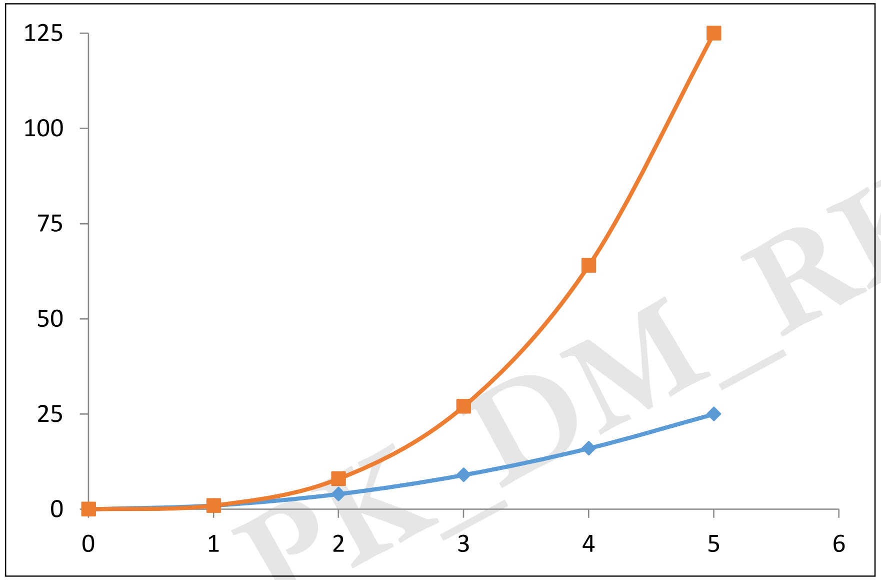
	S1	S2	S3	S4	S5
Maximize f_1	9	8	12	11	16
Maximize f_2	2	5	1	3	2



