Decision Trees

Root/ each internal node: test one attribute
Each branch from a node: selects one
value for the attribute
Each leaf node: predict class

Root
Node

Branches
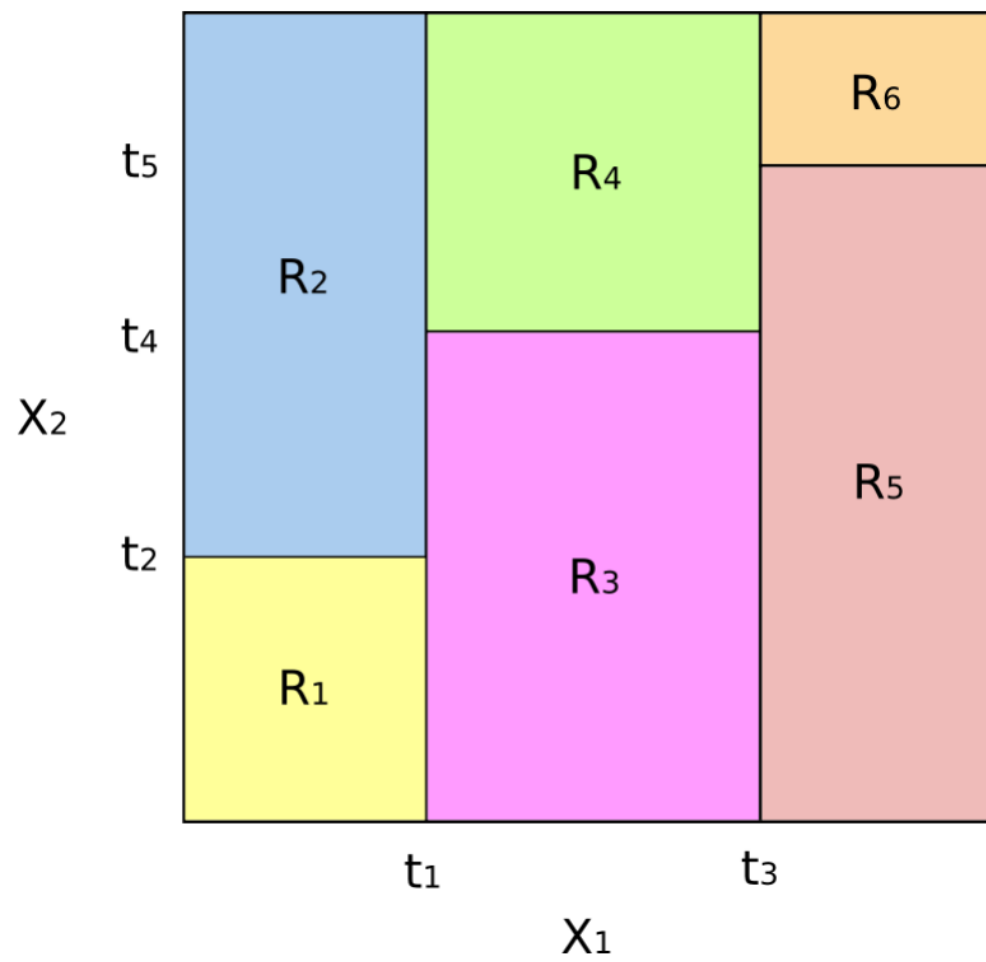
Internal
Node

Internal
Node

Branches

Branches

Leaf
Node

Leaf
Node

Leaf
Node

Leaf
Node

Outlook

Sunny   Overcast   Rain

Humidity   Yes   Wind

High   Normal   Strong   Weak

No   Yes   No   Yes

$X_1 < t_1$

$X_2 < t_2$

