

Bicomponents and the Robustness of Networks to Failure

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We study bicomponents in networks, sets of nodes such that each pair in the set is connected by at least two independent paths, so that the failure of no single node in the network can cause them to become disconnected. We show that standard network models predict there to be essentially no small bicomponents in most networks, but there may be a giant bicomponent, whose presence coincides with the presence of the ordinary giant component, and we find that real networks seem by and large to follow this pattern, although there are some interesting exceptions. We also study the size of the giant bicomponent as nodes in the network fail and find in some cases that our networks are quite robust to failure, with large bicomponents persisting until almost all vertices have been removed.

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The robustness of connections in networks has been studied extensively in the physics community, in part because of its practical importance in settings such as communications networks and epidemiology [1–3]. The typical approach is to consider the largest set of vertices in a network that are connected to one another by at least one path, the so-called *giant component*, and examine how the set's size varies as vertices are removed from the network. This can be thought of as a simple model for the performance of, for instance, a communication network such as the Internet under failure of vertices. The vertices removed can be chosen at random, as if failing because of random technical faults, or in a targeted fashion, as if an adversary were deliberately removing them in an effort to destroy network connectivity.

In real-world situations, however, it is often considered inadequate to rely for communication on just a single path between vertices. Most large organizations, for example, connect to the Internet using a strategy called *multihoming*, in which they maintain two or more independent data connections to the network so that the failure of any one connection will not leave them disconnected. More generally, the connection between two vertices is considered robust if there are at least two independent paths between them, so that the failure of no single vertex in the network can cause the two to be disconnected. This leads us to an obvious generalization of previous approaches to robustness, in which we focus on the set or sets of vertices in a network that are connected by at least two paths. Such sets are called *bicomponents*. In this Letter we study the robustness of networks, including both real and model networks, in terms of the size and behavior of their bicomponents.

Two paths connecting the same pair of vertices in a network are said to be *vertex-independent*, or simply *independent*, if they share none of the same vertices other than their starting and ending vertices. A k -component is a maximal subset of the vertices of a network such that every vertex in the subset is connected to every other by k independent paths. For the special cases $k = 2, 3$ the

k -components are also called *bicomponents* and *tricomponents*, respectively. Note that not all vertices need belong to a k -component for $k \geq 2$, which contrasts with the case for ordinary components ($k = 1$), where every vertex belongs to a component. Note also that for any given network the k -components are nested: every bicomponent is, trivially, a subset of an ordinary component, every tricomponent is a subset of a bicomponent, and so forth. In this Letter we concentrate primarily on bicomponents, although we will give some results for $k \geq 3$ where appropriate.

To gain an understanding of the behavior of bicomponents, let us look first at a standard model network, the widely studied “configuration model,” which is a network chosen uniformly at random from the set of all networks with a given degree sequence [4,5]. The probability of an edge falling between two vertices i and j in such a network is $k_i k_j / 2m$, where k_i is the degree of vertex i and m is the total number of edges in the network. The configuration model is a useful guide to the expected qualitative behavior of many network statistics, and, in particular, provides good insight into the behavior of ordinary components (1-components), having at most one giant component of size $O(n)$ and $O(n)$ small components of size $O(1)$, a pattern that is seen in most real-world networks as well.

A first interesting result to note is that, by contrast with the case for 1-components, there are in general (almost) no small bicomponents in the configuration model. To see this consider the small 1-components of the configuration model, which, as shown elsewhere, are generally treelike [4], meaning they are not bicomponents. In order to turn them into bicomponents, we would need to add at least one edge to the tree, thereby closing a loop and creating two paths between some vertices. Since the small components have size $O(1)$ and hence also $O(1)$ pairs of vertices, there are $O(1)$ opportunities to perform such a closure in each small component. Each closure occurs with probability $k_i k_j / 2m = O(n^{-1})$ on a sparse network and hence the total probability of converting a small component into a bicomponent is $O(n^{-1})$. Since there are $O(n)$ small components,

this means that the total number of small bicomponents formed in this way is $O(1)$. Small bicomponents can also be formed out of treelike subsets of the giant component, but a similar argument shows that such bicomponents are also $O(1)$ in number.

Thus in the limit of large network size the probability that a randomly chosen node belongs to a small bicomponent vanishes as $1/n$. Speaking loosely, there are no small bicomponents in the network.