S2, S4, S5 forms the set of non-dominated points – Pareto points

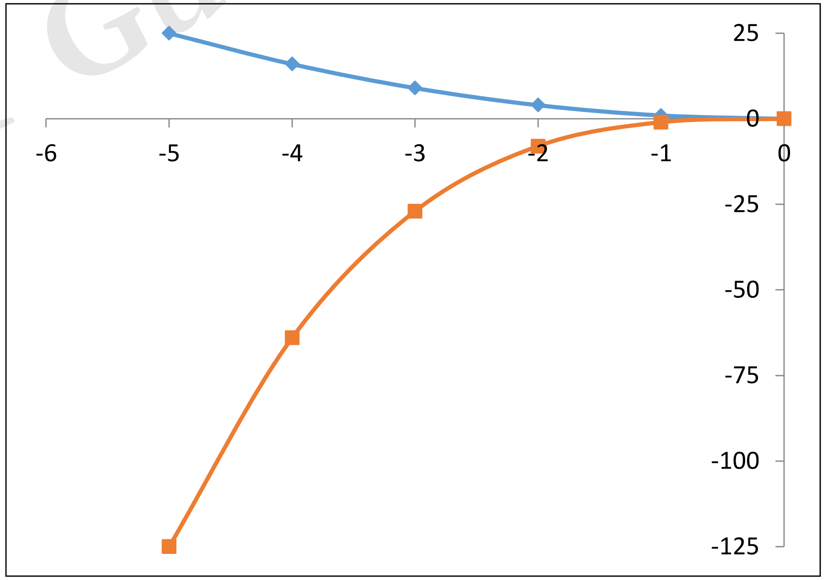
Conflicting objective functions

$$Max \quad f_1(x) = x^3$$
$$Max \quad f_2(x) = x^2$$



$0 \le x \le \infty$

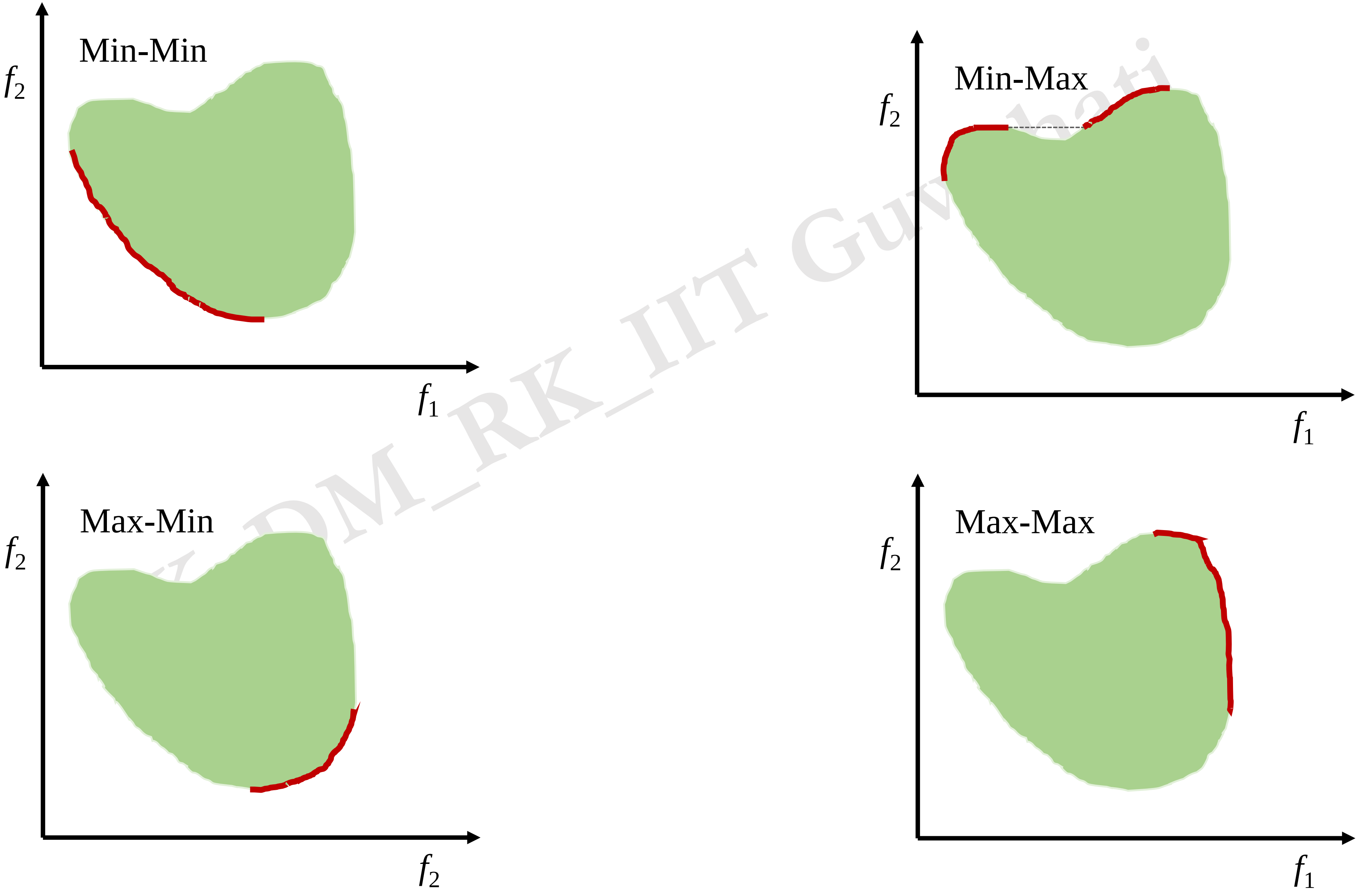
Non-conflicting



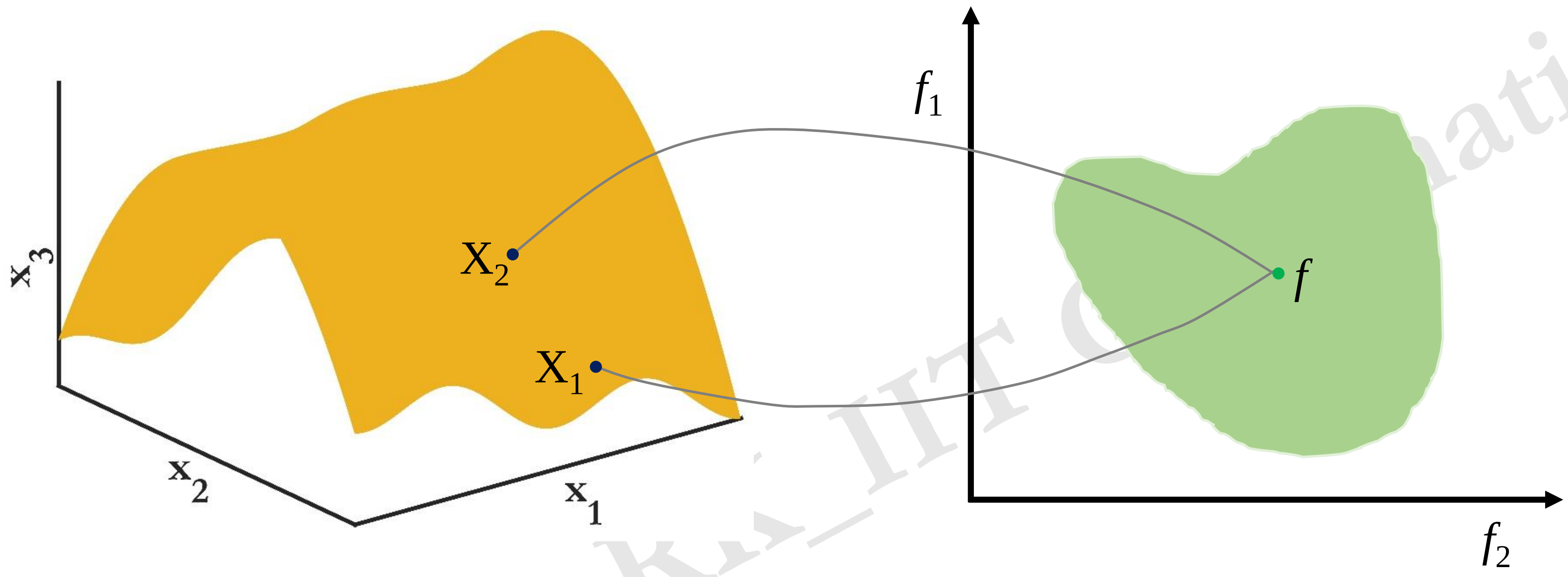
$-\infty \le x \le 0$

Conflicting

Pareto optimal solutions



Realization in multi-objective optimization



X	f_1	f_2
[5 2 1]	125	42
[4 3 5]	125	42

Maxima/minima/saddle point

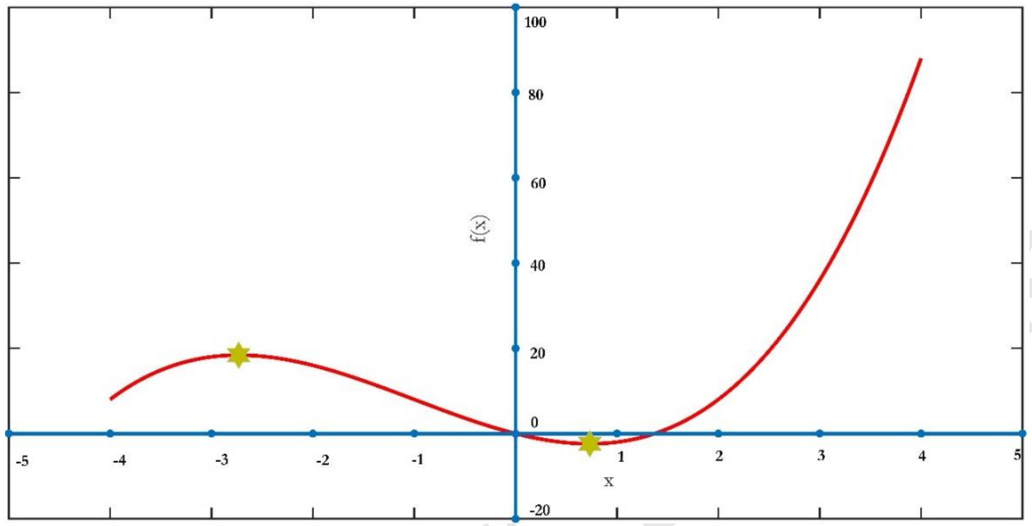
- Maximum or minimum is located at the stationary point.
- Stationary points can be determined by equating the gradient (Jacobian) of the function to zero.
- Second derivative (Hessian) at the stationary point.

Second derivative/Hessian		Nature of extremum
Single variable	Multi variable	
Positive	Positive definite	Minimum
Negative	Negative definite	Maximum
0	Indefinite	Saddle Point (Not possible to decide if min or max)

Single variable function

$f(x) = x^3 + 3x^2 - 6x$ $\quad J = \left[\dfrac{df}{dx}\right] = 3x^2 + 6x - 6$

Number of stationary points will be two



Minima or maxima occurs at the stationary point

Roots of the polynomial $ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x = 0.732$, $x = -2.732$

$$f'' = \frac{d^2 f}{dx^2} = 6x + 6$$

$x = 0.732$, $f'' = 10.392$,
f has minimum at this point ($f = -2.39$)

$x = -2.732$, $f'' = -10.392$,
f has maximum at this point ($f = 18.39$)

Multivariable functions

$$f(x) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 - 2x_1x_3 - 6x_1 - 4x_2 - 2x_3$$

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial f}{\partial x_1} = 6x_1 - 2x_2 - 2x_3 - 6$$

$$\frac{\partial f}{\partial x_2} = -2x_1 + 4x_2 + 2x_3 - 4$$

$$\frac{\partial f}{\partial x_3} = -2x_1 + 2x_2 + 2x_3 - 2$$

$$J = \begin{bmatrix} 6x_1 - 2x_2 - 2x_3 - 6 \\ -2x_1 + 4x_2 + 2x_3 - 4 \\ -2x_1 + 2x_2 + 2x_3 - 2 \end{bmatrix}$$

$$J = 0 \Rightarrow \underbrace{\begin{bmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Stationary point

Multivariable functions

$$J = \begin{bmatrix} 6x_1 - 2x_2 - 2x_3 - 6 \\ -2x_1 + 4x_2 + 2x_3 - 4 \\ -2x_1 + 2x_2 + 2x_3 - 2 \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -2 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_1} = -2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -2 \qquad \frac{\partial^2 f}{\partial x_2^2} = 4 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_3} = -2 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 2 \qquad \frac{\partial^2 f}{\partial x_3^2} = 2$$

$$H = \begin{bmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

$$|H_1| = 6 > 0 \qquad |H_2| = \begin{vmatrix} 6 & -2 \\ -2 & 4 \end{vmatrix} = 20 > 0 \qquad |H_3| = \begin{vmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{vmatrix} = 16 > 0$$

All three principal determinants are positive

Since H is positive definite matrix, the stationary point corresponds to a minima

$J = 0$ ➔ $x_1 = 2, x_2 = 1, x_3 = 2$

Multivariable functions

$$f(x) = x_1^4 + x_1^2 - 2x_1^2 x_2 + 2x_2^2 - 2x_1x_2 + 4.5x_1 - 4x_2 + 4$$

$$\frac{\partial f}{\partial x_1} = 4x_1^3 + 2x_1 - 4x_1x_2 - 2x_2 + 4.5$$

$$\frac{\partial f}{\partial x_2} = -2x_1^2 + 4x_2 - 2x_1 - 4$$

$$J = \begin{bmatrix} 4x_1^3 + 2x_1 - 4x_1x_2 - 2x_2 + 4.5 \\ -2x_1^2 + 4x_2 - 2x_1 - 4 \end{bmatrix} = 0$$

By solving,
$S_1 = (1.941, 3.854)$
$S_2 = (-1.053, 1.028)$
$S_3 = (0.612, 1.493)$

$$\frac{\partial^2 f}{\partial x_1^2} = 12x_1^2 + 2 - 4x_2 \qquad \frac{\partial^2 f}{\partial x_2^2} = 4$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -4x_1 - 2 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -4x_1 - 2$$

$$H = \begin{bmatrix} 12x_1^2 + 2 - 4x_2 & -4x_1 - 2 \\ -4x_1 - 2 & 4 \end{bmatrix}$$

$S_1 = (1.941, 3.854)$

$$H_1 = \begin{bmatrix} 31.794 & -9.764 \\ -9.764 & 4 \end{bmatrix} \qquad \lambda = \begin{bmatrix} 0.9 \\ 34.9 \end{bmatrix}$$

Hessian: positive definite

➔ Minima

$S_2 = (-1.053, 1.028)$

$$H_2 = \begin{bmatrix} 11.194 & 2.212 \\ 2.212 & 4 \end{bmatrix} \qquad \lambda = \begin{bmatrix} 3.4 \\ 11.8 \end{bmatrix}$$

Hessian: positive definite

➔ Minima

$S_3 = (0.612, 1.493)$

$$H_3 = \begin{bmatrix} 0.523 & -4.448 \\ -4.448 & 4 \end{bmatrix} \qquad \lambda = \begin{bmatrix} -2.5 \\ 7.0 \end{bmatrix}$$

Hessian: indefinite

➔ Saddle point

Optimization of chemical processes, Himmelblau

Mathematical Programming Techniques

➢ Advantages

- Guaranteed optimal solutions for well-structured problems (LP, MILP, QP).
- Helpful in debottlenecking, as it provides reasonable insight into the behavior of solutions.
- Does not require multiple runs.
- Requires lower computational resources for certain classes of problems.
- Usually no parameters are to be determined by trial and error.

➢ Drawbacks

- Rigid modeling framework as it requires an explicit model in a definite form.
- Not naturally amenable to multi-objective optimization problems.
- Computational load usually increases significantly with increase in the problem size (combinatorial problems).
- Usually designed to provide one solution.

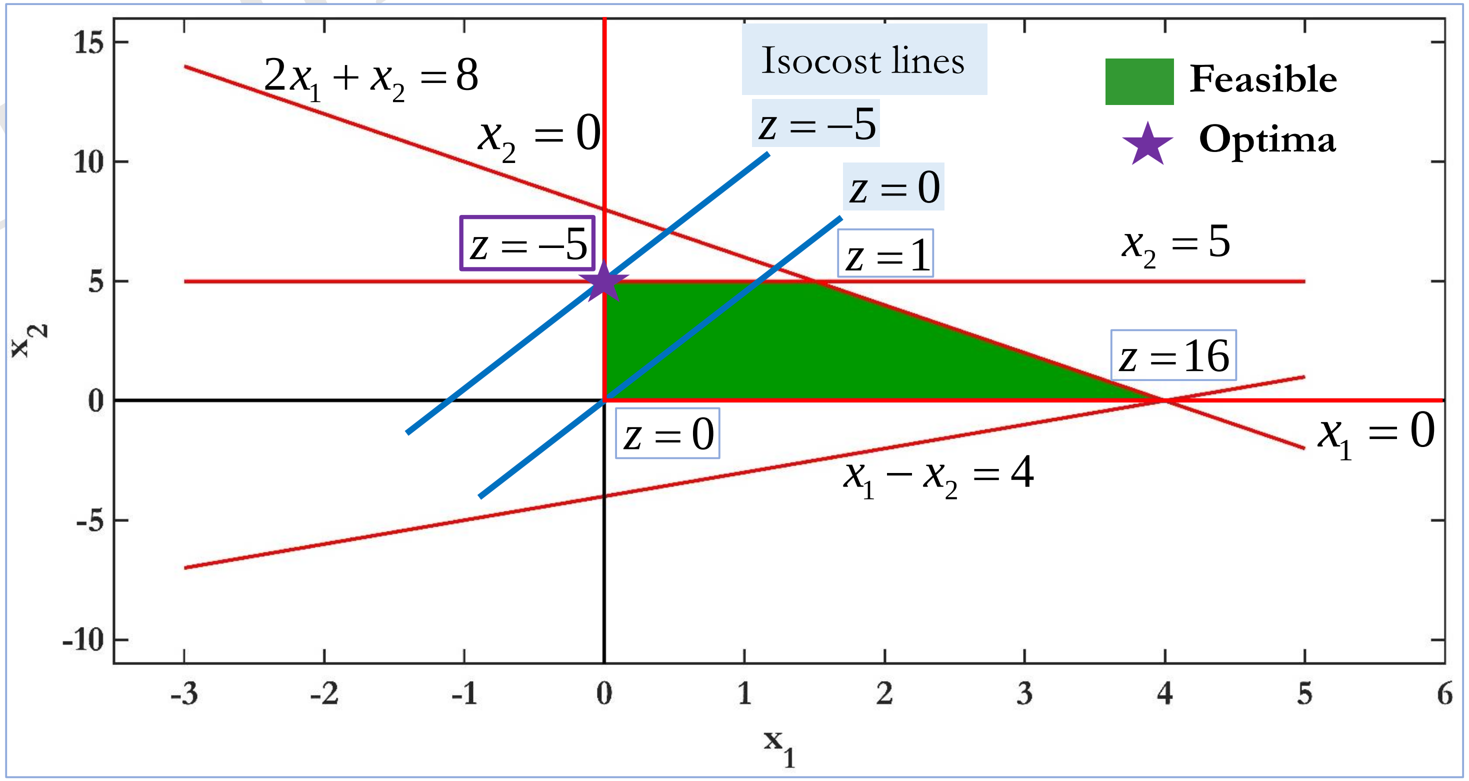
$$\begin{aligned} &\min f(x) \\ &\textit{subject to} \\ &\qquad g(x) \le 0 \\ &\qquad h(x) = 0 \\ &\qquad lb \le x \le ub \end{aligned}$$

Linear Programming (LP)

- Linear objective function and linear constraints
- Optima occurs at vertex
- Can be solved to global optimality
- Algorithms: Simplex, Interior point method

$$\begin{aligned} Min \quad & c^T x \\ s.t. \quad & A_1 x \le b_1 \\ & A_2 x = b_2 \\ & x \ge 0 \end{aligned}$$

$$\begin{aligned} Min \quad & z = 4x_1 - x_2 \\ s.t. \quad & 2x_1 + x_2 \le 8 \\ & x_2 \le 5 \\ & x_1 - x_2 \le 4 \\ & x_1 \ge 0; x_2 \ge 0; \end{aligned}$$



Modelling an LP problem

A farm uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions.