$X_1 < t_3$

R_1

R_2

$X_2 < t_4$

$X_2 < t_5$

R_3

R_4

R_5

R_6

t_5

t_4

t_2

X_2

R_2

R_4

R_6

R_1

R_3

R_5

t_1

t_3

X_1
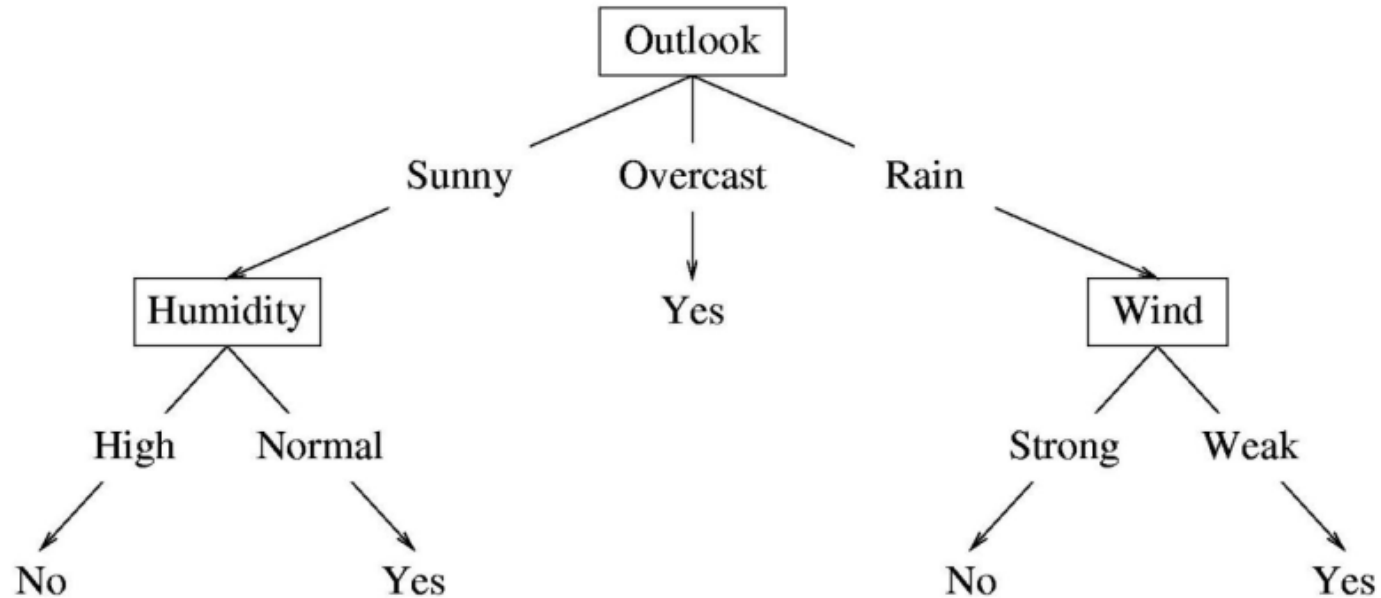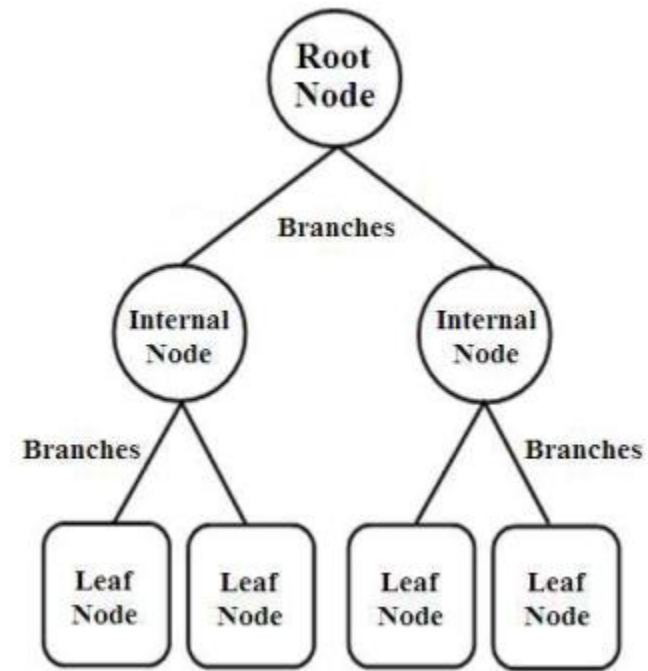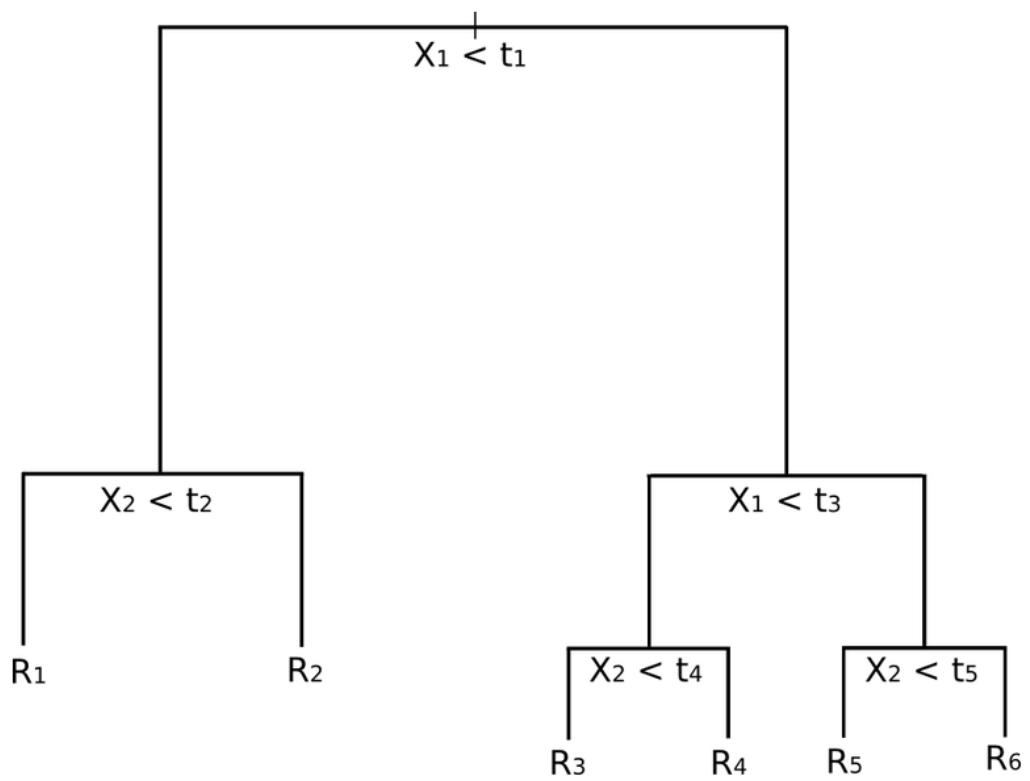
A	B	A xor B
F	F	F
F	T	T
T	F	T
T	T	F
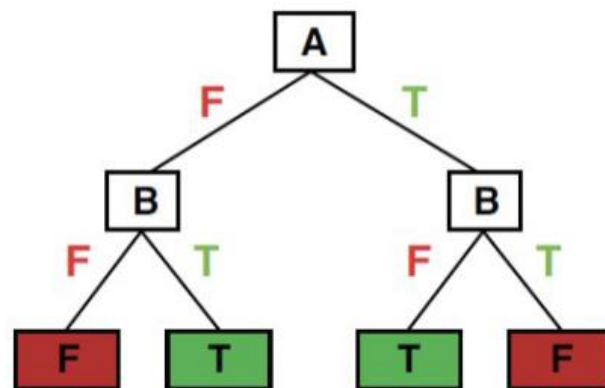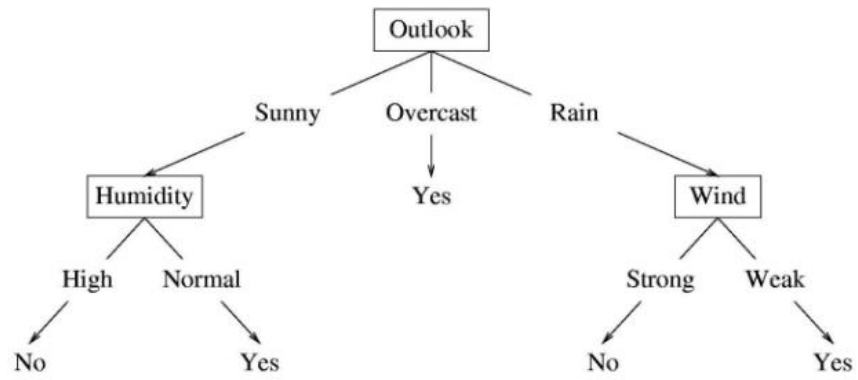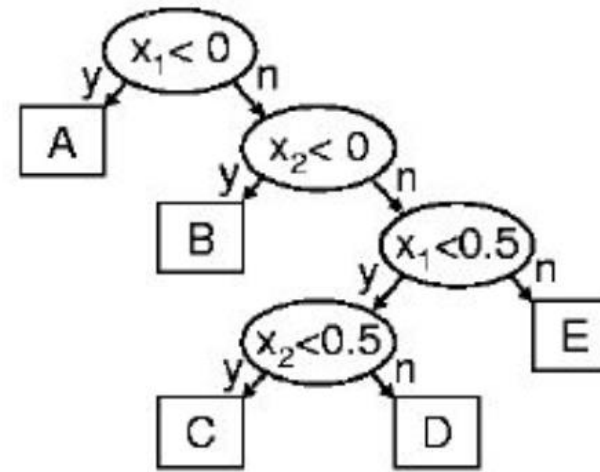
Discrete   Continuous

