Probability for Engineers

2021 IEOR 3658 **Midterm 1 Solutions**

Q. 1 (18 points)**.** In the U.S., about half of the population will get married in their lifetime. About half of first marriages end in divorce. About 80% of divorced people will remarry.

(1) What fraction of people get married at least twice?

(2) What fraction of people get married at most once?

(3) Among those people who get married exactly once in their lifetime, what fraction end up divorced?

Solution. Let A be the event that a person gets married at least once. Let B be the event that they get married and then divorced. Let C be the event that they get married, then divorced, and then married again. The problem gives us

$$\mathbb{P}(A) = 1/2, \qquad \mathbb{P}(B \mid A) = 1/2, \qquad \mathbb{P}(C \mid B) = 4/5.$$

Note also that we have

$$C \subset B \subset A.$$

(To get divorced, you must get married first. To remarry, you must first marry once and then get divorced.)
Note that $B \cap A = B$ since $B \subset A$. Hence, using the multiplication rule, we have

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) = \mathbb{P}(B \mid A)\mathbb{P}(A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Similarly that $C \cap B = C$ since $C \subset B$. Using the multiplication rule again, we have

$$\mathbb{P}(C) = \mathbb{P}(C \cap B) = \mathbb{P}(C \mid B)\mathbb{P}(B) = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}.$$

(1) This is $\mathbb{P}(C)$, which we just saw to be $1/5$.

(2) The event that the person gets married at most once is the same as the event that the do not get married twice, or C^c. The probability is thus

$$\mathbb{P}(C^c) = 1 - \mathbb{P}(C) = 1 - \frac{1}{5} = \frac{4}{5}.$$

(3) The event that the person gets married exactly once is $A \cap C^c$. Hence, the problem is asking for $\mathbb{P}(B \mid A \cap C^c)$. We can calculate this as follows, using the definition of conditional probability:

$$\mathbb{P}(B \mid A \cap C^c) = \frac{\mathbb{P}(B \cap A \cap C^c)}{\mathbb{P}(A \cap C^c)} = \frac{\mathbb{P}(B \cap C^c)}{\mathbb{P}(A \cap C^c)},$$

where the numerator used $A \cap B = B$. Now, since $C \subset B$, we can write $B = (B \cap C) \cup (B \cap C^c)$, which is a union of two disjoint events. Using this and the fact that $C \cap B = C$, we get

$$\mathbb{P}(B) = \mathbb{P}(B \cap C) + \mathbb{P}(B \cap C^c) = \mathbb{P}(C) + \mathbb{P}(B \cap C^c).$$

In other words

$$\mathbb{P}(B \cap C^c) = \mathbb{P}(B) - \mathbb{P}(C).$$

The same logic leads to

$$\mathbb{P}(A \cap C^c) = \mathbb{P}(A) - \mathbb{P}(C).$$

Hence,

$$\mathbb{P}(B \,|\, A \cap C^c) = \frac{\mathbb{P}(B) - \mathbb{P}(C)}{\mathbb{P}(A) - \mathbb{P}(C)} = \frac{\frac{1}{4} - \frac{1}{5}}{\frac{1}{2} - \frac{1}{5}} = \frac{1}{6}.$$

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Q. 2 (18 points)**.** About 1 in every 10 shishito peppers is spicy. Robert does not like spicy food. He bought three peppers from the store, not knowing which (if any) are spicy. Before he starts cooking, Robert takes a tiny bite of each pepper to see if it is spicy. He throws out all of the spicy ones, unless that would leave him empty-handed; if they are all spicy, he reluctantly keeps one; his recipe would be too bland without any peppers. Let X be the number of peppers remaining.

(1) Find the probability mass function (PMF) of the random variable X.

(2) Find the expected value of X.

(3) Find the variance of X.

Solution.

(1) X can take the values 1, 2, or 3, with probabilities

$$\begin{aligned}\mathbb{P}(X=3) &= (9/10)^3,\\ \mathbb{P}(X=2) &= 3(9/10)^2(1/10),\\ \mathbb{P}(X=1) &= (1/10)^3 + 3(9/10)(1/10)^2.\end{aligned}$$

In other words, X is like a Binomial(3, 9/10) random variable, except if $X = 0$ we change it to $X = 1$.