Feedstuff	lb per lb of feedstuff		Cost ($ per lb)
	Protein	Fiber	
Corn	0.09	0.02	0.3
Soyabean meal	0.6	0.06	0.9

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost of feed mix.

Operations research: An introduction, H. A. Taha

Modelling an LP problem

Let x_1 = lb of corn in the daily mix

x_2 = lb of soybean meal in the daily mix

Feedstuff	lb per lb of feedstuff		Cost ($ per lb)
	Protein	fiber	
Corn	0.09	0.02	0.3
Soyabean meal	0.6	0.06	0.9
Dietary requirements	At least 30%	At most 5%	

farms uses at least 800 lb of special feed daily

Objective: determine minimum-cost feed mix.

Minimize $z = 0.3\, x_1 + 0.9\, x_2$

Subject to,

$0.3\, x_1 + 0.9\, x_2 \geq 800$ **Daily demand**

$0.09\, x_1 + 0.6\, x_2 \geq 0.3(x_1 + x_2)$ **Protein requirement in total mix**

$-0.21 x_1 + 0.3\, x_2 \geq 0$

$0.02\, x_1 + 0.06\, x_2 \leq 0.05(x_1 + x_2)$ **Fiber requirement in total mix**

$-0.03\, x_1 + 0.01\, x_2 \leq 0$

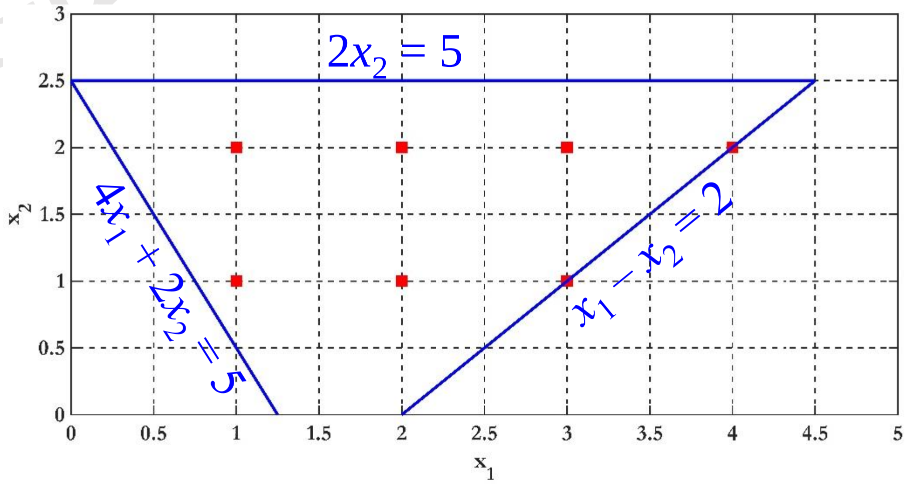
$x_1, x_2 \geq 0$ **Lower bounds of the decision variables**

Operations research: An introduction, H. A. Taha

Integer Linear Programming (ILP)

- All the decision variables are scalars and integers
- Objective function and constraints are linear

$$\begin{aligned} &\textit{Minimize} && 3x_1 + 2x_2 \\ &s.t && 4x_1 + 2x_2 \geq 5 \\ & && 2x_2 \leq 5 \\ & && x_1 - x_2 \leq 2 \\ & && x_1, x_2 \in \mathbb{Z}^+ \end{aligned}$$



- Can be solved to global optimality in sufficient time
- Algorithms: Branch & Bound; Cutting Planes

Modeling of ILP problem

- TOYCO assembles three types of toys: trains, trucks and cars using three operations.

	Assembly time (minutes per unit)			Revenue ($ per unit)
	Operation 1	Operation 2	Operation 3	
Train	1	3	1	3
Truck	2	0	4	2
Car	1	2	0	5
Daily limits of operations (minutes per day)	430	460	420	

- Determine the optimum production rate to maximize profit.

Modeling of ILP problem

- Let x_1 = units of trains to be assembled
  x_2 = units of trucks to be assembled
  x_3 = units of cars to be assembled

Decision variables

All are integer variables

- Maximize profit, $Z = 3\,x_1 + 2\,x_2 + 5\,x_3$

Objective function

- Subject to
  - Limit on operation 1:
    $x_1 + 2\,x_2 + x_3 \le 430$
  - Limit on operation 2:
    $3\,x_1 + 0\,x_2 + 2\,x_3 \le 460$
  - Limit on operation 3:
    $x_1 + 4\,x_2 + 0\,x_3 \le 420$
  - Number of units: $x_1, x_2, x_3 \in \mathbb{Z}^+$

Constraints

Bound constraints

	Assembly time			Revenue
	Operation 1	Operation 2	Operation 3	
Train	1 min	3 min	1 min	\$ 3
Truck	2 min	0 min	4 min	\$ 2
Car	1 min	2 min	0 min	\$ 5
Daily limit on operations	430 min	460 min	420 min	

Mixed Integer Linear Programming (MILP)

- Objective function and constraints are linear.
- At least one decision variables should be an integer.
- At least one decision variable should be continuous.

$$
\begin{aligned}
Max \quad & 4x_1 + 6x_2 + 2x_3 \\
s.t \quad & 4x_1 - 4x_2 \le 5 \\
& -x_1 + 6x_2 \le 5 \\
& -x_1 + x_2 + x_3 \le 5 \\
& x_1, x_2, x_3 \ge 0 \text{ and } x_3 \text{ is integer}
\end{aligned}
$$

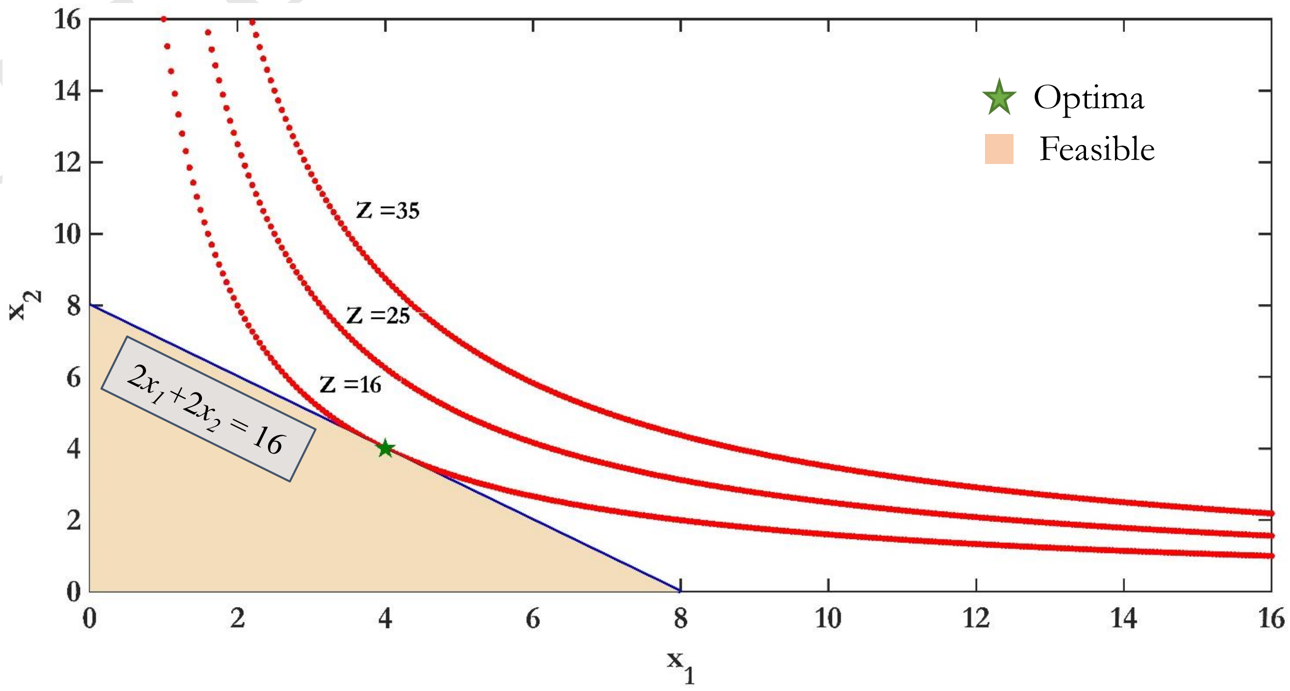
- Can be solved to global optimality (given sufficient amount of time).
- Solve a series of LP problems.
- Algorithms: Branch & Bound; Cutting Planes

Non Linear Programming (NLP)

- Either the objective function, or at least one of the constraints, or both are nonlinear
- Algorithms: Successive Linear Programming, Quadratic programming, Successive Quadratic Programming, Generalized Reduced Gradient method

$$
\begin{aligned}
&Min && f(x) \\
&s.t. && g_j(x) \le b_j \quad j = 1,2,\ldots,m \\
& && h_i(x) = b_i \quad i = 1,2,\ldots,m \\
& && x \in \mathbb{R}
\end{aligned}
$$

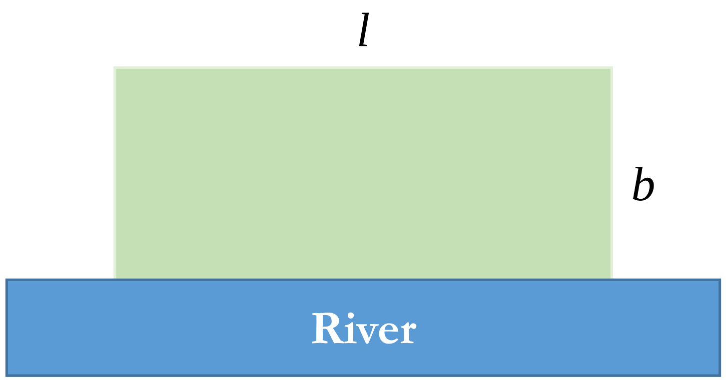
$$
\begin{aligned}
&Maximize && x_1 x_2 \\
&s.t. && 2x_1 + 2x_2 \le 16; \\
& && x_1 \ge 0; x_2 \ge 0;
\end{aligned}
$$



Guaranteed global optimality only if the problem is convex

Modeling of NLP problem

A farmer has 2400 m of fencing and wants to fence off a rectangular field that borders a straight river. No fence is required along the river. What are the dimensions of the field that has the largest area?