There may, however, be a *giant bicomponent*, and moreover it turns out to be possible to calculate the expected size of the giant bicomponent exactly in the limit of large graph size. In order for a vertex to belong to the giant bicomponent at least two of the edges incident on that vertex must lead to the giant bicomponent by independent paths. However, since the giant bicomponent is a subset of the giant 1-component, it follows that any edge that leads to the giant 1-component also leads to the giant bicomponent, so it will suffice that at least two of our vertex's neighbors be in the giant 1-component.

The probability that the neighbor of a vertex belongs to the giant component is straightforward to calculate [5]. Let the degree distribution of our network be p_k , meaning that a randomly chosen vertex has degree k with probability p_k . If we choose an edge at random and follow it to one of the vertices at its ends, then the number of edges incident on that vertex, other than the one we arrived along, follows a different distribution, the so-called *excess degree distribution*:

$$q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}, \quad (1)$$

as shown in [5], for example. Here $\langle k \rangle = \sum_k k p_k$ is the mean degree for the entire network.

Now let u be the probability that upon following an edge we reach a vertex that does not belong to the giant component. In order for this to be the case, it must be that none of the other edges attached to that vertex lead to vertices in the giant component, which happens with probability u^k , where k is the excess degree. Averaging over the distribution q_k of k , we then find that

$$u = \sum_{k=0}^{\infty} q_k u^k = G_1(u), \quad (2)$$

where $G_1(z) = \sum_k q_k z^k$ is the probability generating function for q_k . If the network is to have a giant component this equation must have a nontrivial solution $u < 1$. It is straightforward to show that this occurs if $G_1'(1) > 1$, which leads to the well-known criterion of Molloy and Reed [4] for the existence of the giant component.

Armed with these results we can now write down the probability that a randomly chosen vertex belongs to the giant bicomponent. The probability that it does not is the probability that either zero or one, but not more, of its edges lead to vertices in the giant component, which is

$\sum_k p_k u^k + \sum_k k p_k (1-u) u^{k-1} = G_0(u) + (1-u)G_0'(u)$, where $G_0(z) = \sum_k p_k z^k$ is the probability generating function for p_k . Then the probability S_2 that the vertex is in the giant bicomponent is one minus this quantity:

$$S_2 = 1 - G_0(u) - (1-u)G_0'(u). \quad (3)$$

Alternatively, S_2 is the expected size of the giant bicomponent as a fraction of the size of the entire network.

There can of course be no giant bicomponent when there is no giant 1-component, since the former is a subset of the latter. Equation (3) shows that, in general, the reverse is also true: when there is a giant 1-component there is also a giant bicomponent. Only in the special case where $1 - G_0(u) = (1-u)G_0'(u)$ can the giant bicomponent vanish.

Equation (3) can be generalized straightforwardly to k -components for general k . The general result is

$$S_k = 1 - \sum_{m=0}^{k-1} \frac{(1-u)^m}{m!} \frac{d^m G_0}{dz^m} \Big|_{z=u}, \quad (4)$$

which implies that in fact giant k -components for *all* k exist only when there is a giant 1-component.

Since the presence of large k -components is, as we have said, a desirable property for many kinds of networks, it is instructive to ask how robust that property is to the removal or failure of vertices. We already know, from the results above, that if we start removing vertices from a network, the giant k -components will vanish at the same time the giant component does. Thus the thresholds for the complete destruction of robust connectivity at all orders coincide. This, however, does not tell the whole story.

Let r_k be the probability that a vertex of degree k is operational (i.e., it has not failed or been removed from the network). Common choices for r_k are $r_k = \text{constant}$, which corresponds to uniformly random failure of vertices, or $r_k = \theta(k_{\max} - k)$, where $\theta(x)$ is the Heaviside step function, which corresponds to removal of all vertices with degree $k > k_{\max}$, a form of targeted attack against the best-connected vertices [1].

Then the probability u that an edge does not lead to a vertex in the giant component, if that vertex has excess degree k (or total degree $k+1$), is the probability $1 - r_{k+1}$ that the vertex has been removed plus the probability $r_{k+1} u^k$ that it has not been removed but that none of its other edges lead to the giant component either. Averaging over the distribution q_k of the excess degree, we then find

$$u = 1 - F_1(1) + F_1(u), \quad (5)$$

where $F_1(z) = \sum_k q_k r_{k+1} z^k$. Then, by an argument similar to the one leading to Eq. (4), the expected size of the giant k -component is

$$S_k = F_0(1) - \sum_{m=0}^{k-1} \frac{(1-u)^m}{m!} \frac{d^m F_0}{dz^m} \Big|_{z=u}, \quad (6)$$

where $F_0(z) = \sum_k p_k r_k z^k$.