Which attribute is better to split?

X_1

t   f

Y=t : 4   Y=t : 1
Y=f : 0   Y=f : 3

X_2

t   f

Y=t : 3   Y=t : 2
Y=f : 1   Y=f : 2
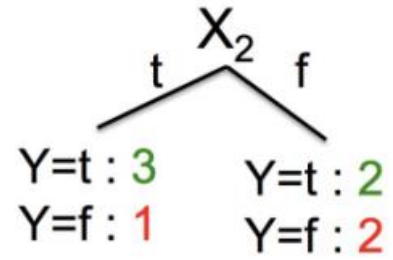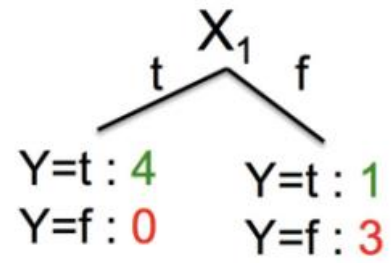
X_1	X_2	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F
F	T	F
F	F	F

- Consider coin flipping

Sequence 1:
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 ... ?

Sequence 2:
0 1 0 1 0 1 1 1 0 1 0 0 1 1 0 1 0 1 ... ?
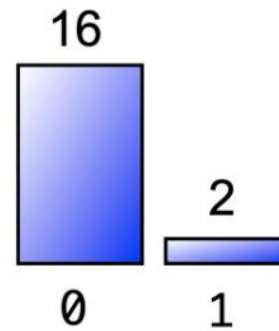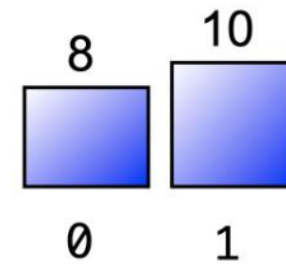
Entropy H:

$$H(X) = - \sum_{x \in X} p(x) \log_2 p(x)$$

8/9

1/9

0   1

4/9   5/9

0   1
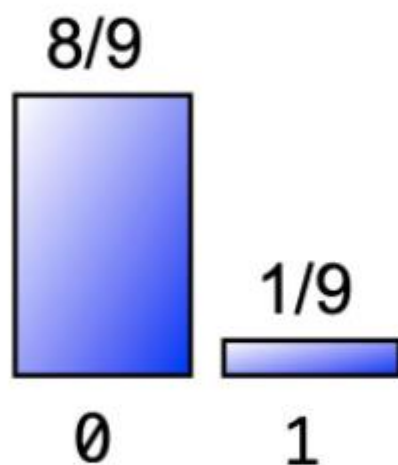
$$-\frac{8}{9} \log_2 \frac{8}{9} - \frac{1}{9} \log_2 \frac{1}{9} \approx \frac{1}{2}$$
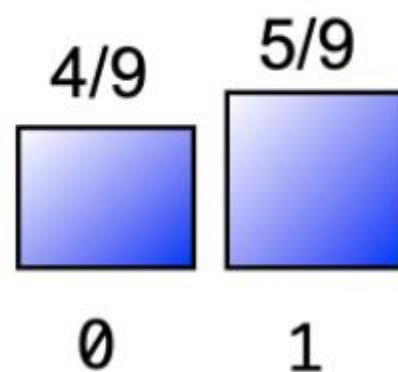
$$-\frac{4}{9} \log_2 \frac{4}{9} - \frac{5}{9} \log_2 \frac{5}{9} \approx 0.99$$