Area of a rectangular region = lb

Objective: Maximize the area, $\max f = lb$

Constraint: Perimeter should not exceed 2400m

$$l + 2b \leq 2400$$

Bounds: $l, b > 0$

Mixed Integer Non Linear Programming (MINLP)

- Either the objective function and/or at least one constraint is non linear.
- At least one decision variable is integer (or binary).
- At least one decision variable is continuous.

$$
\begin{aligned}
Min \quad & Z = 0.5\left(x_1^2 + x_2^2\right) - 3x_1 - x_2 \\
s.t \quad & x_1 + 0.5x_2 \le 3 \\
& x_2 - x_1 \le 0 \\
& 0 \le x_1, x_2 \le 10,\ x_2 \text{ is integer}
\end{aligned}
$$

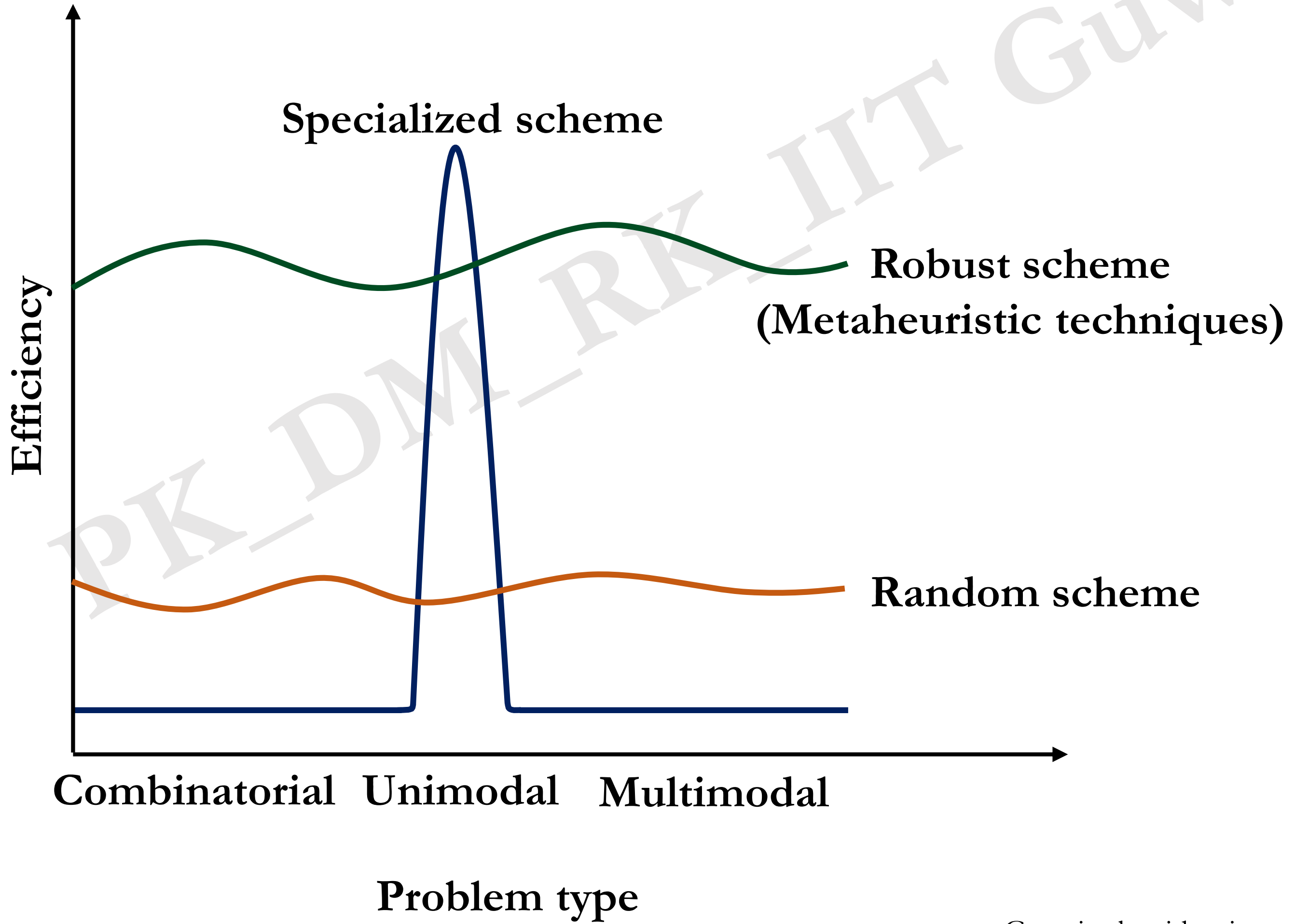
- Can be solved as a series of NLP.
- No guaranteed global optimality.
- Algorithms: Outer Approximation, Branch & Bound

Metaheuristic techniques

- Most of metaheuristic techniques are nature inspired.
- Do not require any information about the physics of the problem.
- Problem need not be postulated in conventional equality and inequality form.
- Suitable to solve problems which are multimodal, larger dimension or discontinuous.
- Provides a satisfactory solution to the problems which are difficult to be solved by conventional methods.
- Can solve black-box optimization problems.
- Works with a population of potential solutions.
- Do not guarantee global optima.

Robustness of optimization techniques

- Traditional methods work well in a narrow problem domain
- Random schemes equally inefficient across a broad spectrum
- Robust scheme works satisfactorily across a broad spectrum



Goldberg's view (1989)

Genetic algorithm in search, optimization and machine learning, D. E. Goldberg

Conventional population based algorithms

Algorithm

1) Generate randomly a single solution/a set of solutions
2) Based on the fitness of the current solution/set of solutions, **suggests other solutions with the help of "intelligent" operators**

Decision variables →

← **Fitness**

Problem Statement

1) Evaluate objective function
2) Check for constraint violation
3) Evaluation of fitness (based on 1 & 2)

Exploiting degrees of freedom

Algorithm

1) Generate randomly a single solution/a set of solutions – Partial decision variables

2) Based on the fitness of the current solution/set of solutions, suggest other solutions

Partial set of decision variables →

← **Fitness**

Problem Statement

1) Generate the remaining set of decision variables

2) Evaluate Objective function

3) Check for Constraint Satisfaction

4) Evaluation of fitness (based on 1,2 & 3)

$$\begin{aligned} \textit{Minimize} \quad & 3x_1 + 2x_2 \\ s.t. \quad & 2x_1 + x_2 \geq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1 - x_2 = 2 \\ & x_1, x_2 \in \mathbb{R} \end{aligned}$$

$\Rightarrow$ $\boxed{x_2 = x_1 - 2}$ $\Rightarrow$

$$\begin{aligned} \textit{Minimize} \quad & 5x_1 - 4 \\ s.t. \quad & 3x_1 \geq 7 \\ & 2x_1 \leq 6 \\ & x_1 \in \mathbb{R} \end{aligned}$$

Metaheuristic techniques and correction approach

Metaheuristic techniques

1) Generate **randomly** a single solution/a set of solutions

2) Based on the fitness of the current solution/set of solutions, **suggests other solutions with the help of "intelligent" operators**

$X = [5 \quad 2 \quad 3]$

Decision Variables

Fitness

$f = 48 + penalty$

Optimization problem

Evaluate fitness function

$$\min f = x_1^3 - 8x_1x_2 + x_3$$
$$s.t. \quad x_1 - x_2 \le 0$$

Metaheuristic techniques

1) Generate **randomly** a single solution/a set of solutions

2) Based on the fitness of the current solution/set of solutions, **suggests other solutions with the help of "intelligent" operators**

$X = [5 \quad 2 \quad 3]$

Decision Variables

Fitness & corrected decision variables

$f = -112,\ X = [5 \quad 6 \quad 3]$

Optimization problem

Evaluate fitness function

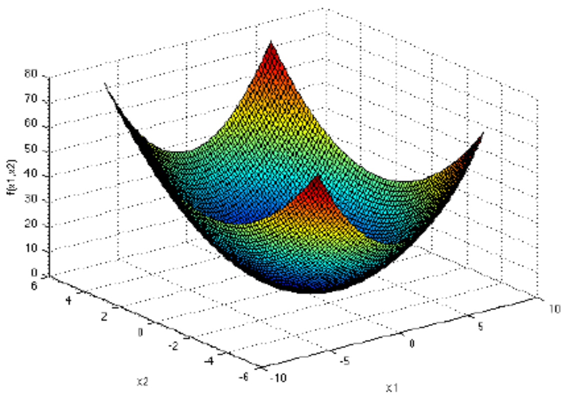
$$\min f = x_1^3 - 8x_1x_2 + x_3$$
$$s.t. \quad x_1 - x_2 \le 0$$

$$Correction: x_2 = \begin{cases} x_2 & x_2 \ge x_1 \\ x_1 + \varepsilon & x_2 < x_1 \end{cases}$$

Let ε be 1

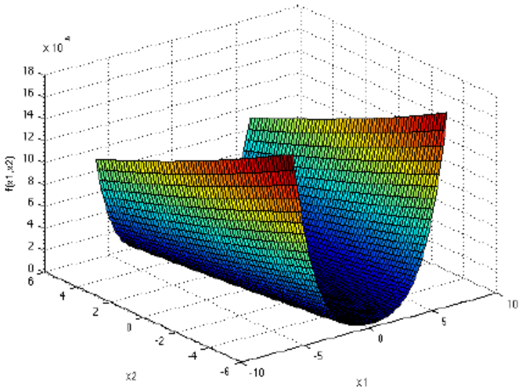
Commonly used benchmark functions for NLP

