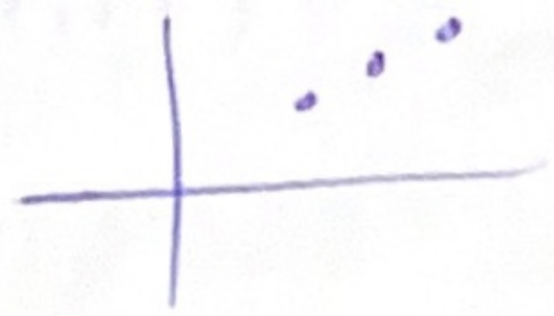
MA201 (Tut 02)

(1) $f: R^2 \to R^2$

$f(x,y) = (x,-y)$



(2) $f(z) = z^3$ $\quad z_1 = 1 \quad z_2 = i \quad y = 1-x$

$$\frac{f(z_1)-f(z_2)}{z_1 - z_2} = f'(c)$$

(1) $f: R^2 \to R^2$

$f(x,y) = (x,-y)$

$f = (f_1, f_2)$

$f'(x_0) = \lim_{x \to 0} \frac{f(x^{*}+h) - f(x)}{h}$ $\times$ not valid in $R^2$

if $f: R^n \to R^n$ defn of derivative is not correct by point by point

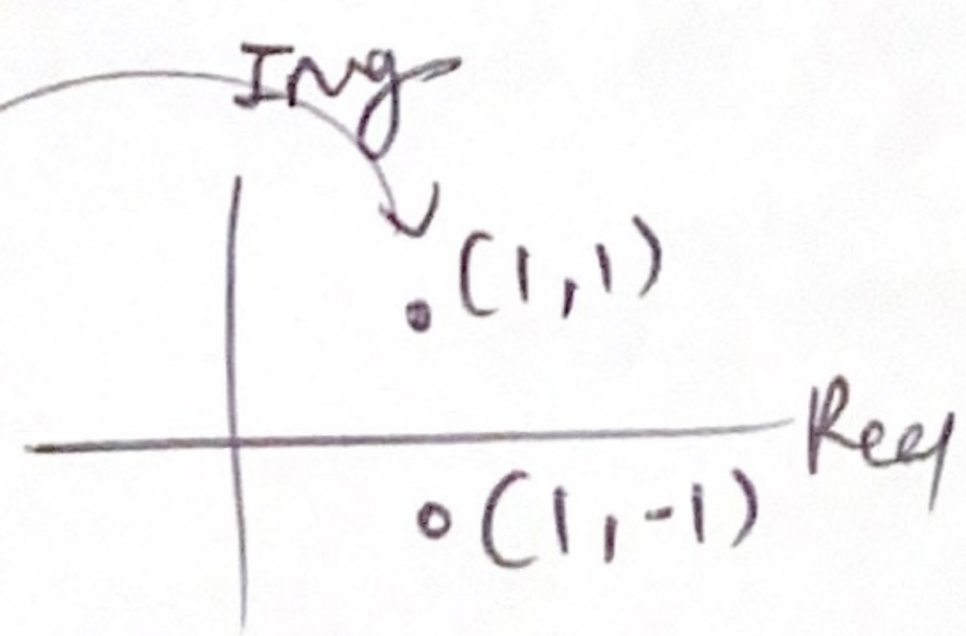
$f' = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix}$ defn of funcn from $R^2 \to R^2$

$f' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \quad \forall x_0 \in R^2$

$\therefore$ ~~⊞~~ $\forall x \in R^2$ we get $f'$ as const value $(-1)$ i.e we claim that it is diff in $R^2$

same funcn in complex plane

$f = z$




$f = \bar{z}$

point is reflected by Real-axis

$f = \bar{z} = x - iy = u + iv$ $\quad u = x, \; v = -y$

if funcn is continous it should satisfy

$u_x = v_y \qquad u_x = 1$

$v_y = -v_x \qquad v_y = +1$ $\quad$ not equal $\quad$ thus, not diff at any point

But what is soln if diff ?