(2) This is

$$\begin{aligned}\mathbb{E}[X] &= 1 \cdot \mathbb{P}(X=1) + 2 \cdot \mathbb{P}(X=2) + 3 \cdot \mathbb{P}(X=3)\\ &= 1 \cdot [(1/10)^3 + 3(9/10)(1/10)^2] + 2 \cdot 3(9/10)^2(1/10) + 3 \cdot (9/10)^3\\ &= 2.701.\end{aligned}$$

Equivalently, this is the mean of a Binomial(3, 9/10) plus [1 times the probability that a Binomial(3, 9/10) equals zero], or $3(9/10) + (1/10)^3$.

(3) The variance is

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[(X - 2.701)^2]\\ &= (1-2.701)^2\mathbb{P}(X=1) + (2-2.701)^2\mathbb{P}(X=2) + (3-2.701)^2\mathbb{P}(X=3)\\ &= (1-2.701)^2\left[(1/10)^3 + 3(9/10)(1/10)^2\right]\\ &\qquad + (2-2.701)^2 3(9/10)^2(1/10) + (3-2.701)^2(9/10)^3\\ &= 0.265599.\end{aligned}$$

Q. 3 (20 points)**.** Let A, B and C be events, with $\mathbb{P}(C) > 0$. The concept of *conditional independence* is defined as follows. We say that "A and B are conditionally independent given C" if $\mathbb{P}(A \cap B \,|\, C) = \mathbb{P}(A \,|\, C)\, \mathbb{P}(B \,|\, C)$.

(1) Suppose A and B are independent. Are A^c and B necessarily independent?

(2) Suppose that A, B, and C are **pairwise** independent, and that A and B are conditionally independent given C. Are (A, B, C) necessarily jointly independent?

(3) Suppose that A, B, and C are **jointly** independent. Are A and B necessarily conditionally independent given C?

(4) Suppose A and C are disjoint. Are A and B necessarily conditionally independent given C?

For each part, either explain why the statement must be true, or give a counterexample to illustrate that it is not necessarily true.

Solution.

(1) Yes. Since $A^c \cap B$ and $A \cap B$ are disjoint and $B = (A^c \cap B) \cup (A \cap B)$, we have

$$\begin{aligned}\mathbb{P}(A^c \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(B)(1 - \mathbb{P}(A)) \\ &= \mathbb{P}(B)\mathbb{P}(A^c).\end{aligned}$$

Alternatively, in the case $\mathbb{P}(B) > 0$ so that we can condition on it, you could see this simply by noting that independence of A and B means $\mathbb{P}(A \,|\, B) = \mathbb{P}(A)$, which implies

$$\mathbb{P}(A^c \,|\, B) = 1 - \mathbb{P}(A \,|\, B) = 1 - \mathbb{P}(A) = \mathbb{P}(A^c),$$

which means A^c and B are independent.

(2) Yes. Pairwise independence implies, in particular, that A and C are independent, and that B and C are independent. The independence of A and C implies $\mathbb{P}(A \,|\, C) = \mathbb{P}(A)$, and the independence of B and C implies $\mathbb{P}(B \,|\, C) = \mathbb{P}(B)$. Use the multiplication rule, then conditional independence, then these two facts:

$$\begin{aligned}\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A \cap B \,|\, C)\mathbb{P}(C) = \mathbb{P}(A \,|\, C)\mathbb{P}(B \,|\, C)\mathbb{P}(C) \\ &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).\end{aligned}$$

(3) Yes. Joint independent implies $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$, and so

$$\mathbb{P}(A \cap B \,|\, C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)}{\mathbb{P}(C)} = \mathbb{P}(A)\mathbb{P}(B).$$

Joint independence implies pairwise independence; in particular, as in the previous part, A and C are independent, and B and C are independent. The independence of A and C implies $\mathbb{P}(A \mid C) = \mathbb{P}(A)$, and the independence of B and C implies $\mathbb{P}(B \mid C) = \mathbb{P}(B)$. From the above equation we thus deduce

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C).$$

(4) Yes. Since A and C are disjoint, $A \cap C = \emptyset$, and so

$$\mathbb{P}(A \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(C)} = 0.$$

But if A and C are disjoint, then $A \cap B$ and C are also disjoint, so

$$\mathbb{P}(A \cap B \mid C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(C)} = 0.$$

This implies the conditional independence property

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C),$$

since both sides are equal to zero.