Sphere Function



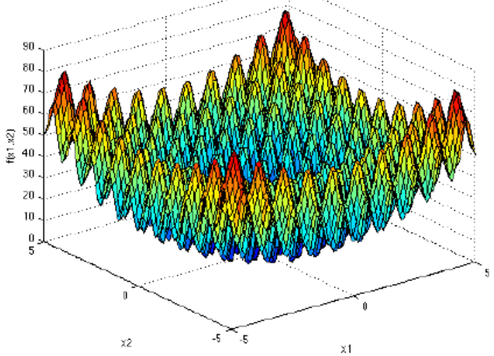
$$f(x) = \sum_{i=1}^{d} x_i^2$$

Rosenbrock Function



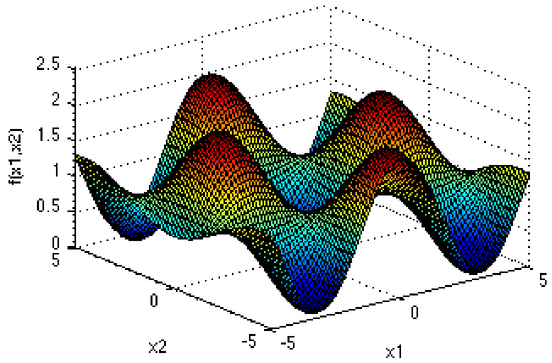
$$f(x) = \sum_{i=1}^{d-1} [100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]$$

Rastrigin Function



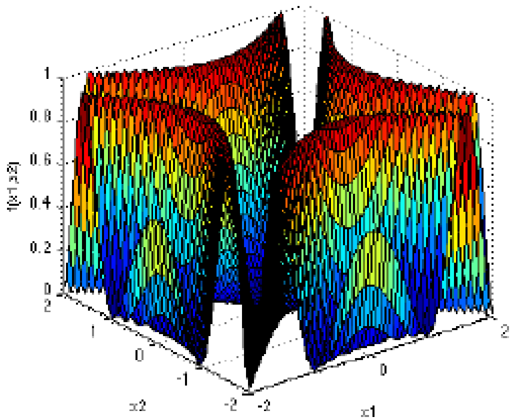
$$f(x) = 10d + \sum_{i=1}^{d} \left[x_i^2 - 10\cos(2\pi x_i) \right]$$

Griewank Function



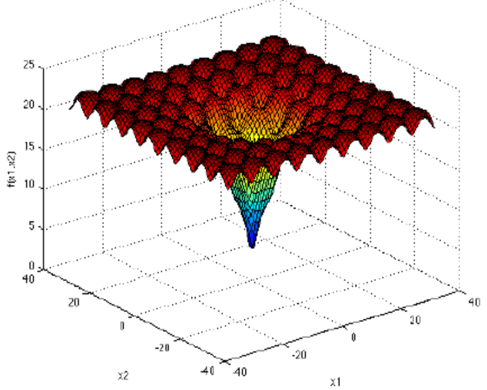
$$f(x) = \sum_{i=1}^{d} \frac{x_i^2}{4000} - \prod_{i=1}^{d} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

Schaffer Function N.2



$$f(x) = 0.5 + \frac{\sin^2(x_1^2 - x_2^2) - 0.5}{[1 + 0.001(x_1^2 + x_2^2)]^2}$$

Ackley Function

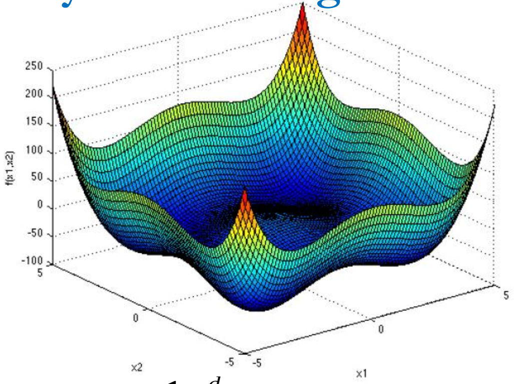


$$f(x) = -20\, e^{\left(-0.2\sqrt{\frac{1}{d}\sum_{i=1}^{d} x_i^2}\right)} - e^{\left(\frac{1}{d}\sum_{i=1}^{d}\cos(2\pi x_i)\right)} + 20 + 2.7183$$

https://www.sfu.ca/~ssurjano/optimization.html

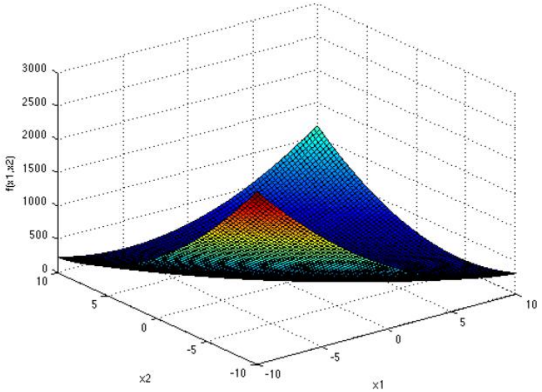
Commonly used benchmark functions for NLP

Styblinski-Tang Function



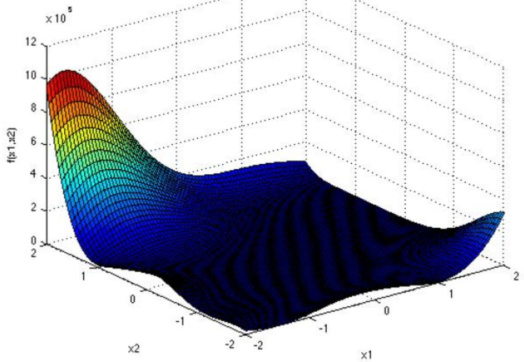
$$f(x) = \frac{1}{2}\sum_{i=1}^{d}\left(x_i^4 - 16x_i^2 + 5x_i\right)$$

Booth Function



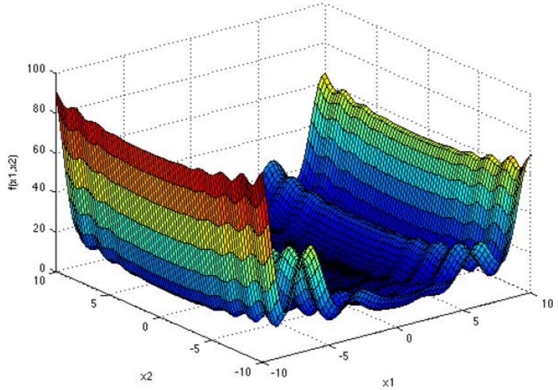
$$f(x) = \left(x_1 + 2x_2 - 7\right)^2 + (2x_1 + x_2 - 5)^2$$

Goldstein-Price Function



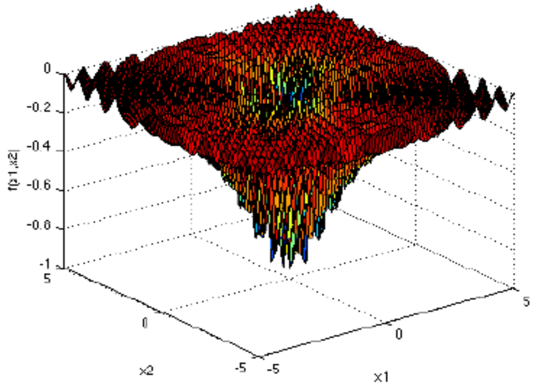
$$f(x) = \left[1 + \left(x_1 + x_2 + 1\right)^2\left(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2\right)\right]$$
$$\times\left[30 + \left(2x_1 - 3x_2\right)^2\left(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2\right)\right]$$

Levy Function



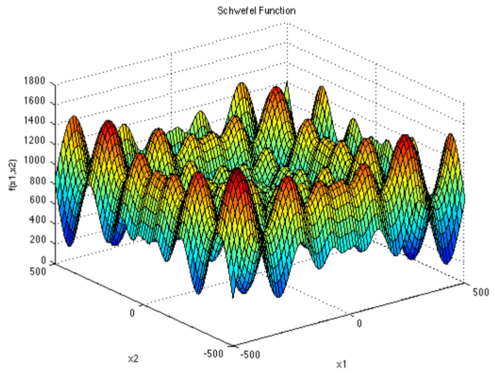
$$f(x) = \sin^2(\pi\omega_1) + \sum_{i=1}^{d-1}(\omega_i - 1)^2\left[1 + 10\sin^2(\pi\omega_i + 1)\right]$$
$$+ (\omega_d - 1)^2\left[1 - \sin(2\pi\omega_d)\right] \quad where\ \omega_i = 1 + \frac{x_i - 1}{4}$$

Drop-Wave Function



$$f(x) = -\frac{1 + \cos\left(12\sqrt{x_1^2 + x_2^2}\right)}{0.5(x_1^2 + x_2^2) + 2}$$

Schwefel Function

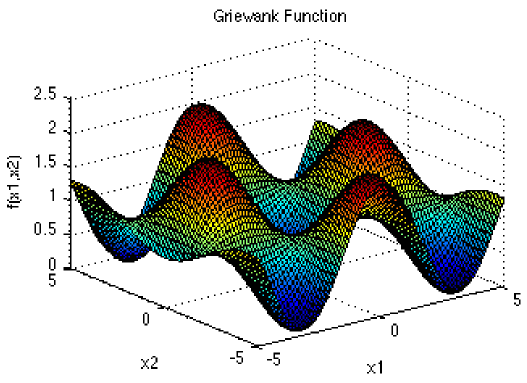


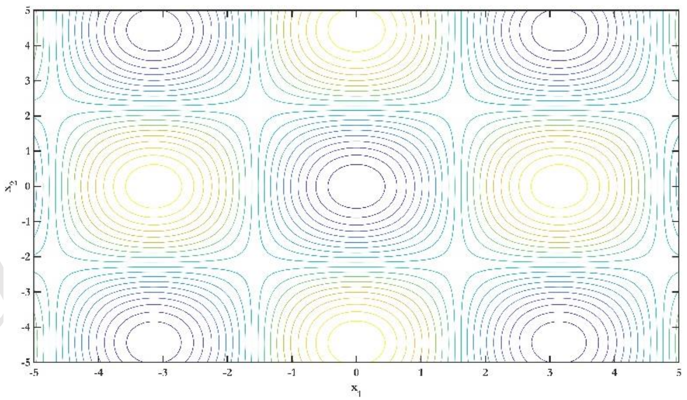
$$f(x) = 418.9829d - \sum_{i=1}^{d} x_i \sin\sqrt{|x_i|}$$