(2) $f = z^3, \; z_1 = 1, \; z_2 = i$ $\quad$ (we have to prove they are unequal)

$\dfrac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$ $\quad$ if we show mod $\neq$ mod of $f'(c)$ we can prove this

$\rightarrow \left|\dfrac{f(1) - f(i)}{1 - i}\right| = 1$ $\qquad f = z^3$

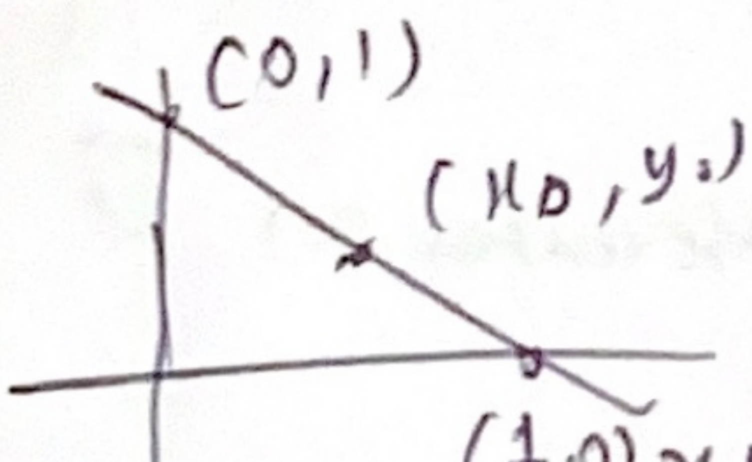
$f' = 3z^2$

doubt $\rightarrow$ $f(1) = 1$ $\quad |f'(c)| = |3c^2| = \left|3\left(\frac{1}{\sqrt{2}}\right)^2\right| = \frac{3}{2}$

$f(i) = i^3 = -i$

$1 - (-i) = 1 + i$

line given is $\qquad |f'(c)| \geq 1.5$




$\sqrt{x_0^2 + y_0^2} \geq \frac{1}{\sqrt{2}}$

but $| \quad | = 1$

thus there doesn't exist any $c$ for which

$|f| = |f'(c)|$

(3)

(3) $f(z)$ $f: \mathbb{C} \to \mathbb{R}$ → Real valued funcⁿ ?
[Range is real]

$f'(z_0) =$

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$\frac{z_1}{z_2}$ is defined thus, can be written in real sense