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$
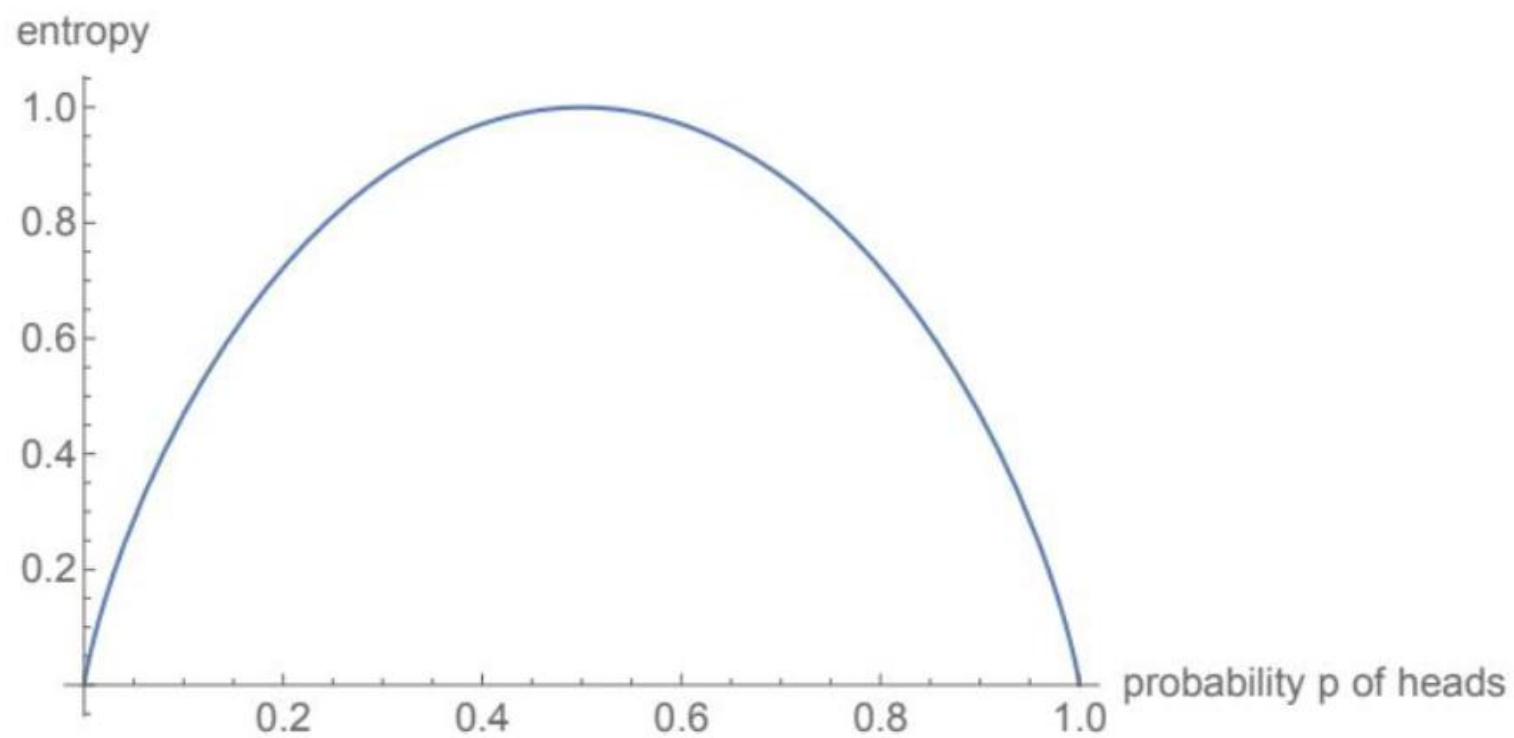
- **High Entropy:**

    - Variable has a uniform like distribution
    - Flat histogram
    - Values sampled from it are less predictable

- **Low Entropy:**

    - Distribution of variable has many peaks and valleys
    - Histogram has many lows and highs
    - Values sampled from it are more predictable

Compare

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

Entropy $H(X)$ of a random variable X

$$H(X) = -\sum_{i=1}^{n} P(X = i) \log_2 P(X = i)$$

Specific conditional entropy $H(X/Y=v)$ of X given $Y=v$:

$$H(X|Y = v) = -\sum_{i=1}^{n} P(X = i|Y = v) \log_2 P(X = i|Y = v)$$

Conditional entropy $H(X/Y)$ of X given Y :

$$H(X|Y) = \sum_{v \in values(Y)} P(Y = v)H(X|Y = v)$$

Mututal information (aka Information Gain) of X and Y :