The percolation transition at which the giant 1-component in a network is destroyed is typically second-order in nature. For k -components with $k \geq 2$ on the other hand, we can show using the results above that the transition is of higher order. Consider, for instance, the case of uniform random failure of vertices, for which $r_k = \phi$ independent of k . Then Eq. (5) becomes $u = 1 - \phi + \phi G_1(u)$ and, defining $\epsilon = 1 - u$ and making use of $G_1(1) = 1$, we find $\epsilon = \phi[\epsilon G_1'(1) - \frac{1}{2}\epsilon^2 G_1''(1)] + O(\epsilon^3)$ or, rearranging,

$$\epsilon = \frac{2[\phi G_1'(1) - 1]}{\phi G_1''(1)}. \quad (7)$$

In other words, $1 - u$ is linear in $\phi - \phi_c$, where $\phi_c = 1/G_1'(1)$ is the critical occupation probability [2].

Now consider Eq. (6) for the size of the giant k -component and let us rewrite it as follows. Performing a Taylor expansion of $F_0(z)$ about $z = u$ we find

$$F_0(z) = \sum_{m=0}^{\infty} \frac{(z-u)^m}{m!} \left. \frac{d^m F_0}{dz^m} \right|_{z=u}. \quad (8)$$

Setting $z = 1$ and making use of (6), we then find that

$$S_k = \sum_{m=k}^{\infty} \frac{(1-u)^m}{m!} \left. \frac{d^m F_0}{dz^m} \right|_{z=u}. \quad (9)$$

The leading order term in this expression is $O(1-u)^k$ and hence, by Eq. (7), $S_k = O(\phi - \phi_c)^k$.

Thus the system displays a (potentially) infinite set of continuous phase transitions marking the appearance of the giant components for different values of k , each occurring at the same point ϕ_c but each of different order, the transition for any given k being of $(k+1)$ th order in ϕ . This means that although, for example, the giant bicomponent does appear at the same moment as the ordinary giant component, its appearance is a third-order transition and hence it initially grows in size at a far slower rate than the giant component so that, in practice, the network does not achieve a significant level of robustness in the sense considered here until considerably above ϕ_c . This behavior is illustrated in the left-hand panel of Fig. 1, where we show the size of the largest 1- and 2-components in random graphs with two different degree distributions, along with simulation results for the same networks. The second- and third-order natures of the phase transitions can be clearly seen.

Armed with the theoretical insights afforded by the configuration model, let us turn now to the behavior of bicomponents in real-world networks. All bicomponents in a network can be found in time $O(n)$ using the depth-first search based algorithm of Hopcroft and Tarjan [6]. Table I summarizes the results of applying this algorithm to a variety of previously documented networks. The table reveals some interesting features. First we note that, with two exceptions, the networks all have large giant bicom-

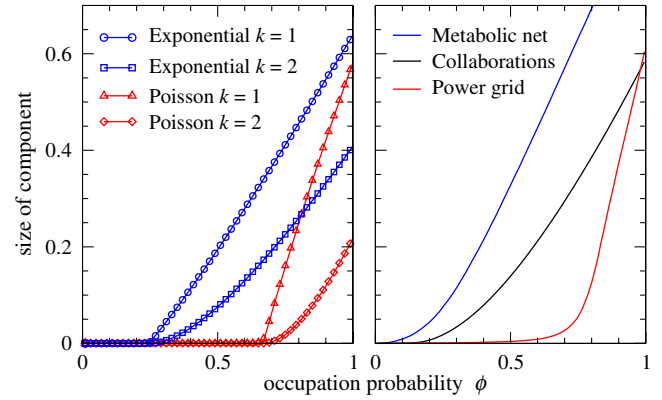


FIG. 1 (color online). Left panel: Size of giant component ($k = 1$) and giant bicomponent ($k = 2$) as vertices are randomly removed from random graphs with exponential ($e^{-\lambda k}$ with $\lambda = 0.4$) and Poisson (mean 1.5) degree distributions. Solid lines are the analytic solutions; points are average numerical results for 100 simulations each on networks with 10^6 vertices. Error bars are smaller than the points in all cases. Right panel: Size of giant bicomponent as vertices are randomly removed from three real-world networks, a metabolic network for *C. Elegans* [9], a collaboration network of scientists working in condensed matter physics [10], and the Western States Power Grid of the United States [8].