doesn't exist

exist $(=0)$

~~(=0)~~ show if exist

$f = u + i(0)$

$u_x = v_y = 0$

$u_y = -v_x = 0$

$u: \mathbb{R}^2 \to \mathbb{R}$ $u_x = 0$ $u_y = 0$

that implies $u = \text{const} = c$

$f = c + i(0)$

i.e $f$ is a const funcⁿ

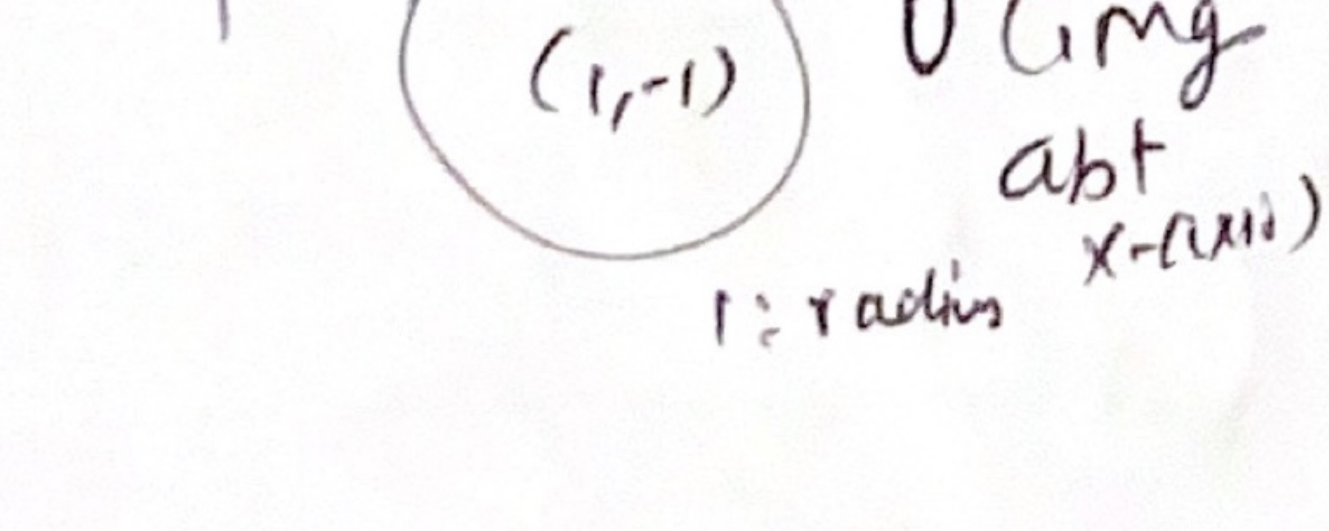
derivative of a const funcⁿ is zero..

(4) $U$: open set $f: U \to \mathbb{C}$ is differentiable

$\bar{U} = \{\bar{z} : z \in U\}$

Show that $g(z) = \overline{f(\bar{z})}$ is differentiable on $\bar{U}$

if




Prove:

$$\lim_{w \to w_0 \in \bar{U}} \frac{g(w) - g(w_0)}{w - w_0} =$$

$$= \lim_{z_0 \in U} \frac{g(\bar{z}) - g(\bar{z}_0)}{\bar{z} - \bar{z}_0}$$

$$= \overline{\lim_{z_0 \in U} \frac{f(z) - f(z_0)}{z - z_0}} \quad \left(\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\right)$$