$$I(X, Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$
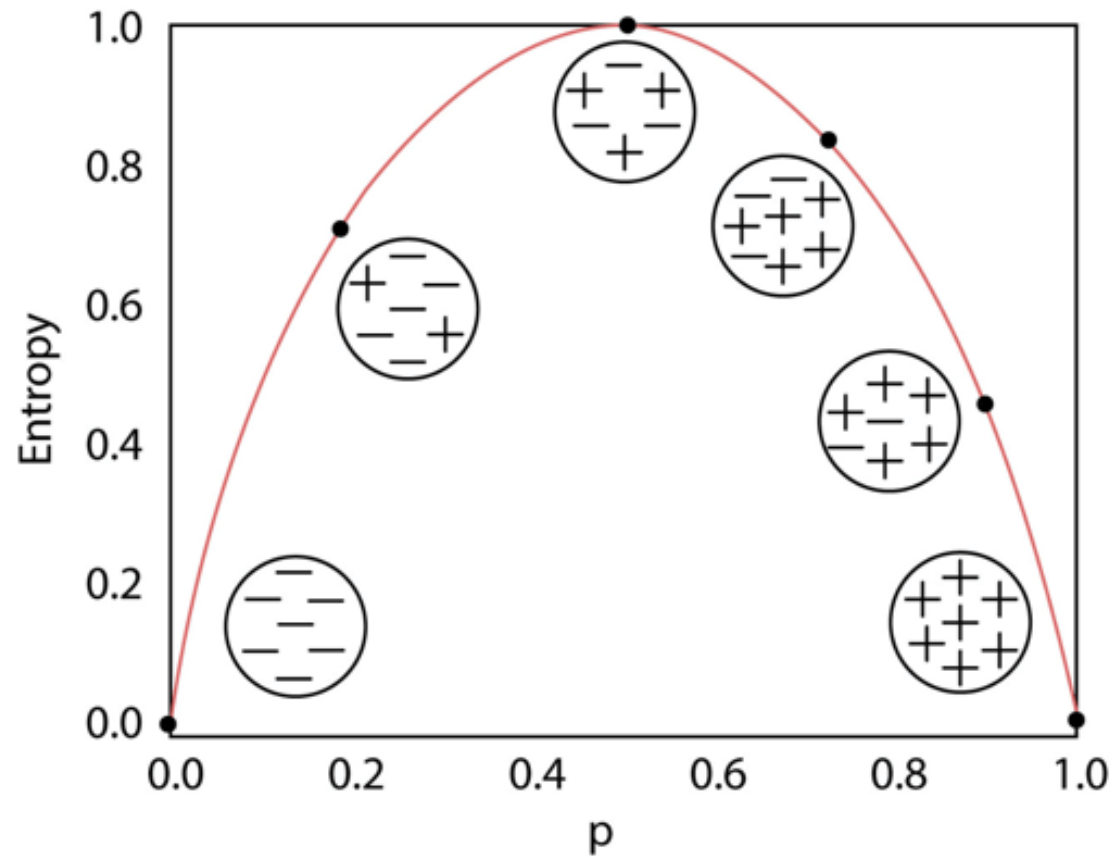
If all the elements belong to a single class, then it can be called pure.

Information Gain = entropy(parent) − [average entropy(children)]

child entropy $-\left(\dfrac{13}{17}\cdot\log_2\dfrac{13}{17}\right)-\left(\dfrac{4}{17}\cdot\log_2\dfrac{4}{17}\right)=0.787$

Entire population (30 instances)

17 instances

child entropy $-\left(\dfrac{1}{13}\cdot\log_2\dfrac{1}{13}\right)-\left(\dfrac{12}{13}\cdot\log_2\dfrac{12}{13}\right)=0.391$

parent entropy $-\left(\dfrac{14}{30}\cdot\log_2\dfrac{14}{30}\right)-\left(\dfrac{16}{30}\cdot\log_2\dfrac{16}{30}\right)=0.996$
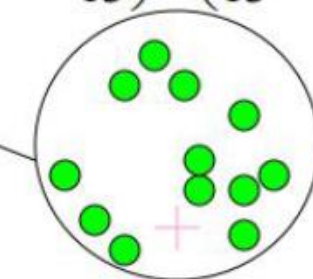
13 instances

(Weighted) Average Entropy of Children = $\left(\dfrac{17}{30}\cdot0.787\right)+\left(\dfrac{13}{30}\cdot0.391\right)=0.615$

Information Gain= 0.996 - 0.615 = 0.38

Day	Outlook	Temp	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Day	Outlook	Temp	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Attribute: Outlook

Values (Outlook) = Sunny, Overcast, Rain

$S = [9+, 5-]$ $Entropy(S) = -\frac{9}{14}log_2\frac{9}{14} - \frac{5}{14}log_2\frac{5}{14} = 0.94$

$S_{Sunny} \leftarrow [2+, 3-]$ $Entropy(S_{Sunny}) = -\frac{2}{5}log_2\frac{2}{5} - \frac{3}{5}log_2\frac{3}{5} = 0.971$

$S_{Overcast} \leftarrow [4+, 0-]$ $Entropy(S_{Overcast}) = -\frac{4}{4}log_2\frac{4}{4} - \frac{0}{4}log_2\frac{0}{4} = 0$

$S_{Rain} \leftarrow [3+, 2-]$ $Entropy(S_{Rain}) = -\frac{3}{5}log_2\frac{3}{5} - \frac{2}{5}log_2\frac{2}{5} = 0.971$

$$Gain(S, Outlook) = Entropy(S) - \sum_{v \in (Sunny, Overcast, Rain)} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$Gain(S, Outlook)$$

$$= Entropy(S) - \frac{5}{14}Entropy(S_{Sunny}) - \frac{4}{14}Entropy(S_{Overcast})$$

$$- \frac{5}{14}Entropy(S_{Rain})$$

$$Gain(S, Outlook) = 0.94 - \frac{5}{14}0.971 - \frac{4}{14}0 - \frac{5}{14}0.971 = 0.2464$$

Day	Outlook	Temp	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Attribute: Temp