https://www.sfu.ca/~ssurjano/optimization.html

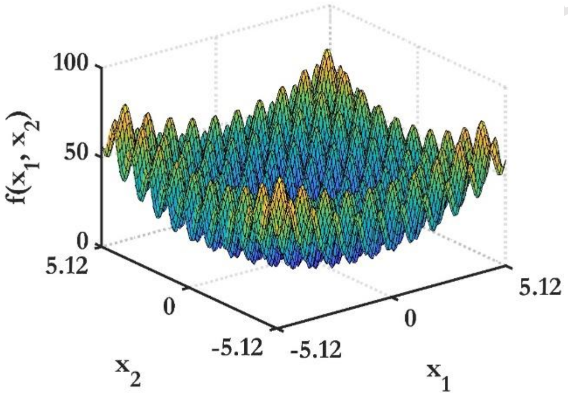
Contour plots of benchmark functions

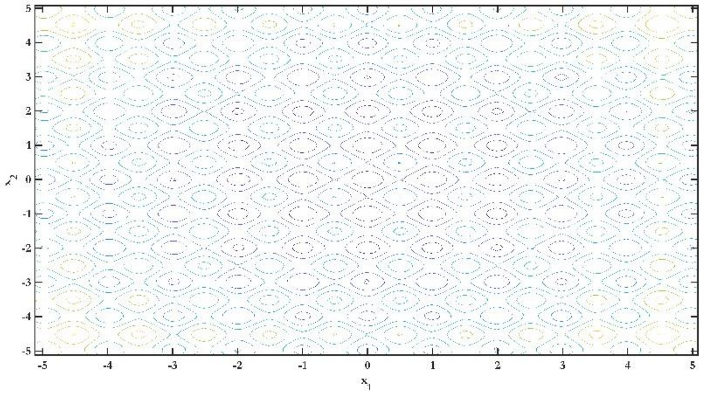
Griewank Function



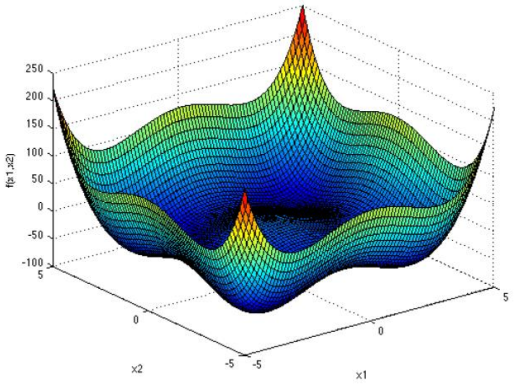


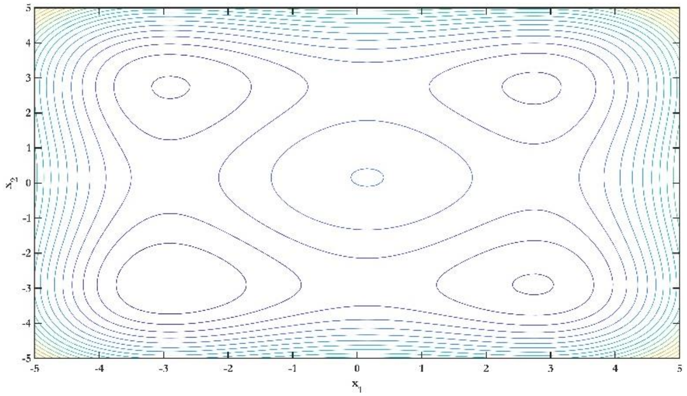
Rastrigin Function





Styblinski-Tang Function





Commonly used benchmark functions for NLP

Table A.1. Unimodal test functions.

Name	Function	D	Range	f_{opt}
Sphere	$f_1(x) = \sum_{i=1}^{n} x_i^2$	30	$[-100,100]^n$	0
Schwefel 2.22	$f_2(x) = \sum_{i=1}^{n} \lvert x_i \rvert + \prod_{i=1}^{n} \lvert x_i \rvert$	30	$[-10,10]^n$	0
Schwefel 1.2	$f_3(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} x_j \right)^2$	30	$[-100,100]^n$	0
Schwefel 2.21	$f_4(x) = \max_i \{ \lvert x_i \rvert, 1 \leq i \leq n \}$	30	$[-100,100]^n$	0
Rosenbrock	$f_5(x) = \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i)^2 \right) + (x_i - 1)^2$	30	$[-30,30]^n$	0
Step	$f_6(x) = \sum_{i=1}^{n} (x_i + 0.5)^2$	30	$[-100,100]^n$	0
Quartic	$f_7(x) = \sum_{i=1}^{n} i x_i^4 + random[0,1)$	30	$[-1.28,1.28]^n$	0

Name	Definition	Domain / Characterisric	f*
Easom's function	$f(x) = -(-1^n)\left(\prod_{i=1}^{n} cos^2(x_i)\right) exp\left[-\sum_{i=1}^{n}(x_i - \pi)^2\right]$	[−2 π, 2 π]/U	−1
Michalewicz's function	$f(x) = -\sum_{i=1}^{n} \sin(x_i) \left[sin\left(\frac{i x_i^2}{\pi}\right)\right]^{2.10}$	[0, π]/M	*
Xin-She Yang's function	$f(x) = \left(\sum_{i=1}^{n} \lvert x_i \rvert\right) exp\left[-\sum_{i=1}^{n} \sin(x_i^2)\right]$	[−2 π, 2 π] /M	0
Zakharov's function	$f(x) = \sum_{i=1}^{n} x_i^2 + \left(\frac{1}{2}\sum_{i=1}^{n} i x_i\right)^2 + \left(\frac{1}{2}\sum_{i=1}^{n} i x_i\right)^4$	[−5,10] /U	0

Table A.2. Multimodal test functions.

Name	Function	D	Range	f_{opt}
Schwefel	$f_8(x) = -\sum_{i=1}^{n} \left(x_i \sin\left(\sqrt{\lvert x_i \rvert}\right)\right)$	30	$[-500,500]^n$	−12 569.5
Rastrigin	$f_9(x) = \sum_{i=1}^{n} \left(x_i^2 - 10 \cos(2\pi x_i) + 10\right)^2$	30	$[-5.12,5.12]^n$	0
Ackley	$f_{10}(x) = -20 \exp\left(-0.2\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - \exp\left(\frac{1}{n}\sum_{i=1}^{n} \cos 2\pi x_i\right) + 20 + e$	30	$[-32,32]^n$	0
Griewank	$f_{11}(x) = \frac{1}{4000}\sum_{i=1}^{n}(x_i - 100)^2 - \prod_{i=1}^{n} \cos\left(\frac{x_i - 100}{\sqrt{i}}\right) + 1$	30	$[-600,600]^n$	0
Penalized	$f_{12}(x) = \frac{\pi}{n} 10 \sin 2(\pi y_1) + \sum_{i=1}^{n-1}(y_i - 1)^2 [1 + 10 \sin 2(\pi y_i + 1)] + (y_n - 1)^2 + \sum_{i=1}^{30} u(x_i, 10, 100, 4)$	30	$[-50,50]^n$	0
Penalized2	$f_{13}(x) = 0.1 \sin 2(3\pi x_1) + \sum_{i=1}^{29}(x_i - 1)^2 p[1 + \sin 2(3\pi x_{i+1})] + (x_n - 1)^2 [1 + \sin 2(2\pi x_{30})] + \sum_{i=1}^{30} u(x_i, 5, 10, 4)$	30	$[-50,50]^n$	0

Manta ray foraging optimization: An effective bio-inspired optimizer for engineering applications, Engineering applications of artificial intelligence, 87, 2020
A novel modified bat algorithm hybridizing by differential evolution algorithm, Expert systems and applications, 141(1), 2020

Constrained engineering problems

B.3. Welded beam design

Consider variable $\vec{x} = [x_1, x_2, x_3, x_4] = [h, l, t, b]$.

Minimize $f_3(\vec{x}) = 1.10471x_1^2x_2 + 0.04811x_3x_4(14 + x_2)$.

Subject to

$g_1(\vec{x}) = \tau(\vec{x}) + \tau_{\max} \le 0,\ g_2(\vec{x}) = \sigma(\vec{x}) + \sigma_{\max} \le 0,\ g_3(\vec{x}) = \delta(\vec{x}) + \delta_{\max} \le 0,$
$g_4(\vec{x}) = x_1 - x_4 \le 0,\ g_5(\vec{x}) = P - P_c(\vec{x}) \le 0, g_6(\vec{x}) = 0.125 - x_1 \le 0,$
$g_7(\vec{x}) = 0.10471x_1^2 + 0.04811x_3x_4(14 + x_2) - 5 \le 0.$

where

$\tau(\vec{x}) = \sqrt{(\tau')^2 + 2\tau'\tau''\frac{x_2}{2R} + (\tau'')^2},\ \tau' = \frac{P}{\sqrt{2}x_1x_2},\ \tau'' = \frac{MR}{J}, M = P\left(L + \frac{x_2}{2}\right),$
$R = \sqrt{\frac{x_2^2}{4} + \left(\frac{x_1+x_3}{2}\right)^2},\ J = 2\left\{\sqrt{2}x_1x_2\left[\frac{x_2^2}{4} + \left(\frac{x_1+x_3}{2}\right)^2\right]\right\}, \sigma(\vec{x}) = \frac{6PL}{x_4x_3^2},\ \delta(\vec{x}) = \frac{4PL^3}{Ex_4x_3^3},$
$P_c(\vec{x}) = \frac{4.013E\sqrt{\frac{x_3^2x_4^6}{36}}}{L^2}\left(1 - \frac{x_3}{2L}\sqrt{\frac{E}{4G}}\right).$

where $P = 6000$ lb, $L = 14$ in, $E = 30 \times 10^6$ psi, $G = 12 \times 10^6$ psi, $\tau_{\max} = 13\,600$ psi, $\sigma_{\max} = 30\,000$ psi, $\delta_{\max} = 0.25$ in.

Variable range $0.1 \le x_1 \le 2,\ 0.1 \le x_2 \le 10,\ 0.1 \le x_3 \le 10, 0.1 \le x_4 \le 2$.

B.7. Belleville spring design

Consider variable $\vec{x} = [t, h, D_i, D_e]$.

Minimize: $f_7(\vec{x}) = 0.07075\pi(D_e^2 - D_i^2)t$.

Subject to

$g_1(\vec{x}) = S - \frac{4E\delta_{\max}}{(1-\mu^2)\alpha D_e^2}\left[\beta\left(h - \frac{\delta_{\max}}{2}\right) + \gamma t\right] \ge 0,$
$g_2(\vec{x}) = \left(\frac{4E\delta_{\max}}{(1-\mu^2)\alpha D_e^2}\left[\left(h - \frac{\delta}{2}\right)\left((h - \delta)t + t^3\right)\right]\right)_{\delta=\delta_{\max}} - P_{\max} \ge 0,$
$g_3(\vec{x}) = \delta_1 - \delta_{\max} \ge 0, g_4(\vec{x}) = H - h - t \ge 0,$
$g_5(\vec{x}) = D_{\max} - D_e \ge 0, g_6(\vec{x}) = D_e - D_i \ge 0, g_7(\vec{x}) = 0.3 - \frac{h}{D_e - D_i} \ge 0.$

where $\alpha = \frac{6}{\pi \ln K}\left(\frac{K-1}{K}\right)^2, \beta = \frac{6}{\pi \ln K}\left(\frac{K-1}{\ln K} - 1\right), \gamma = \frac{6}{\pi \ln K}\left(\frac{K-1}{2}\right),$

$P_{\max} = 5400$ lb, $\delta_{\max} = 0.2$ in, $S = 200\,000$ psi, $E = 30 \times 10^6$ psi, $\mu = 0.3$, $H = 2$ in, $D_{\max} = 12.01$ in, $K = D_e / D_i$, $\delta_1 = f(a)h$, $a = h / t$.