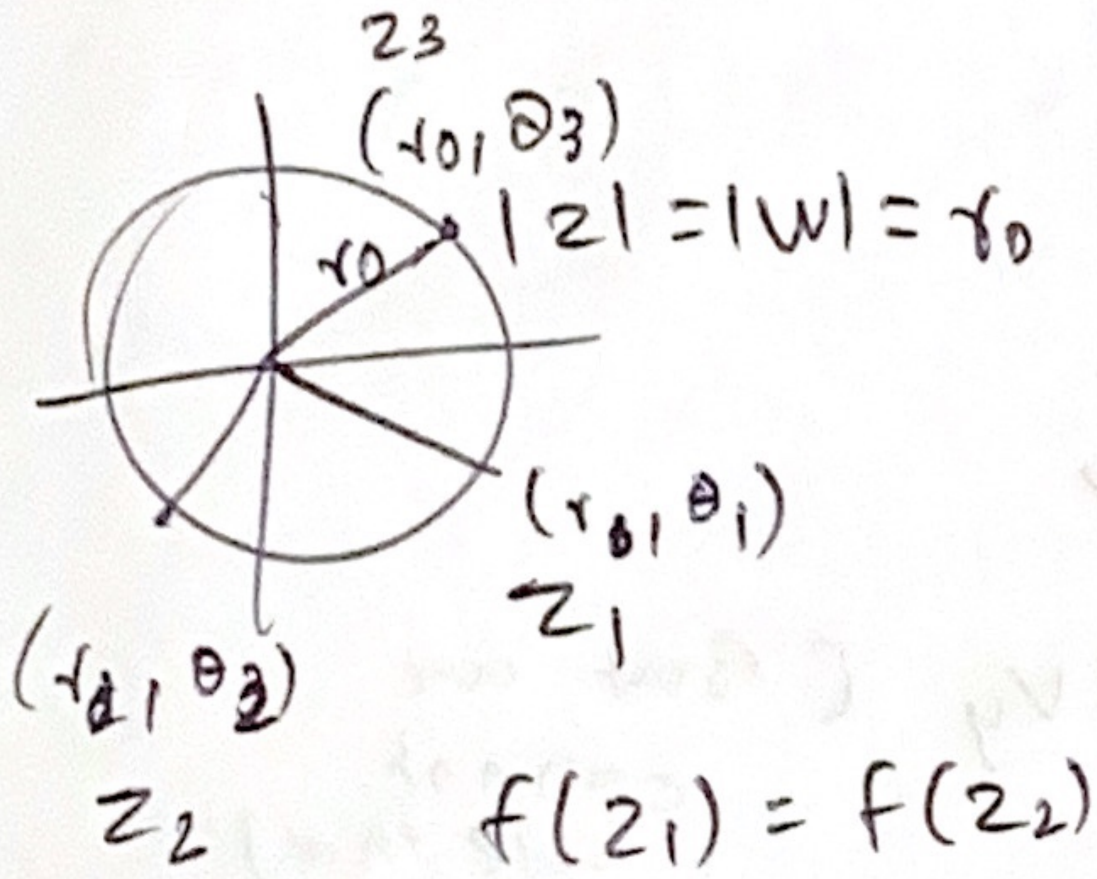
$$= \overline{f'(z_0)}$$

(5) Utube

(6) $f(z) = f(w)$ $\quad |z| = |w|$

imp

$f : D \to C$ $\quad f(z) = f(w)$ when $|z| = |w|$




$f(z_1) = f(z_2) = f(z_3)$

$f = f(r, \theta) = f(r)$

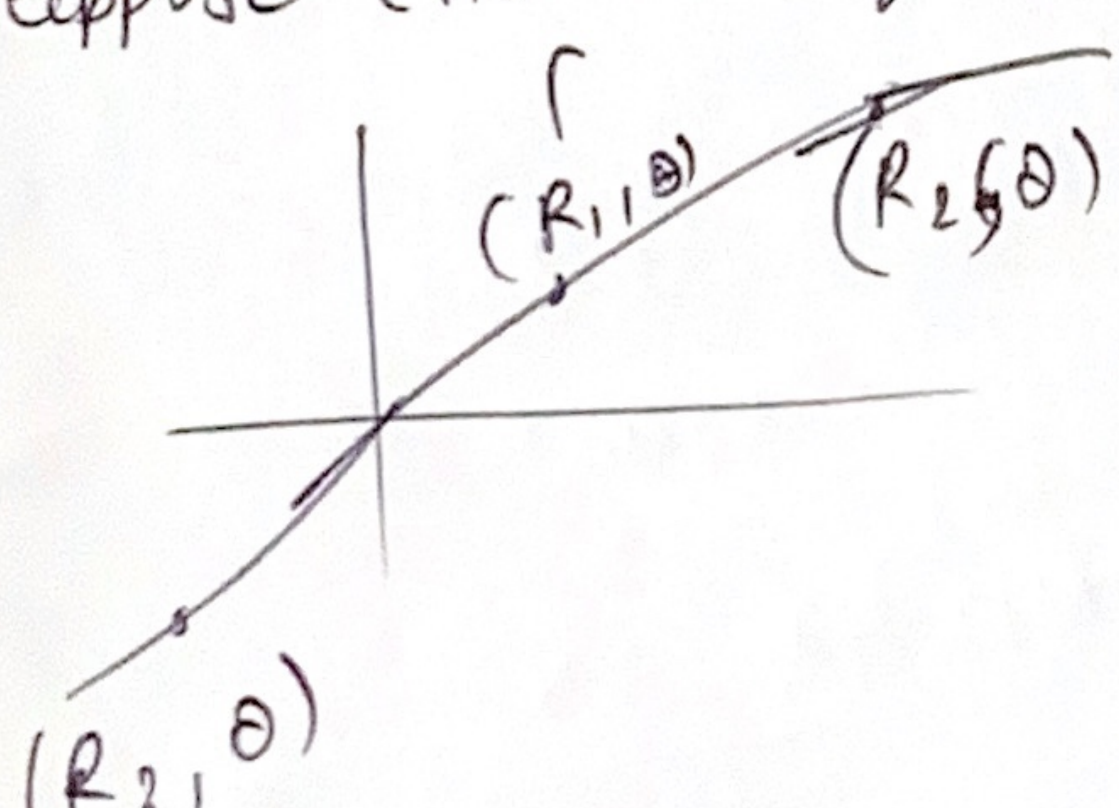
funcn is independent of $\theta$

$U = C_1$ $\quad$ use CR

$V = C_2$ $\quad$ eqns in polar coordinate

Geometrical meaning/-

Suppose (not rel to que) f (independent of r)




$f = f(\theta)$

(7) $f = u + iv$

$u(x,y) = \phi(x)$ $v(x,y) = \psi(y)$

Prove that : $f(z) = az + b$ for some $a \in C, b \in C$

analytic func$^n$ ? func$^n$ is differentiable in the neighbourhood of point

CR equation

$U_x = V_y$ $U_y = -V_x$

$\phi'(x) = \psi'(y)$

$\forall z \in \mathbb{C}$

if $y = 0$ $\phi'(x) = \text{const} = c_1 \; \forall y$

if $x = 0$ $\psi'(y) = c_2 \; \forall x$

but $c_1 = c_2$ Since $\phi'(x) = \psi'(y)$

$f' = u + iv$

$f' = u_x + iv_y$ ( But we cannot use that now/-

$\phi(x) = cx + c_2$

$\psi(y) = cy + c_3$

$f = u + iv$

$f = cx + c_2 + i(cy + c_3)$

$= c(x+iy) + c_2 + ic_3$

$= az + b$ $a = c, b = c_2 + ic_3$

Tut 03

(1) $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$= \underbrace{e^x \cos y}_{\text{Real}} + i \underbrace{e^x \sin y}_{\text{img}}$

$e^z \in \mathbb{R}$ $\quad e^x \sin y = 0$