$Values\ (Temp) = Hot, Mild, Cool$

$S = [9+, 5-]$

$Entropy(S) = -\frac{9}{14}log_2\frac{9}{14} - \frac{5}{14}log_2\frac{5}{14} = 0.94$

$S_{Hot} \leftarrow [2+, 2-]$

$Entropy(S_{Hot}) = -\frac{2}{4}log_2\frac{2}{4} - \frac{2}{4}log_2\frac{2}{4} = 1.0$

$S_{Mild} \leftarrow [4+, 2-]$

$Entropy(S_{Mild}) = -\frac{4}{6}log_2\frac{4}{6} - \frac{2}{6}log_2\frac{2}{6} = 0.9183$

$S_{Cool} \leftarrow [3+, 1-]$

$Entropy(S_{Cool}) = -\frac{3}{4}log_2\frac{3}{4} - \frac{1}{4}log_2\frac{1}{4} = 0.8113$

$$Gain\ (S, Temp) = Entropy(S) - \sum_{v \in (Hot, Mild, Cool)} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$Gain(S, Temp)$$

$$= Entropy(S) - \frac{4}{14} Entropy(S_{Hot}) - \frac{6}{14} Entropy(S_{Mild})$$

$$- \frac{4}{14} Entropy(S_{Cool})$$

$$Gain(S, Temp) = 0.94 - \frac{4}{14}1.0 - \frac{6}{14}0.9183 - \frac{4}{14}0.8113 = 0.0289$$

Day	Outlook	Temp	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Attribute: Humidity

Values (Humidity) = High, Normal

$S = [9+, 5-]$

$$Entropy(S) = -\frac{9}{14}\log_2\frac{9}{14} - \frac{5}{14}\log_2\frac{5}{14} = 0.94$$

$S_{High} \leftarrow [3+, 4-]$

$$Entropy(S_{High}) = -\frac{3}{7}\log_2\frac{3}{7} - \frac{4}{7}\log_2\frac{4}{7} = 0.9852$$

$S_{Normal} \leftarrow [6+, 1-]$

$$Entropy(S_{Normal}) = -\frac{6}{7}\log_2\frac{6}{7} - \frac{1}{7}\log_2\frac{1}{7} = 0.5916$$

$$Gain\ (S, Humidity) = Entropy(S) - \sum_{v \in \{High, Normal\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

$Gain(S, Humidity)$

$$= Entropy(S) - \frac{7}{14} Entropy(S_{High}) - \frac{7}{14} Entropy(S_{Normal})$$

$$Gain(S, Humidity) = 0.94 - \frac{7}{14} 0.9852 - \frac{7}{14} 0.5916 = 0.1516$$

Day	Outlook	Temp	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Attribute: Wind

Values $(Wind) = Strong, Weak$

$S = [9+, 5-]$

$$Entropy(S) = -\frac{9}{14} \log_2 \frac{9}{14} - \frac{5}{14} \log_2 \frac{5}{14} = 0.94$$

$S_{Strong} \leftarrow [3+, 3-]$

$Entropy(S_{Strong}) = 1.0$

$S_{Weak} \leftarrow [6+, 2-]$

$$Entropy(S_{Weak}) = -\frac{6}{8} \log_2 \frac{6}{8} - \frac{2}{8} \log_2 \frac{2}{8} = 0.8113$$

$$Gain(S, Wind) = Entropy(S) - \sum_{v \in \{Strong, Weak\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$Gain(S, Wind) = Entropy(S) - \frac{6}{14} Entropy(S_{Strong}) - \frac{8}{14} Entropy(S_{Weak})$$