Q. 4 (18 points)**.** Suppose you roll five (standard, six-sided) dice.

(1) Find the probability that all five numbers are distinct.

(2) Find the probability that there are more even numbers than odd numbers.

(3) Find the probability that the sum of the five numbers equals 7.

Solution.

(1) This is $(5/6)(4/6)(3/6)(2/6) = 5/54 \approx .09$. The first roll can be any number. For the second number there are 5 choices remaining that do not match the first. For the third number there are 4 choices remaining, for the fourth number there are 3 choices remaining, and for the fifth number there are 3 choices remaining.

(2) The answer is 1/2. There are are several ways to derive this. The easiest way to is recognize that the complementary event, that there are more odd numbers than even numbers, must clearly have the same probability, by symmetry. If an event and its complement have the same probability, then that probability must be 1/2. That is, if $\mathbb{P}(E) = \mathbb{P}(E^c) = 1/2$, then

$$1 = \mathbb{P}(E) + \mathbb{P}(E^c) = \mathbb{P}(E) + \mathbb{P}(E) = 2\mathbb{P}(E).$$

Alternatively, you could let X denote the number of even numbers rolled. Then X is Binom(5,1/2), and the probability of interest is

$$\begin{aligned}\mathbb{P}(X \geq 3) &= \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) \\ &= \binom{5}{3}(1/2)^5 + \binom{5}{4}(1/2)^5 + \binom{5}{5}(1/2)^5 \\ &= (1/2)^5(10 + 5 + 1) = 16/32 = 1/2.\end{aligned}$$

Alternatively, you could arrive to a similar expression by a more direct counting argument.

(3) The answer is $15(1/6)^5 = 5/2592 \approx .0019$. There are two ways this event can happen: You can roll four 1's and one 3, or you can roll three 1's and two 2's. The probability of rolling four 1's and one 3 is $5(1/6)^5$, since there are $5 = \binom{5}{1}$ ways to choose which one of the five rolls should be the 3. The probability of rolling three 1's and two 2's is $10(1/6)^5$, since there are $\binom{5}{2} = 10$ ways to choose which two of the five rolls should be the 3. Add up these two scenarios to get $(5 + 10)(1/6)^5 = 15(1/6)^5$.

Q. 5 (16 points)**.** Suppose you post a video on your YouTube channel. Each person who views your video (independently) **likes** it with probability $1/2$, or **dislikes** it with probability $1/10$. Everyone else, who neither likes nor dislikes the video, is said to **ignore** it. On average, about three people **like** your video per hour.

(1) For a single person who views your video, what is the conditional probability that they **like** the video given that they do not **ignore** it?

(2) What is a good choice of distribution to model the numbers of **likes** your video gets within the first hour?

(3) What is a good choice of distribution to model the numbers of **likes** within the first 100 people who view your video?

(4) What is a good choice of distribution to model the numbers of **likes** your video gets before the first **dislike**?

Each answer should specify the name and parameter(s) of a distribution.

Solution.

(1) Let L and D be the events that they like the video and dislike the video, respectively. The event that they do not ignore the video is $L \cup D$. Then $\mathbb{P}(L) = 1/2$ and $\mathbb{P}(D) = 1/10$, whereas $\mathbb{P}(L \cup D) = (1/2) + (1/10) = 6/10$. Thus

$$\mathbb{P}(L \mid L \cup D) = \frac{\mathbb{P}(L \cap (L \cup D))}{\mathbb{P}(L \cup D)} = \frac{\mathbb{P}(L)}{\mathbb{P}(L \cup D)} = \frac{1/2}{6/10} = \frac{5}{6}.$$

(2) The best choice is Poisson(3), which models the number of "arrivals" within some unit of time, when the average number of arrives per unit time is 3.

(3) The best choice is Binomial(100,1/2), since there are 100 people, and each **likes** your video with probability $1/2$.

(4) The best choice is Geometric(5/6). The choice of Geometric is natural, as this is counting the number of successes (likes) until the first failure (dislike). But the correct choice of parameter requires some thought. The $2/5$ of people who neither like nor dislike the video are irrelevant and should be removed from the count, and the remaining probabilities of liking/disliking should be conditioned on doing one of those two. That is, we should use the conditional like probability from part (1).