$\sin y = 0 \quad y = n\pi \quad n \in \mathbb{Z}$

$z = x + in\pi$

$e^z \in i\mathbb{R}$ $\quad e^x \cos y = 0$

$y = (2n+1)\pi/2$

$z = x + i(2n+1)\frac{\pi}{2} \quad n \in \mathbb{Z}$

(2) $\sinh(\mathrm{Im}(z)) \leq |\sin(z)| \leq \cosh(\mathrm{Im}(z))$

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$

$\cosh^2 y = 1 + \sinh^2 y$
some calcu

$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$

$|\sin z| \leq \sqrt{1 + \sinh^2 y} = \cosh(y) = \cosh(\mathrm{Im}(z))$ —(1)

$|\sin z|^2 = \sin^2 x + \sinh^2 y \geq \sinh^2 y$

$|\sin z| \geq \sinh y = \sinh(\mathrm{Im}(z))$ —(2)

from (1) & (2)

$\sinh(\mathrm{Im}(z)) \leq |\sin(z)| \leq \cosh(\mathrm{Im}(z))$

ST $|\sin(z)| \to \infty$ as $Im(z) \to \infty$

i.e $|\sin(z)| \to \infty$ as $y \to \infty$

we for
$|\sinh(z)| \to \infty$ $|\cosh(z)| \to \infty$

$\infty \le | \; | \le \infty$ ✓

(3) $e^z = 1$ (find z for which $e^z = 1$)

$e^x e^{iy} = 1$

$e^x(\cos y + i\sin y) = 1 + i0$

$e^x \cos y = 1$ $e^x \sin y = 0$

$e^x \to$ non-zero function

$\sin y = 0$

$y = n\pi$ $n \in Z$

~~$\cos y$~~

if $y = n\pi$ $\cos y = \pm 1$ $-1$ not all $\because e^x \in (0, \infty)$

~~$e^x(+1)$~~

this imply, $y = 2n\pi$, $n \in Z$

$e^x = 1$ $\boxed{x = 0}$

$z = 0 + i(2n\pi)$, $n \in Z$

$e^z = i$ (find z)

