Dynamic Time Warping

- The DTW is an elastic matching algorithm that is used to compare time series data, that are to be matched.

- To align two sequences under certain constraints, such that the distance between these two sequences is as small as possible.

- Distance between Same-length Sequences

$$L_p - norm(x, y) = \left(\sum_{i=1}^{n} \left(x_i - y_i \right)^p \right)^{\frac{1}{p}} = \begin{cases} \sum_{i=1}^{n} \left| x_i - y_i \right|, \ p = 1 \\ \sqrt{\sum_{i=1}^{n} \left(x_i - y_i \right)^2}, \ p = 2 \\ \max_i \left| x_i - y_i \right|, \ p = \infty \end{cases}$$

$$\vec{x} = \left[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \right]$$

$$\vec{y} = \left[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9 \right]$$

$$\vec{x} = [x_1, x_2, \cdots x_m], \vec{y} = [y_1, y_2, \cdots y_n]$$

We want to find an alignment path $\{(p_1, q_1), (p_2, q_2), \cdots (p_k, q_k)\}$ (subject to the alignment constraints), such that the distance along the path is minimized:

$$dist(x, y) = \min_{(p_i, q_i), i=1 \cdots k} \sum_{i=1}^{k} \left(x_{p_i} - y_{q_i} \right)^2,$$

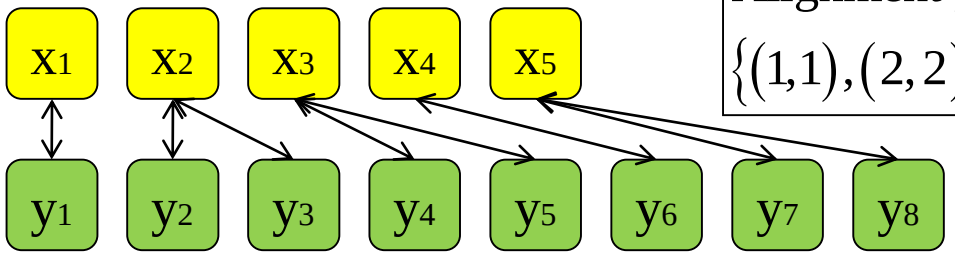
with the bounary conditions:

$$\begin{cases} \text{Anchored beginning: } (p_1, q_1) = (1,1) \\ \text{Anchored end: } (p_k, q_k) = (m, n) \end{cases}$$

Temporal constraints

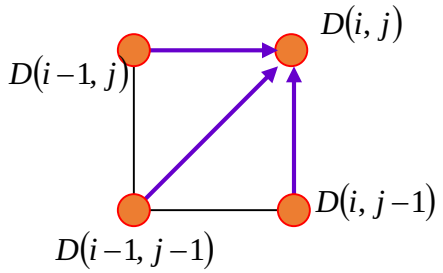
$$p_1 < p_2 < \cdots < p_k, q_1 < q_2 < \cdots < q_k$$

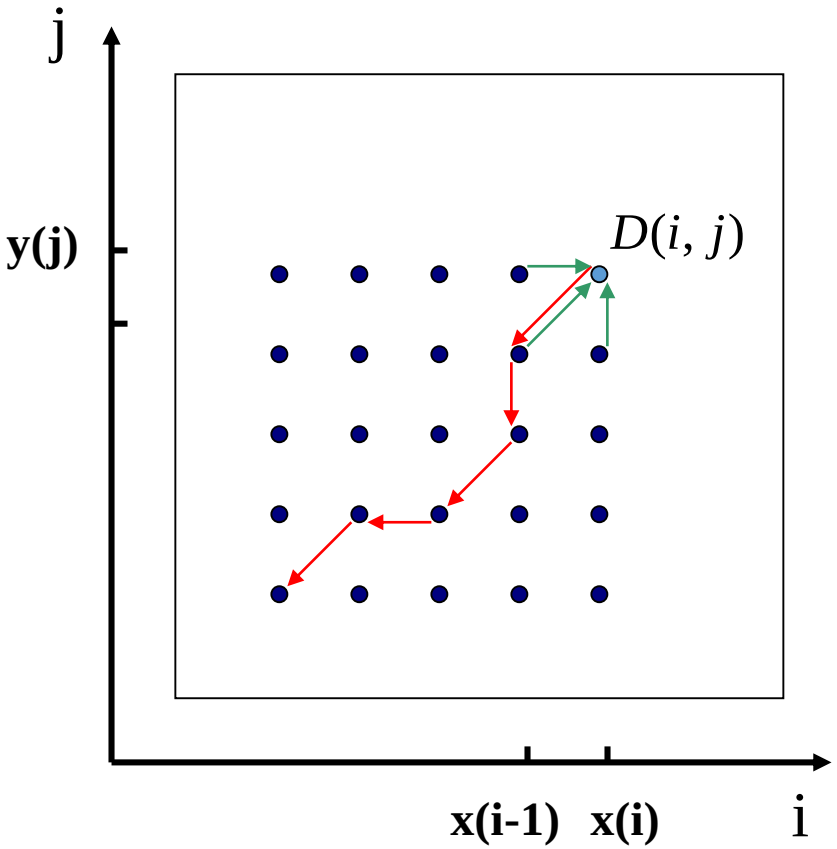
- 1-to-1, 1-to-many, or many-to-1 mapping
- No skip-over



Alignment path =

$$\{(1,1),(2,2),(2,3),(3,4),(3,5),(4,6),(5,7),(5,8)\}$$





x, y: input vector/matrix
Local paths: 0-45-90 degrees

DTW formulation:

1. Optimum-value function $D(i, j)$:

The DTW distance between $x(1:i)$ and $y(1:j)$

2. Recurrent equation for $D(i, j)$:

$$D(i, j) = \|x(i) - y(j)\| + \min \begin{Bmatrix} D(i, j-1) \\ D(i-1, j-1) \\ D(i-1, j) \end{Bmatrix},$$

with $D(1,1) = \|x(1) - y(1)\|$.

3. Answer $= D(len(x), len(y))$