Strictly speaking, the Geometric distribution we learned in class would *include the first dislike* in the count, which does not perfectly fit what the question is asking. The better choice would be the Geometric shifted down by one, to start from zero rather than one.

Q. 6 (10 points)**.** Arturo gets a COVID test. It says negative. As an extra precaution, he gets a second COVID test. It says positive. **Find the probability that Arturo actually has COVID.**

You may assume the following:

- About 1/10 of people have COVID.
- The results of Arturo's first and second tests are conditionally independent given that he has COVID. That is, the event that the first test is x is conditionally independent of the event that the second is y given that Arturo has COVID, for any $x, y \in \{\text{positive, negative}\}$. See Q3 for the definition of conditional independence.
- The results of Arturo's first and second tests are also conditionally independent given that he **does not** have COVID.
- The false positive rate is 1/1000, for both of his tests.
- The false negative rate is 2/1000, for both of his tests.

For partial credit, identify the three relevant events and the given probabilities.

Note: These numbers are fictional, and your answer should be based on these fictional numbers, not on actual COVID statistics that you might find elsewhere.

Solution. Let C be the event that Arturo has COVID. Let A be the event that the first test says positive. Let B be the event that the second test says positive. We are looking for $\mathbb{P}(C \mid A^c \cap B)$. We are given the following:

- $\mathbb{P}(C) = 1/10$.
- $\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C)$, and all variants where A or B is replaced by A^c or B^c.
- $\mathbb{P}(A \cap B \mid C^c) = \mathbb{P}(A \mid C^c)\mathbb{P}(B \mid C^c)$, and all variants where A or B is replaced by A^c or B^c.
- $\mathbb{P}(A \mid C^c) = \mathbb{P}(B \mid C^c) = 1/1000$.
- $\mathbb{P}(A^c \mid C) = \mathbb{P}(B^c \mid C) = 2/1000$.

The definition of conditional probability gives

$$\mathbb{P}(C \mid A^c \cap B) = \frac{\mathbb{P}(A^c \cap B \cap C)}{\mathbb{P}(A^c \cap B)}.$$

For the numerator, use the multiplication rule and the given conditional independence:

$$\begin{aligned}
\mathbb{P}(A^c \cap B \cap C) &= \mathbb{P}(A^c \cap B \mid C)\mathbb{P}(C) \\
&= \mathbb{P}(A^c \mid C)\mathbb{P}(B \mid C)\mathbb{P}(C) = \mathbb{P}(A^c \mid C)\big(1 - \mathbb{P}(B^c \mid C)\big)\mathbb{P}(C) \\
&= \frac{2}{1000}\left(1 - \frac{2}{1000}\right)\frac{1}{10} \\
&= \frac{2 \cdot 998}{1000 \cdot 1000 \cdot 10}.
\end{aligned}$$

For the denominator, use the law of total probability

$$\begin{aligned}
\mathbb{P}(A^c \cap B) &= \mathbb{P}(A^c \cap B \mid C)\mathbb{P}(C) + \mathbb{P}(A^c \cap B \mid C^c)\mathbb{P}(C^c) \\
&= \mathbb{P}(A^c \mid C)\mathbb{P}(B \mid C)\mathbb{P}(C) + \mathbb{P}(A^c \mid C^c)\mathbb{P}(B \mid C^c)\mathbb{P}(C^c) \\
&= \frac{2}{1000}\left(1 - \frac{2}{1000}\right)\frac{1}{10} + \left(1 - \frac{1}{1000}\right)\frac{1}{1000}\left(1 - \frac{1}{10}\right) \\
&= \frac{2 \cdot 998}{1000 \cdot 1000 \cdot 10} + \frac{999 \cdot 9}{1000 \cdot 1000 \cdot 10}.
\end{aligned}$$

The answer is thus

$$\mathbb{P}(C \mid A^c \cap B) = \frac{\frac{2 \cdot 998}{1000 \cdot 1000 \cdot 10}}{\frac{2 \cdot 998}{1000 \cdot 1000 \cdot 10} + \frac{999 \cdot 9}{1000 \cdot 1000 \cdot 10}} = \frac{2 \cdot 998}{2 \cdot 998 + 999 \cdot 9} \approx 0.1817.$$