$y = (2n+1)\pi/2$

$y = (4n+1)\pi/2$, $n \in Z$ [bcoz of sin]

$e^x \sin y = 1$

$e^x = 1$ $x = 0$

$z = (0, (4n+1)\pi/2)$, $n \in Z$

(c)

$e^{z-1} = 1$

$e^{x-1} e^{iy} = 1$

$e^{x-1} [\cos y + i \sin y] = 1 + i(0)$

$e^{x-1} \cos y = 1 \qquad e^{x-1} \sin y = 0$

$y = n\pi, n \in \mathbb{Z}$

$e^{x-1} \cos y = 1 \Rightarrow y = 2n\pi, n \in \mathbb{Z}$

$e^{x-1} = 1 = e^0 \qquad x - 1 = 0$

$x = 1$

$z = (1, 2n\pi)$

(4) $\log(3-2i)$

$\log(z) = \log|z| + i(\alpha + 2n\pi), n \in \mathbb{Z}$

$\log(3-2i) = \log\sqrt{13} + i(\tan^{-1}(\frac{-2}{3}) + 2n\pi), n \in \mathbb{Z}$

(ii) $\mathrm{Log}(i) = \log|i| + i\frac{\pi}{2}$

$= \log(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$

expected in qu

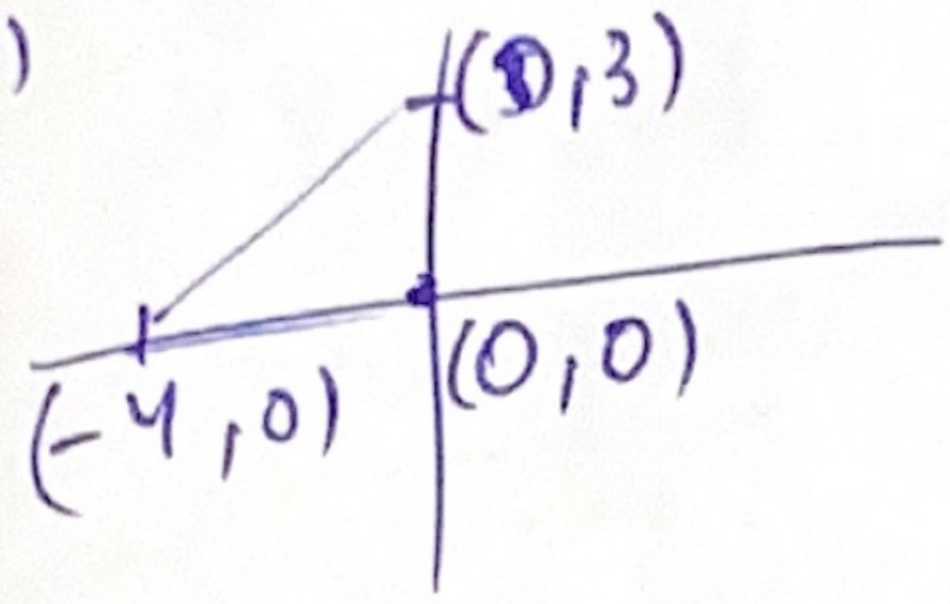
(iii) $i^{(-i)} = e^{\log i^{(-i)}} = e^{-i \log i} = e^{-i[\log|i| + i(\frac{\pi}{2} + 2n\pi)]}$, $n \in \mathbb{Z}$

$= e^{-i[i(\frac{\pi}{2} + 2n\pi)]}$

$= e^{\pi/2 + 2n\pi} \quad n \in \mathbb{Z}$

try $i^i \rightarrow e^{-(\pi/2 + 2n\pi)} \quad n \in \mathbb{Z}$

(5)