Values of $f(a)$ vary as shown in Table B.1.

Variable range $0.01 \le t \le 6,\ 0.05 \le h \le 0.5,\ 5 \le D_i \le 15,\ 5 \le D_o \le 15$.

B.4. Speed reducer design

Consider variable $\vec{x} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7] = [b, m, z, l_1, l_2, d_1, d_2]$.

Minimize

Minimize $f_4(\vec{x}) = 0.7854x_1x_2^2(3.3333x_3^2 + 14.9334x_3 - 43.0934)$
$-1.508x_1(x_6^2 + x_7^2) + 0.7854x_1(x_4x_6^2 - x_5x_7^2)$.

Subject to $g_1(\vec{x}) = \frac{27}{x_1x_2^2x_3} - 1 \le 0, g_2(\vec{x}) = \frac{397.5}{x_1x_2^2x_3^2} - 1 \le 0,$
$g_3(\vec{x}) = \frac{1.93x_4^3}{x_2x_6^4x_3} - 1 \le 0, g_4(\vec{x}) = \frac{1.93x_5^3}{x_2x_7^4x_3} - 1 \le 0,$
$g_5(\vec{x}) = \frac{\left(\left(\frac{745x_4}{x_2x_3}\right)^2 + 16.9\times10^6\right)^{0.5}}{110x_6^3} - 1 \le 0,$
$g_6(\vec{x}) = \frac{\left(\left(\frac{745x_5}{x_2x_3}\right)^2 + 157.5\times10^6\right)^{0.5}}{85x_7^3} - 1 \le 0,$
$g_7(\vec{x}) = \frac{x_2x_3}{40} - 1 \le 0, g_8(\vec{x}) = \frac{5x_2}{x_1} - 1 \le 0,$
$g_9(\vec{x}) = \frac{x_1}{12x_2} - 1 \le 0, g_{10}(\vec{x}) = \frac{1.5x_6 + 1.9}{x_4} - 1 \le 0,$
$g_{11}(\vec{x}) = \frac{1.1x_7 + 1.9}{x_5} - 1 \le 0.$

Variable range $2.6 \le x_1 \le 3.6,\ 0.7 \le x_2 \le 0.8,\ 17 \le x_3 \le 28,$
$7.3 \le x_4 \le 8.3,$
$7.3 \le x_5 \le 8.3,\ 2.9 \le x_6 \le 3.9,\ 5.0 \le x_7 \le 5.5.$

B.2. Pressure vessel design

Consider variable $\vec{x} = [x_1, x_2, x_3, x_4] = [T_s, T_h, R, L]$.

Minimize $f_2(\vec{x}) = 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3$.

Subject to $g_1(\vec{x}) = -x_1 + 0.0193x_3 \le 0,\ g_2(\vec{x}) = -x_2 + 0.00954x_3 \le 0,$
$g_3(\vec{x}) = -\pi x_3^2x_4 - \frac{4}{3}\pi x_3^3 + 1296000 \le 0,\ g_4(\vec{x}) = x_4 - 240 \le 0.$

Variable range $0 \le x_1 \le 99, 0 \le x_2 \le 99, 10 \le x_3 \le 200, 10 \le x_4 \le 200$.

B.1. Tension/compression spring design

Consider variable $\vec{x} = [x_1, x_2, x_3] = [d, D, N]$.

Minimize $f_1(\vec{x}) = (x_3 + 2)x_2x_1^2$.

Subject to $g_1(\vec{x}) = 1 - \frac{x_2^3x_3}{71785x_1^4} \le 0,$
$g_2(\vec{x}) = \frac{4x_2^2 - x_1x_2}{12566(x_2x_1^3 - x_1^4)} + \frac{1}{5108x_1^2} - 1 \le 0,$
$g_3(\vec{x}) = 1 - \frac{140.45x_1}{x_2^2x_3} \le 0,\ g_4(\vec{x}) = \frac{x_1 + x_2}{1.5} - 1 \le 0.$

Variable range $0.05 \le x_1 \le 2, 0.25 \le x_2 \le 1.3, 2 \le x_3 \le 15$.

B.8. Hydrostatic thrust bearing design

Consider variable $\vec{x} = [R, R_o, \mu, Q]$.

Minimize $f_8(\vec{x}) = \frac{QP_o}{0.7} + E_f$.

Subject to

$g_1(\vec{x}) = W - W_s \ge 0,\ g_2(\vec{x}) = P_{\max} - P_o \ge 0,$
$g_3(\vec{x}) = \Delta T_{\max} - \Delta T \ge 0,\ g_4(\vec{x}) = h - h_{\min} \ge 0,$
$g_5(\vec{x}) = R - R_o \ge 0,\ g_6(\vec{x}) = 0.001 - \frac{\gamma}{gP_o}\left(\frac{Q}{2\pi Rh}\right) \ge 0, g_7(\vec{x}) = 5000 - \frac{W}{\pi(R^2 - R_o^2)} \ge 0.$

where $W = \frac{\pi P_o}{2}\frac{R^2 - R_o^2}{\ln(R/R_o)}, P_o = \frac{6\mu Q}{\pi h^3}\ln(R / R_o), E_f = 9336Q\gamma C\Delta T$

$\Delta T = 2(10^P - 560), P = \frac{\log_{10}\log 10(8.122\times10^6\mu + 0.8) - C_1}{n}, h = \left(\frac{2\pi N}{60}\right)^2\frac{2\pi\mu}{E_f}\left(\frac{R^4}{4} - \frac{R_o^4}{4}\right),$
$\gamma = 0.0307, C = 0.5, n = -3.55, C_1 = 10.04, W_s = 101\,000, P_{\max} = 1000, h_{\min} = 0.001,$
$\Delta T_{\max} = 50, g = 386.4, N = 750.$

Variable range $1 \le R \le 16,\ 1 \le R_o \le 16,\ 10^{-6} \le \mu \le 16 \times 10^{-6}, 1 \le Q \le 16$.

Result comparison

Functions		Min	GA	DE	PSO	BA	PBA	SOS
Beale	Mean	0	0	0	0	1.88E−05	0	0
	StdDev		0	0	0	1.94E−05	0	0
Easom	Mean	−1	−1	−1	−1	−0.99994	−1	−1
	StdDev		0	0	0	4.50E−05	0	0
Matyas	Mean	0	0	0	0	0	0	0
	StdDev		0	0	0	0	0	0
Bohachevsky1	Mean	0	0	0	0	0	0	0
	StdDev		0	0	0	0	0	0
Booth	Mean	0	0	0	0	0.00053	0	0.03382
	StdDev		0	0	0	0.00074	0	0.12870
Michalewicz2	Mean	−1.8013	**−1.8013**	**−1.8013**	−1.57287	**−1.8013**	**−1.8013**	**−1.8013**
	StdDev		0	0	0.11986	0	0	0
Schaffer	Mean	0	0.00424	0	0	0	0	0
	StdDev		0.00476	0	0	0	0	0
Six Hump Camel Back	Mean	−1.03163	**−1.03163**	**−1.03163**	**−1.03163**	**−1.03163**	**−1.03163**	**−1.03163**
	StdDev		0	0	0	0	0	0
Boachevsky2	Mean	0	0.06829	0	0	0	0	0
	StdDev		0.07822	0	0	0	0	0
Boachevsky3	Mean	0	0	0	0	0	0	0
	StdDev		0	0	0	0	0	0
Shubert	Mean	−186.73	**−186.73**	**−186.73**	**−186.73**	**−186.73**	**−186.73**	**−186.73**
	StdDev		0	0	0	0	0	0
Colville	Mean	0	0.01494	0.04091	0	1.11760	0	0
	StdDev		0.00736	0.08198	0	0.46623	0	0
Michalewicz5	Mean	−4.6877	−4.64483	−4.68348	−2.49087	**−4.6877**	**−4.6877**	**−4.6877**
	StdDev		0.09785	0.01253	0.25695	0	0	0
Zakharov	Mean	0	0.01336	0	0	0	0	0
	StdDev		0.00453	0	0	0	0	0

Michalewicz5	Mean	−4.6877	−4.64483	−4.68348	−2.49087	**−4.6877**	**−4.6877**	**−4.6877**
	StdDev		0.09785	0.01253	0.25695	0	0	0
Zakharov	Mean	0	0.01336	0	0	0	0	0
	StdDev		0.00453	0	0	0	0	0
Michalewicz10	Mean	−9.6602	−9.49683	−9.59115	−4.00718	**−9.6602**	**−9.6602**	−9.65982
	StdDev		0.14112	0.06421	0.50263	0	0	0.00125
Step	Mean	0	1.17E+03	0	0	5.12370	0	0
	StdDev		76.56145	0	0	0.39209	0	0
Sphere	Mean	0	1.11E+03	0	0	0	0	0
	StdDev		74.21447	0	0	0	0	0
SumSquares	Mean	0	1.48E+02	0	0	0	0	0
	StdDev		12.40929	0	0	0	0	0
Quartic	Mean	0	0.18070	0.00136	0.00116	1.72E−06	0.00678	9.13E−05
	StdDev		0.02712	0.00042	0.00028	1.85E−06	0.00133	3.71E−05
Schwefel 2.22	Mean	0	11.0214	0	0	0	7.59E−10	0
	StdDev		1.38686	0	0	0	7.10E−10	0
Schwefel 1.2	Mean	0	7.40E+03	0	0	0	0	0
	StdDev		1.14E+03	0	0	0	0	0
Rosenbrock	Mean	0	1.96E+05	18.20394	15.088617	28.834	4.2831	1.04E−07
	StdDev		3.85E+04	5.03619	24.170196	0.10597	5.7877	2.95E−07
Dixon–Price	Mean	0	1.22E+03	0.66667	0.66667	0.66667	0.66667	0
	StdDev		2.66E+02	E−9	E−8	1.16E−09	5.65E−10	0
Rastrigin	Mean	0	52.92259	11.71673	43.97714	0	0	0
	StdDev		4.56486	2.53817	11.72868	0	0	0
Griewank	Mean	0	10.63346	0.00148	0.01739	0	0.00468	0
	StdDev		1.16146	0.00296	0.02081	0	0.00672	0
Ackley	Mean	0	14.67178	0	0.16462	0	3.12E−08	0
	StdDev		0.17814	0	0.49387	0	3.98E−08	0

Algorithm	Optimum variables			Optimum weight
	d	*D*	N	
MFO	0.051994457	0.36410932	10.868421862	0.0126669
GSA	0.050276	0.323680	13.525410	0.0127022
PSO [66]	0.051728	0.357644	11.244543	0.0126747
ES [81]	0.051989	0.363965	10.890522	0.0126810
GA [79]	0.051480	0.351661	11.632201	0.0127048
HS [78]	0.051154	0.349871	12.076432	0.0126706
DE [82]	0.051609	0.354714	11.410831	0.0126702
Mathematical optimization [89]	0.053396	0.399180	9.1854000	0.0127303
Constraint correction [88]	0.050000	0.315900	14.250000	0.0128334

Comparison results for cantilever design problem.