$$Gain(S, Wind) = 0.94 - \frac{6}{14} 1.0 - \frac{8}{14} 0.8113 = 0.0478$$

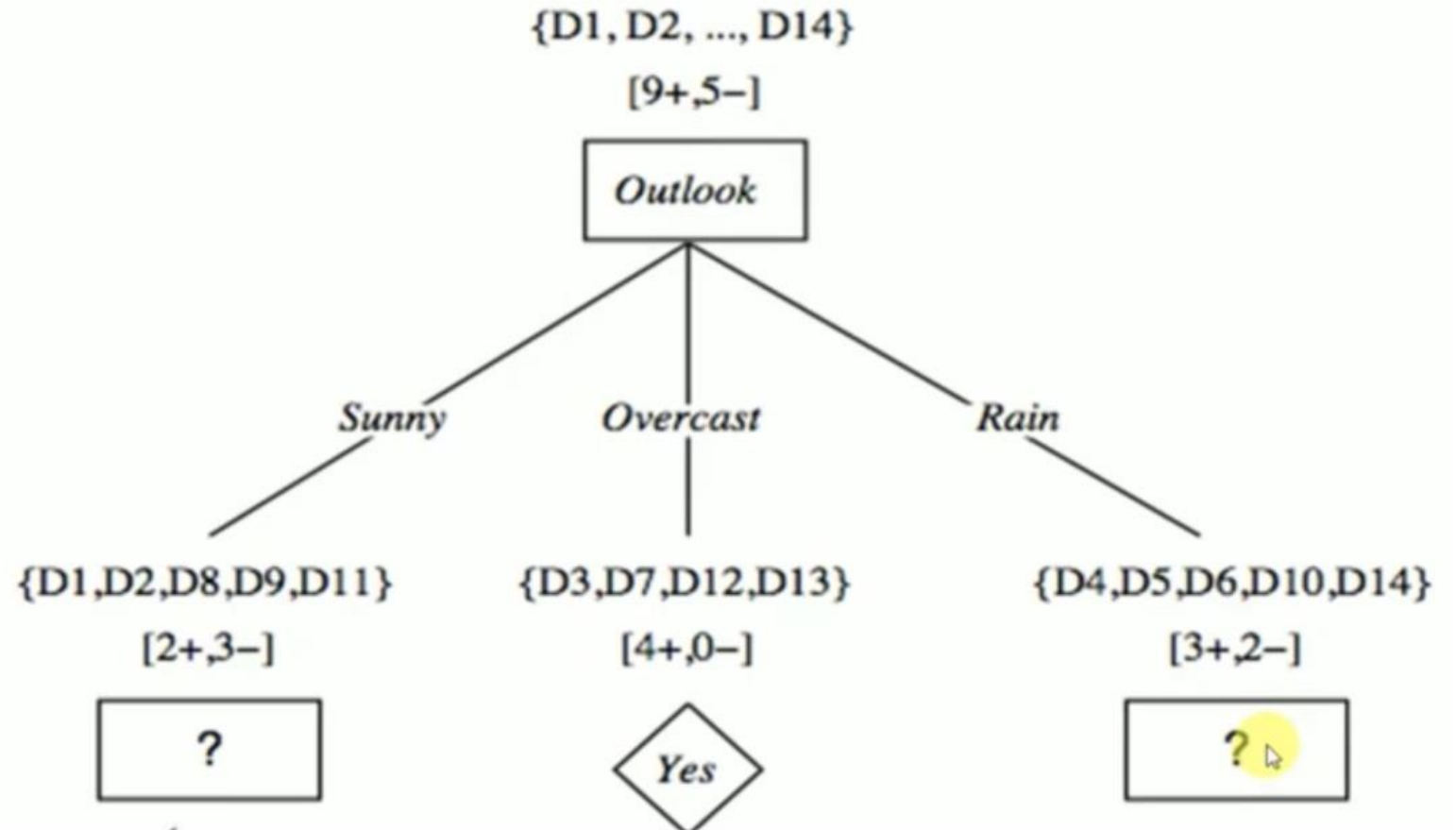| Day	Outlook	Temp	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

$$Gain(S, Outlook) = 0.2464$$

$$Gain(S, Temp) = 0.0289$$

$$Gain(S, Humidity) = 0.1516$$

$$Gain(S, Wind) = 0.0478$$

Day	Outlook	Temp	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

{D1, D2, ..., D14}

[9+,5−]

Outlook

Sunny   Overcast   Rain

{D1,D2,D8,D9,D11}   {D3,D7,D12,D13}   {D4,D5,D6,D10,D14}

[2+,3−]   [4+,0−]   [3+,2−]

?   Yes   ?

Day	Temp	Humidity	Wind	Play Tennis
D1	Hot	High	Weak	No
D2	Hot	High	Strong	No
D8	Mild	High	Weak	No
D9	Cool	Normal	Weak	Yes
D11	Mild	Normal	Strong	Yes

Day	Outlook	Temp	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Attribute: Temp

$Values\ (Temp) = Hot, Mild, Cool$

$S_{Sunny} = [2+, 3-]$ $Entropy(S_{Sunny}) = -\frac{2}{5}log_2\frac{2}{5} - \frac{3}{5}log_2\frac{3}{5} = 0.97$

$S_{Hot} \leftarrow [0+, 2-]$ $Entropy(S_{Hot}) = 0.0$

$S_{Mild} \leftarrow [1+, 1-]$ $Entropy(S_{Mild}) = 1.0$

$S_{Cool} \leftarrow [1+, 0-]$ $Entropy(S_{Cool}) = 0.0$

$$Gain\ (S_{Sunny}, Temp) = Entropy(S) - \sum_{v \in \{Hot, Mild, Cool\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$Gain(S_{Sunny}, Temp)$$

$$= Entropy(S) - \frac{2}{5}Entropy(S_{Hot}) - \frac{2}{5}Entropy(S_{Mild})$$

$$- \frac{1}{5}Entropy(S_{Cool})$$

$$Gain(S_{sunny}, Temp) = 0.97 - \frac{2}{5}0.0 - \frac{2}{5}1 - \frac{1}{5}0.0 = 0.570$$

Day	Temp	Humidity	Wind	Play Tennis
DI	Hot	High	Weak	No
D2	Hot	High	Strong	No
D8	Mild	High	Weak	No
D9	Cool	Normal	Weak	Yes
DI1	Mild	Normal	Strong	Yes

Attribute: Humidity

Values (Humidity) = High, Normal

$$S_{Sunny} = [2+, 3-] \qquad Entropy(S) = -\frac{2}{5} \log_2 \frac{2}{5} - \frac{3}{5} \log_2 \frac{3}{5} = 0.97$$

$$S_{high} \leftarrow [0+, 3-] \qquad Entropy(S_{High}) = 0.0$$

$$S_{Normal} \leftarrow [2+, 0-] \qquad Entropy(S_{Normal}) = 0.0$$

$$Gain\,(S_{Sunny}, Humidity) = Entropy(S) - \sum_{v \in \{High, Normal\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$Gain(S_{Sunny}, Humidity) = Entropy(S) - \frac{3}{5} Entropy(S_{High}) - \frac{2}{5} Entropy(S_{Normal})$$

$$Gain(S_{Sunny}, Humidity) = 0.97 - \frac{3}{5} 0.0 - \frac{2}{5} 0.0 = 0.97$$

Day	Temp	Humidity	Wind	Play Tennis
Dl	Hot	High	Weak	No
D2	Hot	High	Strong	No
D8	Mild	High	Weak	No
D9	Cool	Normal	Weak	Yes
Dl1	Mild	Normal	Strong	Yes