$\left| \int_\gamma (e^z - e^{\bar{z}}) dz \right| \le 60$

$\gamma \to \Delta$

$f = e^z - e^{\bar{z}}$

$|f| = |e^z - e^{\bar{z}}| \le |e^z| + |e^{\bar{z}}| \; |z| = e^x + |z| = 5$

$e^x \quad x \in (0,-4)$ max value $=1$

$|z|$ max $=4$

$\therefore M = 5$

perimeter $= 4+3+5 = 12 \quad L = 12$

$\left| \int_\gamma (e^z - e^{\bar{z}}) dz \right| \le M \times L = 5 \times 12 = 60$

(6) $\int_\gamma |z| \bar{z} \, dz \quad |z| = 2$

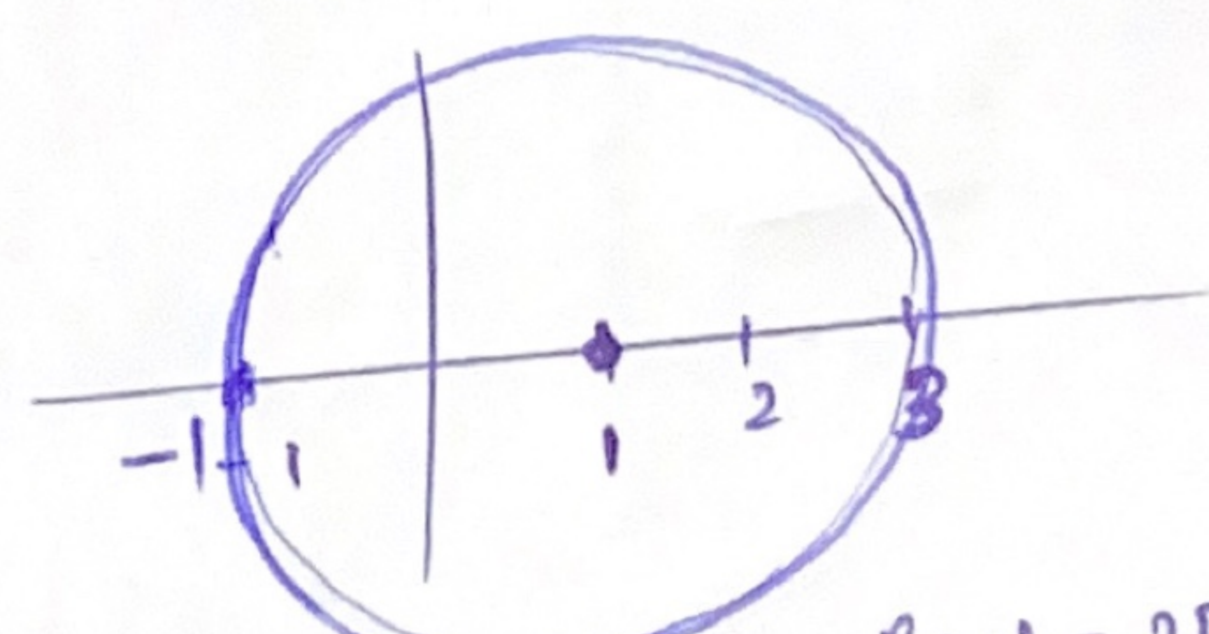
$|z| = 2 \quad$ put $z = 2e^{it}$

$dz = 2ie^{it} dt \quad t \in [0, 2\pi]$

$\int_\gamma (2) \cdot (2e^{-it})(2ie^{it} dt)$

$= \int_\gamma 8i \, dt = 8i \int_0^{2\pi} dt = 8i [2\pi - 0] = 16\pi i$

(7) $\left| \int_{|z-1|=2} \frac{1}{z} dz \right| \le 4\pi$




$f = \frac{1}{z}$

Peri $= 2\pi(2)$

$L = 4\pi$

$M = \max(f) = 1$

$M \times L = 4\pi \times 1 = 4\pi$

(ii) $\left| \int_{|z|=10} \left(\frac{z-1}{z+1}\right) dz \right| \le 2\pi R \left(\frac{R+1}{R-1}\right)$