Algorithm	Optimal values for variables					Optimum weight
	x_1	x_2	x_3	x_4	x_5	
MFO	5.9848717732166	5.31672692429783	4.49733258583062	3.51361646768954	2.16162029338550	1.33998808597181
MMA [86]	6.0100	5.3000	4.4900	3.4900	2.1500	1.3400
GCA_I [86]	6.0100	5.3000	4.4900	3.4900	2.1500	1.3400
GCA_II [86]	6.0100	5.3000	4.4900	3.4900	2.1500	1.3400
CS [72]	6.0089	5.3049	4.5023	3.5077	2.1504	1.33999
SOS [41]	6.01878	5.30344	4.49587	3.49896	2.15564	1.33996

Moth-flame optimization algorithm: A novel nature-inspired heuristic paradigm, Knowledge based systems, 89, 2015
Symbiotic Organisms Search: A new metaheuristic optimization algorithm, Computers and structures, 139, 2014

Benchmark functions

Benchmarks for Evaluation of Evolutionary Algorithms

We organized several competitions on benchmarking evolutionary algorithms. Recently, we also developed **several composition functions** to evaluate evolutionary algorithms and also in the CEC Invited Session / Competition pages listed below.

J. J. Liang, P. N. Suganthan and K. Deb, "Novel Composition Test Functions for Numerical Global Optimization", *IEEE Swarm Intelligence Symposium*, pp. 68-75, June 2005.

CEC'05 Special Session / Competition on Evolutionary Real Parameter single objective optimization

CEC'06 Special Session / Competition on Evolutionary Constrained Real Parameter single objective optimization

CEC'07 Special Session / Competition on Performance Assessment of real-parameter MOEAs

CEC'08 Special Session / Competition on large scale single objective global optimization with bound constraints

CEC'09 Special Session / Competition on Dynamic Optimization (Primarily composition functions were used)

CEC09 Special Session / Competition on Performance Assessment of real-parameter MOEAs

CEC10 Special Session / Competition on large-scale single objective global optimization with bound constraints

CEC10 Special Session / Competition on Evolutionary Constrained Real Parameter single objective optimization

CEC10 Special Session on Niching Introduces novel scalable test problems: B. Y. Qu and P. N. Suganthan, "Novel Multimodal Problems and Differential Evolution with Ensemb
Barcelona, Spain, July 2010.

CEC11 Competition on Testing Evolutionary Algorithms on Real-world Numerical Optimization Problems

CEC2013 Special Session / Competition on Real Parameter Single Objective Optimization

CEC2014 Special Session / Competition on Real Parameter Single Objective Optimization (incorporates expensive function optimization)

CEC2014: Dynamic MOEA Benchmark Problems: Subhodip Biswas, Swagatam Das, P. N. Suganthan and C. A. C Coello, "Evolutionary Multiobjective Optimization in Dynamic E

CEC2015 Special Session / Competition on Real Parameter Single Objective Optimization (incorporates 3 scenarios)
CEC2016 Special Session / Competition on Real Parameter Single Objective Optimization (incorporates 4 scenarios)
CEC2017 Special Session / Competition on Real Parameter Single Objective Optimization (incorporates 3 scenarios)
CEC2018 Special Session / Competition on Real Parameter Single Objective Optimization (incorporates 3 scenarios)
SWEVO (Impact Factor=3.8 in 2017) Special issue on Benchmarking Multi and Many Objective Optimization Algorithms
CEC2019 Special Session / Competition on 100-Digit Challenge on Single Objective Numerical Optimization
GECCO 2019 Workshop & Competition on Numerical Optimization

CEC2020, SEMCCO'20, GECCO'20 Special Session / Competition on Real-World Single Objective Constrained Optimization

CEC2020 GECCO'20, SEMCCO'20 Special Session / Competition on Real Parameter Single Objective Bound Constrained Optimization

CEC2020 GECCO'20, SEMCCO'20 Special Session / Competition on Real Parameter Multimodal Multi-objective Optimization

For more details
https://www.ntu.edu.sg/home/epnsugan/

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Mohit Jain, Vijander Singh, Asha Rani

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Yiying Zhang [a], Zhigang Jin [a], Ye Chen [a, b]

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Dražen Bajer, Bruno Zorić

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August 2019, Volume 31, Issue 8, pp 4049–4083 | Cite as

A multi-objective artificial sheep algorithm

Authors Authors and affiliations
Xinjie Lai, Chaoshun Li, Nan Zhang, Jianzhong Zhou

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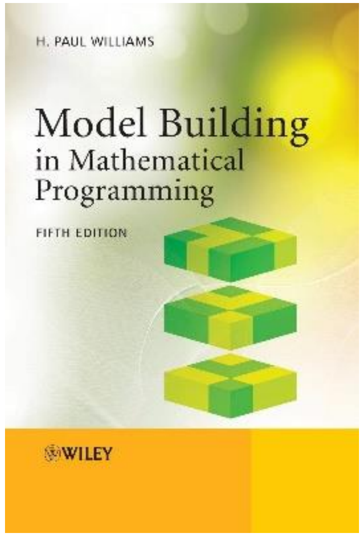
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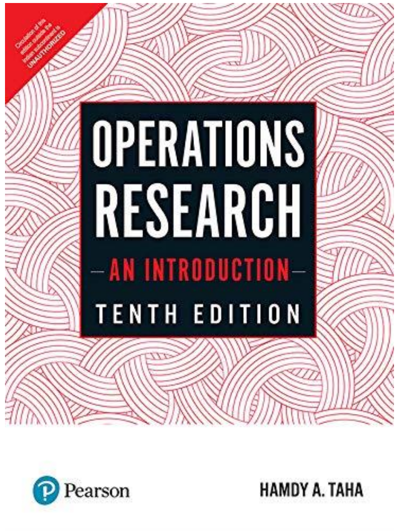
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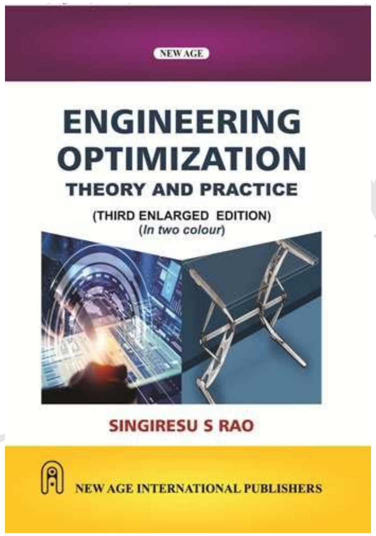
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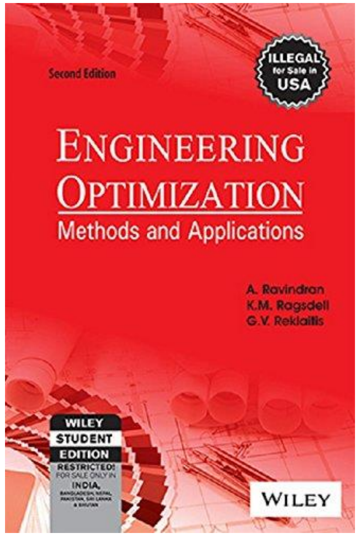
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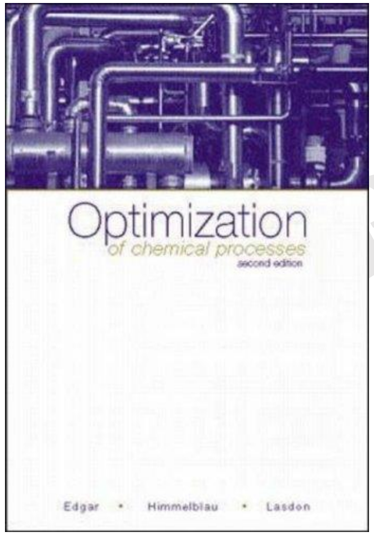
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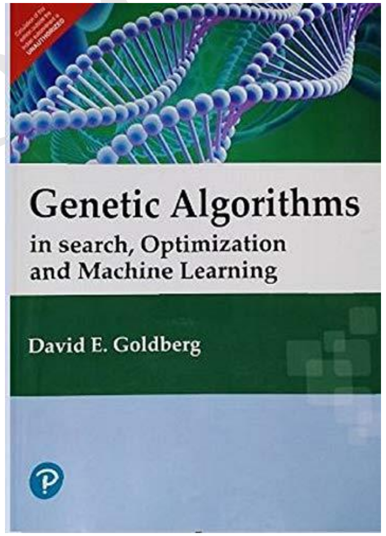
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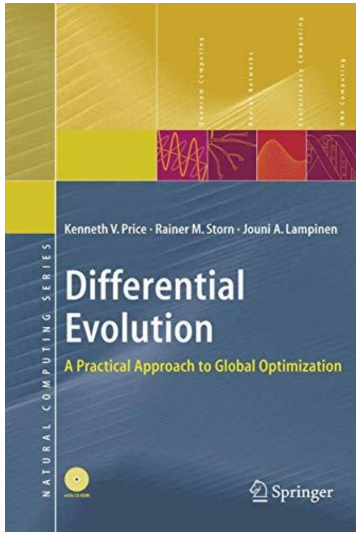
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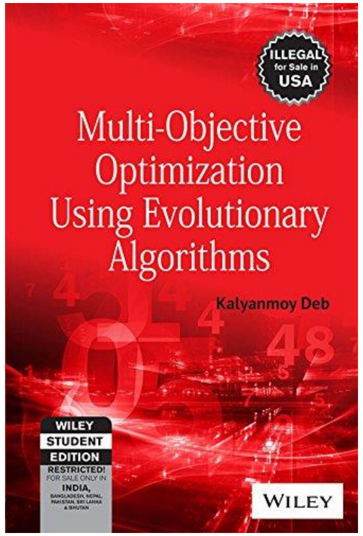
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