Attribute: Wind

Values (Wind) = Strong, Weak

$S_{Sunny} = [2+, 3-]$

$Entropy(S) = -\frac{2}{5}log_2\frac{2}{5} - \frac{3}{5}log_2\frac{3}{5} = 0.97$

$S_{Strong} \leftarrow [1+, 1-]$

$Entropy(S_{Strong}) = 1.0$

$S_{Weak} \leftarrow [1+, 2-]$

$Entropy(S_{Weak}) = -\frac{1}{3}log_2\frac{1}{3} - \frac{2}{3}log_2\frac{2}{3} = 0.9183$

$$Gain(S_{Sunny}, Wind) = Entropy(S) - \sum_{v \in \{Strong, Weak\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

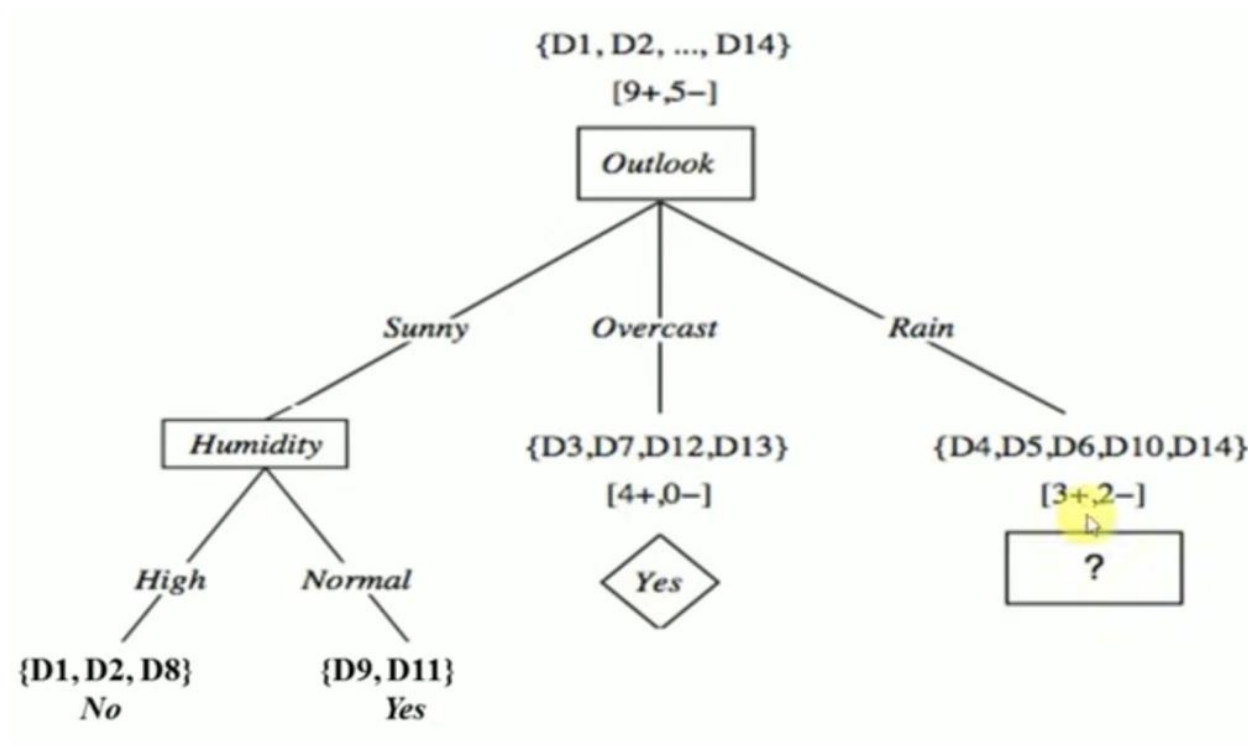$$Gain(S_{Sunny}, Wind) = Entropy(S) - \frac{2}{5}Entropy(S_{Strong}) - \frac{3}{5}Entropy(S_{Weak})$$

$$Gain(S_{sunny}, Wind) = 0.97 - \frac{2}{5}1.0 - \frac{3}{5}0.918 = 0.0192$$

Day	Temp	Humidity	Wind	Play Tennis
D1	Hot	High	Weak	No
D2	Hot	High	Strong	No
D8	Mild	High	Weak	No
D9	Cool	Normal	Weak	Yes
D11	Mild	Normal	Strong	Yes

$$Gain(S_{sunny}, Temp) = 0.570$$

$$Gain(S_{sunny}, Humidity) = 0.97$$

$$Gain(S_{sunny}, Wind) = 0.0192$$

{D1, D2, ..., D14}

[9+,5−]

Outlook

Sunny   Overcast   Rain

Humidity   {D3,D7,D12,D13}   {D4,D5,D6,D10,D14}

[4+,0−]   [3+,2−]

High   Normal   Yes   ?

{D1, D2, D8}   {D9, D11}
No   Yes

Gini Index

The Gini Index or Gini Impurity is calculated by subtracting the sum of the squared probabilities of each class from one

If all the elements belong to a single class, then it can be called pure.

'0' denotes that all elements belong to a certain class or there exists only one class (pure)

'1' denotes that the elements are randomly distributed across various classes (impure).

$$Gini = 1 - \sum_{i=1}^{C} p_i^2$$

The degree of Gini Index varies between 0 and 1,

While building the decision tree , we would prefer to choose the attribute/feature with the least Gini Index as the root node.

The Gini Impurity of a pure node is zero