$L = 2\pi(10) = 20\pi$

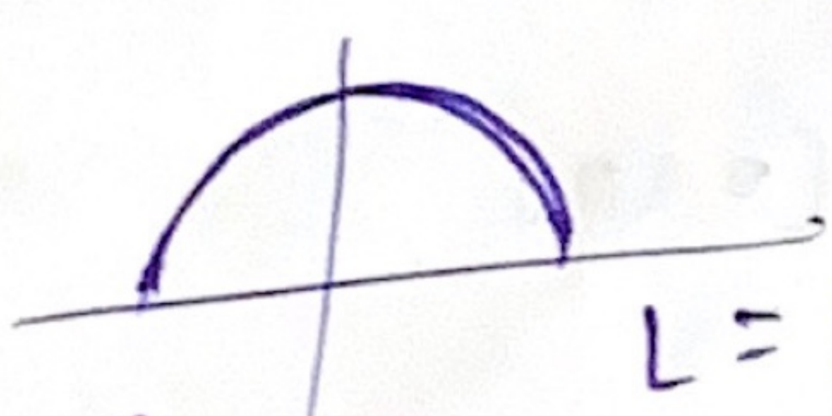
$M = |f(z)| = \left|\frac{z-1}{z+1}\right| \le \frac{|z|+1}{|z|-1} = \frac{10+1}{10-1} = \frac{11}{9}$

$M \times L = \frac{11}{9} \times 20\pi = \frac{220\pi}{9}$

(iii) $\left| \int_{\gamma} \frac{1}{z^4+1} dz \right| \le \frac{3}{80}\pi$

$\gamma(t) = 3e^{it}$

$f(z) = \frac{1}{z^4+1}$



$L = 3\pi$

$M = \left|\frac{1}{z^4+1}\right| \le \frac{1}{|z|^4 - 1}$

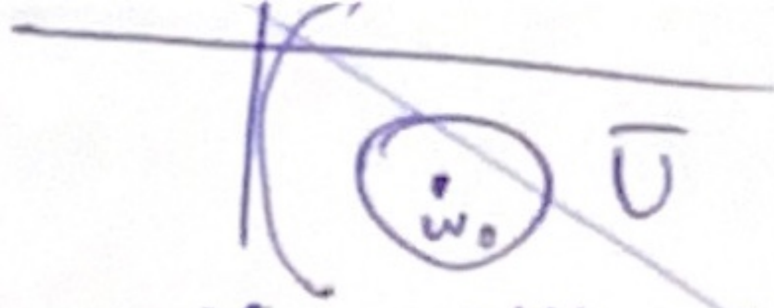
$\cancel{2\pi R}$

$\pi R$

$M \le \frac{1}{81-1} \le \frac{1}{80}$

~~$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0}$~~

~~$\lim_{z \to z_0}$~~




if $w \to w_0 \to z \to z_0$

b/coz $U$ & $\bar{U}$ are conjugate [somehow connected]

Deriving CR eqn

$z = x + iy = r\cos\theta + ir\sin\theta = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$f(z) = u + iv$

$f(re^{i\theta}) = u + iv$ — ①

partial diff wrt $r$,

$f'(re^{i\theta}) \frac{\partial}{\partial r}(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$

$f'(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ — ②

partial diff wrt $\theta$,

$f'(re^{i\theta}) \frac{\partial}{\partial r}(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$f'(re^{i\theta})[rie^{i\theta}] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

put ②

~~$f'(re^{i\theta})$~~

$ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$ir\frac{\partial u}{\partial r} + i^2 r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ , $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Proved

using $x + iy$

~~$x +$~~

$f(z) = u + iv$

$f(x + iy) = u + iv$

likewise