ponents. Certainly the giant bicomponents are smaller than the giant components, but in each case the networks have a substantial fraction of robust connections in the sense considered here. The two exceptions are the collaborations of network scientists and the high-school dating network. The former has quite a small giant component, so the giant

TABLE I. Statistics of a number of real-world networks. The second to fifth columns give the number of vertices in the network, the fractions occupied by the largest component and bicomponent, and the fraction occupied by small bicomponents. The networks are, in order, a snapshot of the Internet topology at the autonomous system (AS) level, the symmetrized web graph of a university web site [7], the Western United States power grid [8], the neural [8] and metabolic [9] networks of the nematode *C. Elegans*, coauthorship networks of physicists [10] and network scientists [11], and friendship and dating networks from a study of U.S. school students [12].

Network	n	S_1	S_2	Small bicomponents
Internet (AS)	22 963	1	0.651	0.012
World Wide Web	325 729	1	0.414	0.076
Power grid	4941	1	0.615	0.062
<i>C. Elegans</i> neural	297	1	0.949	0
<i>C. Elegans</i> metabolic	453	1	0.934	0.049
Physics collaborations	16 726	0.829	0.588	0.243
Network scientists	1589	0.239	0.084	0.634
Friendship	795	0.979	0.940	0
Dating	573	0.503	0.072	0.014

bicomponent cannot be very large. The dating network, however, is clearly anomalous, having a very substantial giant component but a very small giant bicomponent. It is interesting to consider whether there might be sociological reasons for this anomaly.

Second, we note that in all but two cases the networks have no small bicomponents, or very nearly none. This observation agrees well with our calculations for random graphs above, but is otherwise somewhat surprising. It has been observed that most real-world networks contain a high density of short loops [8], a feature that random graphs lack and one that might be expected to give rise to small bicomponents. Our observations imply, however, that in most cases the loops are attached to the giant bicomponent, rather than forming independent bicomponents, and that most portions of the network not in the giant bicomponent are treelike. This agreement with the random graph model stands in contrast with other network properties such as clustering and assortative mixing, for which random graphs match real networks poorly. Note also the two collaboration networks, which provide exceptions to the overall pattern: both have quite significant fractions of their vertices in small bicomponents, a feature possibly rooted in the social structure of scientific research, with research groups which often form bicomponents but which are sometimes not strongly attached to the rest of the network.

It is also interesting to examine the robustness of the bicomponent structure to removal of vertices as we did for our model networks. The Hopcroft–Tarjan algorithm is a poor choice for this calculation, since we would have to perform n runs of the algorithm to find the bicomponents after the removal of each vertex, for a total running time $O(n^2)$. For larger networks this is prohibitive, so instead we employ a different algorithm, similar in spirit to the percolation algorithm of Newman and Ziff [13], that makes use of established methods based on data trees to keep track of 2-connected subgraphs dynamically [14]. Run times for this algorithm are almost linear in the network size and in practice are a factor of 100 or more better than the Hopcroft–Tarjan algorithm for the networks studied here.

The right-hand panel of Fig. 1 shows the results of the application of this algorithm to three of the networks from Table I, the metabolic network, the physicist collaborations, and the power grid. For each network there appears to be a transition point below which the giant bicomponent is destroyed and the network can no longer be said to be robustly connected. For two of the networks the transition appears to be at or close to $\phi = 0$, indicating that the networks are highly robust in the sense considered here: nearly all the vertices have to be removed from the network before the giant bicomponent is destroyed. The third net-

work, the power grid, shows a transition at a higher probability, indicating that this network is relatively fragile to random node removal. On the other hand, we also see in all cases that the transition at which the giant bicomponent appears is a gradual one; the gradient at the transition is shallow—perhaps even zero—so that the giant bicomponent grows very slowly at first above the transition.

Taken together, the analytic and numerical results give an interesting picture of the behavior of network bicomponents. Real-world networks appear to be quite robust in the sense of having large giant bicomponents and moreover the existence of these bicomponents is in some cases (though not all) itself robust to the deletion of vertices. In practice, however, although the giant bicomponent may persist as vertices are removed from the network, its size dwindles rapidly so that large portions of the network lose robust connection considerably before the transition point at which the giant bicomponent finally vanishes. In each of these respects the behavior of our networks is surprisingly similar to the behavior of the standard configuration model, which is exactly solvable for many of the properties of its bicomponents.

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